Problem 1a: Solve the following logistic differential equation for $X(t)$:

$$\frac{dX}{dt} - \alpha X(1 - \frac{X}{\beta}), X(0) = X_0$$

where α and β are constants.

Solve nonlinear ODE using separation of variables:

$$\int \frac{1}{X(1-\frac{1}{\beta}X)} dx = \int \alpha dt$$
$$\int \frac{\beta}{X(\beta-X)} dx = \alpha t + C$$
$$\int \frac{1}{X} + \frac{1}{\beta - X} dx = \alpha t + C$$
$$ln(X) - ln(\beta - X) - \alpha t + C$$
$$ln(\frac{\beta - X}{X}) = -\alpha t - C$$
$$\frac{\beta - X}{X} = e^{-\alpha t} e^{-C}$$
$$X = \frac{\beta}{1 + e^{-\alpha t} e^{-C}}$$

To find a particular solution, let $X(0) = X_0$

$$X_0 = \frac{\beta}{e^{-C} + 1}$$
$$\frac{\beta}{X_0} = e^{-C}$$
$$e^{-C} = \frac{B - X_0}{X_0}$$

Putting it all together,

$$X = \frac{\beta X_0}{e^{-\alpha t}(\beta - X_0) + X_0}$$

b) Find $\lim_{t\to\infty} X(t)$.
Recall $\lim_{t\to\infty} e^{-\alpha t} = 0$, so $\lim_{t\to\infty} X(t) = \frac{\beta X_0}{X_0} = \beta$.

c) Explain what α and β imply in the context of growth dynamics of a population at time t.

The following calculations explain to us that β, also known as the carrying capacity, is a steady state for the population. Any small perturbations to the size of the population will result in the population returning back to its equillibrium state at β. It is clear that the value of α does not effect the state of the population long term. Rather, α merely describes how fast the population will grow over time.

Problem 2a: Solve $\frac{dN}{dt} = bN^2 - aN$ given $N(0)$.

Solve nonlinear ODE using separation of variables:

$$\frac{dN}{dt} = N(bN-a)$$
$$\int \frac{1}{N(bN-a)}dN = \int 1dt$$
$$\int \frac{b}{a(bN-a)} - \frac{1}{aN}dN = t + c$$
$$\frac{1}{a}ln(bN-a) - \frac{1}{a}ln(N) = t + c$$
$$ln(\frac{bN-a}{N}) = at + C$$
$$\frac{bN-a}{N} = e^{at}e^{C}$$
$$b - \frac{a}{N} = e^{at}e^{C}$$
$$N(t) = \frac{a}{b-e^{at}e^{C}}$$

To find a particular solution, use $N(0) = N_0$.

$$N_0 = \frac{a}{b-e^{C}}$$
$$\frac{a}{N_0} = b - e^{C}$$
$$e^{C} = b - \frac{a}{N_0}$$

Putting it all together,

$$N(t) = \frac{a}{b-e^{at}(b-\frac{a}{N_0})}$$

b) Find the long term behavior by calculating $\lim_{t\to\infty} N(t)$, assuming that $N_0 < \frac{a}{b}$.

$\lim_{t\to\infty} N(t) = 0^+$.

Extra Credit What is $\lim_{t\to\infty} N(t)$, assuming that $N_0 > \frac{a}{b}$?

$\lim_{t\to\infty} N(t) = 0^-$.

Problem 3 Let $p(t)$ be the fraction of dextral snails at time t. The equation modeling the number of dextral snails is given by

$$\frac{d}{dt}p = \alpha p(1-p)(p-0.5)$$

which has no left-right bias.

a) Locate the equilibria of p and determine their stability.

By setting $\frac{d}{dt}p = 0$ we see that there are three equilibria: $p = 0, p = 0.5, p = 1$. To check the stability of each of the equilibria, we will test 4 different points. Let $p = .01$, then $\frac{d}{dt}p < 0$ which indicates that $p = 0$ is a stable point since increasing the value of p slightly will decrease the equation back to the stable point. We cannot test a value of p less than 0 since p only takes positive values from $0 \leq p \leq 1$.

Let $p = 0.49$, then $\frac{d}{dt}p < 0$. Let $p = 0.51$, then $\frac{d}{dt}p > 0$. So, $p = 0.5$ is unstable since perturbing slightly towards zero will result in $\frac{d}{dt}p$ decreasing and thus moving further from the equilibrium point and increasing p slightly will result in $\frac{d}{dt}p$ moving to the right, away from the equilibrium point.

Lastly, let $p = 0.99$, then $\frac{d}{dt}p > 0$. Thus, $p = 1$ is a stable point since decreasing p slightly results in $\frac{d}{dt}p$ increasing back toward the stable point. We cannot test a point greater than 1 since the max value of p is 1. In conclusion, $p = 0$ and $p = 1$ are stable points and $p = 0.5$ is unstable.

b) Suppose at time $t = 0$ that $p(0) = 0.5$. That is, there are equal amounts of left and right handed snails. Describe what will happen to the snail population a few million years later.

From the previous part, we found that $p = 0.5$ is an unstable point, meaning that if at any time in the past the fraction of right or left handed snails was slightly perturbed, the fraction of those snails in the total population would move away from $p = 0.5$. Thus, it is safe to say that $\lim_{t\to\infty} p(t) \neq 0.5$ because any slight decreases in the left handed snail population, per say, would result in those snails going extinct over time. Since left handed snails are exceedingly rare today, we can conclude that at some point in the past the number of left handed snails in the total snail population was decreased, leading $\frac{d}{dt}p$ to move towards the stable point of $p = 1$ where only dextral snails exist.

Problem 4 A fish population with harvesting is modeled by

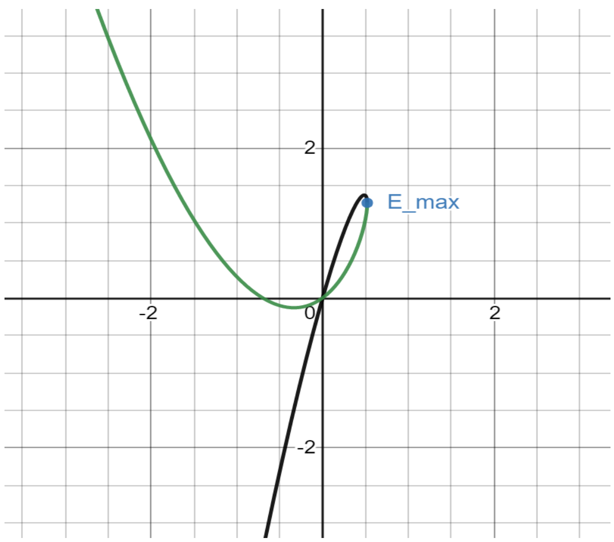
$$\frac{d}{dt}N = rN(\frac{N}{N_c} - 1)(1 - \frac{N}{k}) - qEN$$

a) Find the sustained yield $H(N_3^*)$ as a function of effort E.

Setting $\frac{d}{dt}N = 0$ we solve the equation $N[r(\frac{N}{N_c}-1)(1-\frac{N}{k})-qE] = 0$ for the following three equilibria: $N_1^* = 0$, and $N_2^*, N_3^* = \frac{r(\frac{1}{N_c}+\frac{1}{k})\pm\sqrt{r^2(\frac{1}{N_c}+\frac{1}{k})^2-4\frac{r}{N_ck}(r+qE)}}{\frac{2r}{N_ck}}$. Substituting N_3^* in for $H(N) = qE$, we have

$$H(N_2^*, N_3^*) = qE[\frac{r(\frac{1}{N_c}+\frac{1}{k})\pm\sqrt{r^2(\frac{1}{N_c}+\frac{1}{k})^2-4\frac{r}{N_ck}(r+qE)}}{\frac{2r}{N_ck}}]$$

An approximate plot of the sustained yield (green) and the unsustainable yield (black) are shown below.



b) Their intersection point E_{max} is the point at which the largest effort can be made, beyond which the fish population will head to extinction. To find E_{max} we set $N_2^* = N_3^*$

$$E_{max} = \frac{r(\frac{1}{N_c}+\frac{1}{k})+\sqrt{r^2(\frac{1}{N_c}+\frac{1}{k})^2-4\frac{r}{N_ck}(r+qE)}}{\frac{2r}{N_ck}} = \frac{r(\frac{1}{N_c}+\frac{1}{k})-\sqrt{r^2(\frac{1}{N_c}+\frac{1}{k})^2-4\frac{r}{N_ck}(r+qE)}}{\frac{2r}{N_ck}}$$

Simplifying, we find that

$$E_{max} = \frac{r}{q}[\frac{(\frac{K+N_c}{2})^2}{KN_c} - 1]$$

When $E = E_{max}$ we achieve the highest yield possible while still maintaining the original fish population.

c) Suppose harvesting is done at the effort level $E = E_{max}$ for a while so that the fish population is below what was thought of as sustainable. The government puts in a fishing limit that reduces the effort E to slightly below E_{max}. Can the fishery recover?

Recall that we defined E_{max} to be the absolute maximum rate of harvest before the fish population begins to decline to extinction. Because we over-harvested the fish population, even more than E_{max}, the population decline that the fish are beginning to experience is not fixable with a reduction of the harvesting rate. The fish population will continue to decline until the entire population has died.