Exercise 10.2: Suppose b and c are real numbers. Let r_1 and r_2 be the two real or complex roots of the polynomial $x^2 + bx + c$.

Suppose both r_1 and r_2 are real and distinct. Then, since $r_1 \neq r_2$, $|r_1 - r_2| \geq 0$. It would follow that $(r_1 - r_2)^2 \geq 0$. Now suppose that $r_1 = r_2$. Then $(r_1 - r_2)^2 = (0)^2 = 0$. The last possibility is that r_1 and r_2 are nonreal complex conjugate pairs. Let $r_1 = s + ti$ and $r_2 = s - ti$ for $s, t \in \mathbb{R}$. Then $(r_1 - r_2)^2 = (s + ti - (s - ti))^2 = (2ti)^2 = -4t^2$ is a negative real number.

We will now show that the converse holds true to these observations. Let $\Delta = (r_1 - r_2)^2$. If $\Delta > 0$, then we know that $(r_1 - r_2)^2 > 0$ which implies that both r_1 and r_2 are real numbers such that $r_1 - r_2 \neq 0$, so $r_1 \neq r_2$. It is not possible for only one of r_1, r_2 to be real and the other nonreal since nonreal numbers occur in pairs. Therefore, Δ has two real, distinct roots. Now suppose that $\Delta = 0$, then $(r_1 - r_2)^2 = 0$ and thus we may conclude that $r_1 = r_2$ and Δ has one real root with multiplicity 2. Finally, suppose $\Delta < 0$. Then $(r_1 - r_2)^2 < 0$ which is only possible if r_1 and r_2 are nonreal, complex conjugates. Let $r_1 = s + ti$ and $r_2 = s - ti$ for real numbers s and t. Then $(r_1 - r_2)^2 = (s + ti - (s - ti))^2 = (2ti)^2 = -4t^2 < 0$ because t is real. Therefore, if $\Delta < 0$, Δ has two nonreal conjugate roots. $\square$

Exercise 10.6: What is the reduced cubic equation that you must solve in order to solve the cubic equation?

$$x^3 - 3x^2 - 4x + 12 = 0$$

Let $x = y - \frac{a}{3} = y - \frac{-3}{3} = y + 1$. Substituting into our equation:

$$\begin{gathered}(y+1)^3 - 3(y+1)^2 - 4(y+1) + 12 \\ y^3 + 3y^2 + 3y + 1 - 3y^2 - 6y - 3 - 4y - 4 + 12 \\ y^3 + (3 - 6 - 4)y + (1 - 3 - 4 + 12) \\ y^3 - 7y + 6\end{gathered}$$

The reduced cubic equation is $y^3 - 7y + 6$ $\square$

Exercise 10.10: Solve $y^3 - 7y + 6 = 0$

Let $p = -7$ and $q = 6$. Let $R = (\frac{-7}{3})^3 + (\frac{6}{2})^2 = \frac{-343}{27} + \frac{243}{27} = \frac{-100}{27}$. Then, $\sqrt{R} = \sqrt{\frac{-100}{27}} = \frac{10\sqrt{-1}}{3\sqrt{3}} = \frac{10\sqrt{-3}}{9}$. Applying Cardano's Formula:

$$\begin{aligned} y &= \sqrt[3]{\tfrac{-6}{2} + \tfrac{10}{9}\sqrt{-3}} + \sqrt[3]{\tfrac{-6}{2} - \tfrac{10}{9}\sqrt{-3}} \\ &= \sqrt[3]{-3 + \tfrac{10}{9}\sqrt{-3}} + \sqrt[3]{-3 - \tfrac{10}{9}\sqrt{-3}} \end{aligned}$$

Observe that

$$(1 + \tfrac{2}{3}\sqrt{-3})^2 = (\tfrac{-1}{3} + \tfrac{4}{3}\sqrt{-3})(1 + \tfrac{2}{3}\sqrt{-3}) = -3 + \tfrac{10}{9}\sqrt{-3}$$

and

$$(1 - \tfrac{2}{3}\sqrt{-3})^2 = (\tfrac{-1}{3} - \tfrac{4}{3}\sqrt{-3})(1 - \tfrac{2}{3}\sqrt{-3}) = -3 - \tfrac{10}{9}\sqrt{-3}$$

Therefore, we may conclude that

$$(1 + \tfrac{2}{3}\sqrt{-3})^3 = -3 + \tfrac{10}{9}\sqrt{-3}$$

and

$$(1 - \tfrac{2}{3}\sqrt{-3})^3 = -3 - \tfrac{10}{9}\sqrt{-3}$$

Now, observe that

$$1 + \tfrac{2}{3}\sqrt{-3} = \sqrt[3]{-3 + \tfrac{10}{9}\sqrt{-3}}$$

and

$$1 - \tfrac{2}{3}\sqrt{-3} = \sqrt[3]{-3 - \tfrac{10}{9}\sqrt{-3}}$$

Now we have found one cubic root for each part of Cadano's formula. Substituting these values into the original formula, we can simply see that

$$\begin{aligned} y &= (1 + \tfrac{2}{3}\sqrt{-3}) + (1 - \tfrac{2}{3}\sqrt{-3}) \\ &= 1 + 1 \\ y &= 2 \end{aligned}$$

Now that we know $y = 2$ is a root of $y^3 - 7y + 6$, we can use long division to divide $y - 2$ into $y^3 - 7y + 6$. We find that $y^3 - 7y + 6 = (y-2)(y^2 + 2y - 3)$. Using the quadratic formula, we find the roots of $y^2 + 2y - 3 = (y-1)(y+3)$. Therefore, $y^3 - 7y + 6 = (y-2)(y-1)(y+3)$ and so the roots of the reduced cubic are $y = 1$, $y = 2$ and $y = -3$. $\square$

Exercise 10.22: Let $f(x) = x^4 + bx^3 + cx^2 + dx + e$. Use the change of variable $x = z + a$, for real numbers $a, b, c, d,$ and e.

Substituting $x = z + a$ into $f(x)$ and combining like terms, we create a new function:

$$g(z) = (z+a)^4 + b(z+a)^3 + c(z+a)^2 + d(z+a) + e$$
$$g(z) = z^4 + Bz^3 + Cz^2 + Dz + E$$

Where $B = 4a + b$, $C = 6a^2 + 3ab + c$, $D = 4a^3 + 3a^2b + 2ac + d$, and $E = a^4 + a^3b + a^2c + ad + e$. Observe that if we let $a = \frac{-b}{4}$, then $B = \frac{-4b}{4} + b = 0$ and the B term is eliminated.

Substitute the value $a = \frac{-b}{4}$ into $C, D,$ and E so that they are written in terms of the constants $b, c, d,$ and e to revise the function,

$$g(z) = z^4 + Cz^2 + Dz + E$$

Where $C = \frac{-3b^2}{8} + c$, $D = \frac{b^3}{8} - \frac{bc}{2} + d$, and $E = \frac{-3b^4 + 16b^2c - 64bd}{256} + e$.

Suppose that r is a root of $g(z) = 0$. By reversing the change of variable used previously, $x = r - \frac{b}{4}$, we find that $r = x + \frac{b}{4}$ is also a root of $f(x)$. Therefore, if we can find the roots of $g(z) = 0$, we can find the roots of $f(x) = 0$ by adding $\frac{b}{4}$ to the roots of $g(z)$. $\square$

Exercise 10.26: Find the solutions of the following

1. $z^4 - 3z^2 + 6z - 2 = 0$
By theorem 10.3 in the textbook, we know that the quartic factors as a product of two cubics such that: $z^4 - 3z^2 + 6z - 2 = (z^2 + kz + l)(z^2 - kz + m)$ where k, l, m are real numbers and k is unknown. To find the roots of the quartic, we must first find a real root of the cubic resolvant. From the information above, we can determine that $q = -3$, $r = 6$, and $s = -2$. The formula for the cubic resolvant is $j^3 + (2q)j^2 + (q^2 - 4s)j - r^2$ where $j = k^2$. Using our known values, we find that the cubic resolvant is given by

$$j^3 - 6j^2 + 17j - 36 = 0$$

We can use a guess-and-check method to determine that $j = 4$ is a root of the resolvant $4^3 - 6(4^2) + 17(4) - 36 = 0$. Recall that $j = k^2$ so $k = \pm 2$. Using the formulas provided in the book, we know that $m = \frac{q + k^2 + \frac{r}{k}}{2} = \frac{-3+4+\frac{6}{2}}{2} = 2$ and $l = \frac{q + k^2 - \frac{r}{k}}{2} = \frac{-3+4-\frac{6}{2}}{2} = -1$. We have now determined all of the values necessary to factor the quartic into cubics:

$$\begin{aligned} z^4 - 3z^2 + 6z - 2 &= (z^2 + kz + l)(z^2 - kz + m) \\ &= (z^2 + 2z - 1)(z^2 - 2z + 2) \end{aligned}$$

We can use the quadratic equation to see that the roots of $z^2 + 2z - 1$ are $z = -1 \pm \sqrt{2}$ and the roots of $z^2 - 2z + 2$ are $z = 1 \pm i$.

$$z^4 - 3z^2 + 6z - 2 = (z + 1 + \sqrt{2})(z + 1 - \sqrt{2})(z - 1 + i)(z - 1 - i)$$

Therefore, the quartic factors as a product of four degree one polynomials with two real roots and two complex roots $\square$

2. $z^4 - 10z^2 - 4z + 8 = 0$
Using a similar approach as above, we can see that $q = -10$, $r = -4$, and $s = 8$. The cubic resolvant is given by

$$j^3 - 20j^2 + 68j - 16 = 0$$

for $j = k^2$. Again, we can see that $j = 4$ is a solution by a simple guess-and-check. This implies that $k = \pm 2$. Using the known values of q, r, and s, we find that $m = \frac{-10+4+\frac{-4}{2}}{2} = -4$ and $l = \frac{-10+4-\frac{-4}{2}}{2} = -2$. Therefore

$$z^4 - 10z^2 - 4z + 8 = (z^2 + 2z - 2)(z^2 - 2z - 4)$$

One can expand the right side of the equation to confirm these findings. Using the quadratic equation, we find that the roots of z^2+2z-2 are $z = -1 \pm \sqrt{3}$ and the roots of $z^2 - 2z - 4$ are $z = 1 \pm \sqrt{5}$. Then

$$z^4 - 10z^2 - 4z + 8 = (z + 1 + \sqrt{3})(z + 1 - \sqrt{3})(z - 1 + \sqrt{5})(z - 1 - \sqrt{5})$$

and we see that the quartic has four real roots and factors as four degree one polynomials. $\Box$.

Exercise 10.40:
1. Suppose that $f(x)$ is a polynomial of positive degree in $\mathbb{R}[x]$ and that r is a root of $f(x)$ in $\mathbb{C}$. Suppose that $f(x)$ factors as a product of degree one polynomials in $\mathbb{C}[x]$ where $\alpha_1, \alpha_2, \alpha_3 ..., r$ are the roots of $f(x)$, though not necessarily real:

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)...(x - r) \in \mathbb{C}[x]$$

As a consequence of Gauss's theorem (10.5) in the textbook, we know that nonreal complex roots of a polynomial with real coefficients occur in conjugate pairs of two. Since $f(x)$ has one complex root r, we know that the conjugate of r, call it $\bar{r}$, is also a factor of $f(x) \in \mathbb{C}[x]$. Then

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)...(x - r)(x - \bar{r}) \in \mathbb{C}[x]$$

is the true factorization of $f(x)$.

If we take a look at the conjugate of the polynomial $f(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + ... + c_1 x + c_0$, since $f(x) \in \mathbb{R}$, none of the coefficients of $f(x)$ are complex, therefore, the conjugate of $f(x)$ is precisely $f(x)$: $f(x) = \overline{f(x)}$. Using this information, we may now conclude that if r is a root of $f(x)$, then $\bar{r}$ is also a root of $f(x)$.

2. If r is real, then $r = \bar{r}$ and $(x - r)(x - \bar{r}) = (x - r)^2$ which is a degree two polynomial with real coefficients in $\mathbb{R}[x]$. Assume that r is nonreal. Let $r = s + ti$ and $\bar{r} = s - ti$ for some real numbers s, t. Then

$$\begin{aligned}(x - r)(x - \bar{r}) &= (x - (s + ti))(x - (s - ti)) \\ &= x^2 - 2sx + s^2 + t^2\end{aligned}$$

where x, s, and t are all real numbers. Therefore, $(x - r)(x - \bar{r})$ is a second degree polynomial with real coefficients in $\mathbb{R}[x]$.

3. Suppose that r is nonreal. Because $r \in \mathbb{C}$ is a root of $f(x)$, we know that $(x - r)$ divides $f(x)$ in $\mathbb{C}[x]$. By part (1), we know that $\bar{r} \in \mathbb{C}$ is also a root of $f(x)$, and by Theorem 9.7, we know that $(x - \bar{r})$ also divides $f(x)$ in $\mathbb{C}[x]$. Then, by Theorem 9.9, we may conclude that $(x - r)(x - \bar{r})$ divides $f(x)$ in $\mathbb{C}[x]$.

Suppose we let $h(x) = (x - r)(x - \bar{r})$ and let $g(x)$ be a nonzero polynomial such that $f(x) = g(x)h(x)$. Recall that in part (2), we showed that if r is nonreal, then for $s, t \in \mathbb{R}$, $(x - r)(x - \bar{r}) = (x^2 - 2sx + s^2 + t^2) \in \mathbb{R}[x]$. Then $h(x)$ is actually contained in the subfield $\mathbb{R}[x] \subset \mathbb{C}[x]$ and therefore $h(x) = (x - r)(x - \bar{r})$, which divides $f(x)$, divides $f(x)$ in $\mathbb{R}[x]$.

Now consider $h(x)$, a polynomial of positive degree m. Suppose we factor $h(x)$ into degree one polynomials in $\mathbb{C}[x]$ such that their product is equal to $h(x) = (x - \beta_1)(x - \beta_2)...(x - \beta_m) \in \mathbb{C}[x]$. For every real root, $\beta_k \in \mathbb{R}$, we we leave $(x - \beta_k)$ as is. Now we will look at each $\beta_i \in \mathbb{C}$. Since complex numbers come in pairs of two, we take β_i and its conjugate pair, β_{ij} and we multiply the two polynomials $(x - \beta_i)(x - \beta_{ij})$ together to create a new degree two polynomial. Without loss of generality, we have already seen that the product of two degree one conjugate pairs lies in $\mathbb{R}[x]$, so for every pair of complex roots of $h(x)$, we multiply the two polynomials to create the new degree two polynomial that lies in $\mathbb{R}[x]$. If we leave every $\beta_k \in \mathbb{R}$ alone, and multiply together the conjugate pairs $(x - \beta_i)(x - \beta_{ij})$, we form the polynomial, $h'(x)$ which lies in $\mathbb{R}[x]$. We can do this for every pair of conjugates until the only roots and coefficients left are real. Therefore, we have now shown that $g(x)$ and $h(x)$ divide $f(x)$ in $\mathbb{R}[x]$ and thus, we have shown that if r is nonreal, that $(x - r)(x - \bar{r})$ divides $f(x)$ in $\mathbb{R}[x]$. $\square$