Physics Homework 10

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9.8. Associated B Field

$$\mathbf{E} = E_0(\hat{x} + \hat{y}) sin \frac{2\pi}{\lambda}(z + ct) \tag{1}$$

Using Maxwell's equations in the absence of current

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu_0 \epsilon_0} \nabla \times \mathbf{B} \tag{3}$$

This gives us that

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \begin{pmatrix} E_0 sin \frac{2\pi}{\lambda}(z + ct) \\ E_0 sin \frac{2\pi}{\lambda}(z + ct) \\ 0 \end{pmatrix} = \begin{pmatrix} E_0 \frac{2\pi}{\lambda} sin \frac{2\pi}{\lambda}(z + ct) \\ -E_0 \frac{2\pi}{\lambda} sin \frac{2\pi}{\lambda}(z + ct) \\ 0 \end{pmatrix} \tag{4}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \begin{pmatrix} E_0 sin \frac{2\pi}{\lambda}(z + ct) \\ E_0 sin \frac{2\pi}{\lambda}(z + ct) \\ 0 \end{pmatrix} = \begin{pmatrix} E_0 sin \frac{2\pi}{\lambda}(z + ct) \\ E_0 sin \frac{2\pi}{\lambda}(z + ct) \\ 0 \end{pmatrix} \tag{5}$$

Using the first of these relations we know that $\mathbf{B}$ takes the direction $\hat{x} - \hat{y}$. We also know that each component of $\mathbf{B}$ should be of the form $A \cdot sin \frac{2\pi}{\lambda}(z + ct) + \phi$ in order to satisfy the wave equation:

$$c^2 \nabla^2 \mathbf{B} = \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{6}$$

Computing the time partial:

$$\frac{\partial}{\partial t}(A \cdot sin \frac{2\pi}{\lambda}(z + ct) + \phi) = Ac \cdot cos \frac{2\pi}{\lambda}(z + ct) + \phi \tag{7}$$

And setting this equal to $E_0 \frac{2\pi}{\lambda} sin \frac{2\pi}{\lambda}(z + ct)$, we see that

$$A = E_0 \frac{2\pi}{\lambda c} \tag{8}$$

$$\phi = \frac{\pi}{2} \tag{9}$$

And so the associated **B** field wave is:

$$E_0 \frac{2\pi}{\lambda c}(\hat{x} - \hat{y}) cos \frac{2\pi}{\lambda}(z + ct) \tag{10}$$

9.12 An electromagnetic wave

Given:

$$E_x = 0, \qquad E_y = E_0 sin(kx + \omega t), \qquad E_z = 0 \tag{11}$$

$$B_x = 0, \qquad B_y = 0, \qquad B_z = -\frac{E_0}{c} sin(kx + \omega t) \tag{12}$$

These equations must satisfy

$$c^2 \nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{13}$$

$$c^2 \nabla^2 \mathbf{B} = \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{14}$$

Computing the spatial and temporal second derivatives of both fields we find that:

$$\nabla^2 \mathbf{E} = \begin{pmatrix} 0 \\ E_0 k^2 sin(kx + \omega t) \\ 0 \end{pmatrix} \tag{15}$$

$$\nabla^2 \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0}{c} k^2 sin(kx + \omega t) \end{pmatrix} \tag{16}$$

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \begin{pmatrix} 0 \\ E_0 w^2 sin(kx + \omega t) \\ 0 \end{pmatrix} \tag{17}$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0}{c} w^2 sin(kx + \omega t) \end{pmatrix} \tag{18}$$

We can more or less read off at this point that

$$k^2 c^2 = w^2 \tag{19}$$

$$k = \frac{\omega}{c} \tag{20}$$

Given the specifics that $\omega = 10^{10}s^-1$ and $E_0 = 1\frac{kV}{m}$, it is immediate that $\lambda = .19meters$. One can calculate the power density as follows:

$$\Delta E = (c\Delta t)(A)\frac{dE}{dV} \tag{21}$$

$$P = \frac{\Delta E}{\Delta t} = (A)(c)\frac{dE}{dV} \tag{22}$$

$$\frac{dP}{dA} = \frac{dE}{dV} \cdot c \tag{23}$$

We can calculate the energy density:

$$\frac{dE}{dV} = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B} \tag{24}$$

$$c^2 = \frac{1}{\epsilon_0\mu_0} \tag{25}$$

$$\frac{dE}{dV} = \frac{\epsilon_0 E_0^2}{2}(2sin^2(kx + wt)) \tag{26}$$

$$\frac{d\bar{E}}{dv} = \frac{\epsilon_0 E_0^2}{2} \tag{27}$$

So, the average power density is

$$\frac{d\bar{P}}{dA} = \frac{\epsilon_0 c E_0^2}{2} = 1327\frac{j}{m^2 s} \tag{28}$$

9.15 Field in a box

Given:

$$\mathbf{E} = E_0\hat{z}cos(kx)cos(ky)cos(\omega t) \tag{29}$$

$$\mathbf{B} = B_0(\hat{x}cos(kx)sin(ky) - \hat{y}sin(kx)cos(ky))sin(\omega t) \tag{30}$$

These equations must satisfy:

$$c^2\nabla^2\mathbf{E} = \frac{\partial^2\mathbf{E}}{\partial t^2} \tag{31}$$

$$c^2\nabla^2\mathbf{B} = \frac{\partial^2\mathbf{B}}{\partial t^2} \tag{32}$$

Computing spatial and temporal second derivatives of the fields we find that

$$\nabla^2 \mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ -2E_0 k^2 (cos(kx)cos(ky))cos(\omega t) \end{pmatrix} \tag{33}$$

$$\nabla^2 \mathbf{B} = \begin{pmatrix} 0 \\ -2B_0 k^2 (cos(kx)sin(ky))sin(\omega t) \\ 2B_0 k^2 (sin(kx)cos(ky))sin(\omega t) \end{pmatrix} \tag{34}$$

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \begin{pmatrix} 0 \\ 0 \\ -E_0 \omega^2 cos(kx)cos(ky)cos(\omega t) \end{pmatrix} \tag{35}$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \begin{pmatrix} 0 \\ -B_0 w^2 (cos(kx)sin(ky))sin(\omega t) \\ B_0 w^2 (sin(kx)cos(ky))sin(\omega t) \end{pmatrix} \tag{36}$$

From these we can read off that $\omega = \sqrt{2}ck$, but what about the relative amplitudes of the fields? These can be computed using

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{37}$$

$$\tag{38}$$

Computing both sides gives $E_0 = \sqrt{2}cB_0$

The field existing in such a box would look like stacked layers (in the $\hat{z}$ direction) of a planar standing wave. In each layer, The electric field would oscillate in the z direction (doing so with the highest amplitude at the center of the box), and the electric field would oscillate at the radial direction with greatest amplitude in a circle circumscribed in a box with half the edge lengths of the original box. The magnetic field would not oscillate at all at the center of the box or at its edges.

10.5 Leyden Jar

Approximating the setup as a spherical capacitor, we use plates that have the same area as the surface area of the jar and a separation that is the thickness of the glass. Given:

$$\kappa_{glass} = 4 \tag{39}$$

$$V_{bottle} = .001m^3 \tag{40}$$

$$d = .002m \tag{41}$$

Assuming the bottle was roughly spherical,

$$r_{inner} = (\frac{3V_{bottle}}{4\pi})^{\frac{1}{3}} = .062m \tag{42}$$

$$r_{outer} = r_{inner} + d = .064 \tag{43}$$

$$C = \frac{4\pi\kappa\epsilon_0}{\frac{1}{r_{inner}} - \frac{1}{r_{outer}}} = .883 \quad picoFarads \tag{44}$$

An equivalent way to write the capacitance of the sphere is

$$\frac{4\pi\epsilon_0}{\frac{1}{r} + \frac{1}{r+d}} = \frac{4\pi\epsilon_0 \cdot r(r+d)}{d} = \frac{4\pi\epsilon_0 \cdot r^2(1+\frac{d}{r})}{d} \tag{45}$$

$$\text{Assuming } r >> d \tag{46}$$

$$C \approx \frac{4\pi\epsilon_0 \cdot r^2}{d} \tag{47}$$

$$\tag{48}$$

Assuming the separation stays the same, then an equivalent spherical capacitor in air would have diameter

$$D = 2\sqrt{\frac{C_{jar}d}{4\pi\epsilon_0}} = .252 \quad meters \tag{49}$$

10.7 Hydrogen Chloride Dipole Moment

If the electron of the hydrogen atom were to migrate all the way to the chlorine atom then the dipole moment of the molecule would be, without loss of generality,

$$\mathbf{p} = \mathbf{d}q = 20.48 \cdot 10^{-30}\hat{z} \quad C \cdot m \tag{50}$$

This is ~ 6 times the actual dipole moment of the molecule. Because charge is quantized, we must therefore conclude that the electron from the hydrogen is only displaced one sixth of the distance to the chlorine atom. So, the 'center of negative' charge would be not at the location of the chlorine nucleus but at

$$\frac{(1.28 \quad angstroms)(\frac{e}{6}) + 0}{18e} = .012 \qquad angstroms \tag{51}$$

This is the displacement of the charge center from the chlorine nucleus.

10.22 Force from Induced Dipole

We start with the field of the point charge. Without loss of generality,

$$\mathbf{E} = \frac{kQ}{z^2} \tag{52}$$

Now the field will induce a dipole with charge $\pm q$ and separation d, dipole moment p. The moment will be proportional to the field from the point charge.

$$p = \alpha|\mathbf{E}| = \frac{\alpha k Q}{z^2} \tag{53}$$

$$\phi_{dipole} = \frac{kp}{z^2} = \frac{\alpha k^2 Q}{z^4} \tag{54}$$

$$\mathbf{E}_{dipole} = \nabla\phi_{dipole} = -\frac{4\alpha k^2 Q}{z^5}\hat{z} \tag{55}$$

$$\mathbf{F}_{induced} = Q\mathbf{E}_{dipole} = -\frac{4\alpha k^2 Q^2}{z^5}\hat{z} \tag{56}$$

I looked up a reasonable value for α for the sodium atom (also known as the polarizability), and found it to be $24.11 * 10^{-24}$. Thus,

$$U = \int_0^z \mathbf{F}_{induced} dz = k_B T_{room} \tag{57}$$

$$\frac{\alpha k^2 e^2}{z^4} = k_B T_{room} \tag{58}$$

$$z = (\frac{\alpha k^2 e^2}{k_B T_{room}})^{\frac{1}{4}} = 1.34 \quad angstroms \tag{59}$$