Problem 1 Define $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
a) Show that $\lambda_1 + \lambda_2 = a + d = tr(A)$.

To find the eigenvalues of A, we use $det(A - \lambda I) = 0$:

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$
$$det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = 0$$
$$\lambda^2 - \lambda(a + d) + (ad - bc) = 0$$

Using the quadratic formula, we find the eigenvalues of A:

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$
$$\lambda_1 + \lambda_2 = \frac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2} + \frac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$
$$\frac{a+d}{2} + \frac{a+d}{2}$$
$$\lambda_1 + \lambda_2 = a + d = tr(A) \ \square$$

Similarly,

$$\lambda_1 \cdot \lambda_2 = \frac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \cdot \frac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$
$$\frac{(a+d)^2 - [(a+b)^2 - 4(ad-bc)]}{4}$$
$$\frac{4(ad-bc)}{4}$$
$$\lambda_1 \cdot \lambda_2 = ad - bc = det(A) \ \square$$

b) Suppose that the eigenvalues of A are complex conjugates: $\lambda_{1,2} = \alpha \pm \beta i$.

From the previous part, we found that $\lambda_1 + \lambda_2 = tr(A)$ for any two values of λ. Then

$$tr(A') = \alpha + \beta i + \alpha - \beta i$$
$$tr(A') = 2\alpha \ \square$$

Similarly, the determinant of A' is given by

$$\lambda_1 \cdot \lambda_2 = (\alpha + \beta i)(\alpha - \beta i)$$
$$\alpha^2 + \alpha\beta i - \alpha\beta i - \beta^2 i^2$$
$$det(A') = \alpha^2 + \beta^2 \ \square$$

Problem 2 (Ch. 9, Problem 1) Let $x(t)$ and $y(t)$ be "war potential" of nations A and B. Their potentials are given by

$$\frac{dx}{dt} = r(x_0 - x) + ay,$$
$$\frac{dy}{dt} = r(y_0 - y) + bx$$

where $r = 1/5$ years and $a = b = 1$ year.

a) Suppose $a = b = 0$. Find the equilibrium and determine its stability.

Let $\frac{dx}{dt} = r(x_0 - x) = 0$ and $\frac{dy}{dt} = r(y_0 - y) = 0$. Setting the two equations equal to one another, we have

$$r(x_0 - x) = r(y_0 - y)$$
$$x_0 - x = y_0 - y = 0$$

we achieve the equilibrium point of (x_0, y_0). To determine its stability, we use $f(x, y) = r(x_0 - x)$ and $g(x, y) = r(y_0 - y)$ to find the eigenvalues of the Jacobian matrix.

$$M = \begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}$$
$$\lambda_{1,2} = -r$$

Using the Jacobian matrix we find the eigenvalues of M to be both real and both negative. Therefore the equilibrium point (x_0, y_0) is a stable node.

b) In the presence of interactions between the two nations, find the new equilibrium and determine the condition for stability.

Similarly, set $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. Using the system of equations, we can find the equilibrium point.

$$\begin{cases} r(x_0 - x) + ay = 0, \\ r(y_0 - y) + bx = 0, \end{cases}$$

The system gives $(\frac{r^2x_0 + ary_0}{r^2 - ab}, \frac{r^2y_0 - brx_0}{r^2 - ab})$. The Jacobian matrix M is given by:

$$M = \begin{bmatrix} -r & a \\ b & -r \end{bmatrix}$$

$$det(M - \lambda I) = \lambda^2 + (2r)\lambda + (r^2 - ab) = 0$$
$$\lambda_{1,2} = \frac{-2r \pm \sqrt{(2r)^2 - 4(r^2 - ab)}}{2}$$
$$\lambda_{1,2} = -r \pm \sqrt{ab}$$

Since $\lambda_1 \cdot \lambda_2 = r^2 - ab$, the equilibrium point is stable when $ab < r^2$.

c) For the European arms race of 1909-1914 France was allied with Russia (A), and Germany with Austria-Hungary (B). Would the arms race lead to war?

Using $r = \frac{1}{5}$ and $a = b = 1$, $\lambda_1 < 0$ and $\lambda_2 > 0$. Therefore the arms race would lead to war. Since the equilibrium point is a saddle, if either of the nation begins to equip weaponry to their army, an arms race would begin and devlove into war.

Problem 3 (Ch. 9, Problem 4) A combat model for the Vietnam War is given by

$$\frac{dx}{dt} = -axy$$
$$\frac{dy}{dt} = -bx$$

where $a = c_1 \frac{A_g}{A_x}$ and $b = c_2 p_x$.

a) Derive the condition for a stalemate.

Dividing the two equations and integrating each side, we find

$$\frac{dy \backslash dt}{dx \backslash dt} = \frac{-bx}{-axy}$$
$$\frac{dy}{dx} = \frac{b}{ay}$$
$$\int bdx = \int aydy$$
$$bx(t) - bx_0 = \frac{1}{2}ay(t)^2 - \frac{1}{2}ay_0^2$$
$$\frac{1}{2}ay(t)^2 - bx(t) = \frac{1}{2}ay_0^2 - bx_0 \equiv K$$

There is a stalemate when $K = 0$ or $\frac{1}{2}ay_0^2 = bx_0$.

b) Estimate the ratio in terms of the initial forces (x_0, y_0) for the y-force to prevail.

Since we want the y-force to prevail, we need $y(t) > x(t)$ so that $\frac{1}{2}ay(t)^2 - bx(t) > 0$. From then, it follows that $\frac{1}{2}ay_0^2 - bx_0 > 0$. Thus

$$\frac{1}{2}ay_0^2 - bx_0 > 0$$
$$\frac{1}{2}ay_0^2 > bx_0$$
$$\frac{y_0^2}{x_0} > \frac{2b}{a}$$

Using the information given in the prompt, we know that $c_1 \sim c_2$, $p_x \approx 0.1$, $A_g \sim 2$ sq. ft. and, $A_x = 1000x_0$ sq. ft. (for a 1000 sq. ft. combat area). Then the equality becomes

$$\frac{y_0^2}{x_0} > \frac{2c_2(0.1)}{c_1 \frac{2}{1000x_0}}$$
$$\frac{y_0^2}{x_0} > 0.1(1000)x_0$$
$$(\frac{y_0}{x_0})^2 > 100$$
$$\frac{y_0}{x_0} > 10$$

And so the y-force needs to be 10 times stronger to prevail. If the U.S. force never exceeded its opponents by a ratio of 6, then the U.S. would not prevail since it needed to be 10 times stronger to succeed.

Extra Credit For a more realistic model, a stalemate would occur when there are still enemies remaining on each side. This would be comparable to a truce or a ceasefire, where both nations have taken enough loss to agree to stop shooting and save the remaining soldiers.

Problem 4 (Ch. 9, Problem 5) The total population of humanoids is given by $N(t) = x(t) + y(t)$. The total population satisfies the logistic equation

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \beta N$$

a) Suppose there is no difference in their survival skills. Then the following two equations describe the two populations:

$$\begin{aligned} \frac{dx}{dt} &= xr(1 - \frac{x+y}{K}) - \beta x, \\ \frac{dy}{dt} &= yr(1 - \frac{x+y}{K}) - \beta y \end{aligned}$$

b) Suppose the early humans are slightly better adapted to survival than the Neanderthals. Then the survival equations are given by

$$\begin{aligned} \frac{dx}{dt} &= xr(1 - \frac{x+y}{K}) - \beta x, \\ \frac{dy}{dt} &= yr(1 - \frac{x+y}{K}) - (1-\epsilon)\beta y \end{aligned}$$

In a competition model, equilibrium can only be achieved when either $x = 0$, $y = 0$, or $x = y = 0$. It follows that $P_1 = (0, 0)$ is a fixed point. Suppose $x = 0$, then

$$\begin{gathered} yr(1 - \frac{y}{K}) - (1-\epsilon)\beta y = 0 \\ y(r(1 - \frac{y}{K}) - (1-\epsilon)\beta) = 0 \\ r(1 - \frac{y}{K}) - (1-\epsilon)\beta = 0 \\ r - (1-\epsilon)\beta = \frac{y}{K} \\ y = Kr - (1-\epsilon)\beta K \end{gathered}$$

So $P_2 = (0, Kr - (1-\epsilon)\beta K)$ is a fixed point. Setting $y = 0$,

$$\begin{gathered} xr(1 - \frac{x}{K}) - \beta x = 0 \\ x(r(1 - \frac{x}{K}) - \beta) = 0 \\ r(1 - \frac{x}{K}) - \beta = 0 \\ r - \beta = \frac{rx}{K} \\ x = \frac{K(r-\beta)}{r} \end{gathered}$$

So $P_3 = (\frac{K(r-\beta)}{r}, 0)$ is the last fixed point.

c) Determine the stability of the equilibria

Given $f(x,y) = xr(1 - \frac{x+y}{K}) - \beta x$ and $g(x,y) = yr(1 - \frac{x+y}{K}) - (1-\epsilon)\beta y$, and the fact that $\frac{xyr}{K} = 0$ since at least x or y is 0, the Jacobian matrix M is given by:

$$M = \begin{bmatrix} r - \frac{2xr}{K} - \beta & 0 \\ 0 & r - \frac{2yr}{K} - (1-\epsilon)\beta \end{bmatrix}$$

At the point (0,0), $\lambda_1 = r - \beta$ and $\lambda_2 = r - (1-\epsilon)\beta$. Since $r > \beta$ and $0 < \epsilon << 1$, we may conclude that both $\lambda_{1,2} > 0$ and thus (0,0) is unstable.

At the point $(0, Kr - (1-\epsilon)\beta K)$, we have $\lambda_1 = r - \beta$ and $\lambda_2 = r(1 - 2r) + \beta(1-\epsilon)(2r-1)$. We know that $\lambda_1 > 0$, and $\lambda_2 > 0$ when $\beta > \frac{r}{1-\epsilon}$, and $\lambda_2 < 0$ when $\beta < \frac{r}{1-\epsilon}$. Since $\epsilon << \beta$, it is safe to assume that $\beta < \frac{r}{1-\epsilon}$, so $\lambda_2 < 0$. Thus the point $(0, Kr - (1-\epsilon)\beta K)$ is stable.

At the point $(\frac{K(r-\beta)}{r}, 0)$, we have $\lambda_1 = \beta - r < 0$ and $\lambda_2 = r - (1-\epsilon)\beta > 0$. Thus the point is an unstable saddle.

d) Discuss the implications of the results on the equilibria and their stability.

In part c) we found that both $(0,0)$ and $(\frac{K(r-\beta)}{r}, 0)$ are unstable points, meaning that it is impossible for both the early humans and the Neanderthals to coexist. It is suggested by the second equilibrium point that $y = 0$ is also unstable, meaning that any perturbations will lead towards that stable point $(0, Kr - (1-\epsilon)\beta K)$ where Neanderthals will inevitably go extinct.

e) Show that $\frac{x(t)}{y(t)} = A_0 e^{-\epsilon\beta t}$.

Using the equation given in the prompt,

$$\frac{d}{dt}(x/y) = \frac{1}{y}\frac{dx}{dt} - (\frac{x}{y})\frac{1}{y}\frac{d}{dt}y$$
$$\frac{1}{y}(xr(1 - \frac{x+y}{K}) - \beta x) - \frac{x}{y}(r(1 - \frac{x+y}{K}) - (1-\epsilon)\beta)$$
$$\frac{d}{dt}(x/y) = \frac{-x\beta\epsilon}{y}$$

$$(\frac{y}{x})\frac{d}{dt}(x/y) = -\epsilon\beta$$
$$ln(\frac{x(t)}{y(t)}) = -\epsilon\beta t + C$$
$$\frac{x(t)}{y(t)} = A_0 e^{-\epsilon\beta t}$$

Take $\beta = \frac{1}{30}$ and $t = 10000$. Then $\frac{1}{e} = e^{10000\frac{-\epsilon}{30}} \rightarrow \epsilon = 0.003$. The mortality difference is very small but over time led to the extinction of Neanderthals.

Problem 5 A SIR model is given by

$$\frac{dS}{dt} = -\beta IS, \ (1)$$
$$\frac{dI}{dt} = \beta IS - \alpha I, \ (2)$$
$$\frac{dR}{dt} = \alpha I (3)$$

where $\alpha, \beta > 0$.

a) Use words to explain each of the equations above.

The term IS describes the interactions of infected and susceptible people in a population with β being the rate of infection. α describes the rate at which people recover (or die) from the infection. So (1) describes the number of susceptible people becoming infected, (2) describes the number of currently infected people recovering or dying from the disease, and (3) describes the total number of people in the population who have recovered or died from the disease.

b) Show that $S(t) + I(t) + R(t)$ is independent of time.

If we add each of the three populations together we have $-\beta IS + \beta IS - \alpha I + \alpha I = 0$. The sum of each of the individual populations represents the total population of people and is thus independent of time (assuming the total population stays the same or only changes minimally during the duration of the virus).

c) Show that over time, as $t \to \infty$, $I(t) \to 0$ but $S(t) \to S(\infty) \neq 0$.

In class and in the lecture notes, we determined a critical point S_c in which if $S < S_c$, $I(t)$ is stable, and if $S \geq S_c$, $I(t)$ is unstable. If $I(t) \to 0$, we know that $I(t)$ is stable, and so S can take any value $0 \leq S < S_c$. Therefore, $S(\infty) \neq 0$.

d) Derive a better formula for R_0.

Using equation 9.22 from the lecture notes, we have

$$S_c ln(\frac{S^*}{S_0}) - S^* + S_0 + I_0 = 0$$
$$S_c = \frac{S^* - S_0 - I_0}{ln(\frac{S^*}{S_0})}$$

Assuming that $I(0) \approx 0$, it follows that $S_0 \approx N$. R_0 is defined as $\frac{S_0}{S_c}$, so

$$R_0 = \frac{Nln(\frac{S^*}{S_0})}{S^* - N}$$
$$(\frac{S^*}{N} - 1)^{-1} ln(\frac{S^*}{S_0})$$
$$R_0 = (1 - \frac{S^*}{N})^{-1} ln(\frac{S_0}{S^*})$$

For S^* found in part c).