Signals and Systems Quick Reference
Revision 0.3

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Contents

1 Notation and Definitions

1.1 Notation

1.1.1 Logic

$:=$ means equal by definition

$\Rightarrow\Leftarrow$ means contradiction

1.1.2 Set Definitions

Topology and Basic Sets

∂U is the boundary of U

$\overline{U}$ is the closure of U. i.e. a set plus its boundary.

$\mathring{U}$ is the interior of U i.e. a set plus minus its boundary.

$|U|$ is the size or cardinality of set U

$\mathbb{N} := \bigcup_{k=1}^{\infty}\{k\}$ is the set of natural numbers

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is the set of whole numbers

$\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$

$|\mathbb{N}| = \aleph_0$

Subsets of $\mathbb{Z}$ and $\mathbb{N}_0$

Given $k, a \in \mathbb{Z}$, $k\mathbb{Z} + a := \{kn + a : n \in \mathbb{Z}\} = \{a, -k + a, k + a, 2k + a, -2k + a, \ldots\}$

Given $k, a \in \mathbb{N}_0$, $k\mathbb{N}_0 + a := \{kn + a : n \in \mathbb{N}_0\} = \{a, k + a, 2k + a, 3k + a, \ldots\}$

I extend interval notation to the integers. Instead of a comma, two dots are used between the upper and lower bound when it's set of integers. For example: $[a \,..\, b) := [a, b) \cap \mathbb{Z}$.

Other Set definitions

$\mathbb{F}$ is a general field. It can be either $\mathbb{R}$ or $\mathbb{C}$

$\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann Sphere

$\{a_i\}_{i=k}^{n} := \{a_i\}_{i\in[k \,..\, n]}$ is an indexed set. Informally, for indexed sets $\{a_1, a_2\} \neq \{a_2, a_1\}$.

Algebraic Structure

$(V, \mathbb{F})$ is a vector space

$(V, \mathbb{F}, \|\cdot\|)$ is a metric space

$(V, \mathbb{F}, \langle\cdot, \cdot\rangle)$ is an inner product space

$U \leq W$ means U is a vector subspace of W

$U < W$ means U is a proper vector subspace of W

1.1.3 Linear Algebra

$[0]$ is the matrix with all entries equal to 0

$\boldsymbol{A} \sim \boldsymbol{B}$ means $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$

$\text{diag}\{a_i\}_{i=1}^{n} := \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \boldsymbol{I}$

$\bigoplus_{i=1}^{n} \boldsymbol{A}_k = \boldsymbol{A}_1 \oplus \cdots \oplus \boldsymbol{A}_n := \text{diag}\{\boldsymbol{A_1}, \cdots, \boldsymbol{A}_n\}$

Basis Representation

$\mathcal{E}_U := \{\hat{\mathbf{u}}_i\}_{i=1}^{n}$ is the standard basis for U

$\mathcal{B}_U := \{\vec{\boldsymbol{u}}_i\}_{i=1}^{n}$ is a basis for U

$[\mathcal{B}_U]$ is the matrix with columns corresponding to elements of the basis $\mathcal{B}_U$

$\mathcal{E}_n := \{\vec{\boldsymbol{e}}_i\}_{i=1}^{n}$ is the standard basis for $\mathbb{R}^n$

$\vec{\boldsymbol{x}}_{\mathcal{B}_U}$ is the vector $\vec{\boldsymbol{x}} \in U$ wrt basis $\mathcal{B}_U$

$\boldsymbol{P}_{\mathcal{B}_U \to \mathcal{B}_W} := [\mathcal{B}_U]^{-1}[\mathcal{B}_W]$ is the change of basis matrix from basis $\mathcal{B}_U$ to basis $\mathcal{B}_W$

$\mathfrak{L}(U, W)$ is the set of linear operators that maps $(U, \mathbb{F})$ to $(W, \mathbb{F})$

$[\mathcal{T}]_{(\mathcal{B}_U, \mathcal{B}_W)}$ is the matrix representation of linear operator $\mathcal{T}$ with vectors in the domain wrt $\mathcal{B}_U$ and vectors in the codomain wrt $\mathcal{B}_W$

Eigenvalues and Eigenvectors

$\chi_{\boldsymbol{A}}(\lambda) := \det(\boldsymbol{A} - \lambda \boldsymbol{I})$

$\Lambda(\boldsymbol{A})$ is the spectrum (set of all eigenvalues) of $\boldsymbol{A}$.

$\mu_{\boldsymbol{A}}(\lambda)$ is the minimal polynomial for the matrix $\boldsymbol{A}$

$\alpha_{\boldsymbol{A}}(\lambda)$ is the algebraic multiplicity of eigenvalue λ for the matrix $\boldsymbol{A}$

$\gamma_{\boldsymbol{A}}(\lambda)$ is the geometric multiplicity of eigenvalue λ for the matrix $\boldsymbol{A}$

Definiteness

$\boldsymbol{A} \succ 0 \iff \boldsymbol{A}$ is positive definite,

$\boldsymbol{A} \succeq 0 \iff \boldsymbol{A}$ is positive semidefinite

$\boldsymbol{A} \prec 0 \iff \boldsymbol{A}$ is negative definite,

$\boldsymbol{A} \preceq 0 \iff \boldsymbol{A}$ is negative semidefinite

1.1.4 Calculus

Nth Derivative/Antiderivative

$\mathcal{D}^n f(t) := f^{(n)}(t)$

$\mathcal{D}^{-n} f(t) := \underbrace{\int \cdots \int}_{\text{n times}} f(t)\, dt \ldots dt$

Accumulation

$$\mathcal{A}^k f(t) := \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{\text{k times}} f(\tau)\, d\tau \cdots d\tau$$

$$\mathcal{A}^k f[n] := \underbrace{\sum_{n=-\infty}^{n} \cdots \sum_{k=-\infty}^{n}}_{\text{k times}} f[k]$$

Jacobian and Parital Derivatives:

$\partial_{x_1}^n f(x_1, \ldots, x_n) := \dfrac{\partial^n f}{\partial x^n}$

$\partial_{x_1}^{-n} f := \underbrace{\int \cdots \int}_{\text{n times}} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_1$

If mixed partials commute we can write:

$\partial_{x_1, \ldots, x_k}^{(k_1, \ldots, k_n)} f(x_1, \ldots, x_n) := \dfrac{\partial^K f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$, where $K = k_1 + \cdots + k_n$

$$\left[\mathcal{D}\vec{\boldsymbol{f}}(x_1, \ldots, x_n)\right] = \left[\partial_{x_1}\vec{\boldsymbol{f}} \quad \ldots \quad \partial_{x_n}\vec{\boldsymbol{f}}\right] := \begin{bmatrix} \partial_{x_1} f_1 & \ldots & \partial_{x_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_n & \ldots & \partial_{x_n} f_n \end{bmatrix}$$

1.1.5 Complex Numbers

$(a + bj)^* := a - bj$ is the complex conjugate

$\Re\{a + bj\} := a$

$\Im\{a + bj\} := b$

$\boldsymbol{A}^{\mathsf{H}} := (\boldsymbol{A}^*)^{\mathsf{T}} = \left(\boldsymbol{A}^{\mathsf{T}}\right)^*$

1.1.6 Linear Systems Theory

1.1.7 Digital Signal Processing

Delta Operator

$\Delta x[n] := x[n+1] - x[n]$

$\Delta^k x[n] := \Delta^{k-1} x[n+1] - \Delta^{k-1} x[n]$

Circular Convolution

$$f(t) \circledast g(t) := \int\limits_{\theta_0}^{\theta_0+2\pi} f(\tau)g(t-\tau)\,d\tau,\ \theta_0 \in \mathbb{R}$$

Useful Functions/Equations:

$$\operatorname{sgn}(x) := \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$$

$$\underset{[a,b]}{\operatorname{clamp}}(x) := \min(\max(x,a),b)$$

$$\operatorname{Arg}(\theta) := \theta - 2\pi \left\lfloor \frac{\theta - \theta_0}{2\pi} \right\rfloor \text{ (i.e. } \theta \in \mathbb{R} \text{ maps to } \hat{\theta} \in [\theta_0,\ \theta_0 + 2\pi])$$

1.2 Function Definitions

Name	Definition
(Heaviside) unit step function	$u_H(t) := \begin{cases} 1 \text{ if } x > 0 \\ 0, \text{ otherwise} \end{cases}$
Discrete rectangular function	$\text{rect}[x] := u_H[x] - u_H[x-1]$
Discrete Triangular function	$\text{tri}[n] := (1 - \lvert x \rvert)\text{rect}\left[\frac{1}{2}(x+1)\right]$
Continuous rectangular function	$\text{rect}(x) := u_H\left(x + \frac{1}{2}\right) - u_H\left(x - \frac{1}{2}\right)$
Triangular function	$\text{tri}(x) := (1 - \lvert x \rvert)\text{rect}\left(\frac{1}{2}x\right)$
ReLU/Ramp function	$\text{relu}(x) := u_H(x)x$
(Normalized) sinc	$\text{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$
Dirichlet kernel	$\text{diric}_N(x) := \frac{\sin\left(\left(N + \frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$
(Dirac) comb function	$W_T(x) := \sum_{k=-\infty}^{\infty} \delta(t - kT)$

Note:

Discrete functions can only be scaled by integer factors.

2 Identities

2.1 Trig Identities

Co-Function Identities

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$

Supplement Angle Identities

$\sin(\pi - \theta) = \sin(\theta)$
$\sin(\pi + \theta) = -\sin(\theta)$

$\cos(\pi - \theta) = -\cos(\theta)$
$\cos(\pi + \theta) = -\cos(\theta)$

$\tan(\pi - \theta) = -\tan(\theta)$
$\tan(\pi + \theta) = \tan(\theta)$

Negative Angle Identities

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\tan(-\theta) = -\tan(\theta)$

Additional and Subtraction Identities

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

Product Identities

$\sin(x)\cos(y) = \frac{1}{2}(\sin(x + y) + \sin(x - y))$

$\cos(x)\sin(y) = \frac{1}{2}(\sin(x + y) - \sin(x - y))$

$\cos(x)\cos(y) = \frac{1}{2}(\cos(x + y) - \cos(x - y))$

$\sin(x)\sin(y) = \frac{1}{2}(\cos(x - y) - \cos(x + y))$

Superposition

$A_1\cos(\theta) + A_2\sin(\theta) = A\cos(\theta - \phi)$

$A = \sqrt{A_1^2 + A_2^2}\,,\ \phi = \tan^{-1}\left(\frac{A_2}{A_1}\right)$

Power Reduction Formula

$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$

$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$

Double Angle Identities

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta)
\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

Half Angle Identities

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$

Sum Identities

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\cos(x) - \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

Complex Exponential Identities

$$\sin(z) = \frac{1}{2j}\left(e^{jz} - e^{-jz}\right)$$

$$\cos(z) = \frac{1}{2}\left(e^{jz} + e^{-jz}\right)$$

2.2 Other Identities

Convolutions

$f(x) \star g(x) = f(x) * g(-x)$

$\mathrm{rect}(x) * \mathrm{rect}(x) = \mathrm{tri}(x)$

Dirichlet Kernel

$\mathrm{diric}_N(x) := \dfrac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} = \displaystyle\sum_{k=-N}^{N} e^{jkx} = 1 + 2\sum_{k=1}^{N} \cos(kx)$

$\lim_{N \to \infty} \mathrm{diric}_N(Tx) = W_{\frac{1}{T}}(x)$

Accumulation

$\mathcal{A}\{\Delta x[t]\} = \Delta\{\mathcal{A}x[t]\} = x[t]$

$\mathcal{A}\{\mathcal{D}x(t)\} = \mathcal{D}\{\mathcal{A}x(t)\} = x(t)$

3 Linear Systems Theory

3.1 Continuous Time Fourier Transform

$$X\left(e^{j\omega}\right) = \mathcal{F}\{x(t)\} := \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad x(t) = \mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right)\right\} := \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(e^{j\omega}\right)e^{j\omega t}d\omega$$

3.1.1 CTFT Properties

Property	Time Domain	Frequency Domain
Linearity	$c_1x_1(t) + c_2x_2(t)$	$c_1X_1(e^{j\omega}) + c_2X_2(e^{j\omega})$
Time Shift	$x(t - t_0)$	$X(e^{j\omega})e^{-j\omega t_0}$
Time Reversal	$x(-t)$	$X(e^{-j\omega})$
Time Scale	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(e^{\frac{j\omega}{a}}\right)e^{-\frac{j\omega}{a}}$
Time Scale + Shift	$x(at - t_0)$	$\frac{1}{\lvert a\rvert}X\left(e^{\frac{j\omega}{a}}\right)e^{-\frac{j\omega t_0}{a}}$
Complex Conjugation	$x^*(t)$	$X^*(e^{-j\omega})$
Frequency Shift	$x(t)e^{-j\omega_0}$	$X\left(e^{j(\omega-\omega_0)}\right)$
Convolution	$x_1(t) * x_2(t)$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(e^{j\omega}) * X_2(e^{j\omega})$
Time Differentiation	$\mathcal{D}_t^n\{x(t)\}$	$(j\omega)^nX(e^{j\omega})$
Causal Accumulation	$\mathcal{A}_t\{x(t)u_H(t)\}$	$\frac{1}{j\omega}X(e^{j\omega}) + \pi X(0)\delta(\omega)$
Frequency Differentiation	$t^nx(t)$	$j^n\mathcal{D}_\omega^n\{Xe^{j\omega}\}$
Parseval's Theorem	$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1\left(e^{j\omega}\right)X_2^*\left(e^{j\omega}\right)d\omega$
Duality	$X(e^{j\omega t})$	$2\pi x(\omega)$

3.1.2 CTFT Table

$f(t)$	$F\left(e^{j\omega}\right)$
0	constant ω_0
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
$\sin(\omega_0 t)$	$-j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$
$\cos(\omega_0)u(t)$	$\frac{\pi}{2}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) + \frac{j\omega}{\omega_0^2 - \omega^2}$
$\sin(\omega_0)u(t)$	$\frac{\pi}{2j}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) + \frac{\omega_0}{\omega_0^2 - \omega^2}$
$e^{-at}\cos(\omega_0)u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$
$e^{-at}\sin(\omega_0)u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$
$\text{rect}\left(\frac{t}{T}\right)$, $T > 0$	$T\text{sinc}\left(\frac{T}{2\pi}\omega\right)$
$\text{tri}\left(\frac{t}{T}\right)$, $T > 0$	$T\text{sinc}^2\left(\frac{T}{2\pi}\omega\right)$
$A\text{sinc}\left(\frac{1}{2\pi A}t\right)$	$\text{rect}(A\omega)$
$A\text{sinc}^2\left(\frac{1}{2\pi A}t\right)$	$\text{tri}(A\omega)$
x^n	$2\pi j^n \delta^{(n)}(\omega)$
$e^{-at}u(t)$, $a > 0$	$\frac{1}{a + j\omega}$
$t^n e^{-at}u(t)$, $a > 0$	$\frac{1}{(a + j\omega)^{n+1}}$
$e^{-a\lvert t\rvert}$, $a > 0$	$\frac{2a}{a^2 + \omega^2}$
$e^{-\frac{t^2}{2\sigma^2}}$	$\sqrt{2\pi\sigma^2}e^{-\frac{\sigma^2}{2}\omega^2}$

3.2 Laplace Transform

$$X(s) = \mathcal{Z}\{x(t)\} := \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \qquad x(t) = \mathcal{Z}^{-1}\{X(s)\} := \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

3.2.1 Laplace Transform Properties

Property	$x[n]$	$X(z)$
Linearity	$c_1x_1(t) + c_2x_2(t)$	$c_1X_1(s) + c_2X_2(s)$
Time Shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0}$
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$
Time Transformation	$x(at-t_0)u(t-t_0)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)e^{-\frac{st_0}{a}}$
Frequency Shift	$e^{at}x(t)$	$X(s-a)$
1st Time Derivative	$x'(t)$	$sX(s) - x(0^-)$
2nd Time Derivative	$x''(t)$	$s^2X(s) - sx(0^-) - x'(0^-)$
General Time Derivative (One-Sided)	$\mathcal{D}_t^n\{x(t)u(t)\}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}\mathcal{D}_t^{k-1}\{x\}(0^-)$
General Time Derivative (Two-Sided)	$\mathcal{D}_t^n\{x(t)\}$	$s^nX(s)$
1st Frequency Derivative	$tx(t)$	$-X'(s)$
General Frequency Derivative	$t^nx(t)$	$(-1)^n\mathcal{D}_s^n\{X(s)\}$
Accumulation	$\mathcal{A}_t^n\{x(t)\}$	$s^{-n}X(s)$
Frequency Integration	$\frac{1}{t^n}x(t)$	$\mathcal{I}_s^n\{X(s)\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} X_1(u)X_2(s-u)\,du$
Complex Conjugation	$x^*(t)$	$X^*(s^*)$

Initial Value Theorem:

$x(t)$ is causal $\implies x(0^+) = \lim_{s\to\infty} sX(s)$

Final Value Theorem:

$x(t)$ is causal and stable $\implies \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

3.2.2 Laplace Transform Table

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	$\mathbb{C}$
$\delta(t-t_0)$	e^{-st_0}	$\mathbb{C}$
$u_H(t)$	$\frac{1}{s}$	$\Re\{s\} > 0$
$u_H(t-t_0)$	$\frac{1}{s}e^{-st_0}$	$\Re\{s\} > 0$
$\operatorname{rect}\left(\frac{t}{T}\right)$	$\frac{1-e^{-Ts}}{s}$	$\Re\{s\} > 0$
$\operatorname{relu}(t)$	$\frac{1}{s^2}$	$\Re\{s\} > 0$
$t^n u_H(t)$, $n \in \mathbb{N}_0$	$\frac{n!}{s^{n+1}}$	$\Re\{s\} > 0$
$t^z u_H(t)$, $\Re\{z\} > -1$	$\frac{\Gamma(z+1)}{s^{z+1}}$	$\Re\{s\} > 0$
$t^{\frac{1}{n}} u_H(t)$, $n \in \mathbb{N}_0$	$\frac{\Gamma\left(\frac{1}{n}+1\right)}{s^{\frac{1}{n}+1}}$	$\Re\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\Re\{s\} > -a$
$e^{-a\|t\|}$	$\frac{2a}{(a^2-s^2)}$	$-a < \Re\{s\} < a$
$(1-e^{-at})u(t)$	$\frac{a}{s(s+a)}$	$\Re\{s\} > 0$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\Re\{s\} > 0$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$	$\Re\{s\} > 0$
$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\Re\{s\} > -a$
$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\Re\{s\} > -a$

4 Digital Signal Processing

4.1 Discrete Time Fourier Transform

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} := \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\} := \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

4.1.1 DTFT Properties

Property	Time Domain	Frequency Domain
Linearity	$c_1x_1[n] + c_2x_2[n]$	$c_1X_1 + c_2X_2(e^{j\omega})$
Time Shift	$x[n-k]$	$X(\omega)e^{-j\omega k}$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Scale	$x[an]$	$\frac{1}{\|a\|}X\left(e^{\frac{j\omega}{a}}\right)$
Time Shift + Scale	$x[an-k]$	$\frac{1}{\|a\|}X\left(e^{\frac{j\omega}{a}}\right)e^{-\frac{j\omega k}{a}}$
Complex Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Frequency Shift	$x[n]e^{-j\omega_0}$	$X\left(e^{j(\omega-\omega_0)}\right)$
Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Modulation	$x_1[n]x_2[n]$	$\frac{1}{2\pi}X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$
First Difference	$\Delta x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
Accumlation	$\mathcal{A}_n\{x[n]\}$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$
Frequency Differentiation	$(-jt)^k x[n]$	$\mathcal{D}_\omega^k\{F(e^{j\omega})\}$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1\left(e^{j\omega}\right)X_2^*\left(e^{j\omega}\right)d\omega$
Duality	$X(e^{2\pi n})$	$2\pi x(-\omega)$

4.1.2 DTFT Symmetries

$x[n]$	$X(z)$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\Re\{x[n]\}$	$X_e(e^{j\omega})$
$j\Im\{x[n]\}$	$X_o(e^{j\omega})$
$x_e[n]$	$\Re\{X(e^{j\omega})\}$
$x_o[n]$	$j\Im\{X(e^{j\omega})\}$
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
Any real $x[n]$	$\Re\{X(e^{-j\omega})\} = \Re\{X(e^{j\omega})\}$
Any real $x[n]$	$\Im\{X(e^{-j\omega})\} = -\Im\{X(e^{j\omega})\}$
Any real $x[n]$	$\lvert X(e^{-j\omega})\rvert = \lvert X(e^{j\omega})\rvert$
Any real $x[n]$	$\arg\left(X(e^{-j\omega})\right) = -\arg\left(X(e^{j\omega})\right)$

4.1.3 DTFT Table

$x[n]$	$X(e^{j\omega})$
0	constant ω_0
1	$2\pi W_{2\pi}(\omega)$
$\delta[n]$	1
$u_H[n]$	$\frac{1}{1-e^{-j\omega}} + \pi W_{2\pi}(\omega)$
$a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$
$\mathrm{rect}\left[\frac{t}{N}\right],\ \ N \in \mathbb{Z} \setminus \{0\}$	$\mathrm{diric}_{\frac{N}{2}}(\omega)$
$\mathrm{sinc}[\omega_c n],\ \omega_c \leq \pi$	$\mathrm{rect}\left(\frac{1}{\omega_c}\omega\right)$
$e^{j\omega_0 n}$	$2\pi W_{2\pi}(\omega - \omega_0)$
$\cos[\omega_0 n + \phi]$	$\pi e^{j\phi} W_{2\pi}(\omega - \omega_0) + \pi e^{-j\phi} W_{2\pi}(\omega + \omega_0)$
$\sin[\omega_0 n + \phi]$	$\frac{\pi}{j} e^{j\phi} W_{2\pi}(\omega - \omega_0) - \frac{\pi}{j} e^{-j\phi} W_{2\pi}(\omega + \omega_0)$

4.2 Z Transform

$$X(z) = \mathcal{Z}\{x[n]\} := \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad x[n] = \mathcal{Z}^{-1}\{X(z)\} := \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$$

4.2.1 Z Transform Properties

Property	$x[n]$	$X(z)$
Linearity	$c_1x_1[n] + c_2x_2[n]$	$c_1X_1(z) + c_2X_2(z)$
Time Shifting	$x[n-k]$	$X(z)z^{-k}$
Time Reversal	$x[-n]$	$X(z^{-1})$
Z-Scaling	$z_0^n x[n]$	$X(z_0^{-1}z)$
Backward First Difference	$\Delta x[n-1]$	$(1-z^{-1})X(z)$
Forward First Difference	$\Delta x[n]$	$(z-1)X(z) - zx[0]$
Complex Conjugation	$x^*[n]$	$X^*(z^*)$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$
Z Differentiation	$n^k x[n]$	$(-z)^k\mathcal{D}_z^k\{X(z)\}$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$	$\frac{1}{2\pi j}\oint_C X_1(z)X_2^*\big((z^*)^{-1}\big)z^{-1}\,dz$

Initial Value Theorem:

$x[n]$ is causal $\implies x[0] = \lim_{z\to\infty} X(z)$

Final Value Theorem:

$x[n]$ is causal $\implies \lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

4.2.2 Z Transform Table

$x[n]$	$X(z)$	ROC
$\delta[n-k]$	z^{-k}	$\mathbb{C}$
$u_H[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u_H[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$nu_H[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
$n^2u_H[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| > 1$
$-nu_H[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| < 1$
$-n^2u_H[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| < 1$
$a^nu_H[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$na^nu_H[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$n^2a^nu_H[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| > \|a\|$
$-na^nu_H[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$-n^2a^nu_H[-n-1]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| < \|a\|$
$\binom{n+k-1}{k-1}a^nu[n]$	$\frac{1}{(1-az^{-1})^k}$	$\|z\| > \|a\|$
$(-1)^k\binom{-n-1}{k-1}a^nu[-n-k]$	$\frac{1}{(1-az^{-1})^k}$	$\|z\| < \|a\|$
$\cos(\omega_0 n)u_H[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
$\sin(\omega_0 n)u_H[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
$a^n\cos(\omega_0 n)u_H[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$
$a^n\sin(\omega_0 n)u_H[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$

4.3 Sampling, Compression, and Expansion

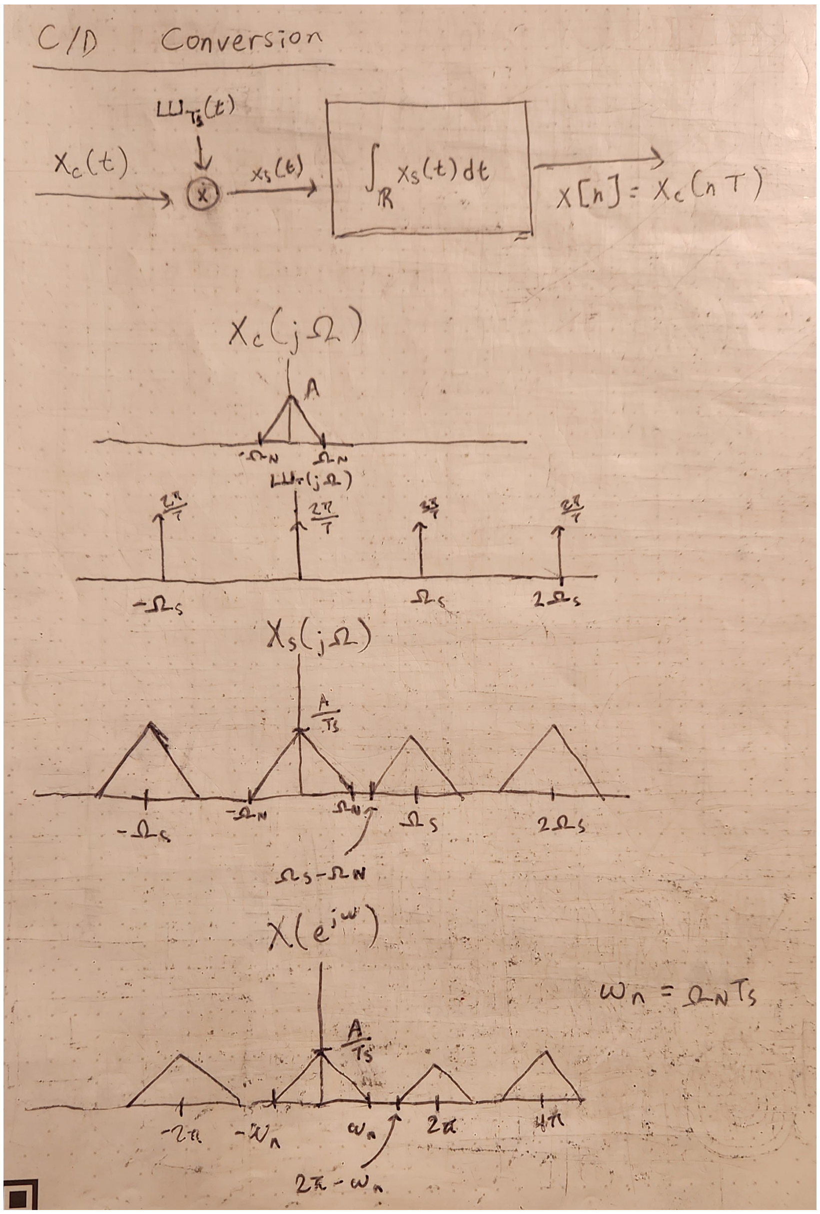
4.3.1 C/D Conversion

Shanon-Nyquist Sampling Theorem:

Suppose X is a bandlimited signal with bandwidth $2\Omega_n$

$\forall \Omega : |\Omega| > \Omega_N$, $X(j\Omega) = 0$.

Perfect reconstruction of the signal requires: $\Omega_s = \dfrac{2\pi}{T_s} > 2\Omega_N$.

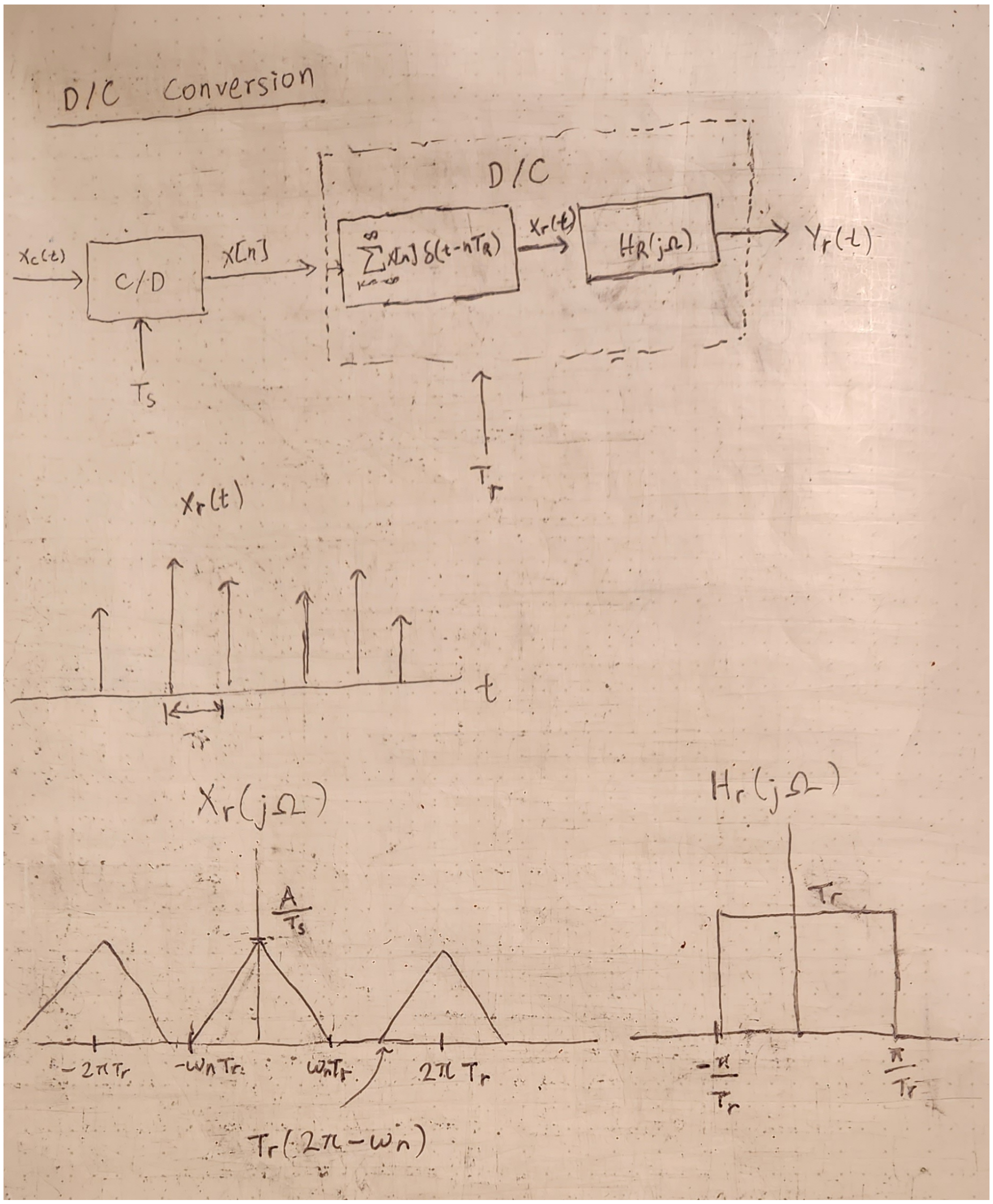


4.3.2 D/C Conversion

Mismatch in Reconstruction and Sampling Period:

Assuming no aliasing effects, if $T_r \neq T_s$

$y_r(t) = \frac{T_r}{T_s} x_c\left(\frac{T_s}{T_r} t\right)$



4.3.3 Compression and Decimation

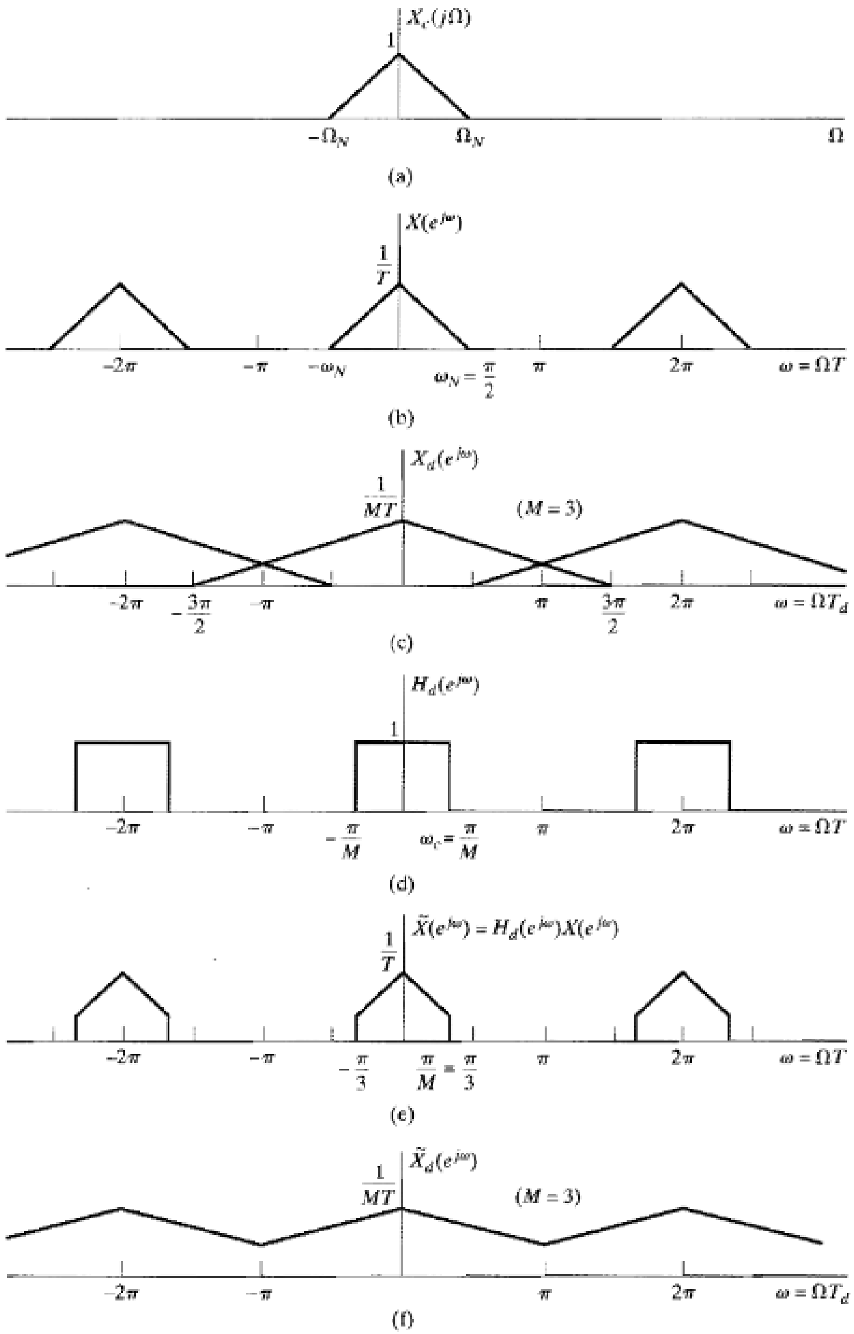


Figure 4.21 (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.

4.3.4 Expansion and Interpolation

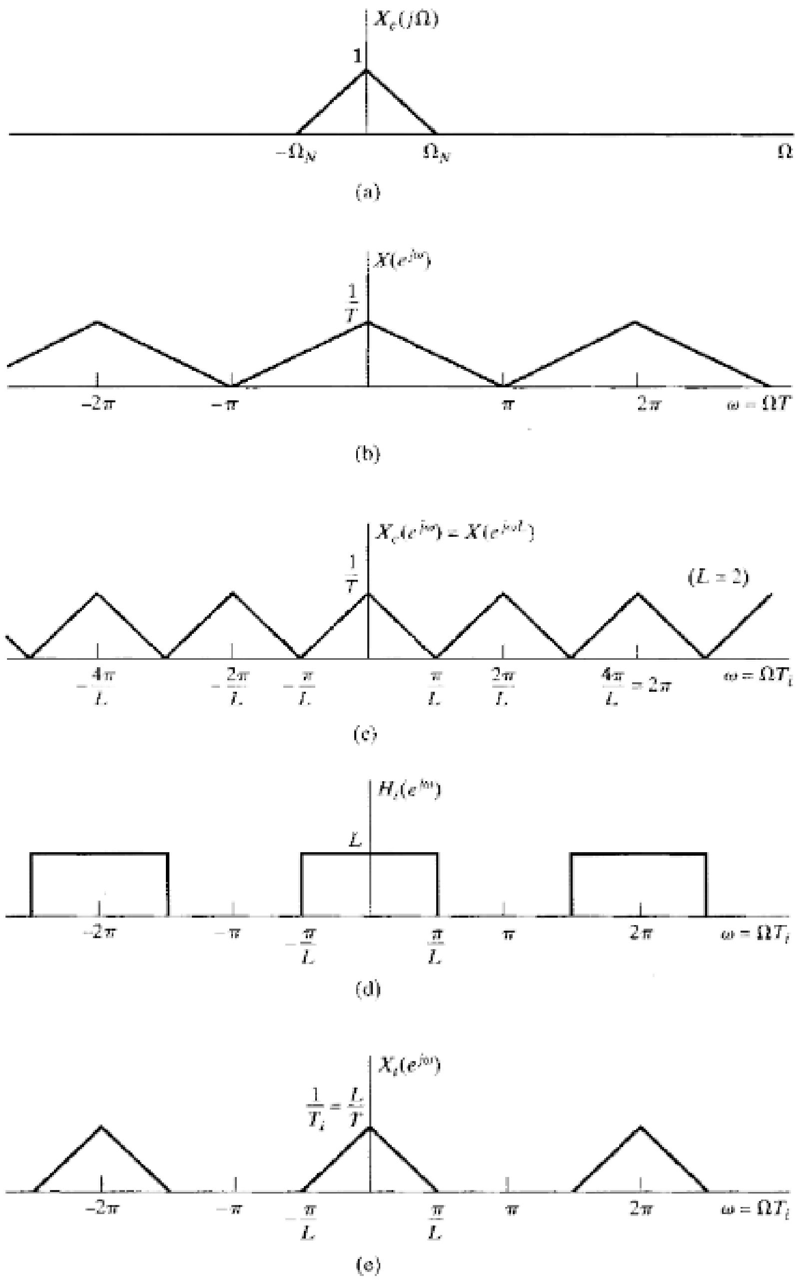


Figure 4.24 Frequency-domain illustration of interpolation.

4.3.5 Change Sampling Rate by Noninteger Factor

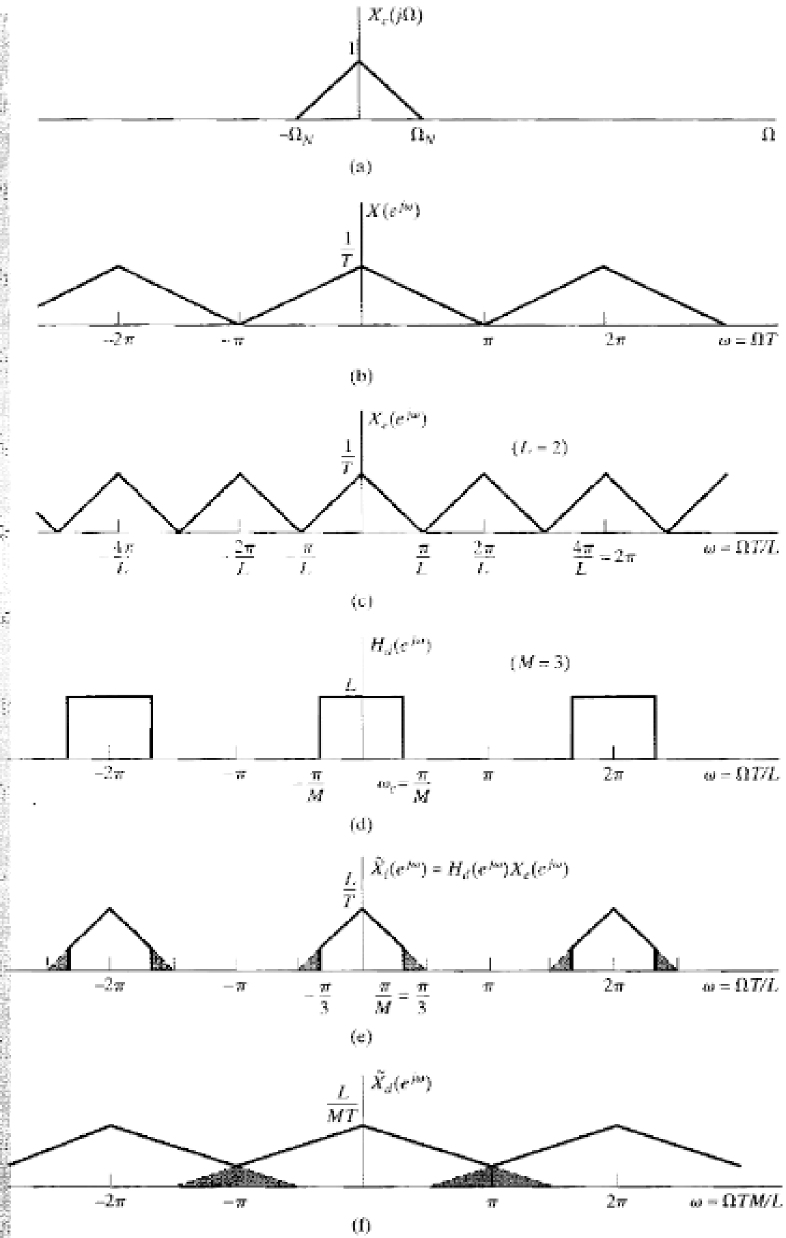


Figure 4.30 Illustration of changing the sampling rate by a noninteger factor.

4.3.6 Polyphase Decompositions

Noble Identities:

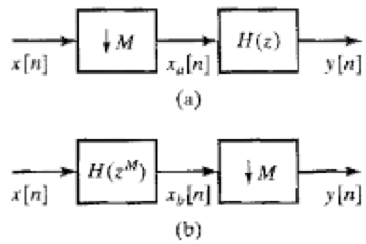


Figure 4.31 Two equivalent systems based on downsampling identities.

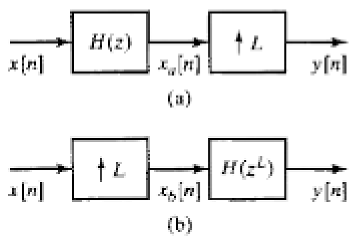


Figure 4.32 Two equivalent systems based on upsampling identities.

Polyphase Decomposition for Compression:

In the below diagram $e_k[n] = h[nM + k]$

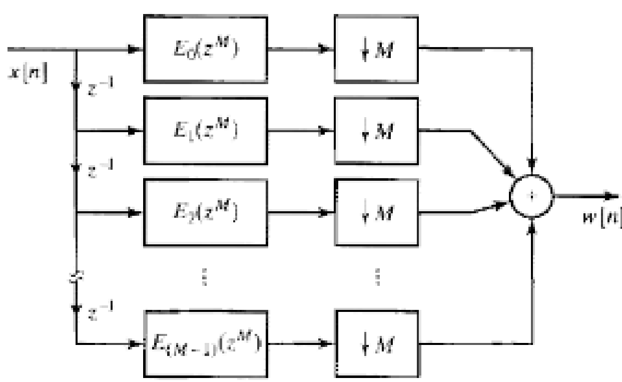


Figure 4.39 Implementation of decimation filter using polyphase decomposition.

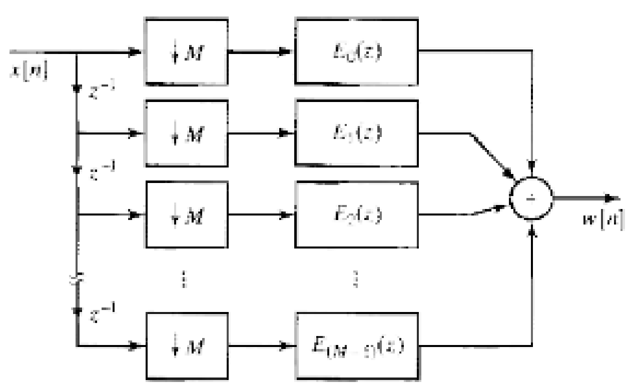


Figure 4.40 Implementation of decimation filter after applying the downsampling identity to the polyphase decomposition.