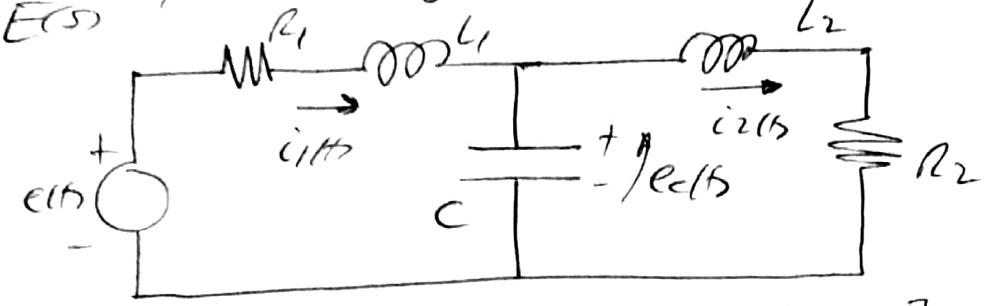
Ejemplos de Resolución de Sistemas Lineales

Ejemplo #1 Obtenga las transferencias $\frac{I_1(s)}{E(s)}$; $\frac{I_2(s)}{E(s)}$ y $\frac{E_c(s)}{E(s)}$ para el siguiente circuito eléctrico



CIN

Se considera: $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} i_1(t) \\ i_2(t) \\ e_c(t) \end{bmatrix}$

$$e = R_1 x_1 + L_1 \dot{x}_1 + x_3 \longrightarrow \dot{x}_1 = -\frac{R_1}{L_1} x_1 - \frac{1}{L_1} x_3 + \frac{1}{L_1} e$$

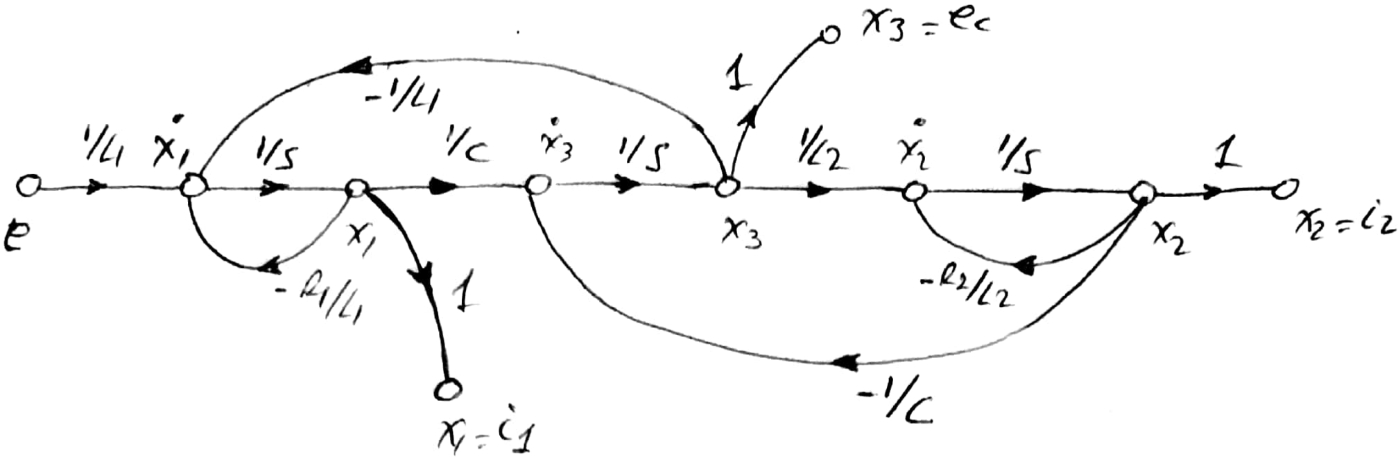
$$x_3 = L_2 \dot{x}_2 + R_2 x_2 \longrightarrow \dot{x}_2 = -\frac{R_2}{L_2} x_2 + \frac{1}{L_2} x_3$$

$$C \dot{x}_3 = x_1 - x_2 \longrightarrow \dot{x}_3 = \frac{1}{C} x_1 - \frac{1}{C} x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} e \qquad (1)$$

$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ e_c(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} e$$

De las Ec. de Estado y Salida, se pueden armar el Diagrama de Flujo de Estados



$$\Delta = 1 + \frac{R_1}{sL_1} + \frac{1}{s^2L_2C} + \frac{R_2}{sL_2} + \frac{1}{s^2L_1C} + \frac{R_1R_2}{s^2L_1L_2} + \frac{R_1}{s^3L_1L_2C} + \frac{R_2}{s^3L_1L_2C}$$

$$\Delta = \frac{s^3L_1L_2C + s^2L_2CR_1 + sL_1 + s^2L_1CR_2 + sL_2 + sCR_1R_2 + R_1 + R_2}{s^3L_1L_2C}$$

$$\Delta = \frac{s^3L_1L_2C + s^2C(L_2R_1 + L_1R_2) + s(L_1 + L_2 + CR_1R_2) + R_1 + R_2}{s^3L_1L_2C}$$

$$\frac{I_1(s)}{E(s)} = \frac{\frac{1}{sL_1}\left(1 + \frac{R_2}{sL_2} + \frac{1}{s^2L_2C}\right)}{\Delta} = \frac{\frac{s^2L_2C + sCR_2 + 1}{s^3L_1L_2C}}{\Delta}$$

$$\frac{I_1(s)}{E(s)} = \frac{s^2L_2C + sCR_2 + 1}{s^3L_1L_2C + s^2C(L_2R_1 + L_1R_2) + s(L_1 + L_2 + CR_1R_2) + R_1 + R_2} \quad (2)$$

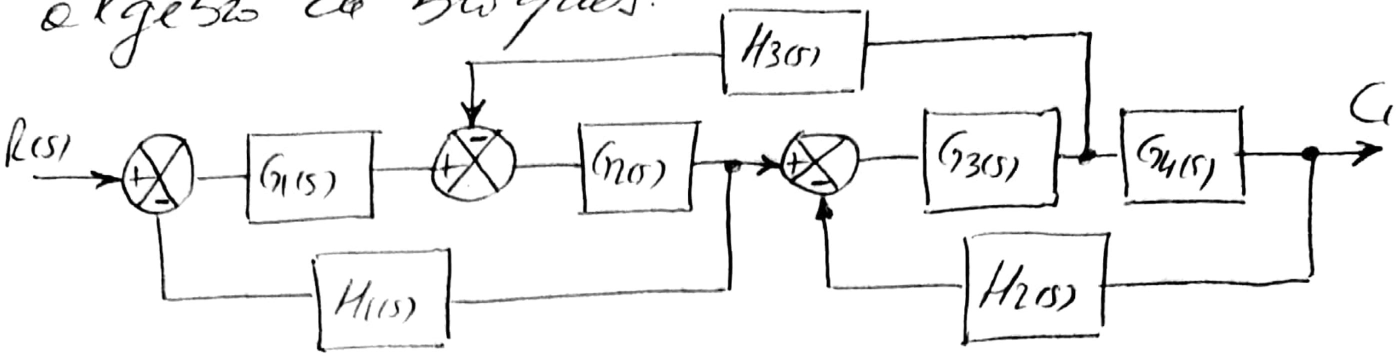
$$\frac{I_2(s)}{E(s)} = \frac{\frac{1}{s^3L_1L_2C}}{\Delta}$$, con lo cual

$$\frac{I_2(s)}{E(s)} = \frac{1}{s^3L_1L_2C + s^2C(L_2R_1 + L_1R_2) + s(L_1 + L_2 + CR_1R_2) + R_1 + R_2} \quad (3)$$

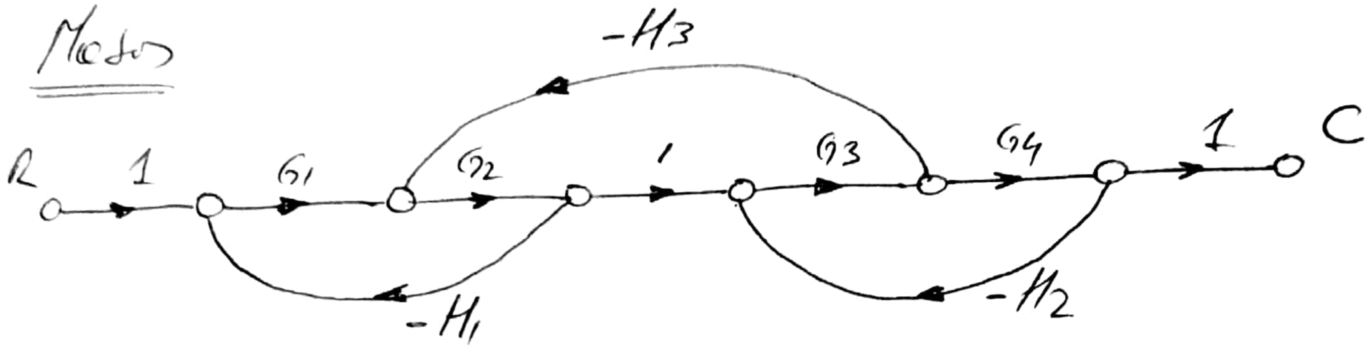
$$\frac{E_C(s)}{E(s)} = \frac{\frac{1}{s^2L_1C}\left(1 + \frac{R_2}{sL_2}\right)}{\Delta} = \frac{\frac{sL_2 + R_2}{s^3L_1L_2C}}{\Delta}$$

$$\frac{E_C(s)}{E(s)} = \frac{sL_2 + R_2}{s^3L_1L_2C + s^2C(L_2R_1 + L_1R_2) + s(L_1 + L_2 + CR_1R_2) + R_1 + R_2} \quad (4)$$

Ejemplo #2 Resolver por Mason y verificar mediante álgebra de bloques.



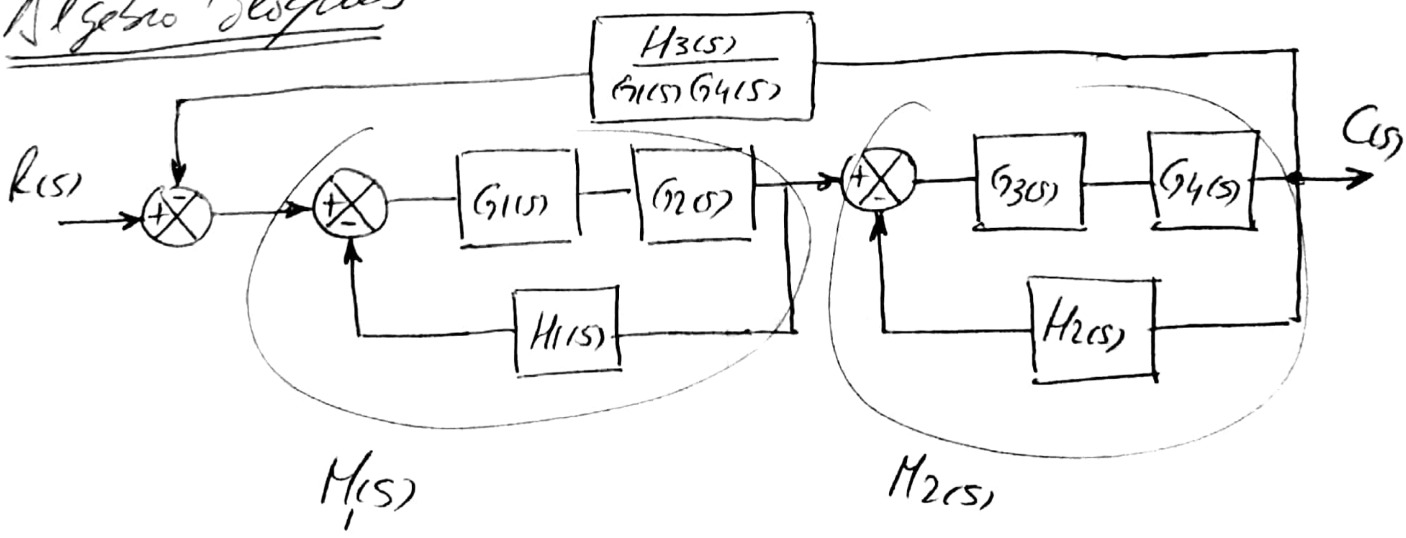
Mason



$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2$$

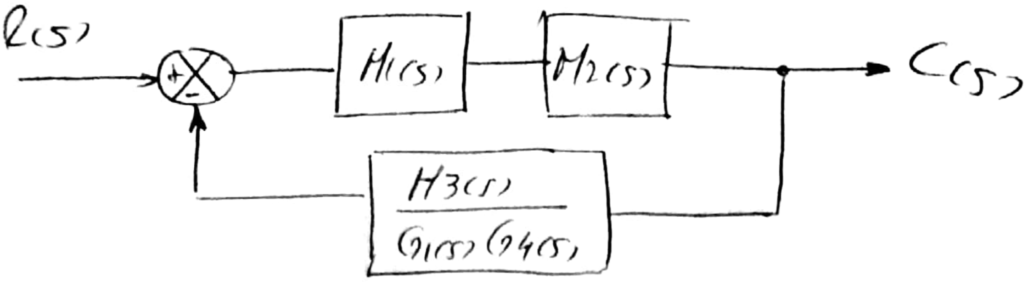
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2H_1 + G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2}$$ S

Álgebra Bloques



$$M_1(s) = \frac{G_1G_2}{1+G_1G_2H_1} \quad ; \quad M_2(s) = \frac{G_3G_4}{1+G_3G_4H_2}$$

En consecuencia, se tiene que:



$$\frac{C(s)}{R(s)} = \frac{M_1M_2}{1+M_1M_2\frac{H_3}{G_1G_4}} = \frac{\frac{G_1G_2}{1+G_1G_2H_1}\cdot\frac{G_3G_4}{1+G_3G_4H_2}}{1+\frac{\cancel{G_1}G_2}{1+G_1G_2H_1}\cdot\frac{G_3\cancel{G_4}}{1+G_3G_4H_2}\cdot\frac{H_3}{\cancel{G_1}\cancel{G_4}}}$$

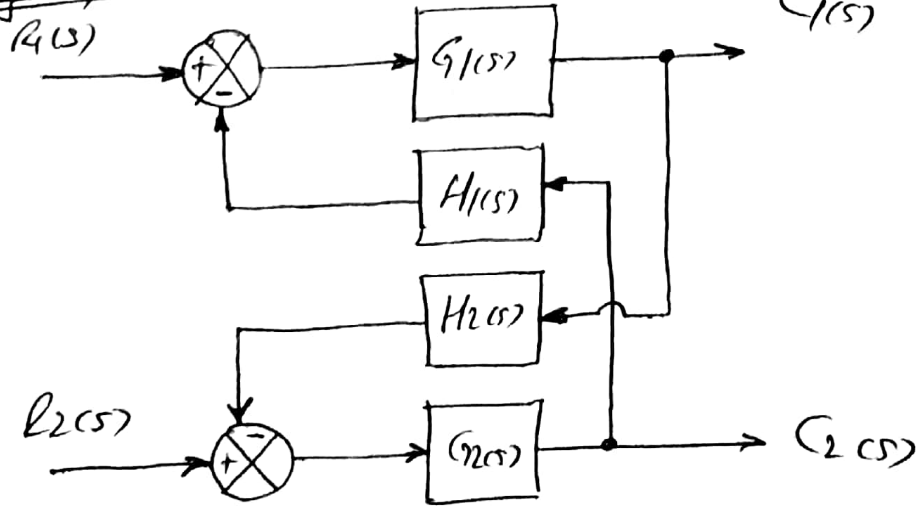
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)+G_2G_3H_3}$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_3G_4H_2+G_1G_2H_1+G_1G_2H_1G_3G_4H_2+G_2G_3H_3}$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_1G_2H_1+G_2G_3H_3+G_3G_4H_2+G_1G_2G_3G_4H_1H_2} \quad (6)$$

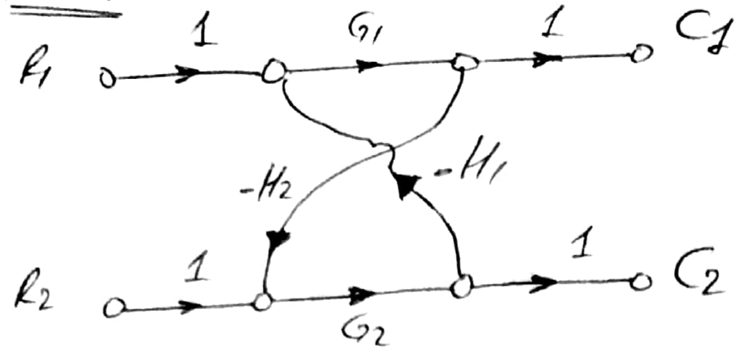
Las Ecs (5) y (6) son iguales.

Ejemplo #3



Calcule $C_1(s)$ y $C_2(s)$.

Mason



$$\Delta = 1 - G_1G_2H_1H_2$$

$$C_1\Big|_{R_1} = \frac{G_1}{\Delta}R_1 \; ; \; C_1\Big|_{R_2} = \frac{-G_1G_2H_1}{\Delta}R_2$$

$$C_1 = C_1\Big|_{R_1} + C_1\Big|_{R_2} = \frac{G_1R_1}{\Delta} - \frac{G_1G_2H_1}{\Delta}R_2 \; ; \text{ con lo cual}$$

$$C_{1(s)} = \frac{G_{1(s)}R_{1(s)} - G_{1(s)}G_{2(s)}H_{1(s)}R_{2(s)}}{1 - G_{1(s)}G_{2(s)}H_{1(s)}H_{2(s)}} \quad (7)$$

$$C_2\Big|_{R_2} = \frac{G_2}{\Delta}R_2 \; ; \; C_2\Big|_{R_1} = \frac{-G_1G_2H_2}{\Delta}R_1 \Rightarrow$$

$$C_{2(s)} = C_2\Big|_{R_2} + C_2\Big|_{R_1} = \frac{G_2R_2}{\Delta} - \frac{G_1G_2H_2R_1}{\Delta} \Rightarrow$$

$$C_{2(s)} = \frac{G_{2(s)}R_{2(s)} - G_{1(s)}G_{2(s)}H_{2(s)}R_{1(s)}}{1 - G_{1(s)}G_{2(s)}H_{1(s)}H_{2(s)}} \quad (8)$$

Laplace:

De la figura del Ejemplo #3, se tiene que:

$$C_1 = G_1(R_1 - H_1C_2) \; ; \; C_1 + G_1H_1C_2 = G_1R_1 \quad (a)$$

$$C_2 = G_2(R_2 - H_2C_1) \; ; \; G_2H_2C_1 + C_2 = G_2R_2 \quad (b)$$

De (a) y (b) $\rightarrow \begin{bmatrix} 1 & G_1H_1 \\ G_2H_2 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} G_1R_1 \\ G_2R_2 \end{bmatrix}$

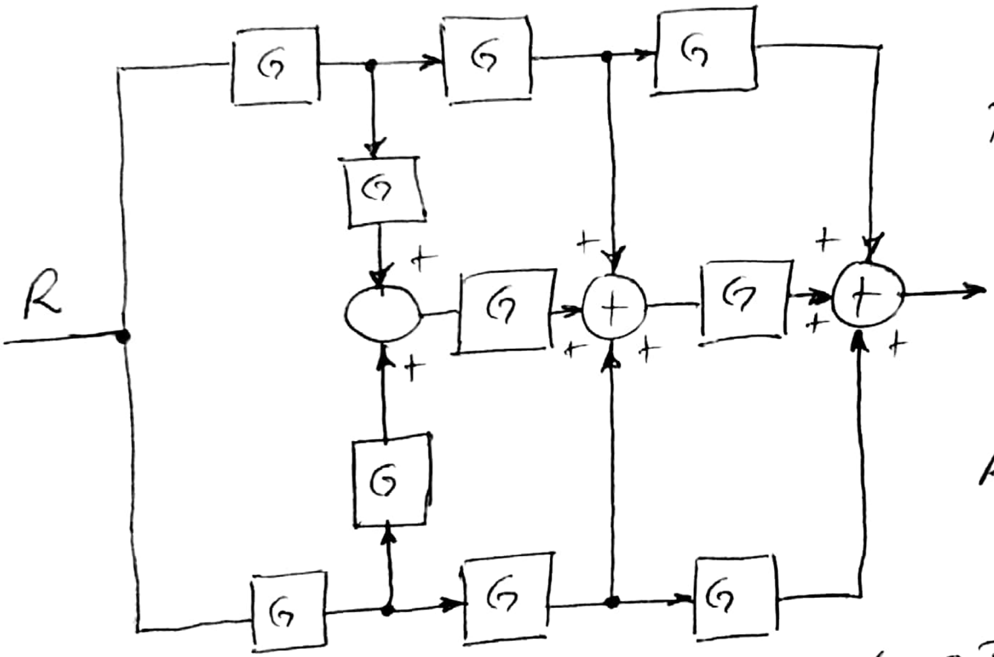
$$C_1 = \frac{\det\begin{bmatrix} G_1R_1 & G_1H_1 \\ G_2R_2 & 1 \end{bmatrix}}{\det\begin{bmatrix} 1 & G_1H_1 \\ G_2H_2 & 1 \end{bmatrix}} = \frac{G_1R_1 - G_1G_2H_1R_2}{1 - G_1G_2H_1H_2} \quad (9)$$

$$C_2 = \frac{\det\begin{bmatrix} 1 & G_1R_1 \\ G_2H_2 & G_2R_2 \end{bmatrix}}{\det\begin{bmatrix} 1 & G_1H_1 \\ G_2H_2 & 1 \end{bmatrix}} = \frac{G_2R_2 - G_1G_2H_2R_1}{1 - G_1G_2H_1H_2} \quad (10)$$

Se observo que (7) = (9) y (8) = (10)

Ejemplo #4

Encuentre $\frac{Y(s)}{R(s)}$ siendo $G(s) = \frac{1}{s+1}$



No hay LAZOS

Todos Caminos directos!!!

$\Delta = 1$

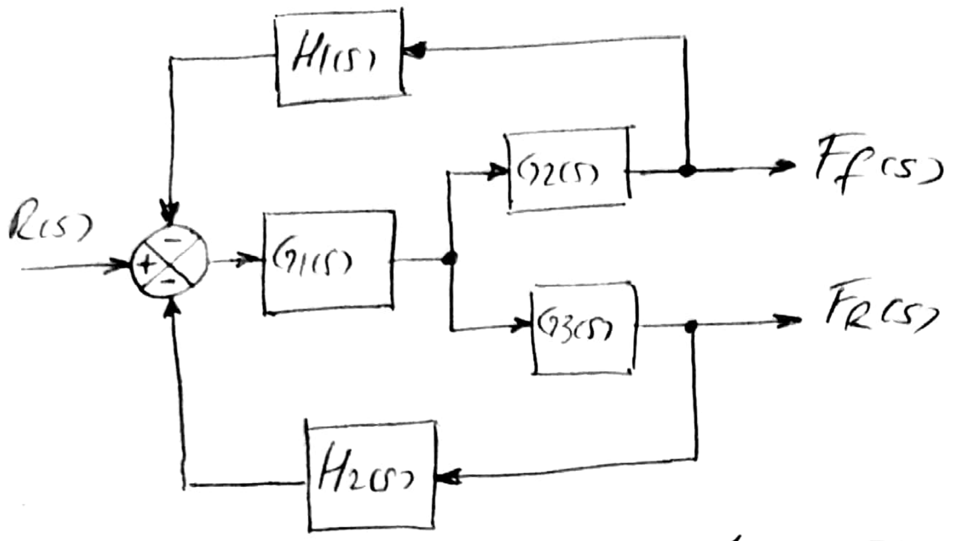
$$\frac{Y}{R} = \frac{\sum P_k \Delta_k}{\Delta} = \frac{G^3 + G^4 + G^3 + G^3 + G^4 + G^3}{1} = 4G^3 + 2G^4$$

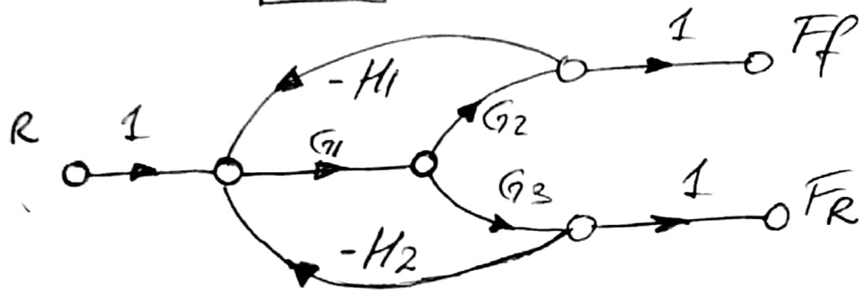
$$\frac{Y}{R} = 2G^3(2+G) = 2 \cdot \frac{1}{(s+1)^3}\left(2 + \frac{1}{s+1}\right) = \frac{2}{(s+1)^3} \cdot \frac{2(s+1)+1}{s+1}$$

$$\frac{Y}{R} = \frac{2}{(s+1)^3} \cdot \frac{2s+3}{s+1} = \frac{2}{(s+1)^3} \cdot \frac{2(s+3/2)}{(s+1)}$$

$$\frac{Y(s)}{R(s)} = \frac{4(s+3/2)}{(s+1)^4} \quad (11)$$

Ejemplo #5. Calcule $\frac{F_f(s)}{R(s)}$ y $\frac{F_R(s)}{R(s)}$





$$\Delta = 1 + G_1G_2H_1 + G_1G_3H_2 \; ; \; \frac{F_f}{R} = \frac{G_1G_2}{\Delta} \; ; \; \frac{F_R}{R} = \frac{G_1G_3}{\Delta}$$

$$\frac{F_f(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1+G_1(s)G_2(s)H_1(s)+G_1(s)\,G_3(s)\,H_2(s)} \qquad (12)$$

$$\frac{F_R(s)}{R(s)} = \frac{G_1(s)\,G_3(s)}{1+G_1(s)G_2(s)H_1(s)+G_1(s)\,G_3(s)\,H_2(s)} \qquad (13)$$

Ejemplo #6 Calcule la Ec. Dif. del Proceso descripto por la siguiente función transferencia, así también como su modelo de estados

$$\frac{Y(s)}{U(s)} = \frac{3(s+2)(s+6)}{s(s+1)(s^2+4s+8)(s+12)}$$

Con ayuda de Matlab (¿Qué instrucciones?), se tiene que:

$$\frac{Y_{(s)}}{U_{(s)}} = \frac{3s^2+24s+36}{s^5+17s^4+72s^3+152s^2+96s}$$

$$(s^5+17s^4+72s^3+152s^2+96s)Y_{(s)} = (3s^2+24s+36)u_{(s)} \xrightarrow{\mathcal{L}^{-1}}$$

$$y^{(V)}_{(t)}+17y^{(IV)}_{(t)}+72y'''_{(t)}+152y''_{(t)}+96y'_{(t)} = 3u''_{(t)}+24u'_{(t)}+36u_{(t)}$$

14

$$Y_{(s)} = \frac{\overset{b_2}{3}s^2+\overset{b_1}{24}s+\overset{b_0}{36}}{s^5+\underbrace{17}_{a_4}s^4+\underbrace{72}_{a_3}s^3+\underbrace{152}_{a_2}s^2+\underbrace{96}_{a_1}s}$$

$$D=[b_5]=[0] \; ; \quad B=[0\;0\;0\;0\;1]^T \; ;$$

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -96 & -152 & -72 & -17 \end{bmatrix}$$

$$C=[(b_0-\cancel{b_5a_0})(b_1-\cancel{b_5a_1})(b_2-\cancel{b_5a_2})(b_3-\cancel{b_5a_3})(b_4-\cancel{b_5a_4})]$$

$$C=[36 \;\; 24 \;\; 3 \;\; 0 \;\; 0] \Longrightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -96 & -152 & -72 & -17 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

15

$$y = [36 \;\; 24 \;\; 3 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + [0]u$$

Ejemplo #7

CI: $V_c(0) = V_{c0}$; $i_L(0) = i_0$

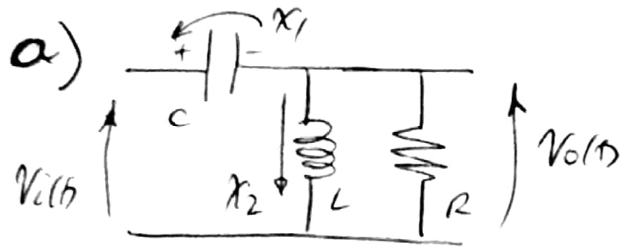
Dado el siguiente circuito, se pide

a) Modelo de Estado

b) $V_o(s)/V_i(s)$ con lo calculado anteriormente

c) Modelo de Flujo de Estados y cálculo de $V_o(s)$

a)



$V_o(t) = V_i - x_1$ (I)

$L\dot{x}_2 = V_o = V_i - x_1$; $\dot{x}_2 = -\frac{1}{L}x_1 + \frac{1}{L}V_i$ (II)

$$C\dot{x}_1 = x_2 + \frac{V_o}{R} = x_2 + \frac{V_i - x_1}{R} = -\frac{x_1}{R} + x_2 + \frac{V_i}{R}$$

$\dot{x}_1 = -\frac{1}{RC}x_1 + \frac{1}{C}x_2 + \frac{1}{RC}V_i$ (III). De (I), (II) y (III)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} V_i$$

16

$$V_o(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1]V_i$$

b) $\frac{V_o(s)}{V_i(s)} = C(SI - A)^{-1}B + D$

$$(SI - A) = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix} = \begin{pmatrix} S + \frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & S \end{pmatrix}$$

$\det(SI - A) = S^2 + \frac{1}{RC}S + \frac{1}{LC}$; $(SI - A)^{-1} = \dfrac{\begin{pmatrix} S & \frac{1}{C} \\ -\frac{1}{L} & S + \frac{1}{RC} \end{pmatrix}}{S^2 + \frac{1}{RC}S + \frac{1}{LC}}$

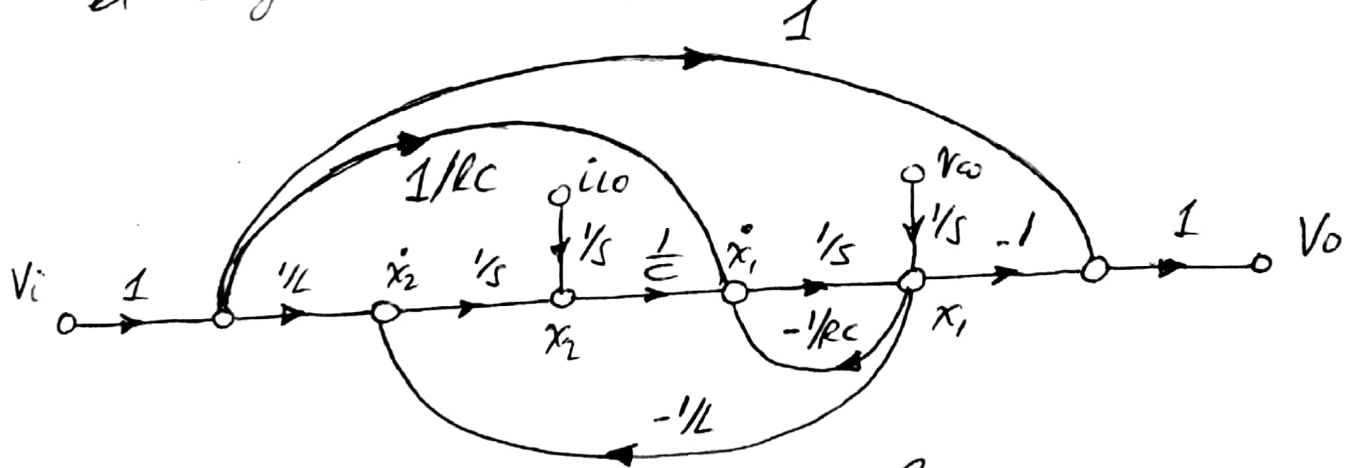
$$\frac{V_o(s)}{V_i(s)} = C(SI-A)^{-1}B+D = \frac{(-1\ \ 0)\begin{pmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s+\frac{1}{RC}\end{pmatrix}\begin{pmatrix}\frac{1}{RC} \\ \frac{1}{L}\end{pmatrix}}{\det(SI-A)} + 1$$

$$\frac{V_o(s)}{V_i(s)} = \frac{(-s\ \ -1/C)\begin{pmatrix}\frac{1}{RC} \\ \frac{1}{L}\end{pmatrix}}{\det(SI-A)} + 1 = \frac{-\frac{1}{RC}s - \frac{1}{LC}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} + 1$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\cancel{-\frac{1}{RC}s} \cancel{-\frac{1}{LC}} + s^2 \cancel{+\frac{1}{RC}s} \cancel{+\frac{1}{LC}}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \longrightarrow$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s^2}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \qquad (17)$$

c) A partir de las Ecs. (16), podemos obtener el diagrama de Flujo de Estados



$$\Delta = 1+\frac{1}{RC}\frac{1}{s}+\frac{1}{LC}\frac{1}{s^2} = \frac{s^2LCR+Ls+R}{LCRs^2}$$

$$V_o/V_i = \frac{\sum_i P_i\Delta_i}{\Delta} = \frac{1\left(1\cancel{+\frac{1}{RC}\frac{1}{s}}\cancel{+\frac{1}{LC}\frac{1}{s^2}}\right)\cancel{-\frac{1}{RC}\frac{1}{s}}\cancel{-\frac{1}{LCs^2}}}{\Delta}V_i(s)$$

$$V_o/V_i = \frac{1.V_i(s)}{\frac{s^2LCR+Ls+R}{LCRs^2}} = \frac{(LCRs^2)V_i(s)}{s^2LCR+Ls+R} = \frac{\cancel{LCR}\,s^2V_i(s)}{\cancel{LCR}\left(s^2+\frac{1}{RC}s+\frac{1}{LC}\right)}$$

Escaneado con CamScanner

$$\frac{V_{O(s)}}{V_{i(s)}} = \frac{s^2 V_{i(s)}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \quad (18)$$

$$\frac{V_{O(s)}}{i_{Lo}} = \frac{-\cancel{\frac{Li_{Lo}}{CS^2}}}{\cancel{\frac{s^2LCR+LS+R}{LCRS^2}}} = \frac{-LRi_{Lo}}{LCR\left(s^2+\frac{1}{RC}s+\frac{1}{LC}\right)}$$

$$\frac{V_{O(s)}}{i_{Lo}} = \frac{-\frac{1}{C}Li_{Lo}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \quad (19)$$

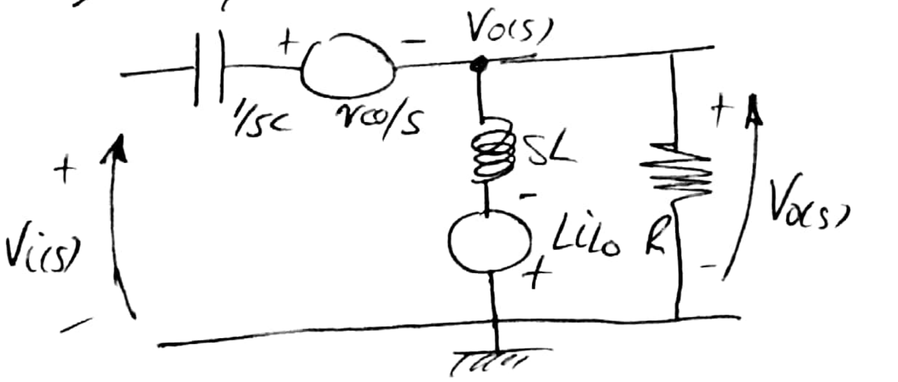
$$\frac{V_{O(s)}}{V_{Co}} = \frac{-\frac{V_{Co}}{s}}{\Delta} = \frac{-\cancel{\frac{V_{Co}}{S}}}{\frac{s^2LCR+LS+R}{LCRS^2}} = \frac{-\cancel{LCR}\,S\,V_{Co}}{\cancel{LCR}\left(s^2+\frac{1}{RC}s+\frac{1}{LC}\right)}$$

$$\frac{V_{O(s)}}{V_{Co}} = \frac{-sV_{Co}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \quad (20)$$

De (18), (19) y (20)

$$V_{O(s)} = -\frac{sV_{Co}+\frac{i_{Lo}}{S}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} + \frac{S^2}{s^2+\frac{1}{RC}s+\frac{1}{LC}}V_{i(s)} \quad (21)$$

De (21) puede observarse claramente la componente Natural (Transitoria), así también como la Forzada.
Se verificará la Ec. (21) por Laplace



Se resuelve por Nodos:
Hay 1 solo nodo que es $V_{O(s)}$

$$V_{O(s)}\left(sC+\frac{1}{SL}+\frac{1}{R}\right) = sC\left(V_{i(s)}-\frac{V_{Co}}{s}\right)-\frac{Li_{Lo}}{SL}$$

$$V_{O(s)} \frac{s^2LCR + R + SL}{SLR} = \cancel{s}C\,\frac{sV_{i(s)} - V_{C0}}{\cancel{s}} - \frac{i_{L0}L}{SL}$$

$$V_{O(s)} = \frac{SLR}{s^2LCR + SL + R} CSV_{i(s)} - \frac{SLRCV_{C0}}{s^2LRC + SL + R} - \frac{\cancel{S}\cancel{L}R}{s^2RLC + SL + R}\,\frac{Li_{L0}}{\cancel{S}\cancel{L}}$$

$$V_{O(s)} = \frac{s^2\cancel{LRC}V_{i(s)}}{\cancel{LCR}\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)} - \frac{S\cancel{LRC}V_{C0}}{\cancel{LRC}\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)} - \frac{L\cancel{R}i_{L0}}{L C\cancel{R}\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

$$V_{O(s)} = -\frac{sV_{C0} + \frac{i_{L0}}{C}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} + \frac{s^2}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} V_{i(s)} \quad (22)$$

Se observa que las Ecs. 21 y 22 son iguales.