Problem 1 A logistic difference equation is given by

$$f(x_n) = rx_n(1-x_n)$$

a) Expand and factor $f(f(x)) = 0$.
Starting at the end goal and working our way backward, we need to show that $f(f(x)) = rx(rx-r+1)(rx^2-(1+r)x+\frac{1+r}{r}) = 0$. If we expand all the terms and use polynomial division we have

$$\begin{gathered} r^3x^4 + r^2x^2 + r^3x^2 - 2r^3x^3 + x - r^2x = 0 \\ r^2x(rx^3 + x + rx - 2rx^2 - 1) + x \\ r^2x(1-x)(-rx^2 + rx - 1) - x \\ rx(r-rx)(rx^2 - rx + 1) - x = 0 = f(f(x)) \ \square \end{gathered}$$

b) Let $x_1 = \frac{1+r+\sqrt{(1+r)(r-3)}}{2r}$ and $x_1 = \frac{1+r-\sqrt{(1+r)(r-3)}}{2r}$. Show that $f(x_1) = x_2$ and $f(x_2) = x_1$.

Let $f(x_n) = rx_n - rx_n^2$.

$$\begin{gathered} f(x_{1,2}) = r(\frac{1+r\pm\sqrt{(1+r)(r-3)}}{2r}) - r(\frac{1+r\pm\sqrt{(1+r)(r-3)}}{2r})^2 \\ \frac{1+r\pm\sqrt{(1+r)(r-3)}}{2} - \frac{(1+r)^2\pm2(1+r)\sqrt{(1+r)(r-3)\pm(1+r)(r-3)}}{4r} \\ \frac{2r+2r^2\pm2r\sqrt{(1+r)(r-3)}}{4r} - \frac{1+2r+r^2\pm(2+2r)\sqrt{(1+r)(r-3)}\pm(1+r)(r-3)}{4r} \\ \frac{-1+r^2\pm2\sqrt{(1+r)(r-3)}\pm(3+r^2-2r)}{4r} \\ \frac{1+r\pm\sqrt{(1+r)(r-3)}}{2r} = x_{1,2} \end{gathered}$$

c) Show that the derivative $\frac{d}{dx}f(f(x)) = f'(x_1)f'(x_2)$.

To take the derivative of $f(f(x))$, we first use chain rule:

$$\begin{gathered} \frac{d}{dx}f(f(x)) = f'(x)f'(f(x)) \\ f'(x_{1,2})f'(f(x_{1,2})) = f'(x_1)f'(x_2) \end{gathered}$$

Assuming $f(x_1) = x_2$ and $f(x_2) = x_1$ from part b). To find the $r > 3$ at which the period 2 orbit becomes unstable, we solve

$$\begin{gathered} |f'(x_1)f'(x_2)| = 1 \\ f'(x_1)f'(x_2) = (r-(1+r+\sqrt{(r+1)(r-3)}))(r-(1+r-\sqrt{(r+1)(r-3)}) \\ (-1-\sqrt{(r+1)(r-3)}))(-1+\sqrt{(r+1)(r-3)}) \\ |1-(r+1)(r-3)| = 1 \end{gathered}$$

The quantity above is equal to 1 when $(r+1)(r-3)=0$ and when $(r+1)(r-3)=2$. When $(r+1)(r-3)=0$ we find $r_1=-1$ and $r_2=3$. When $(r+1)(r-3)=2$ we have $r_3=1+\sqrt{6}$ and $r_4=1-\sqrt{6}$. Since $r_3=1+\sqrt{6}$ is the only value greater than 3, the period 2 orbit will become unstable when $r=1+\sqrt{6}\approx 3.45$.

Problem 2 A fishery gives a model for the population size versus the number of offspring that survive as

$$N_{n+1} = \frac{RN_n}{1+[\frac{R-1}{K}]N_n}$$

a) Find the equilibria and determine their stability.

To find the equilibria, we need to find the points N^* at which $N^* = f(N^*)$. The points that satisfy these conditions are $N^* = 0$ and $N^* = K$. To determine their stability, we first need to compute

$$f'(N) = \frac{(1+[\frac{R-1}{K}]N_n)R - RN_n(\frac{R-1}{K})}{(1+[\frac{R-1}{K}]N_n)^2}$$
$$f'(0) = \frac{R}{1} = R > 1$$
$$f'(K) = \frac{R^2-R^2+R}{R^2} = \frac{1}{R} < 1$$

By the stability tests, the point $N^* = 0$ is unstable and the point $N^* = K$ is stable.

b) Find an exact, closed form solution by using the substitution $X_n = \frac{1}{N_n}$.

Using the substitution $X_n = \frac{1}{N_n}$ into the original equation, we have

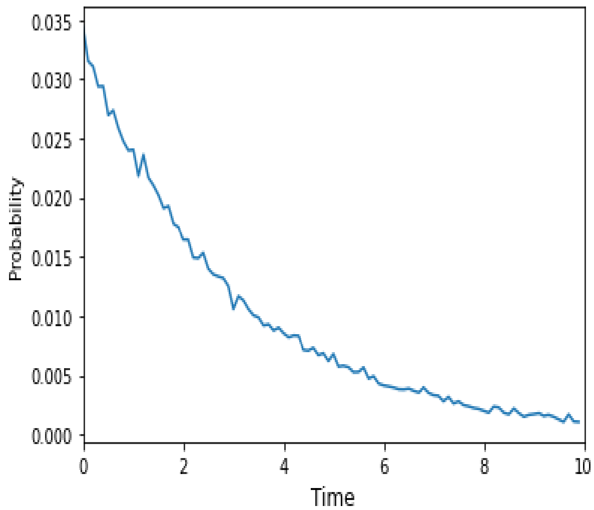
$$\frac{1}{X_{n+1}} = \frac{1+\frac{R-1}{KX_n}}{R} \cdot \frac{1}{X_n}$$
$$\frac{1}{X_{n+1}} = \frac{X_n}{R} + \frac{1}{K} - \frac{1}{RK}$$

To begin to solve this iteratively, we will use the substitution $B = \frac{1}{K} - \frac{1}{RK}$ for simplicty:

$$X_0 = \frac{X_1}{R} + B \rightarrow X_1 = X_0R - BR$$
$$X_1 = \frac{X_2}{R} + B \rightarrow X_2 = X_2R - BR \rightarrow X_2 = X_0R^2 - BR^2 - BR$$
$$X_2 = \frac{X_3}{R} + B \rightarrow X_3 = X_2R - BR \rightarrow X_3 = X_0R^3 - BR^3 - BR^2 - BR$$
$$.$$
$$.$$
$$X_n = X_0R^{n+1} - BR[1 + R + R^2 + ... + R^n]$$
$$X_n = X_0R^{n+1} - (\frac{R}{K} - \frac{1}{K})\frac{1-R^n}{1-R} \square$$

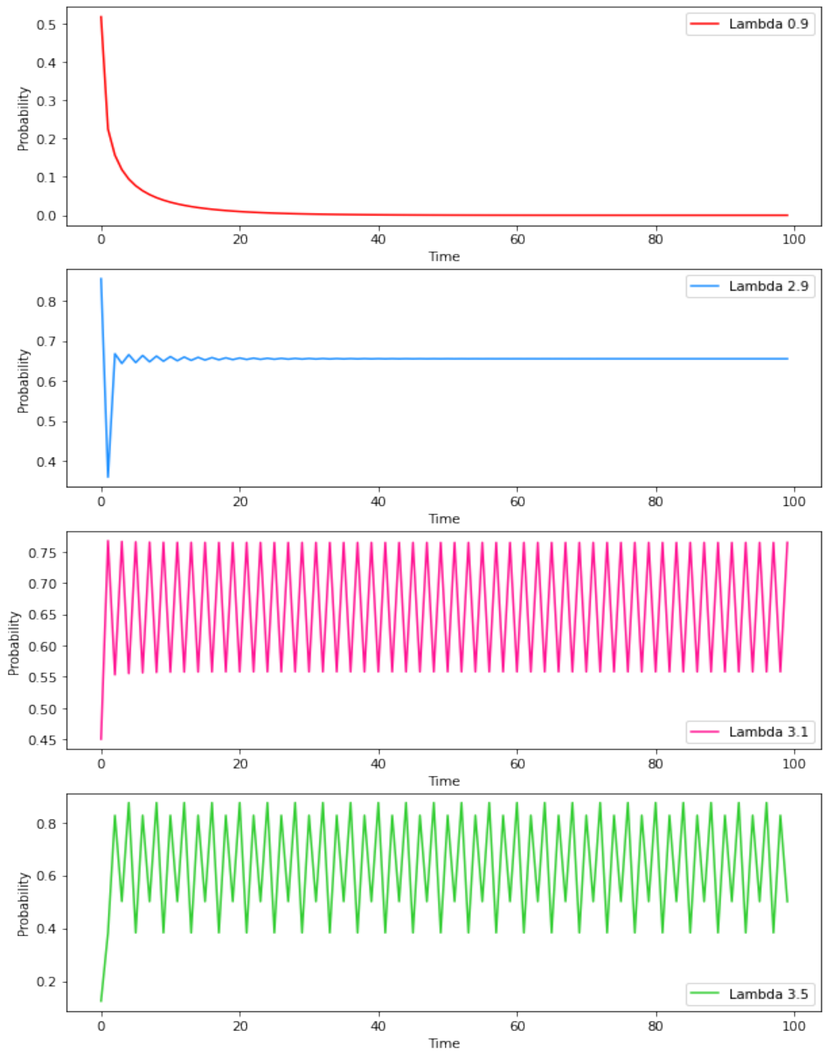
Problem 3:

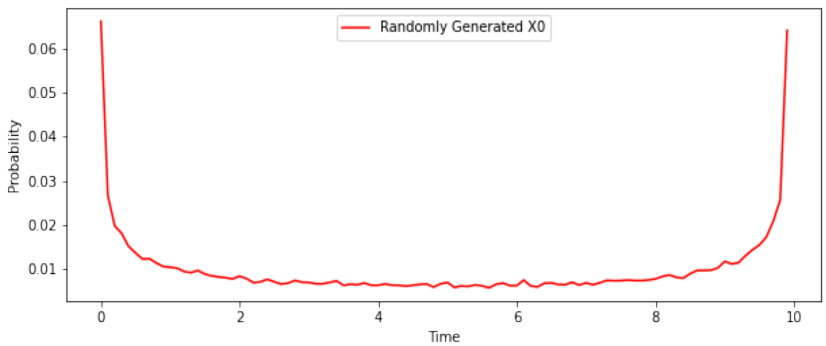
a)

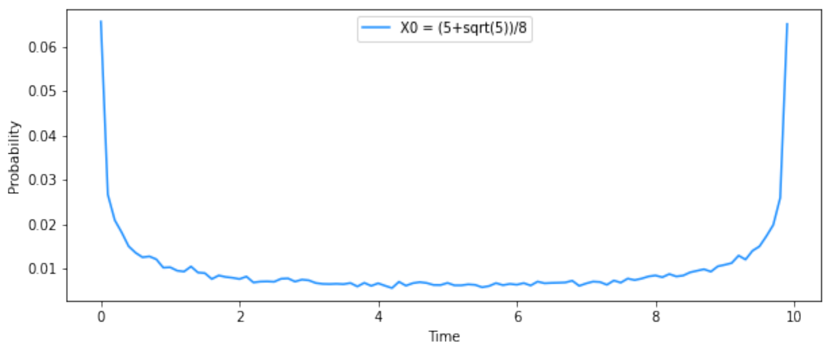


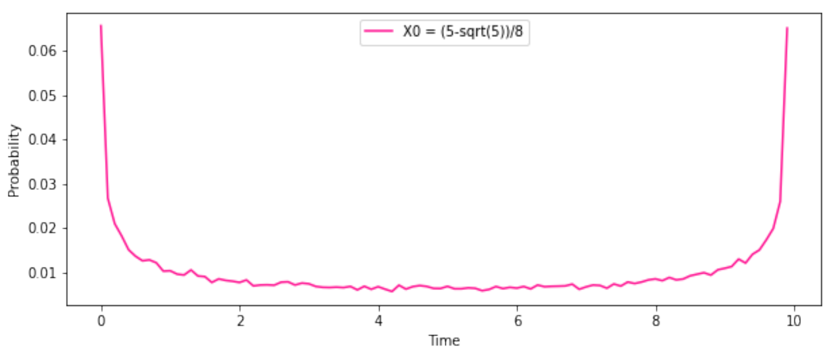
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
Probability
0
2
4
6
8
10
Time

b)









There are no other special values because a polynomial of degree 2 can only have at most 2 distinct roots .

Problem 4: A classical chemistry kinetics problem models differenct concentrations over time:

$$\frac{dc_A(t)}{dt} = -k_1 c_A c_B + k_{-1} c_C,$$
$$\frac{dc_B(t)}{dt} = -k_1 c_A c_B + k_{-1} c_C,$$
$$\frac{dc_C(t)}{dt} = k_1 c_A c_B - k_{-1} c_C,$$

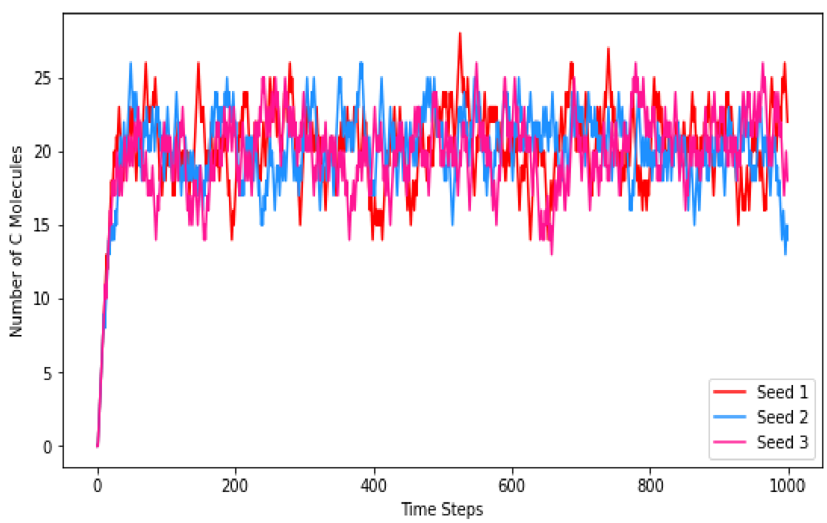
For k_1 and k_{-1} rate constants.

a) Show that $\frac{d}{dt}(c_A(t) + c_C(t)) = 0$ and $\frac{d}{dt}(c_B(t) + c_C(t)) = 0$.

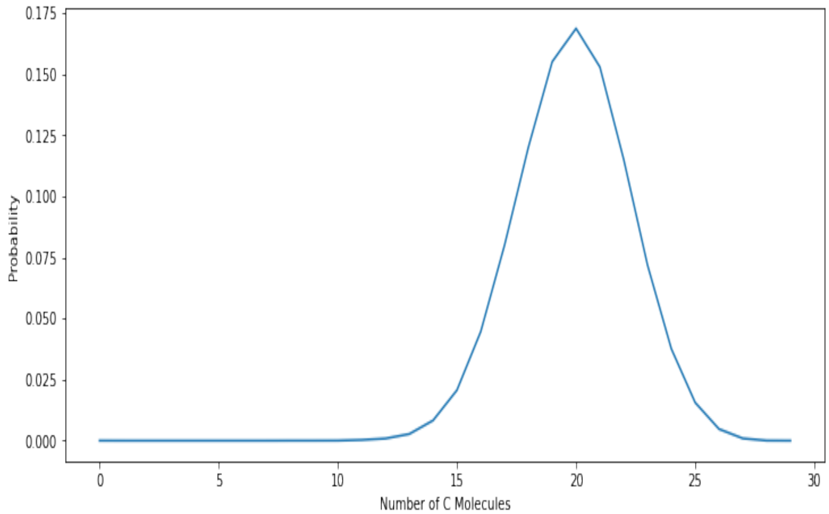
$$\frac{d}{dt}(c_A(t) + c_C(t)) = (-k_1 c_A c_B + k_1 c_A c_B - k_{-1} c_C + k_{-1} c_C) = 0$$
$$\frac{d}{dt}(c_B(t) + c_C(t)) = (-k_1 c_A c_B + k_1 c_A c_B - k_{-1} c_C + k_{-1} c_C) = 0$$

The change in concentration over time being zero makes sense because an equal number of atoms from substance A and B are being used to create one molecule of substance C, so no particles are lost in the transaction and the change in the number of atoms over time will be zero.

b)



c)



d) The equilibirum of the chemical reaction is when

$$\frac{k_{-1}}{k_1} = \frac{(N_A - n^*)(N_B - n^*)}{n^*}$$

Using the given values, we find that $n^* = 20$. From the graph above, it is easy to see that 20 is the mean number of atoms of C in the reaction, which supports our findings of $n^* = 20$ being an equilibrium point.