Habib University
CS-113 Discrete Mathematics
Spring 2018
Homework 1

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Due: 2nd February, 2018

1. (a) $\boxed{\text{5 points}}$ You are given four cards, each of which has a number on one side and a letter on another. You place them on a table in front of you and the four cards read *B 5 2 J*. Which cards would you have to turn over, to test the following rule? Explain your choice.

If there is a 5 on one side, there is a J on the other side.

Solution: In order to test the given statement logically, we have to interpret the given statements as logical propositions.
Let,

p: Number 5 on one side of card
q: Letter J on other side of card.

In the lights of logical reasoning, we only have to test two cards in order to prove the validity of given statement.

Actual Statement:

$$p \to q$$

Therefore, if we flip the card labelled "5" and we see that on the other side there is a letter "J", the statement is proved.

Contrapositive:

$$\neg q \to \neg p$$

We know that a statement and its contrapositive are equivalent. Therefore, the statement is not tested valid for all the cases. There is still one case left where this statement can be proved in-valid. In order to be fully sure about the validity of given statement we will have to check for the possibility that there must not exist be a number "5" on the other side of the card if the card is labelled "J" on one side.

Hence the above statement can be proved valid or in-valid by testing it for two scenarios.

(b) $\boxed{\text{5 points}}$ Express the following statement with a propositional function $P(x)$, where $x \,\varepsilon \{1, 2, 3, 4, 5\}$, without using quantifiers, instead using only logical operators $\neg$, $\vee$, and $\wedge$.

$$\exists x(\neg P(x)) \wedge \forall x((x \neq 2) \to P(x))$$

Solution:
In order to remove the quantifiers, we must write the given statement for all possible entries

of the given domain. We know, in general, that the mathematical representation of "for all" involves the operator "$\wedge$" between all the propostions for the given domain values. Similarly, "there exists" corresponds to the operator "$\vee$".

Therefore, the given statement can be written as:

$$\exists x(\neg P(x)) \wedge \forall x(\neg(x=2) \rightarrow P(x))$$
$$\exists x(\neg P(x)) \wedge \forall x((x=2) \vee P(x))$$
$$(\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5)) \wedge (((1=2) \vee P(1)) \wedge ((2=2) \vee P(2)) \wedge ((3=2) \vee P(3)) \wedge ((4=2) \vee P(4)) \wedge ((5=2) \vee P(5)))$$

Simplifying the above result on the basis of mathematical reasoning we know. For example '1' is not equal to '2' so we write False (F) in place of '1=2'. Similarly, we write True (T) in place of '2=2'.

$$(\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5)) \wedge ((F \vee P(1)) \wedge (T \vee P(2)) \wedge (F \vee P(3)) \wedge (F \vee P(4)) \wedge (F \vee P(5)))$$

Applying some logical simplification techniques to decrease the number of terms. We know that a logical OR statement is always true even if only one statement is true, while a logical AND statement is always false even if only one statement is false.
Therefore,

$$(\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5)) \wedge (P(1)) \wedge (T) \wedge (P(3)) \wedge (P(4)) \wedge (P(5))$$

A "True" in the series of many propostions connected via AND operator can be simplified further.

$$(\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5)) \wedge (P(1)) \wedge (P(3)) \wedge (P(4)) \wedge (P(5))$$

(c) $\boxed{\text{5 points}}$ Let $F(x, y)$ be the statement *x can fool y*, where the domain consists of all people in the world. Use quantifiers to express the following statement:

There are exactly two people whom everybody can fool.

Solution:

F(x, y): x can fool y

So,

$$\exists x \exists y(\forall z(F(z,x) \wedge F(z,y) \wedge (x \neq y) \wedge (\forall p(F(z,p) \rightarrow ((p=x) \vee (p=y)))))$$

2. $\boxed{\text{10 points}}$ Show that $(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r)$ is a tautology, without use a truth table. Show your reasoning.

Solution: The above proof can be done in two ways. The first one is an analytical proof where the statements can be proved using some logical arguments. While the other proof is a formal proof where the statement will be proved using logical rules and simplification techniques.

Thinking analytically, if $p \lor q$ is taken true, then from $p \lor q$, either p is true or q is true.
In the next statement, if $\neg p \lor r$ is taken true, then either $\neg p$ is true or r. Looking at both the statements on a whole, we see that both p and $\neg p$ can not be true simultaneously. Hence, the possiblity of existing p or $\neg p$ as truth values gets nullified. Hence, the actual answer must lie in either $\underline{q \text{ or } r}$

The other way of proving is following:

$$(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r)$$
$$\neg((p \lor q) \land (\neg p \lor r)) \lor (q \lor r)$$
$$\neg(p \lor q) \lor \neg(\neg p \lor r) \lor (q \lor r)$$
$$(\neg p \land \neg q) \lor (p \land \neg r) \lor (q \lor r)$$

Using "simplification property" on the above statement:

$$(\neg p) \lor (p) \lor (q \lor r)$$
$$\neg p \lor p \lor q \lor r$$

and since, $\neg p \lor p$ is always a tautology, so:

$$T \lor q \lor r$$

Since, either term OR'ed with a T is always true. Therefore:

$$T(Tautology)$$

3. $\boxed{\text{20 points}}$ Suppose A, B, C are sets. And $C \neq \emptyset$. Show that if $A \times C = B \times C$, then $A = B$.

Solution: We start off using some general sets such that:

$$A = \{a_1, a_2, a_3, ..., a_n\}$$
$$B = \{b_1, b_2, b_3, ..., b_n\}$$

and,

$$C = \{c_1, c_2, c_3, ..., c_n\}$$

By the definition of cross-product in sets,

$$A \times C = \{(a_1, c_1), (a_2, c_2), (a_3, c_3), ..., (a_n, c_n)\}$$

and,

$$B \times C = \{(b_1, c_1), (b_2, c_2), (b_3, c_3), ..., (b_n, c_n)\}$$

The results of $A \times C$ and $B \times C$ can only be equal if the corresponding values are equal. For example, $a_1 = b_1, a_2 = b_2, a_3 = b_3 ... a_n = b_n$, which in turn means that the set A must be equal to the set B.

Moreover for infinite sets A, B and C, we know that if one-one correspondence exists between two sets then the cardinalities of domain and co-domain must be equal in every case where the sets are countable i.e. the set of should have exact same number of distinct mappings in its co-domain as the number of elements. Hence, in this case the function of cross product should be bijective. From the argument it follows that in (i) if the cardinalities are equal and the corresponding elements are also equal then A must be equal to B.
Hence, proved.

4. For S in the domain of a function f^*, let $f^*(S) = \{f(x) : x \in S\}$ where function $f(x)$ is in the domain of $x \in S$. Let C and D be subsets of the domain of f^*.

   (a) $\boxed{\text{20 points}}$ Show that $f^*(C \cap D) \subseteq f^*(C) \cap f^*(D)$.

   Solution: Let $f^*(x)$ be any function such that $y = f^*(x)$. Let's assume that the x, which is the domain of $f^*(x)$, actually belongs to $(C \cap D)$. So, if $x \in (C \cap D)$ then it is confirmed that x belongs to both C and D. Mathematically, $x \in C$ and $x \in D$.
   At this point we know the status of x that it belongs to both the given sets, C and D. Our actual assumed function was $y = f^*(x)$; and if that is the case, then it must be true that $y(which is f^*(x))$ belongs to set C and D (since x is a subset of C as well as D). Therefore, from the argument it follows that $f^*(x)(f^*(C \cap D))$ must be the subset of intersection of two functions which are composed of a super-set for x (belongs to $C \cap D$). Hence, the whole argument proves that $f^*(C \cap D) \subseteq f^*(C) \cap f^*(D)$.

   (b) $\boxed{\text{5 points}}$ Give an example where equality does not hold in the previous part.

   Solution: Now that we have proved the statement for the sets which have some elements in common, we will prove this part using the same rule and if it gets violated, then we reach the conclusion that the equality does not hold for disjoint sets. In order to prove this part, we have to assume that C and D are completely distinct sets and their intersection gives us a nullset. Let's assume it for a special function where $y = x^2$. For a given set of C, let's say $C = \{-1\}$. Similarly, the set D is assumed to be $D = \{1\}$. If the function is applied on the intersection $C \cap D$, then it is a null set. On the other hand if the function is applied on individual sets first and then the intersection is taken, the answer will be the set $\{1\}$. This proves that the given equality does not hold in all cases.

5. Five guests were invited to an exclusive party at Dr. Harvey's mansion. However during the long dark night, the scandalous Dr. Harvey was found dead in his office. The trouble is, every member of the party went to his office, each at a different time, with a different motive, and a different weapon. There was a fingerprint, a footprint, a blood drop, a note and a strand of hair found at the murder site. Forensics revealed that the suspect who entered the room at 10.30 pm did it. From their statements below, can you work out who killed Dr. Harvey?

   Colonel Woody - "I never left a footprint because a woman did. I entered the room before the person who took in the poison. I must say though, she was in there for a quarter of an hour before someone else went in."

   Lady Rose - "Okay I admit it! I took in the revolver, even though my motive wasn't revenge. A man entered the room after me and his motive was either rage or blackmail. They all took their sweet time about it, but no man took more than 40 minutes."

   Sir Tarantino - "I did leave a fingerprint, but that doesn't explain why Professor Selma lost a hair, does it? Oh yes, the person who entered seventy minutes before me took in the lead pipe. Then there was the man with the dagger!"

   Reverend Spacey - "I entered after a woman, who did not take in a rope, because the last person to visit him did. I was in there for over 35 minutes, confronting Dr. Harvey over my motive, which may I say wasn't greed or blackmail."

   Professor Selma - "Yes, you caught me! My motive was jealousy, but it wasn't as bad as that man's blackmail motive, who entered at five minutes past ten. I entered before another man who left the incriminating clue of the blood drop."

   (a) $\boxed{\text{20 points}}$ For each suspect, write their testimony in the form of several well-defined propositions, using quantifiers wherever possible, in addition to negations, disjunctions and conjunctions. You may use the following statements to construct your propositions:

   Clue(x, y) : suspect x left clue y
   Weapon(x, y) : suspect x used weapon y
   Motive(x, y) : suspect x had motive y
   Order(x,y,z) : suspect x entered y minutes before suspect z

   Solution: Let 'm' be the set of all males and 'f' be the set of all females i.e.

   $$f = \{Selma, Rose\}$$
   $$m = \{Woody, Tarantino, Spacey, Harvey\}$$

   Moreover, let 'x' be the set of all persons.

   $$x = \{Woody, Tarantino, Spacey, Harvey, Selma, Rose\}$$

   From the above mentioned statements of all the five persons, we can write the predicate functions for all those statements. Following the same format for the predicate formation as given in the question, we get:

   <u>Colonel Woody:</u>

   (1). $\exists m((Clue(m, footprint) \wedge \neg(m = Woody))$
   (2). $\exists! f(Clue(f, footprint))$
   (3). $\exists! f(Weapon(f, poison))$
   (4). $\exists! f \exists! x(Order(f, 15, x))$

Lady Rose:

(5). $Weapon(Rose, Revolver)$
(6). $\exists! f(\neg(Motive(f, Revenge) \wedge f = Rose)$
(7). $\exists! m(Order(Rose, 40, m))$
(8). $\exists! m(Motive(m, Rage) \vee Motive(m, Blackmail)$

Sir Tarantino:

(9). $Clue(Tarantino, Fingerprint))$
(10). $Clue(Selma, hair))$
(11). $\exists! m \exists x(Order(x, 70, m) \wedge m = Tarantino)$
(12). $\exists! x(Weapon(x, leadpipe))$
(13). $\exists! m(Weapon(m, dagger))$

Reverend Spacey:

(14). $\exists! f \neg(Weapon(f, Rope))$
(15). $\exists! m((\neg Motive(m, greed) \vee \neg Motive(m, blackmail)) \wedge m = Spacey)$
(16). $\exists! f(Order(Spacey, 35, f))$

Professor Selma:

(17). $\exists! m(Motive(m, blackmail))$
(18). $\exists! m(Clue(m, blooddrop))$
(19). $Motive(Selma, Jealousy)$.

(b) $\boxed{\text{10 points}}$ Use your propositions from the previous part to fill in the following table:

	Weapon	Motive	Clue	Time
Woody	Lead pipe	Blackmail	Note	10:05 pm
Rose	Revolver	Rage	Footprint	11:20
Tarantino	Rope	Greed	Fingerprint	11:50 pm
Spacey	Dagger	Revenge	Blood drop	10:45 pm
Selma	Poison	Jealousy	Hair	10:30 pm