Q1. For a function $f(x)$, its Fourier transform can be defined as

$$g(y) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xy} dx.$$

Use FFT, calculate (approximately) the Fourier transform of

$$f(x) = [\cos(6\pi x) + \sin(7\pi x)] e^{-\pi x^2}.$$

Suggestions: Truncate x to $[-L, L)$ for $L = 20$ with $N = 2m = 4096$ points, truncate y to $[-m/(2L), (m-1)/(2L)]$ with also $2m$ points, plot $f(x)$ on $[-2, 2]$ and plot real and imaginary parts of g on $[-7, 7]$.

Hints: We need to approximate the integral relation and write it to something like Eq. (5) of lecture note. Define $x_j = jL/m$ for $j = -m, -m+1, ..., 0, 1, ..., m-1$, define $y_k = k/(2L)$ for $k = -m, -m+1, ..., m-1$. Notice that $x_j y_k = jk/(2m) = jk/N$. We can approximate $g(y)$ as

$$g(y_k) = \frac{L}{m} \sum_{j=-m}^{m-1} f(x_j) e^{-i2\pi x_j y_k} = \frac{L}{m} \sum_{j=-m}^{m-1} f(x_j) e^{-i2\pi jk/N}, \quad k = -m, -m+1, ..., m-1.$$

The above is somewhat similar to Eq. (5) of the lecture note, but the ranges for j and k are different. For DFT, we should have indices from 0 to $N-1$. Now, suppose j is negative, we can add N to it and define $j+N$ as the new j. Notice that since $e^{-i2\pi k} = 1$ for any integer k, we have $e^{-i2\pi jk/N} = e^{-i2\pi (j+N)k/N}$. Similarly, for a negative k, we can replace it by $k+N$. This implies that if we can define

$$\tilde{\mathbf{f}} = \begin{bmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_{N-1} \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_{-m}) \\ \vdots \\ f(x_{-2}) \\ f(x_{-1}) \end{bmatrix}, \quad \tilde{\mathbf{g}} = \begin{bmatrix} \tilde{g}_0 \\ \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{N-1} \end{bmatrix} = \begin{bmatrix} g(y_0) \\ g(y_1) \\ \vdots \\ g(y_{m-1}) \\ g(y_{-m}) \\ \vdots \\ g(y_{-2}) \\ g(y_{-1}) \end{bmatrix}$$

then

$$\tilde{g}_k = \frac{L}{m} \sum_{j=0}^{N-1} \tilde{f}_j e^{-i2\pi jk/N}, \quad k = 0, 1, ..., N-1.$$