Problem 1 Let T_1 have the probability density function $f_1(t)$ and cumulative density function $F_1(t)$. Similarly, let T_2 have the probability density function $f_2(t)$ and cumulative density function $F_2(t)$. Assume T_1 and T_2 are statistically independent. Define $T^* = max\{T_1, T_2\}$.

a) The cdf of T^* is given by

$$\begin{gathered} P\{T^* < t\} = P\{T_1 < t, T_2 < t\} \\ = P\{T_1 < t\} \cdot P\{T_2 < t\} \\ F_{T^*}(t) = F_1(t) \cdot F_2(t) \end{gathered}$$

The pdf of T^* is found by differentiating $F_{T^*}(t)$:

$$\begin{gathered} \tfrac{d}{dt}F_{T^*}(t) = \tfrac{d}{dt}F_1(t) \cdot F_2(t) \\ f_{T^*}(t) = f_1(t)F_2(t) + f_2(t)F_1(t) \end{gathered}$$

b) In the equation above, the term $f_1(t)F_2(t)$ describes the area under the curve of T_1 (the pdf of T_1) multiplied by the cdf of T_2, and for the second term, vice versa. Essentially, the two terms are equal to the product of one variable's pdf and the other variable's cdf.

c) Assuming T_1 and T_2 are exponentially distributed with means τ_1 and τ_2, find the mean time for T^*.

In homework 1, we proved that for a random time variable T, $E[T] = \int_0^\infty 1 - F_T(t)$. Therefore,

$$\begin{gathered} E[T^*] = \int_0^\infty 1 - F_{T^*} dt \\ \int_0^\infty 1 - F_1(t)F_2(t) dt \\ \int_0^\infty 1 - (1 - e^{\frac{t}{\tau_1}})(1 - e^{\frac{t}{\tau_2}}) dt \\ \int_0^\infty e^{\frac{-t}{\tau_1}} + e^{\frac{-t}{\tau_2}} - e^{\frac{-t}{\tau_1} - \frac{t}{\tau_2}} dt \\ [-\tau_1 e^{\frac{-t}{\tau_1}} - \tau_2 e^{\frac{-t}{\tau_2}} - \tfrac{\tau_1\tau_2}{\tau_1+\tau_2} e^{\frac{-t(\tau_1+\tau_2)}{\tau_1\tau_2}}]_0^\infty \\ E[T^*] = \tau_1 + \tau_2 + \tfrac{\tau_1\tau_2}{\tau_1+\tau_2} \end{gathered}$$

Problem 2 The probability of a population at time t having n individuals is given by

$$\frac{dp_n(t)}{dt} = (n-1)bp_{n-1}(t) - (b+d)np_n(t) + (n+1)dp_{n+1}(t)$$

a) The coefficient of p_{n-1} is in terms of $(n-1)b$ because in order to have n people at time t we need $n-1$ people times the birth rate to give birth and increase the population. The coefficient of p_{n+1} is in terms of $(n+1)d$ because we need to account for the number of people dying in order to maintain the probability of n people at time $t+1$. The term $(n-1)bp_{n-1}$ is positive because new babies are being born which accounts for a positive change in the population, and the term $(n+1)dp_{n+1}$ is also positive because the term represents the number of people who must die in order for the population to maintain its current state. The term dnp_n is negative because it describes the number of people that will die and decrease the size of the population whereas bnp_n is negative because it represents the possibility of moving out a state where there are n people at time t.

b)

$$\frac{dp_0(t)}{dt} = -bp_0 + dp_1(t)$$

Where $-bp_0$ represents people moving out of the population and $dnp_1(t)$ is the number of people dying at time 1.

$$\frac{dp_1(t)}{dt} = bp_0 - (b+d)p_1(t) + 2dp_2(t)$$

Where $-(b+d)p_1(t)$ is the probability of the number of new births and new deaths at time 1, and $2dp_2(t)$ is the probability that $2dp_2(t)$ people will be dying at time 2.

c)

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} p_n(t)\right) = \frac{d}{dt}(p_0(t) + p_1(t) + p_2(t)...)$$
$$= dp_1(t) - (b+d)p_1(t) + 2dp_2(t) + bp_1(t) - 2(b+d)p_2(t) + 3dp_3(t) + ... + = 0$$

When we write each term $p_n(t)$ for $0 < n$ we find that each term cancels one another out. This is because the total probability is always 1 and will not change with time.

Problem 3 Let the waiting time for death for the k^{th} individual be T_k with probability density function $f_{T_k} = \lambda e^{-\lambda t}$ with expected value $F[T_k] = \frac{1}{\lambda}$.

a) Find the pdf of $f_{T^*}(t) = min\{T_1, T_2, ..., T_n\}$.

$$F_{T^*}(t) = P\{T^* > t\} = P\{T_1 > t\}P\{T_2 > t\}P\{T_3 > t\}...$$
$$P\{T_i > t\} = 1 - \int_0^t \lambda e^{-\lambda t} = e^{-\lambda t}$$
$$P\{T^* > t\} = (e^{-\lambda t})(e^{-\lambda t})(e^{-\lambda t})...$$
$$F_{T^*}(t) = e^{-\lambda nt}$$
$$f_{T^*}(t) = \frac{d}{dt}F_{T^*}(t) = \frac{d}{dt}e^{-\lambda nt}$$
$$f_{T^*}(t) = \lambda n e^{\lambda nt}$$

b) To find the expected value for T^*, we can use the expectation formula for a continuous random variable:

$$E[T^*] = \int_0^\infty \lambda n t e^{-\lambda nt} dt$$
$$[-te^{-\lambda nt} - \frac{1}{\lambda n}e^{-\lambda nt}]_0^\infty$$
$$0 - (-\frac{1}{\lambda n})$$
$$\frac{1}{\lambda n} \square$$