HW#4
MAE 598

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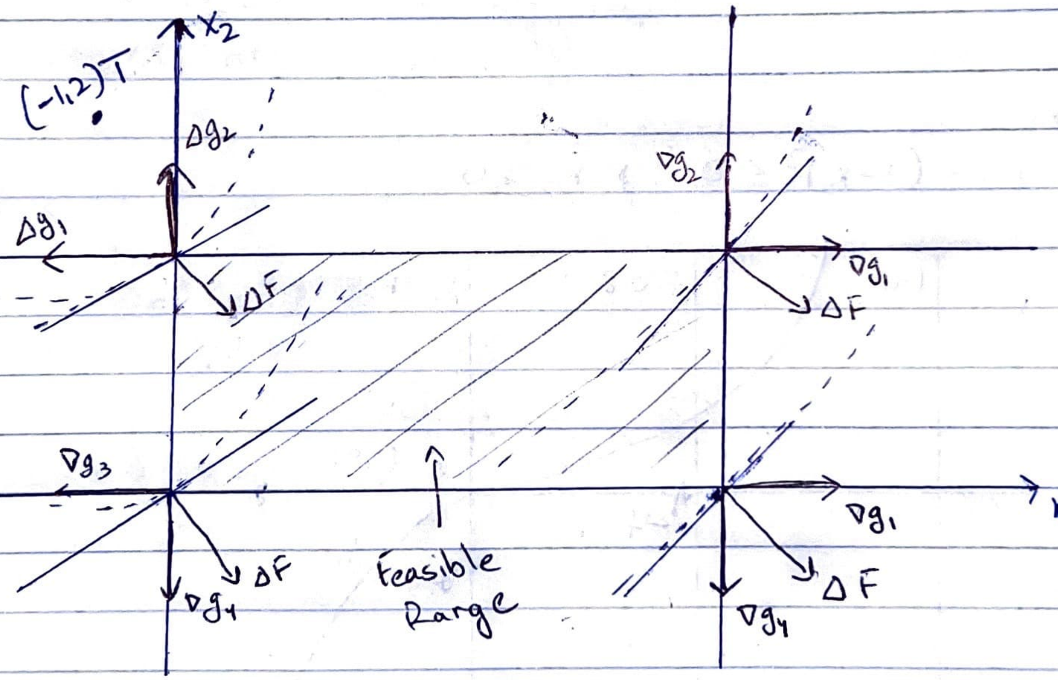
Problem 1

$$\min f(x) = (x_1+1)^2 + (x_2-2)^2$$

s.t. $g_1 = x_1 - 2 \le 0$ $\quad g_2 = x_2 - 1 \le 0$

$g_3 = -x_1 \le 0$ $\quad g_4 = -x_2 \le 0$



At (0,0); ~~or~~ (2,0) ~~&~~ (2,1) feasible direction exists but for (0,1) there is no feasible direction.
∴ The optimum soln from the graph ~~is~~ at (0,1).

now applying KKT.
since point (0,1) is on g_2 & g_3 ∴ $\mu_1 = \mu_4 = 0$ + $\mu_2 = \mu_3 > 0$
∴ $\Delta F - \mu^T \nabla g = 0^T$

$$\left.\begin{pmatrix} 2(x_1+1) \\ 2(x_2-2) \end{pmatrix} + \begin{pmatrix} -\mu_3 \\ \mu_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}\right|_{(x_1,x_2)=(0,1)}$$

$$= \begin{pmatrix} 2 - \mu_3 \\ -2 + \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \mu_2 = 2$ & $\mu_3 = 2$

$\therefore$ KKT conditions are statisfied at $(0,1)$.

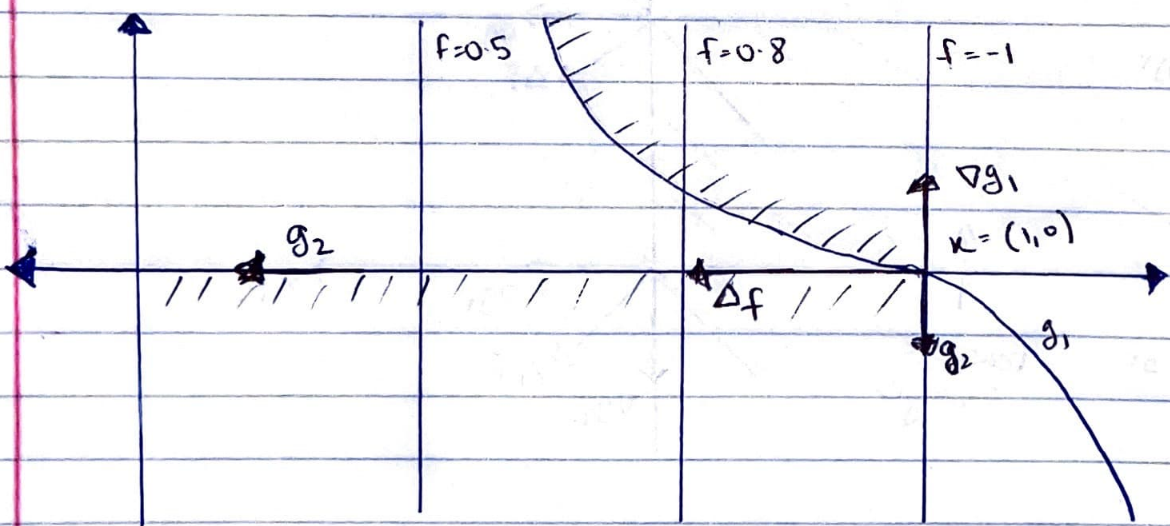
The hessian of the lagrangian is +ve definite everywhere.

Thus $(0,1)$ is the global minimum.

Problem 2

$$\min f = -x_1$$

$$\text{s.t} \quad g_1 = x_2 - (1-x_1)^3 \leq 0 \quad \& \quad x_2 \geq 0$$



$(1,0)$ is the optimal solution from the graph.

now applying KKT

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 3(1-x_1^2) \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\mu_1 + (x_2 - (1-x_1)^3) = 0 \quad \therefore \mu_1 \geq 0$

$-\mu_2 x_2 = 0 \quad \therefore \mu_2 \geq 0$

at $(1,0)$

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-1=0$ not possible

$\mu_1 = \mu_2$

No sol^n exists at $(1,0)$ because it is not a regular point as the given constraints are linearly dependent at $(1,0)$.

Problem 3

max $f = x_1x_2 + x_2x_3 + x_1x_3$

s.t $h = x_1 + x_2 + x_3 - 3 = 0$

using reduced gradient

let $d_1 = x_1$, $d_2 = x_2$ and $s_1 = x_3 = 6 - x_1 - x_2$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} + \frac{\partial f}{\partial s}\frac{ds}{dd} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0$$

at optimum sol^n : $\dfrac{\partial f}{\partial d} = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \end{bmatrix}$

$\dfrac{\partial f}{\partial s} = x_1 + x_2$ $\quad \left(\dfrac{\partial h}{\partial s}\right)^{-1} = 1$ $\quad \dfrac{\partial h}{\partial d} = 11$

hence $\dfrac{\partial f}{\partial d} = 0$

$\therefore$ $x_1 = x_2 = x_3 = 1$ & hessian is +ve definite

$(1,1,1)$ is the optimal solution.

checking 2nd order

$$\frac{d^2 f}{dd^2} = \left(I, \frac{ds}{dd}^T\right) \begin{pmatrix} \frac{\partial^2 f}{\partial d^2} & \frac{\partial^2 f}{\partial s \partial d} \\ \frac{\partial^2 f}{\partial s \partial d} & \frac{\partial^2 f}{\partial s^2} \end{pmatrix} \begin{pmatrix} I \\ \frac{ds}{dd} \end{pmatrix} + \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial^2 s}{\partial d^2}\right)$$

$$\frac{ds}{dd} = -\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right) = -1 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial d^2} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial s \partial d} = \begin{bmatrix} -1 & -1 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial d \partial s} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial s^2} = [0]$$

$$\frac{\partial^2 s}{\partial d^2} = [0]$$

$$\frac{d^2 f}{dd^2} = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + 0$$

$$= 2 > 0$$

Problem 4

$$\min \quad f = x_1^2 + x_2^2 + x_3^2$$

$$\text{s.t} \quad h_1 = x_1^2/4 + x_2^2/5 + x_3^2/25 - 1 = 0$$

$$h_2 = x_1 + x_2 - x_3 = 0$$

let $d = x_1$

$s = [x_2, x_3] - [0, 0]$

$$\frac{\partial f}{\partial d} = 2x_1 \qquad \frac{\partial h}{\partial s} = \begin{bmatrix} 2x_2/5 & 2x_3/25 \\ 1 & -1 \end{bmatrix}$$

$$\frac{\partial f}{\partial s} = \begin{bmatrix} 2x_2 \\ 2x_3 \end{bmatrix} \qquad \frac{\partial h}{\partial d} = \begin{bmatrix} x_1/2 \\ 1 \end{bmatrix}$$

Problem 4

In [1]:
```
#Importing Libraries
import numpy as np
import math
```

In [2]:
```
#Defining the Functions
obj = lambda x: x[0]**2 + x[1]**2 + x[2]**2

#Calculating the gradients
Phpd = lambda x: np.array([[x[0]/2.0], [1.0]])
Pfps = lambda x: np.array([2.0*x[1], 2.0*x[2]])
Pfpd = lambda x: 2.0*x[0]
Phps = lambda x: np.array([[2.0*x[1]/5.0, 2*x[2]/25.0],[1.0,-1.0]])
Dfdd = lambda x: Pfpd(x) - np.matmul(np.matmul(Pfps(x),\
                        np.linalg.inv(Phps(x))), Phpd(x))
```

In [3]:
```
#Line Search
def xevl(x,a,dfdd):
    d_evl= (x[0]-a*dfdd)[0]
    x1 = np.linalg.inv(Phps(x))
    x2 = np.matmul(x1,Phpd(x))
    y = np.transpose([Dfdd(x)])
    x3 = np.matmul(x2,y)
    s_evl= x[1:3] + a* np.transpose(x3)[0]
    return np.append(d_evl,s_evl)

def linesearch(dfdd, x):
    a=1
    b=0.5
    t=0.2
    while obj(xevl(x,a,dfdd)) > (obj(x) - a*t* dfdd**2):
        a=b*a
    return a
```

In [4]:
```
#Solution
def solve(x):
    while np.linalg.norm(np.array([ [ x[0]**2/4 + x[1]**2/5 + x[2]**2/25 -1 ]\
                                , [x[0]+x[1]-x[2] ] ])) > e:
        phps=Phps(x)
        s1= np.transpose( np.transpose([x[1:3]]) - \
                           np.matmul( np.linalg.inv(phps),\
             np.array([ [ x[0]**2/4 + x[1]**2/5 + x[2]**2/25 -1 ],\
                         [x[0]+x[1]-x[2] ] ])))
        x=np.append(x[0:1], np.transpose(s1[0]))
    return x

x1=0
x3= 1/12 * ( (600-170*(x1**2))**(1/2) +10*x1)
x2= x3-x1
x0=np.array([x1, x2, x3])
e=10**(-3)
x_str=[x0]
err=[]


while np.linalg.norm(Dfdd(x_str[-1])) > e:
    x=x_str[-1]
    dfdd=Dfdd(x)
    err.append(math.log(np.linalg.norm(dfdd)))
    a= linesearch(dfdd, x)
    dk= x[0]- a*dfdd
    s0= x[1:3] + a* np.transpose(  np.matmul(np.matmul(np.linalg.inv(Phps(x)),\
                                         Phpd(x)), np.transpose(dfdd)) )
    k0=np.append(dk,s0)
    x = solve(k0)
    x_str.append(x)
```

In [5]:
```
#Answer
print('The local soln exists at ' +str(x_str[-1]))
```

```
The local soln exists at [-1.5739877   1.37736099 -0.19662671]
```

Problem 5

Let

$S = \{0, 1, \ldots, N\}$ be the set of N sites where pt. 0 is the start & return.

C_{ij} = cost of moving i, j
considering no paths taken with $c_{ij} = \infty$

Let

$x_{ij} = 1$ if $\exists$ a path $i \to j$ otherwise
$x_{ij} = 0$ (no path)

$\therefore$ we get

$$\min \sum_{i=1}^{N} \sum_{j \neq 1, j=1}^{N} C_{ij} x_{ij}$$

s.t $\quad c_{ij} \neq \infty$ at any point.

$$\sum_{i=1, i \neq j} x_{ij} = 1$$