Representación de SLIT SISO

Existen distintas formas de representación y análisis de sistemas lineales e invariantes en el tiempo, y particularmente en los sistemas de única entrada y única salida. Entre ellos:

a) Ecuaciones Diferenciales

b) Función Transferencia.

c) Métodos Operacionales (Nodos, Mallas, Norton, etc.).

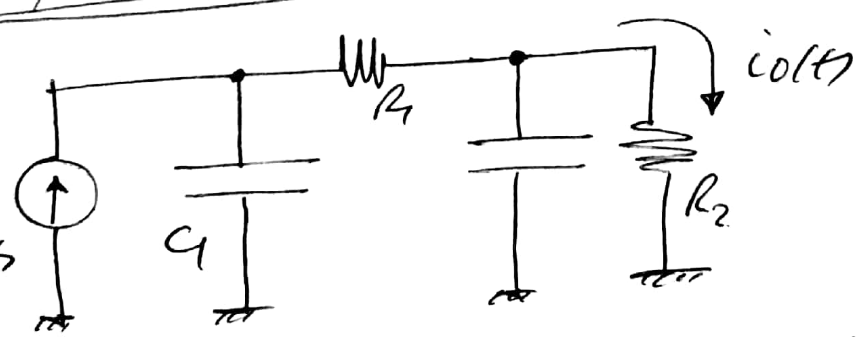
d) Variables de Estado Natural.

e) Variables de Estado Canónicas.

f) Diagrama en Bloques.

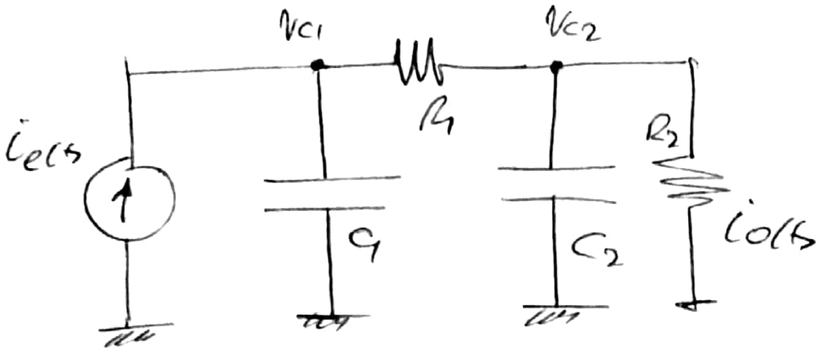
De manera tal de poder comparar cada uno de ellos, se presenta un ejemplo de análisis de caso físico:

Ejemplo de Análisis.

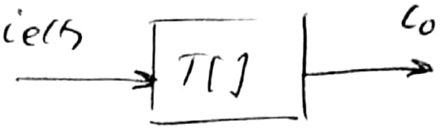


Dado el siguiente circuito eléctrico donde la entrada del mismo es la corriente $i_e(t)$ y la salida la corriente en la resistencia R_2, es decir, $i_0(t)$, analice las distintas representaciones del sistema:

a) Ecuaciones Diferenciales



SISO



$D = \frac{d}{dt}$ (Operador Derivada)

$D^{-1} = \frac{1}{D} = \int_{-\infty}^{t} (\ )\,dt$ (Operador Integral)

$$\begin{cases} i_e(t) = C_1 D V_{c1}(t) + \dfrac{V_{c1}(t) - V_{c2}(t)}{R_1} \\ \dfrac{V_{c1}(t) - V_{c2}(t)}{R_1} = C_2 D V_{c2}(t) + i_o(t) \end{cases} ; \quad i_o(t) = \frac{V_{c2}(t)}{R_2} \rightarrow V_{c2}(t) = R_2\, i_o(t)$$

$$\frac{V_{c1}(t)}{R_1} - \frac{V_{c2}(t)}{R_1} = C_2 D V_{c2}(t) + i_o(t) \ ; \quad \frac{V_{c1}(t)}{R_1} = C_2 D V_{c2}(t) + \frac{V_{c2}(t)}{R_1} + i_o(t)$$

$$\frac{V_{c1}(t)}{R_1} = \left(C_2 D + \frac{1}{R_1}\right) V_{c2}(t) + i_o(t) = \left(C_2 D + \frac{1}{R_1}\right) R_2\, i_o(t) + i_o(t)$$

$$\frac{V_{c1}(t)}{R_1} = C_2 R_2 D\, i_o(t) + \left(1 + \frac{R_2}{R_1}\right) i_o(t) \ ; \quad V_{c1}(t) = R_1\left[C_2 R_2 D\, i_o(t) + \left(1 + \frac{R_2}{R_1}\right) i_o(t)\right]$$

$$i_e(t) = C_1 D V_{c1}(t) + \frac{V_{c1}(t)}{R_1} - \frac{V_{c2}(t)}{R_1}$$; reemplazando $V_{c1}(t)$ y $V_{c2}(t)$ $\rightarrow$

$$i_e(t) = C_1 D R_1\left[C_2 R_2 D\, i_o(t) + \left(1 + \frac{R_2}{R_1}\right) i_o(t)\right] + \cancel{\frac{1}{R_1}}\, \cancel{R_1}\left[C_2 R_2 D\, i_o(t) + \left(1 + \frac{R_2}{R_1}\right) i_o(t)\right] - \frac{R_2}{R_1} i_o(t)$$

$$i_e(t) = C_1 C_2 R_1 R_2 D^2 i_o(t) + C_1 R_1\left(1 + \frac{R_2}{R_1}\right) D\, i_o(t) + C_2 R_2 D\, i_o(t) + i_o(t) + \cancel{\frac{R_2}{R_1} i_o(t) - \frac{R_2}{R_1} i_o(t)}$$

$$i_e(t) = C_1 C_2 R_1 R_2 D^2 i_o(t) + (C_1 R_1 + C_1 R_2 + C_2 R_2) D\, i_o(t) + i_o(t) \rightarrow$$

$$D^2 i_o(t) + \frac{C_1 R_1 + C_1 R_2 + C_2 R_2}{C_1 C_2 R_1 R_2} D\, i_o(t) + \frac{1}{C_1 C_2 R_1 R_2} i_o(t) = \frac{1}{C_1 C_2 R_1 R_2} i_e(t)$$

$$\boxed{i_o''(t) + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_1 C_2}\right) i_o'(t) + \frac{1}{R_1 R_2 C_1 C_2} i_o(t) = \frac{1}{R_1 R_2 C_1 C_2} i_e(t)} \quad (1)$$

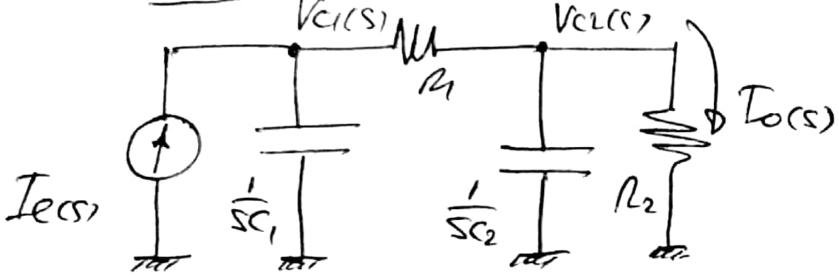
b) Función Transferencia

Transformando por Laplace m.a.m ① se tiene que:

$$S^2 I_{o(s)} + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}\right) s\, I_{o(s)} + \frac{1}{R_1R_2C_1C_2} I_{o(s)} = \frac{1}{R_1R_2C_1C_2} I_{e(s)}$$

$$\Rightarrow \boxed{\frac{I_{o(s)}}{I_{e(s)}} = \frac{\frac{1}{R_1R_2C_1C_2}}{S^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}\right) s + \frac{1}{R_1R_2C_1C_2}}} \quad ②$$

c) Método Operacional de Nodos.



$$I_{o(s)} = \frac{V_{C2(s)}}{R_2}$$

$$\begin{bmatrix} SC_1 + G_1 & -G_1 \\ -G_1 & G_1 + G_2 + SC_2 \end{bmatrix} \cdot \begin{bmatrix} V_{C1(s)} \\ V_{C2(s)} \end{bmatrix} = \begin{bmatrix} I_{e(s)} \\ 0 \end{bmatrix}$$

$$\Delta Y_{(s)} = (SC_1 + G_1)(G_1 + G_2 + SC_2) - G_1^2$$

$$\Delta Y_{(s)} = S^2C_1C_2 + SC_1G_1 + SC_1G_2 \cancel{+ G_1^2} + G_1G_2 + SC_2G_1 \cancel{- G_1^2}$$

$$\Delta Y_{(s)} = S^2C_1C_2 + S(C_1G_1 + C_1G_2 + C_2G_1) + G_1G_2$$

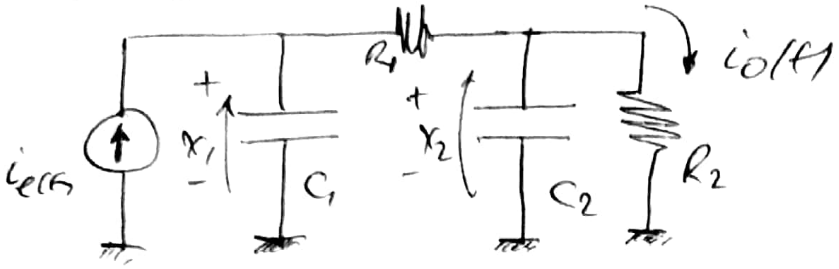
$$V_{C2(s)} = \frac{\Delta V_{C2(s)}}{\Delta Y_{(s)}} = \frac{\det \begin{bmatrix} SC_1 + G_1 & I_{e(s)} \\ -G_1 & 0 \end{bmatrix}}{\Delta Y_{(s)}} = \frac{G_1 I_{e(s)}}{\Delta Y_{(s)}}$$

$$I_{o(s)} = \frac{V_{C2(s)}}{R_2} = \frac{1}{R_2}\frac{G_1}{\Delta Y_{(s)}} I_{e(s)} = \frac{G_1 \cdot I_{e(s)}}{R_2\left(S^2C_1C_2 + (C_1G_1 + C_1G_2 + C_2G_1)S + G_1G_2\right)}$$

$$\frac{I_{o(s)}}{I_{e(s)}} = \frac{\frac{1}{R_1R_2}}{S^2C_1C_2 + \left(\frac{C_1}{R_1} + \frac{C_1}{R_2} + \frac{C_2}{R_1}\right)S + \frac{1}{R_1R_2}}$$

$$\boxed{\frac{I_{o(s)}}{I_{e(s)}} = \frac{\frac{1}{R_1R_2C_1C_2}}{S^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}\right)S + \frac{1}{R_1R_2C_1C_2}}}$$ ③

d) Variables de Estado Naturales



$i_{e(t)} = C_1\dot{x}_1 + \frac{x_1 - x_2}{R_1}$; $C_1\dot{x}_1 = -\frac{x_1-x_2}{R_1} + i_e = -\frac{1}{R_1}x_1 + \frac{1}{R_1}x_2 + i_e$

$\dot{x}_1 = -\frac{1}{R_1C_1}x_1 + \frac{1}{R_1C_1}x_2 + \frac{1}{C_1}i_e$; $\frac{x_1 - x_2}{R_1} = C_2\dot{x}_2 + \frac{x_2}{R_2}$

$C_2\dot{x}_2 = \frac{x_1}{R_1} - \left(\frac{1}{R_1}+\frac{1}{R_2}\right)x_2$; $\dot{x}_2 = \frac{1}{R_1C_2}x_1 - \frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)x_2$

$i_{o(t)} = \frac{x_2}{R_2}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-\frac{1}{R_1C_1} & \frac{1}{R_1C_1}\\ \frac{1}{R_1C_2} & -\frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{1}{C_1}\\ 0\end{bmatrix}}_{B} i_e$$

$$i_{o(t)} = \underbrace{\begin{bmatrix}0 & 1/R_2\end{bmatrix}}_{C}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0\end{bmatrix}}_{D} i_{e(t)}$$

Recordando que $\frac{I_{o(s)}}{I_{e(s)}} = C(SI-A)^{-1}B + D \longrightarrow$

$$(SI-A) = \begin{pmatrix}S & 0\\ 0 & S\end{pmatrix} - \begin{pmatrix}-\frac{1}{R_1C_1} & \frac{1}{R_1C_1}\\ \frac{1}{R_1C_2} & -\frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\end{pmatrix} = \begin{pmatrix}S+\frac{1}{R_1C_1} & -\frac{1}{R_1C_1}\\ -\frac{1}{R_1C_2} & S+\frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\end{pmatrix}$$

$$\det(SI-A) = \left(S+\frac{1}{R_1C_1}\right)\left(S+\frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right) - \frac{1}{R_1^2C_1C_2}$$

$$\det(SI-A) = S^2 + S\left(\frac{1}{R_1C_2}+\frac{1}{R_2C_3}+\frac{1}{R_1C_1}\right)+\frac{1}{R_1C_1C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-\frac{1}{R_1^2C_1C_2}$$

$$\det(SI-A) = S^2 + \left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_2}\right)S+\cancel{\frac{1}{R_1^2C_1C_2}}+\frac{1}{R_1R_2C_1C_2}\cancel{-\frac{1}{R_1^2C_1C_2}}$$

$$\det(SI-A) = S^2 + \left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_2}\right)S+\frac{1}{R_1R_2C_1C_2}$$

$$(SI-A)^{-1} = \frac{\text{Mat Adj}(SI-A)}{\det(SI-A)} = \frac{\begin{array}{cc} a_{11} & a_{12} \\ \begin{pmatrix} S+\frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & +\frac{1}{R_1C_1} \\ +\frac{1}{R_1C_2} & S+\frac{1}{R_1C_1} \end{pmatrix} \\ a_{21} & a_{22} \end{array}}{\det(SI-A)}$$

$$\frac{I_o(s)}{I_e(s)} = \left(0 \quad 1/R_2\right)\frac{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\frac{1}{C_1} \\ 0\end{pmatrix}}{\det(SI-A)} + 0$$

$$\frac{I_o(s)}{I_e(s)} = \frac{\left(0 \quad 1/R_2\right)\begin{pmatrix} a_{11}/C_1 \\ a_{21}/C_1\end{pmatrix}}{\det(SI-A)} = \frac{a_{21}/R_2C_1}{\det(SI-A)}$$

$$\boxed{\frac{I_o(s)}{I_e(s)} = \frac{\frac{1}{R_1R_2C_1C_2}}{S^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_2}\right)S+\frac{1}{R_1R_2C_1C_2}}} \quad (4)$$

e) Variables de Estado Canónicas

Dada una transferencia de orden "n"

$$M(s) = \frac{b_nS^n+b_{n-1}S^{n-1}+\dots+b_2S^2+b_1S+b_0}{S^n+a_{n-1}S^{n-1}+\dots+a_2S^2+a_1S+a_0}$$

El modelo "Canónico Controlable" de Variable de Estado de la transferencia M(s), estará dado por el siguiente conjunto de matrices:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \cdots & 0 \\ & & \vdots & & & \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix};\quad B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix};\quad D=[b_n]$$

$$C=\left[(b_0-b_na_0)\ (b_1-b_na_1)\ (b_2-b_na_2)\ \cdots\ (b_{n-1}-b_na_{n-1})\right]$$

En nuestro caso:

$$\frac{I_o(s)}{I_e(s)}=\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_2}\right)s+\frac{1}{R_1R_2C_1C_2}}=\frac{b_0}{s^2+a_1s+a_0}$$

$$A=\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix};\quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix};\quad C=\begin{bmatrix} b_0 & 0 \end{bmatrix};\quad D=[0]$$

$$(sI-A)=\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}-\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}=\begin{pmatrix} s & -1 \\ a_0 & s+a_1 \end{pmatrix}$$

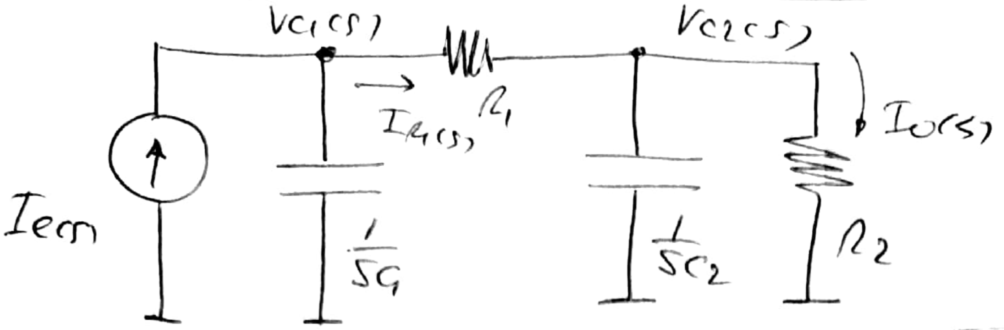
$$\det(sI-A)=s^2+a_1s+a_0;\quad (sI-A)^{-1}=\frac{\begin{pmatrix} s+a_1 & 1 \\ -a_0 & s \end{pmatrix}}{\det(sI-A)}$$

$$\frac{I_o(s)}{I_e(s)}=C(sI-A)^{-1}B+D=(b_0\ \ 0)\frac{\begin{pmatrix} s+a_1 & 1 \\ -a_0 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\det(sI-A)}$$

$$\frac{I_o(s)}{I_e(s)}=\frac{(b_0\ \ 0)\begin{pmatrix} 1 \\ s \end{pmatrix}}{\det(sI-A)}=\frac{b_0}{s^2+a_1s+a_0},\ \text{es decir}$$

$$\boxed{\frac{I_o(s)}{I_e(s)}=\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_2}\right)s+\frac{1}{R_1R_2C_1C_2}}}$$ (5)

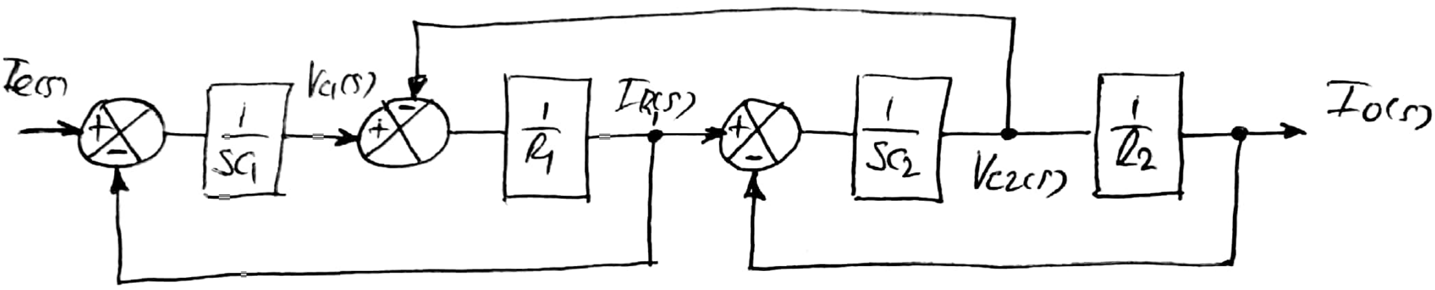
1) Diagrama de Bloques



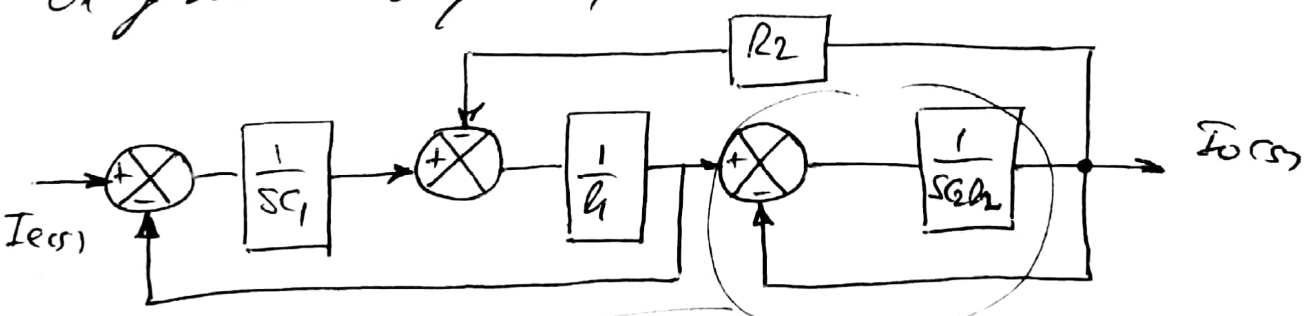
$$I_e(s) = sC_1 V_{c1}(s) + I_{R1}(s) \rightarrow V_{c1}(s) = \frac{1}{sC_1}\left(I_e(s) - I_{R1}(s)\right)$$

$$I_{R1}(s) = \frac{1}{R_1}\left(V_{c1}(s) - V_{c2}(s)\right) \; ; \; I_{R1}(s) = sC_2 V_{c2}(s) + I_o(s)$$

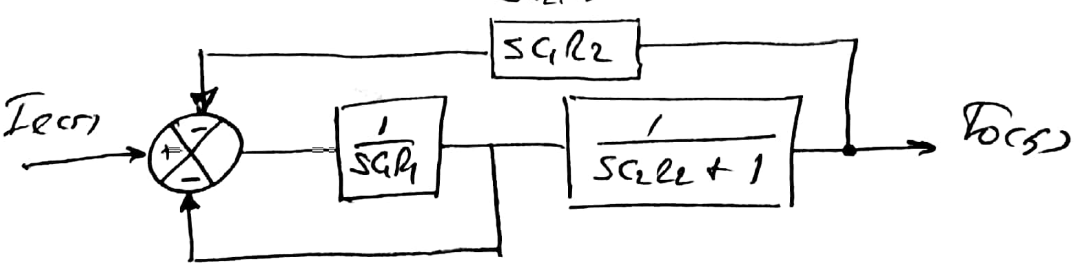
$$V_{c2}(s) = \frac{1}{sC_2}\left(I_{R1}(s) - I_o(s)\right) \; ; \; I_o(s) = \frac{V_{c2}(s)}{R_2}$$

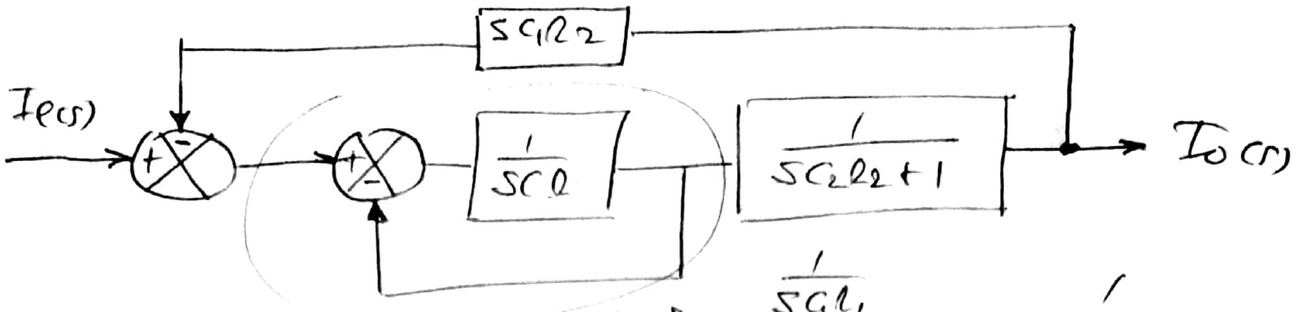


Este diagrama de bloques se puede simplificar con álgebra de bloques, es decir:

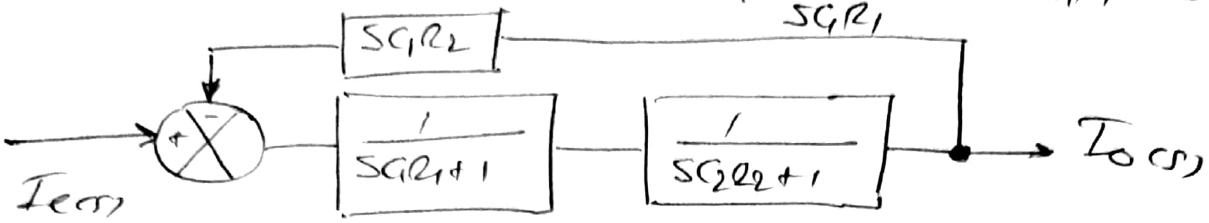


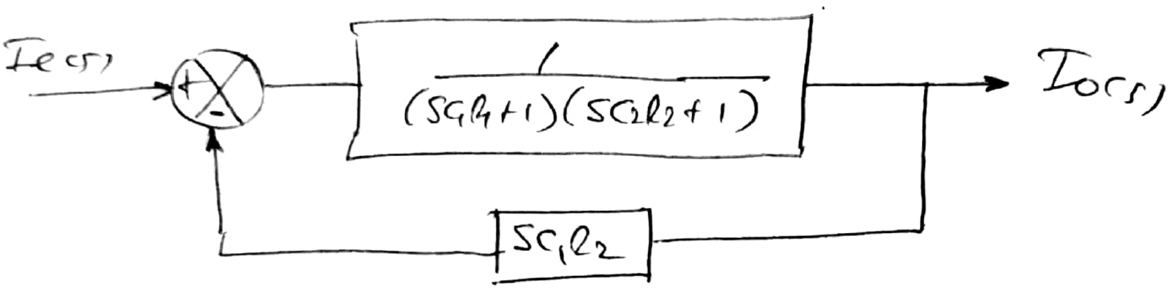
$$\frac{\frac{1}{sC_2R_2}}{1 + \frac{1}{sC_2R_2}} = \frac{1}{sC_2R_2 + 1}$$





$$\frac{\frac{1}{sC_1R_1}}{1+\frac{1}{sC_1R_1}} = \frac{1}{sC_1R_1+1}$$





$$\frac{I_o(s)}{I_e(s)} = \frac{\frac{1}{(sC_1R_1+1)(sC_2R_2+1)}}{1+\frac{sC_1R_2}{(sC_1R_1+1)(sC_2R_2+1)}}$$

$$\frac{I_o(s)}{I_e(s)} = \frac{1}{(sC_1R_1+1)(sC_2R_2+1)+sC_1R_2} = \frac{1}{s^2C_1C_2R_1R_2+s(C_1R_1+C_2R_2)+1+sC_1R_2}$$

$$\frac{I_o(s)}{I_e(s)} = \frac{1}{s^2C_1C_2R_1R_2+s(C_2R_2+C_1R_1+C_1R_2)+1} = \frac{\frac{1}{C_1C_2R_1R_2}}{s^2+\frac{C_2R_2+C_1R_1+C_1R_2}{C_1C_2R_1R_2}s+\frac{1}{C_1C_2R_1R_2}}$$

$$\boxed{\frac{I_o(s)}{I_e(s)} = \frac{\frac{1}{R_1R_2C_1C_2}}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_2}\right)s+\frac{1}{R_1R_2C_1C_2}}} \quad (6)$$

Investigue el uso de las funciones Matlab:
tf; ode45; tf2ss; ss2tf; conv; lsim.