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Take Home Exam

Reduced Cubic: $f(y) = y^3 + 6y - 20$
$R = \frac{p}{3}^3 + \frac{q}{2}^2$
$R = \frac{6}{3}^3 + \frac{-20}{2}^2$
$R = 2^3 + (-10)^2 = 108$
$\sqrt{R} = \sqrt{108} = 6\sqrt{3}$

a) Cardano's Formula:
$y = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$
$y = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$

Let $A = 10 + 6\sqrt{3}$ and $B = 10 - 6\sqrt{3}$

b) Formula: $(a + b\sqrt{3})^3 = a(a^2 + 9b^2) + 3b(a^2 + b^2)\sqrt{3}$

Observe:
$A = (10 + 6\sqrt{3})^3 = a(a^2 + 9b^2) + 3b(a^2 + b^2)\sqrt{3}$ and
$B = (10 - 6\sqrt{3})^3 = a(a^2 + 9b^2) + 3b(a^2 + b^2)\sqrt{3}$

Using the formula, we can equate coefficients on the left and right side of the equation and form the system of equations:

$$\begin{aligned} 10 &= a_1^3 + 9a_1b_1^2 \\ 6 &= 3a_1^2b_1 + 3b_1^3 \end{aligned}$$

For the coefficients of A, and

$$\begin{aligned} 10 &= a_2^3 + 9a_2b_2^2 \\ -6 &= 3a_2^2b_2 + 3b_2^3 \end{aligned}$$

For B. To solve for a_1, a_2, b_1 and b_2, it is easy to see that if we let $a_1 = b_1 = 1$ and $a_2 = 1$ and $b_2 = -1$ both systems of equalities have been met. Therefore, we may let $A = 1 + \sqrt{3}$ and $B = 1 - \sqrt{3}$ $\square$.

c) Let $A = 1 + \sqrt{3}$ and $B = 1 - \sqrt{3}$.

Then we can verify that $AB = (1+\sqrt{3})(1-\sqrt{3}) = -2 = \frac{-p}{3}$, and so, $r_1 = A+B = 1+\sqrt{3}+1-\sqrt{3} = 2$. Using the formulas: $r_2 = \omega A + \omega^2 B$ and $r_3 = \omega^2 A + \omega B$ with $\omega = (\frac{-1}{2} + \frac{i\sqrt{3}}{2})$ and $\omega^2 = (\frac{-1}{2} - \frac{i\sqrt{3}}{2})$, we find that

$$\begin{gathered} r_2 = (\tfrac{-1}{2} + \tfrac{i\sqrt{3}}{2})(1+\sqrt{3}) + (\tfrac{-1}{2} - \tfrac{i\sqrt{3}}{2})(1-\sqrt{3}) \\ \tfrac{-1-\sqrt{3}}{2} + i\tfrac{3+\sqrt{3}}{2} + \tfrac{-1+\sqrt{3}}{2} + i\tfrac{3-\sqrt{3}}{2} \\ r_2 = -1+3i \\ \text{and} \\ r_3 = (\tfrac{-1}{2} - \tfrac{i\sqrt{3}}{2})(1+\sqrt{3}) + (\tfrac{-1}{2} + \tfrac{i\sqrt{3}}{2})(1-\sqrt{3}) \\ \tfrac{-1-\sqrt{3}}{2} + i\tfrac{-3-\sqrt{3}}{2} + \tfrac{-1+\sqrt{3}}{2} + i\tfrac{-3-\sqrt{3}}{2} \\ r_3 = -1-3i \end{gathered}$$

Therefore, $r_1 = 2$, $r_2 = -1+3i$, and $r_3 = -1-3i$ $\square$

d) Suppose instead we were to long divide $y - r_1 = y - 2$ into the reduced cubic, $f(y) = y^3 + 6y - 20$. We see that

$$y^3 + 6y - 20 = (y-2)(y^2 + 2y + 10)$$

Using the quadratic formula,

$$\begin{gathered} y = \tfrac{-2 \pm \sqrt{2^2 - 4(10)}}{2} \\ \tfrac{-2 \pm \sqrt{-36}}{2} \\ \tfrac{-2 \pm 6i}{2} \\ y = -1+3i \text{ and } y = -1-3i \end{gathered}$$

Using this method we have verified that the three roots of the reduced cubic are $2, -1+3i$, and $-1-3i$ $\square$.

e) Let $g(x) = x^3 + 3x^2 + 9x - 13$. Suppose we want to find the reduced cubic of $g(x)$, $g(y)$. We begin with the substitution $x = y - \frac{3}{3} = y - 1$. Substituting and simplifying, we see that

$$\begin{gathered} (y-1)^3 + 3(y-1)^2 + 9(y-1) - 13 \\ y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 + 9y - 9 - 13 \\ y^3 + 6y - 20 \end{gathered}$$

Therefore, the reduced cubic of $g(x)$ is equal to $f(y)$ which we have already solved the roots for. The roots of the reduced cubic $g(y)$ are $2, -1+3i$ and $-1-3i$. To find the roots of $g(x)$ we use the substitution $x = y - 1$ to find that the roots of $g(x)$ are $2-1, -1+3i-1$, and $-1-3i-1$. Therefore, the roots of $g(x)$ are $1, -2+3i$ and $-2-3i$ $\square$.

Let $h(z) = z^4 + 2z^2 + 5z + 11$ be a reduced quartic.

a) Using the equation $j^3 + 2qj^2 + (q^2 - 4s)j - r^2 = 0$, we find that the cubic resolvant of $h(z)$ is

$$R(j) = j^3 + 2(2)j^2 + (2^2 - 4(11))j - 5^2$$
$$R(j) = j^3 + 4j^2 - 40j - 25$$

Where $j = k^2$.

b) In order to find a root of the cubic resolvant, we can use the rational root test to find a root instead of repeating the same steps as above. According to the rational root test, the possible roots of $R(j)$ are: 25,5, and 1. We can plug the value $j = 5$ into $R(j)$ to see that $R(5) = 125+100-200-25 = 0$. So $j = 5 = k^2$ is a root.

c) Using the formulas provided,

$$m = \frac{q+k^2+\frac{r}{k}}{2}$$
$$m = \frac{2+5+\frac{5}{\sqrt{5}}}{2}$$
$$m = \frac{7+\sqrt{5}}{2}$$

and

$$l = \frac{q+k^2-\frac{r}{k}}{2}$$
$$l = \frac{2+5-\frac{5}{\sqrt{5}}}{2}$$
$$l = \frac{7-\sqrt{5}}{2}$$

The factorization of $h(z)$ is as follows:

$$h(z) = (z^2 + \sqrt{5}z + \tfrac{7-\sqrt{5}}{2})(z^2 - \sqrt{5}z + \tfrac{7+\sqrt{5}}{2})$$

d) Let $k(x) = x^4 + 4x^3 + 8x^2 + 13x + 19$ be a quartic polynomial. Suppose we wanted to find the reduced quartic of $k(x)$, $k(z)$. We will use the substitution $x = z - \frac{4}{4} = z - 1$. Substituting and simplifying:

$$k(z) = (z-1)^4 + 4(z-1)^3 + 8(z-1)^2 + 13(z-1) + 19$$
$$z^4 - 4z^3 + 6z^2 - 4z + 1 + 4z^3 - 12z^2 + 12z - 4 + 8z^2 - 16z + 8 + 13z - 13 + 19$$
$$k(z) = z^4 + 2z^2 + 5z + 11$$

Therefore, we see that the reduced quartic of $k(x)$ is equal to $h(z)$, which we have already found the factorization for. To find the factorization of $k(x)$, we use the reverse substitution $z = x + 1$ and plug into the factorization of $h(z)$ to find that

$$\begin{aligned} k(x) &= ((x+1)^2 + \sqrt{5}(x+1) + \tfrac{7-\sqrt{5}}{2})((x+1)^2 - \sqrt{5}(x+1) + \tfrac{7=\sqrt{5}}{2}) \\ &(x^2 + 2x + 1 + \sqrt{5}x + \sqrt{5} + \tfrac{7-\sqrt{5}}{2})(x^2 + 2x + 1 - \sqrt{5}x - \sqrt{5} + \tfrac{7+\sqrt{5}}{2}) \\ k(x) &= (x^2 + (2+\sqrt{5})x + \tfrac{9+\sqrt{5}}{2})(x^2 + (2-\sqrt{5})x + \tfrac{9-\sqrt{5}}{2}) \ \square \end{aligned}$$