MAE-598
HW-5

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Problem 2

Consider a moon lander with state $[h, v, m]^T$ with dynamics

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \alpha(t)/m(t) \\ \dot{m}(t) = -K\alpha(t) \end{cases}$$

h- altitude
v- velocity
m- mass of moon lander

$\alpha(t) \in [0,1]$ is the thrust
K is a constant fuel burning rate.

Ans $\min\limits_{\alpha(t)} \quad p(\alpha) = \int_0^T \alpha(t)\,dt$ fixed end state i.e $h(T)=0 \quad v(T)=0$

$$\frac{\partial h}{\partial t} = v \qquad \frac{\partial v}{\partial t} = -g + \frac{\alpha}{m} \qquad \frac{\partial m}{\partial t} = -K\alpha$$

let $H = -l + \lambda^T f$

$$= -\alpha + \lambda_1 v + \lambda_2\left(-g + \frac{\alpha}{m}\right) + \lambda_3(-K\alpha)$$

$$\therefore \frac{d\lambda_1}{dt} = \frac{-\partial H}{\partial h} = 0 \quad - (1)$$

$$\frac{d\lambda_2}{dt} = \frac{-\partial H}{\partial v} = -\lambda_1 \quad - (2)$$

$$\frac{d\lambda_3}{dt} = \frac{-\partial H}{\partial m} = \frac{\alpha\lambda_2}{m^2} \quad - (3)$$

$$\alpha^* = \underset{\alpha[0,1]}{\operatorname{argmax}}\, H \qquad \cancel{= \operatorname{argmax}\left(-1 + \frac{\lambda_2}{m} - \lambda_3 K\right) + \lambda_1 g}$$

$$= \operatorname{argmax}\Big(\underbrace{-1 + \frac{\lambda_2}{m} - \lambda_3 K}_{b} + \lambda_1 v - \lambda_2 g\Big)$$

$\therefore \alpha^* = \begin{cases} 0 & b \le 0 \\ 1 & b > 0 \end{cases}$

$\alpha^* = 0 \quad t \in [0, t^*]$

$\alpha = 1 \quad t \in [t^*, T]$

from ①,②,③ we have

$\frac{d\lambda_1}{dt} = 0 \quad \lambda = a_1$ (constant

$\left[a_n \to \text{constant} \right]$

$\frac{d\lambda_2}{dt} = -C_1 \quad \therefore \lambda_2 = -a_1 t + a_2$

$\frac{d\lambda_3}{dt} = \frac{\alpha}{m^2} (-a_1 t + a_2)$

b/w $t \in [0, t^*]$

$\begin{bmatrix} \dot{h} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} v \\ -g \\ 0 \end{bmatrix}$ $\therefore$ we have

$v = -gt + a_4$ at $t = 0$

$m = m_0 \quad \therefore a_4 = v_0$

$v = v_0 - gt$

$\frac{dh}{dt} = -gt + v_0$

$h = \frac{-gt^2}{2} + v_0 t + a_6 \quad \therefore h_0 = a_6$

$h(0) /=/ \therefore h = h_0 - \frac{gt^2}{2} + v_0 t$

b/w $t \in [t^*, t]$

$\begin{bmatrix} \dot{h} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} v \\ -g + 1/m \\ -k \end{bmatrix}$ $\therefore dm/dt = -k$

$\therefore m = -kt + a_7$

at $t = t^*$ $m = m_0$ $\therefore m = m_0 - k(t - t^*)$

&

$$\frac{dv}{dt} = -g + \frac{1}{m} = -g + \frac{1}{m_0 - k(t - t^*)}$$

$\therefore v = -g(t - t)$

$\therefore$ As we know $b = -1 + \frac{\lambda^2}{m} - \lambda_3 k$ $\cancel{+\lambda_1 v - \lambda_2 g}$

$$\therefore \frac{db}{dt} = \frac{1}{m}\frac{d\lambda_2}{dt} - \frac{1}{m^2}\frac{dm}{dt}\lambda_2 - k\frac{d\lambda_3}{dt}$$

$$= \frac{1}{m}(-a_1) + \frac{k\alpha}{m^2}\lambda_2 - \frac{k\alpha\lambda_2}{m^2}$$

$$\therefore \frac{db}{dt} = -\frac{a_1}{m}$$

$$\Rightarrow b = -\frac{a_1}{m}t + a_3$$

4, $m \geq 0$

$\therefore \frac{db}{dt} \leq 0$ $\therefore$ monotonicaly deercasing.