Problem 1: $n_k(t)$: number of nodes with k links at time t
$\frac{kp_k(t)}{2m}$: probability of attachment by a new node to any existing node with k links
$\frac{kp_k(t)}{2}$: number of nodes that gain one additional link by this new additional node

a) If we add one more node to the network, how many nodes now have k links? We want to find the number of nodes at time $t+1$ with k links which is given by $n_k(t+1)$. Then, we need to subtract the number of nodes with k links at time t, because at time $t+1$, nodes with k links will have $k+1$, links so we want to subtract them from the equation describing the number of nodes with k links.

$n_k(t+1) - n_k(t)$

Similarly, when we add the additional node, the existing nodes at time t with $k-1$ links will gain one additional link for a total of k links at time $t+1$. This is given by $\frac{(k-1)p_{k-1}(t)}{2}$. The existing nodes at time t that already have k links will gain an additional link and then have $k+1$ links, so we need to subtract them from the total number of nodes with k links.

$n_k(t+1) - n_k(t) = \frac{(k-1)p_{k-1}(t)}{2} - \frac{kp_k(t)}{2}$

b) As time becomes very large, the probability $p_k(t)$ becomes nearly independent of t, so we can treat it as though it is a constant and not a function of time:

$$p_k = \tfrac{1}{2}(k-1)p_{k-1} - \tfrac{1}{2}kp_k$$
$$2p_k + kp_k = (k-1)p_{k-1}$$
$$p_k(k+2) = p_{k-1}(k-1)$$
$$\frac{p_k}{p_{k-1}} = \frac{k-1}{k+2}$$

c) Solve the first order difference equation derived in part b). Previously, we saw that $p_k = \frac{k-1}{k+2}p_{k-1}$. Solving iteratively,

$$p_2 = \frac{k-1}{k+2} = \frac{1}{4}p_1$$
$$p_3 = \frac{k-2}{k+1} = \frac{3}{5}p_2$$
$$p_4 = \frac{k-3}{k} = \frac{3}{6}\frac{2}{5}p_2$$

$$p_5 = \frac{k-4}{k-1} = \frac{4}{6}\frac{3}{6}\frac{2}{5}p_2$$
$$.$$
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$$.$$
$$p_k = \frac{(k-1)(k-2)(k-3)(k-4)...2}{(k+2)(k+1)k(k-1)(k-2)...5}p_2$$
$$p_k = \frac{(k-1)!}{(k+2)(k+1)k(k-1)!}p_2$$

Most of the terms in the above equation cancel one another out, leaving only the first three terms on the denominator. The first three terms, $(p_3, p_4, p_5) = \frac{4}{6}\frac{3}{6}\frac{2}{5}p_2$ simplify to $\frac{24}{1}p_2$ since the first three terms do not cancel out any factors in the denominator, so we multiply their numerators together. Thus,

$$p_k = \frac{24}{(k+2)(k+1)k}p_2$$

Problem 2:

a) Probability of a node with degree k being attached from the new node:

$$1 - (1 - p_k)^m$$

b) Let $p_k << 1$. The binomial expansion of $1 - (1 - p_k)^m$ is given by:

$$\begin{aligned} 1 - (1 - mp_k + \tbinom{n}{2}p_k^2 - \tbinom{n}{3}p_k^3 + \tbinom{n}{4}p_k^4 - ....) \\ = 1 - 1 + mp_k + 0 \\ = mp_k \end{aligned}$$

Because $p_k << 1$, as p_k^n grows larger, the terms will become smaller and insignificant. If p_k is sufficiently small, the binomial expansion of $(1 - p_k)^m$ will be represented well enough by the first two terms. Hence, P(node with degree k being attached from a new node)$=mp_k$.

Problem 3a: For a school with only 3 grades, the following equation

$$\begin{bmatrix} x_0(n+1) \\ x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 1-\mu_0 & 0 & 0 \\ \mu_0 & 1-\mu_1 & 0 \\ 0 & \mu_1 & 1-\mu_2 \end{bmatrix} \begin{bmatrix} x_0(n) \\ x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

describes the grade population dynamics of a school with only 3 grades, where μ_i is the percent of students in grade i with satisfactory grades and b_i is the number of new students incoming to grade i from other schools.

b) Using a more compact notation, represent the equation above with $x(n+1) = Ax(n) + b$. Solving iteratively,

$$x(1) = Ax(0) + b$$
$$x(2) = Ax_1 + b = A[Ax(0) + b] + b = A^2x(0) + b(I + A)$$
$$x(3) = Ax_2 + b = A[A^2x(0) + b(I + A)] + b = A^3x(0) + b(I + A + A^2)$$
$$\vdots$$
$$x(n) = A^n x(0) + b(I + A + A^2 + ...A^{n-1})$$
$$x(n) = A^n x(0) + b(\tfrac{I-A^n}{I-A})$$

using the identity $(I + A + A^2 + ... + A^k)(I - A) = I - A^{k+1}$ provided in the homework notes.

c) As n goes to infinity, show that $\lim_{n\to\infty} x(n) = (I - A)^{-1}b$.
Since A is a matrix, we can use eigendecomposition to write $A = VDV^{-1}$ where D is the diagonal matrix of the eigenvalues of A. Because A is a lower triangular matrix, its eigenvalues are precisely the elements on its diagonal, so

$$D = \begin{bmatrix} 1-\mu_0 & 0 & 0 \\ 0 & 1-\mu_1 & 0 \\ 0 & 0 & 1-\mu_2 \end{bmatrix}$$

An interesting property of eigendecomposition is that $A^n = VD^nV^{-1}$ where

$$D^n = \begin{bmatrix} (1-\mu_0)^n & 0 & 0 \\ 0 & (1-\mu_1)^n & 0 \\ 0 & 0 & (1-\mu_2)^n \end{bmatrix}$$

which implies that $A^n = 0$ since $(1 - \mu) < 1$, so $\lim_{n\to\infty}(1 - \mu)^n = 0$. Therefore,

$$x(n) = b(\tfrac{I}{I-A}) = b(I - A)^{-1}$$

Problem 4: $T(t)$: uninfected T-cells
$T^*(t)$: infected T-cells
$V(t)$: population of the HIV virus
δ : rate of clearance of infected cells by the body
k : rate constant for the infection of the T-cells by the virus

$$\frac{dT^*}{dt} = kVT - \delta T^*, \text{ setting } k = 0,$$
$$\frac{dT^*}{dt} = -\delta T^*$$
$$T^*(t) = T^*(0)e^{-\delta t}$$

Substituting $T^*(t) = T^*(0)e^{-\delta t}$ into the equation $P(t) = N\delta T^*(t)$,

$$P(t) = N\delta T^*(0)e^{-\delta t}$$
$$\frac{dV}{dt} = P - cV(t)$$
$$\frac{dV}{dt} + cV(t) = N\delta T^*(0)e^{-\delta t}$$

Solve for homogenous: let $\mu = e^{\int c dt} = e^{ct}$ and $D = N\delta T^*(0)$

$$e^{ct}\frac{dV}{dt} + cV(t)e^{ct} = De^{ct-\delta t}$$
$$\int \frac{d}{dt}e^{ct}V(t)dt = D\int e^{ct-\delta t}dt$$
$$e^{ct}V(t) = \frac{D}{c-\delta}e^{ct-\delta t}$$
$$V(t) = \frac{D}{c-\delta}e^{-\delta t} + \frac{\gamma}{e^{ct}}$$

To find a particular solution, use $V(0) = \frac{N\delta T^*(0)}{c} = \frac{D}{c}$

$$V(0) = \frac{D}{c-\delta} + \gamma = \frac{D}{c}$$
$$\gamma = D(\frac{1}{c} - \frac{1}{c-\delta})$$

Therefore

$$V(t) = \frac{N\delta T^*(0)}{c-\delta}e^{-\delta t} + N\delta T^*(0)[\frac{1}{c} - \frac{1}{c-\delta}]e^{-ct}$$

Problem 5a: Given that $x_i = \frac{X_i}{\sum_{i=1}^n X_i}$ is the fraction of the subpopulation X_i among the total population,

$$\frac{dx_i(t)}{dt} = \frac{\sum_{j=1}^n X_j \frac{dX_i(t)}{dt} - X_i \sum_{j=1}^n \frac{dX_j(t)}{dt}}{(\sum_{j=1}^n X_j)^2}$$
$$= \frac{\sum_{j=1}^n X_j r_i X_i - \sum_{j=1}^n X_i r_j X_j}{(\sum_{j=1}^n X_j)^2}$$
$$\frac{dx_i(t)}{dt} = \frac{X_i \sum_{j=1}^n (r_i - r_j)X_j}{(\sum_{j=1}^n X_j)^2}$$

b) Show that the per capita growth rate for the total population at time t is $\sum_{i=1}^n r_i x_i(t)$.

$$\frac{\frac{dP(t)}{dt}}{P(t)} = \frac{\sum_{i=1}^n \frac{dX_i}{dt}}{\sum_{i=1}^n X_i}$$
$$\bar{r} = \frac{\sum_{i=1}^n r_i X_i}{\sum_{i=1}^n X_i}$$
$$= \sum_{i=1}^n r_i x_i(t)$$

By the definition of x_i.

c) Show that $\sum_{i=1}^n (\frac{dx_i(t)}{dt}) = 0$.
Recall that $\frac{dx_i(t)}{dt} = \frac{X_i \sum_{j=1}^n (r_i - r_j) X_j}{(\sum_{j=1}^n X_j)^2}$. Then if we sum from $i = 1$ to n, we have

$$\sum_{i=1}^n (\frac{X_i \sum_{j=1}^n (r_i - r_j) X_j}{(\sum_{j=1}^n X_j)^2})$$
$$= \frac{\sum_{i=1}^n \sum_{j=1}^n X_i (r_i - r_j) X_j}{\sum_{i=1}^n (\sum_{j=1}^n X_j)^2} = 0$$

because $\sum_{i=1}^n \sum_{j=1}^n X_i X_j (r_i - r_j) = 0$.

Show that $\frac{d}{dt}(\sum_{i=1}^n r_i x_i(t)) \geq 0$.
From the previous parts combined, we know that

$$\sum_{i=1}^n \frac{dx_i(t)}{dt} = \frac{\sum_{i=1}^n r_i X_i^2 - \sum_{i=1}^n X_i^2 r_i}{(\sum_{i=1}^n X_i)^2}$$

So,

$$\frac{d}{dt}(\sum_{i=1}^n r_i x_i(t)) = \frac{\sum_{i=1}^n r_i^2 X_i^2 - \sum_{i=1}^n X_i^2 r_i^2}{(\sum_{i=1}^n X_i)^2}$$
$$= \frac{\sum_{i=1}^n X_i r_i^2}{(\sum_{i=1}^n X_i)^2} - (\frac{\sum_{i=1}^n X_i r_i}{\sum_{i=1}^n X_i})^2$$
$$= \frac{\sum_{i=1}^n X_i (r_i - \bar{r})^2}{\sum_{i=1}^n X_i} \geq 0$$

The last equation describes the variance among the different subpopulations. Since the quantity is squared and measures the difference from the mean, it is always positive when there are variations among r_i.

d) The last equation of part c) describes how the change in the per capita growth rate of the entire population is never less than the average change in the per capita growth rate of the subpopulations. This implies that the success of the entire population is dependent on how successful each of the subpopulations are; if only one of the subpopulations is successful, then the entire population will likely fail, but if the majority of subpopulations are successful, the entire population will succeed.