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Math 301 Spring 22
Homework 3

Page 32 Problem 4
Theorem 2.11 states that for integers m, n, their greatest common divisor is the product of the prime factors that m and n share in common.
$m = 119790 = 2 * 3 * 3 * 5 * 11 * 11 * 11$
$n = 42900 = 2 * 2 * 3 * 5 * 5 * 11 * 13$
$\gcd(m, n) = 2 * 3 * 5 * 11 = 330$

119790=42900(2)+33990
42900=33990(1)+8910
33990=8910(3)+7260
8910=7260(1)+1650
7260=1650(4)+660
1650=660(2)+330
660=330(2)+0 $\square$

Page 32 Problem 6
$m = 9797 = 97 * 101$
$n = 14507 = 89 * 163$
Because m and n share no common prime factors, their greatest common divisor is $(m, n) = 1$

14507=9797(1)+4710
9797=4710(2)+377
4710=377(12)+186
377=186(2)+5
186=5(37)+1
5=5(1)+0 $\square$

Page 33 Problem 13: Let p be a prime integer such that $p|a^n$ for integers a and n. Theorem 2.8 in the textbook states that if $p|a^n$ then $p|a$. By definition, this means that $a = pq$ for $q \in \mathbb{Z}$. If we raise both sides of the equation to the nth power, then $a^n = (pq)^n = p^n q^n$ by exponential distribution laws. Thus, by definition, we have shown that $p^n|a^n$ by a factor of q^n.

Page 35 Problem 3: Let n be a positive integer greater than or equal to 2. Suppose that $\sqrt[n]{n}$ is a rational number. Then there exist $a, b \in \mathbb{Z}$ such that $\sqrt[n]{n} = \frac{a}{b}$ and where $(a, b) = 1$. Then, $n = \frac{a^n}{b^n}$ which tells us that $nb^n = a^n$. Observe that $a \neq 1$ because $nb^n = 1$ clearly has no integer solutions since $n \geq 2$. Thus, if a is an integer that is not 1, it is at least 2. Since $a \geq 2$, it has a prime divisor, call it p, for $p \geq 2$. If $p|a$, then by the previous problem $p^n|a^n$. If p^n divides a^n, it must also divide nb^n. Since $(a, b) = 1$, it follows that $(a^n, b^n) = 1$ by Lemma 1 below. Because a^n and b^n are relatively prime, we may conclude using the theorem from class that $p^n|n$. However, note that $2 < 2^n \leq p^n$ by Lemma 2, so clearly $n < p^n$ and thus it is impossible for p^n to divide n. Therefore we have reached a contradiction and $\sqrt[n]{n}$ is irrational.

Lemma 1: Suppose that $a, b \in \mathbb{Z}$ and $(a, b) = 1$. By theorem 2.11, this means that a and b share no common prime divisors. Let $a = p_1 p_2 ... p_k$ and $b = q_1 q_2 ... q_i$ be the prime factorizations of a and b and $p \neq q$. Then for a positive integer n, $a^n = (p_1 p_2 ... p_k)^n = p_1^n p_2^n ... p_k^n$ and similarly $b^n = q_1^n q_2^n ... q_i^n$. Observe that since $p \neq q$, a^n and b^n still share no common prime divisors. Therefore, $(a^n, b^n) = 1$.

Lemma 2: If $n \geq 2$, then $2^n > n$. Observe that for $n = 2$, $2^2 = 4 > 2$. Suppose that for some $k > 2$, that $2^k > k$. Then $2^{k+1} = 2 * 2^k$. Using the inequality from the inductive step, we know that $2 * 2^k > 2 * k = k + k > k + 1$. Therefore $2^{k+1} > k + 1$ and we have proven the lemma using induction.

Page 43 Problem 5: Suppose that p is a prime number such that $2^p - 1$ is also prime. Although we cannot say whether 2^{p-1} is prime or composite, we can observe that $2^{p-1} \neq 2^p - 1$ because the only solution to $2^{n-1} = 2^n - 1$ is $n = 1$, and by the hypothesis, $p \geq 2$ because p is prime. Therefore, if $2^p - 1$ is prime and $2^{p-1} \neq 2^p - 1$, then $(2^{p-1}, 2^p - 1) = 1$. Thus, by definition of a multiplicative function, $\sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1)$. The divisors of 2^{p-1} are $\{1, 2, 4, ..., 2^{p-1}\}$ and the sum of these divisors can be represented by the geometric series $\sum_{k=0}^{p-1} 2^k = 2^{p-1+1} - 1 = 2^p - 1$. Thus $\sigma(2^{p-1}) = 2^p - 1$. Next, because $2^p - 1$ is prime, its only divisors a 1 and itself, so the sum of divisors is $2^p - 1 + 1 = 2^p$. Combining our findings, we have

$$\sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1)$$
$$\sigma(2^{p-1})\sigma(2^p - 1) = (2^p - 1)(2^p)$$

$$(2^p-1)2(2^{p-1})$$
$$\sigma(2^{p-1}(2^p-1)) = 2(2^p-1)(2^{p-1})$$

Therefore $2^{p-1}(2^p-1)$ is perfect. $\Box$

Page 43 Problem 7: Find an integer n such that $\sigma(n) = 546$.
Let $m = 546$, then $m = \prod \frac{p^{k+1}-1}{p-1} = \sigma(n)$. Then for each divisor d,

$$\sigma(n) = \frac{p^{k+1}-1}{p-1} = 546$$
$$p^{k+1} - 1 = 546p - 546$$
$$p^{k+1} = 546p - 545 \ (1)$$
$$p(p^k - 546) = 545$$

From the last equation we see that $p|545$, so we need to check a few cases to find the values of p that work. Observe 545=6*7*13. Let $d = 6$, then equation (1) tells us $p^{k+1} = 6p-5$. If we let $p = 5$, then $5^{k+1} = 6(5)-5 = 25$ which tells us that 5 is a divisor with multiplicity 1. If we let $d = 7$, then $p^{k+1} = 7p-6$. Similarly, we can observe that $p = 2$ is a solution with multiplicity 2 since $2^{2+1} = 7(2)-6 = 8$. Lastly, let $d = 13$ and $p = 3$, then $3^{2+1} = 13(3)-12 = 27$. Thus we have found that $\sigma(2^2)\sigma(3^2)\sigma(5) = 7*13*6 = 546 = \sigma(4*9*5)$ and so $n = 180$.

Page 43 Problem 8: Define the function $f(n) = n^2$, then $f(n)$ is multiplicative since $f(yz) = (yz)^2 = y^2z^2 = f(y)f(z)$. Next, define the funcion $\sigma_2(n) = \sum_{d|n} f(n)$ which is the sum of the squares of the divisors of an integer n. By theorem 2.15 in the textbook, $\sigma_2(n)$ is multiplicative because $f(n)$ is multiplicative.

Let $n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ be a prime factorization of an integer n. Then $n^2 = (p_1^{a_1}p_2^{a_2}...p_k^{a_k})^2 = p_1^{2a_1}p_2^{2a_2}...p_k^{2a_k}$. Since this is a prime factorization, each of the factors are relatively prime with one another, and when computing $\sigma_2(n)$, we can use the definition of a multiplicative function. So, $\sigma_2(n) = \sigma(p_1^{a_1})\sigma(p_2^{a_2})...\sigma(p_k^{a_k})$. For any given p_i, $1 \le i \le k$, the squared sum of its divisors of are:

$$\sigma_2(m) = 1 + p_i^2 + p_i^4 + ... + p_i^{2a_i}$$
$$\sigma_2(m)(p_i^2) = p_i^2 + p_i^4 + ... + p_i^{2a_i+2}$$
$$\sigma_2(m)p_i^2 - \sigma_2(m) = p_i^{2a_i+2} - 1$$
$$\sigma_2(m) = \frac{p_i^{2a_i+2}-1}{p_i^2-1}$$

Since this is the general form for just one prime divisor, it follows that

$$\sigma_2(n) = \frac{p_i^{2a_1+2}-1}{p_1^2-1}\frac{p_i^{2a_2+2}-1}{p_2^2-1}...\frac{p_k^{2a_k+2}-1}{p_k^2-1}$$