Emma Deckers
Math 301 Homework 8

Page 48 Problem 8: $9x + 11y = 79$

$$\begin{gathered} 11 = 9(1) + 2 \\ 9 = 2(4) + 1 \\ 1 = 9 - 4(11 - 9) \\ 1 = 11(-4) + 9(5) \\ 79 = 11(-316) + 9(395) \\ X = -316 + 9t, \\ Y = 395 - 11t \end{gathered}$$

Let $t = 35$, then $(X, Y) = (-1, 10)$. If we let $t = 36$, then $(X, Y) = (8, -1)$. Therefore there are no solutions in which $9x + 11y = 79$ is satisfied by only positive integers.

Page 148 Problem 3: Find all integer solutions to $x^3 - y^3 = 19$

$$(x - y)(x^2 + xy + y^2) = 19$$

Case 1: $x - y = 1$, $x^2 + xy + y^2 = 19$.

$$\begin{gathered} x = 1 + y \\ (1 + y)^2 + (1 + y)y + y^2 = 19 \\ 3y^2 + 3y - 18 = 0 \\ y_{1,2} = \tfrac{-3 \pm 15}{6} \\ y_1 = 2,\ y_2 = -3 \\ x_1 = 3,\ x_2 = -2 \end{gathered}$$

Case 2: $x - y = 19$, $x^2 + xy + y^2 = 1$.

$$\begin{gathered} x = 19 + y \\ (19 + y)^2 + (19 + y)y + y^2 = 1 \\ 3y^2 + 57y + 360 = 0 \\ y_{1,2} = \tfrac{-57 \pm \sqrt{-1071}}{6} \\ \text{No integer solutions} \end{gathered}$$

The only solutions to $x^3 - y^3 = 19$ are $(3, 2)$ and $(-2, -3)$.

Page 148 Problem 4: Find all integer solutions to $x^2 - dy^2 = 1$ if d is a perfect square.
Let $d = n^2$ for some $n \in \mathbb{Z}$. The equation above factors as $(x-ny)(x+ny) = 1$.
Case 1: $x - ny = 1$, $x + ny = 1$

$$\begin{aligned} x - ny - (x + ny) &= 1 - 1 \\ -2ny &= 0 \\ y &= 0 \\ x &= 1 \end{aligned}$$

Case 2: $x - ny = -1$, $x + ny = -1$

$$\begin{aligned} x - ny - (x + ny) &= -1 - (-1) \\ -2ny &= 0 \\ y &= 0 \\ x &= -1 \end{aligned}$$

The only possible solutions to $x^2 - dy^2 = 1$ are $(1, 0)$ and $(-1, 0)$.

Page 151 Problem 3: We can solve $x^2 - 11y^2 = 3$ using congruence equations because taking the equation with each of the various modulos (mod 3,4 and 11) will never result in the x^2 term and the y^2 term both being zero with the given modulus.

Mod 3: $x^2 + y^2 \equiv 0 \pmod 3$
Without loss of generality, assume $(x, y) = 1$. Since $3 \equiv 3 \pmod 4$, we can use Theorem 3.30 to find $x \equiv y \equiv 0 \pmod 3$. This tells us 3—$x$ and $3|y$. This is a contradiction because $(x, y) = 1$. therefore there are no solutions mod 3

Mod 4: $x^2 + y^2 \equiv 3 \equiv -1 \pmod 4$. Theorem 3.29 tells us that -1 is not a square mod 4, so both $x, y > 0$ otherwise $x^2 \equiv -1 \pmod 4$ or $y^2 \equiv -1 \pmod 4$ have no solutions. Since we are using modular arithmetic, we can test each case using $x, y = \{1, 2, 3\}$.

Case 1 $x = 1$:
$y^2 + 1 \equiv 3 \pmod 4$

$$y=1,\ 1+1 \equiv 2 \pmod 4$$
$$y=2,\ 4+1 \equiv 1 \pmod 4$$
$$y=3,\ 9+1 \equiv 2 \pmod 4$$

No solutions

Case 2 $x=2$:
$y^2+4 \equiv 3 \pmod 4$

$$y=1,\ 1+4 \equiv 1 \pmod 4$$
$$y=2,\ 4+4 \equiv 0 \pmod 4$$
$$y=3,\ 9+4 \equiv 1 \pmod 4$$

No solutions

Case 3 $x=3$:
$y^2+9 \equiv 3 \pmod 4$

$$y=1,\ 1+9 \equiv 2 \pmod 4$$
$$y=2,\ 4+9 \equiv 1 \pmod 4$$
$$y=3,\ 9+9 \equiv 2 \pmod 4$$

No solutions
In each possible case, there are no solutions mod 4.

Mod 11: $x^2 \equiv 3 \pmod{11}$
Mod 11, there are two solutions to $x^2 \equiv 3 \pmod{11}$: $x=5$ and $x=6$ since $25 \equiv 3 \pmod{11}$ and $36 \equiv 2 \pmod{11}$. So $(5,0)$ and $(6,0)$ are two solutions to $x^2-11y^2=3$ mod 11.

Page 151 Problem 8: $x^2-7y^2=3z^2$
Looking at the equation mod 7, we have $x^2 \equiv 3z^2 \pmod 7$. If we let $x,z=\{1,2,3,4,5,6\}$, the posible values are $x^2=\{1,2,4\} \pmod 7$ and $3z^2=\{3,5,6\} \pmod 7$. Therefore there are no non-trivial solutions to $x^2 \equiv 3z^2 \pmod 7$ so there are no integer solutions to $x^2-7y^2=3z^2$.

Prage 155 Problem 2: $x^2+3y^2=z^2$.
Assume that x,y,z are primitive such that $(x,y,z)=1$. Without loss of generality, assume that x is odd and y is even. Let $y=3y'$ such that $3y'^2=\frac{z^2-x^2}{9}=(\frac{z-x}{3})(\frac{z+x}{3})$. If $3y'^2=\frac{1}{9}(z+x)(z-x)$ and $(z+x,z-x)=1$,

it follows from Euclids Lemma that $9|z+x$ or $9|z-x$.

Case 1: $9|z-x$.
If $9|z-x$, then $3y'^2 = (\frac{z-x}{9})(z+x)$. Since $\gcd(\frac{z-x}{9}, z+x) = 1$, it follows from Theorem 2.12 that there exist positive integers a, b such that $\frac{z-x}{9} = a^2$ and $z + x = b^2$. Then $\frac{z-x}{3} = 3a^2$ and $3y'^2 = 3a^2b^2 \to y' = ab$. So $y = 3y' = 3ab$,

$$z = 9a^2 + x$$
$$\frac{z+x}{3} = b^2$$
$$9a^2 + 2x = 3b^2$$
$$x = \frac{3b^2-9a^2}{2},\ z = \frac{3b^2+9a^2}{2}$$

Thus $(\frac{3b^2-9a^2}{2}, 3ab, \frac{3b^2+9a^2}{2})$ is a general solution.

Case 2: $9|z+x$
If $9|z+x$ then we have $\frac{z+x}{3} = 3a^2$ and $\frac{z-x}{3} = b^2$ which again gives $y = 3ab$. Similarly,

$$z = 9a^2 - x$$
$$\frac{z-x}{3} = b^2$$
$$9a^2 - 2x = 3b^2$$
$$x = \frac{9a^2-3b^2}{2},\ z = \frac{9a^2+3ba^2}{2}$$

Thus $(\frac{9a^2-3b^2}{2}, 3ab, \frac{9a^2+3ba^2}{2})$ is a general solution.

Page 161 Problem 3: $x^4 + 4y^4 = z^2$, $x, y, z > 0$.
Rewriting the equation above, we have

$$(x^2)^2 + (2y^2)^2 = z^2$$

So we can consider this equations as a pythagorean triple $(x^2, 2y^2, z)$. Suppose that $(x^2, 2y^2, z) = d$ for some $d > 1$. Then $d|x^2$ implies $d|x^4$, $d|2y^2$ implies $d|4y^4$, and $d|z$ implies $d|z^2$. Therefore $(x^4, 4y^4, z^2)$ is at least $d > 1$ so the triple is not primitive. Assume that the triple is primitive, that is, $(x^2, 2y^2, z) = 1$. Without loss of generality, assume x is odd and y is even. Then we can use Theorem 5.2 to find $x^2 = m^2 - n^2$, $2y^2 = 2mn$ and $z = m^2 + n^2$ with $m, n \in \mathbb{Z}_+$ and $(m, n) = 1$. For $2y^2 = 2mn \to y^2 = mn$, we see that both m and n are perfect squares. Let $m = s^2$ and $n = t^2$ for $s, t \in \mathbb{Z}_+$. Then $z^2 = s^4 + t^4$ which has no solutions by Theorem 5.3. Therefore $x^4 + 4y^4 = z^2$ has no solutions.

Page 161 Problem 4: Let a primitive right triangle have sides $a^2+b^2=c^2$. Without loss of generality, assume a is odd and b is even such that $a=p^2-q^2$ and $b=2pq$ for $p,q\in\mathbb{Z}_+$ and $(p,q)=1$. The area of a triangle A is given by $A^2=\frac{ab}{2}=\frac{2pq(p^2-q^2)}{2}=pq(p^2-q^2)$. In order for A^2 to be square, both pq and $p^2-q^2=(p+q)(p-q)$ must both be square. Let $p+q=U^2$, and $p-q=V^2$. Since p and q are relatively prime, both p and q must each be squares, $p=P^2$ and $q=Q^2$ for $P,Q\in\mathbb{Z}$. Observe:

$$\begin{gathered}p+q=(p-q)+2q\\ U^2=V^2+2Q^2\\ 2Q^2=U^2-V^2\\ 2Q^2=(U+V)(U-V)\end{gathered}$$

with $gcd(U,V)=2$. Then one of the following factors must take the form $2R^2$ and the other must take the form $4S^2$, so we can write $U=R^2+2S^2$, $\pm V=R^2-2S^2$, and $Q=2RS$ for $R,S\in\mathbb{Z}$. Consequently, observe

$$\begin{gathered}p=\tfrac{1}{2}(p+q+p-q)\\ P^2=\tfrac{1}{2}(U^2+V^2)\\ \tfrac{1}{2}[(R^2+2S^2)^2+(R^2-2S^2)^2]\\ \tfrac{1}{2}(R^4+4R^2S^2+4S^4+R^4-4R^2S^2+4S^4)\\ \tfrac{1}{2}(2R^4+8S^4)\\ P^2=R^4+4S^4\end{gathered}$$

which by the previous problem has no solution. Therefore there is no right triangle with integral sides whose area is a perfect square.