TWO-DIMENSIONAL HEAT TRANSFER ANALYSIS

Finite element analysis software tools

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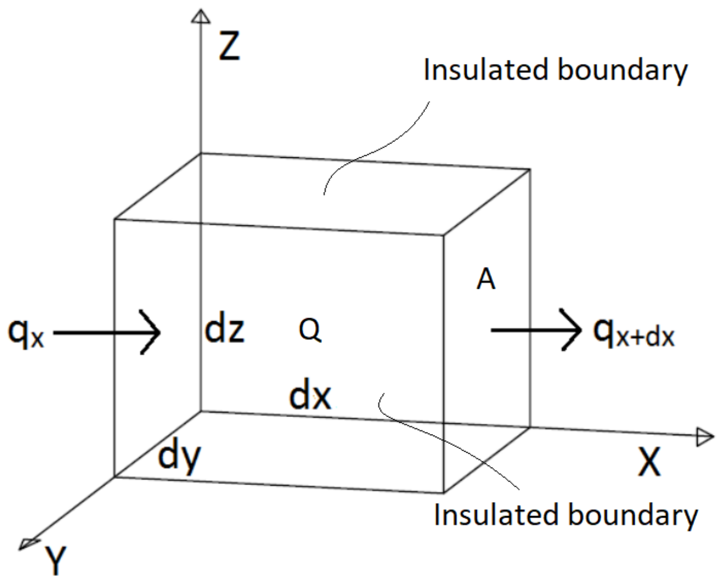
Governing differential equations of heat conduction/ diffusion problem

A. One-Dimensional Heat Conduction (Without Convection)



<u>Control volume for one-dimensional heat conduction</u>

We start with the control volume shown above. By conservation of energy, we have

$$E_{in} + E_{generated} = E_{stored} + E_{out} \qquad \text{Eq A1}$$

$$q_x A dt + QAdxdt = \Delta U + q_{x+dx} A dt \qquad \text{Eq A2}$$

q_x is the heat conducted (heat flux) into the control volume at surface edge x, in units of kW/m^2

q_{x+dx} is the heat conducted out of the control volume at the surface edge x+dx.

ΔU is the change in stored energy. The change in stored energy can be expressed by

$$\Delta U = specific\ heat\ \times mass \times change\ in\ temperature$$

$$\Delta U = c(\rho A dx)dT \qquad \text{Eq A3}$$

Q is the internal heat source (heat generated per unit time per unit volume is positive or a heat sink, heat drawn out of the volume, is negative) kW/m^3

A is the cross-sectional area perpendicular to heat flow q

By Fourier's law of heat conduction

$$q_x = -K_{xx}\frac{dT}{dx} \qquad \text{Eq A4}$$

K_{xx} is the thermal conductivity in the x direction in kW/m.degC

dT/dx is the temperature gradient in degC/m

Eq A4 states that the heat flux in the x direction is proportional to be gradient of temperature in x direction. The minus sign implies that, heat flow is positive in the direction opposite the direction of temperature increase.

$$q_{x+dx} = -K_{xx}\frac{dT}{dx}\bigg|_{x\,+\,dx} \qquad \text{Eq A5}$$

By Taylor series expansion for any function f(x), we have

$$f_{x+dx} = f_x + \frac{df}{dx} + \frac{d^2 f}{dx^2}\frac{dx^2}{2} + \cdots$$

Therefore, using first two term of Taylor series, Eq A5 becomes

$$q_{x+dx} = -\left[K_{xx}\frac{dT}{dx} + \frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)dx\right] \qquad \text{Eq A6}$$

Substituting in Eq A2

$$\begin{aligned} -K_{xx}\frac{dT}{dx}Adt + QAdxdt &\\ &= c(\rho Adx)dT \\ &\quad - \left[K_{xx}\frac{dT}{dx} + \frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)dx\right]Adt \end{aligned}$$

Simplifying

$$\left[\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)dx\right]Adt + QAdxdt = c(\rho Adx)dT$$

Dividing by Adxdt gives

$$\left[\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right)\right] + Q = c\rho\frac{\partial T}{\partial t} \qquad \text{Eq A7}$$

Eq A7 is the one-dimensional heat conduction equation.

For steady state, any differentiation with respect to time is equal to zero, so Eq A7 becomes

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right) + Q = 0 \qquad \text{Eq A8}$$

For constant thermal conductivity and steady state, Eq A7 becomes

$$K_{xx}\frac{d^2T}{dx^2} + Q = 0 \qquad \text{Eq A9}$$

The boundary conditions are of the form

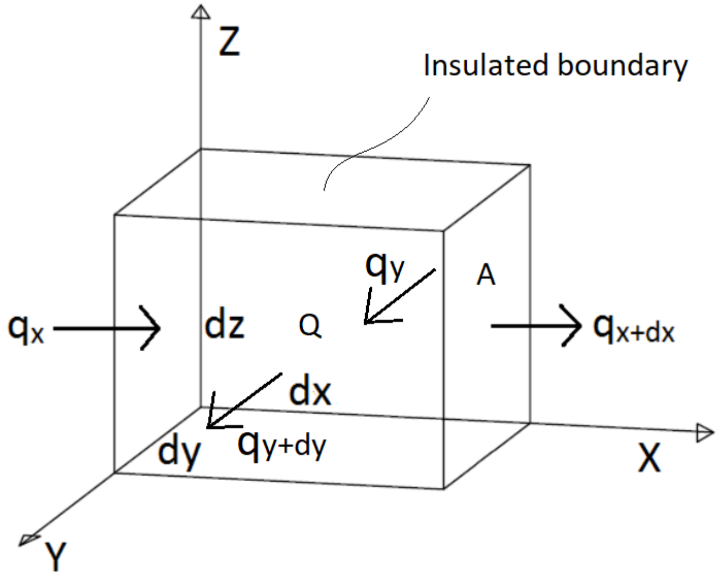
Dirichlet BC $\qquad T = T_B \qquad$ on Surface S_1

Where TB represents a known boundary temperature and S_1 is a surface where the temperature is known.

Neumann BC $\qquad q_x^* = -K_{xx}\frac{dT}{dx} = constant \qquad$ on surface S_2

Where heat flux or temperature gradient q_x is known on surface S_2

B. Two-Dimensional Heat Conduction (Without Convection)



Control volume for two-dimensional heat conduction

Similar to one dimensional case, the two-dimensional steady state heat conduction equation is shown below

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{dT}{dy}\right)+Q=0 \qquad \text{Eq B1}$$

with boundary conditions

Dirichlet BC $\qquad T = T_B \qquad$ on Surface S_1

Where TB represents a known boundary temperature and S_1 is a surface where the temperature is known.

Neumann BC $\qquad q_n^* = K_{xx}\frac{dT}{dx}C_x + K_{yy}\frac{dT}{dy}C_y = constant \qquad$ on surface S_2

Where heat flux or temperature gradient q_x is known on surface S_2

C. Heat Transfer with Convection

Heat flow by convective heat transfer is given by Newton's law of cooling

$$q_h = h(T - T_\infty) \qquad \text{Eq C1}$$

Where

h is the heat transfer or convection coefficient in kW/m^2

T is the temperature of the solid surface at the solid/ fluid interface

T_∞ is the temperature of the fluid (free-stream fluid temperature)

$$q_x A dt + Q A dx dt = c\rho A dx dT + q_{x+dx} A dt + q_h P dx dt \qquad \text{Eq C2}$$

P is he perimeter around the constant cross-sectional area A

Now the one-dimensional heat conduction equation with convection is

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right) + Q = c\rho\frac{\partial T}{\partial t} + \frac{hP}{A}(T - T_\infty) \qquad \text{Eq C3}$$

The steady state differential equation for the same is

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right) + Q = \frac{hP}{A}(T - T_\infty) \qquad \text{Eq C4}$$

With boundary conditions (1) specified temperature on S_1, (2) temperature gradient – heat flux on S_2 and (3) loss of heat by convection on surface S_3

Robin BC $$-K_{xx}\frac{dT}{dx} = h(T - T_\infty)$$ on surface S_3

Similarly, the two-dimensional heat conduction equation with convection is

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{dT}{dy}\right)+Q = c\rho\frac{\partial T}{\partial t}+\frac{2h}{b}(T-T_{\infty}) \qquad \text{Eq D1}$$

b is the thickness of the 2D element, 2h is due convection happening on the two side of the element.

The steady state differential equation for the 2D heat conduction equation with convection is

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{dT}{dx}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{dT}{dy}\right)+Q=\frac{2h}{b}(T-T_{\infty}) \qquad \text{Eq D2}$$

D. Principle of minimum potential energy and heat conduction matrix

Derivation of element conduction matrix and equations

Temperature gradient matrix {g}

$$[g]=\left\{\frac{dT}{dx}\right\}=\frac{d}{dx}[N][t]=[B][t] \qquad \text{Eq F1}$$

$$[B]=\frac{d}{dx}[N]$$

$$[D]=[K_{xx}] \qquad \text{Eq F2}$$

B matrix is like strain displacement matrix and D matrix is material property matrix

For conservative systems, of all temperature fields, those corresponds to equilibrium extremize the total potential energy. If the extreme condition's total potential energy is minimum, the equilibrium state is stable. In other words, from a super imposed external temperature, a structure will lose its potential energy by conduction to attain equilibrium. This state's potential energy will be minimum.

Total Potential energy

$$\pi = U+\Omega_Q+\Omega_q+\Omega_h$$

From principle of minimum potential energy

$$\partial\pi = 0 \qquad \text{Eq. F3}$$

Total heat conduction in the infinitesimal element

$$U = \frac{1}{2}\iiint_V \left[K_{xx}\left(\frac{dT}{dx}\right)^2\right] dV$$

$$U = \frac{1}{2}\iiint_V [g]^T[D][g]\, dV$$

$$U = \frac{1}{2}[t]^T \iiint_V [B]^T[D][B]\, dV[t] \qquad \text{Eq. F4}$$

Internal heat source

$$\Omega_Q = -\iiint_V QT\, dV$$

$$\Omega_Q = -\iiint_V [t]^T[N]^T Q\, dV \qquad \text{Eq. F5}$$

Heat flow (flux)

$$\Omega_q = -\iint_{S_2} q^* T\, dS$$

$$\Omega_q = -\iint_{S_2} [t]^T[N]^T q^*\, dS \qquad \text{Eq. F6}$$

Convection loss

$$\Omega_h = \frac{1}{2}\iint_{S_3} h(T - T_\infty)^2\, dS$$

$$\Omega_h = \frac{1}{2}\iint_{S_3} h([t]^T[N]^T - T_\infty)^2\, dS$$

$$\Omega_h = \frac{1}{2}\iint_{S_3} h[[t]^T[N]^T[N][t] - ([t]^T[N]^T + [N][t])T_\infty + T_\infty^2]\, dS \qquad \text{Eq. F7}$$

Total potential energy

$$\pi = \frac{1}{2}[t]^T \iiint_V [B]^T[D][B]\,dV[t] - [t]^T \iiint_V [N]^T Q\,dV - [t]^T \iint_{S_2} [N]^T q^*\,dS + \frac{1}{2}\iint_{S_3} h[[t]^T[N]^T[N][t] - ([t]^T[N]^T + [N][t])T_\infty + T_\infty^2]\,dS \quad \text{Eq. F8}$$

Minimizing the total potential energy

$$\frac{\partial\pi}{\partial t} = \iiint_V [B]^T[D][B]\,dV[t] - \iiint_V [N]^T Q\,dV - \iint_{S_2} [N]^T q^*\,dS + \iint_{S_3} h[N]^T[N][t]\,dS - \iint_{S_3} h[N]^T T_\infty\,dS = 0 \quad \text{Eq. F9}$$

$$\iiint_V [B]^T[D][B]\,dV[t] + \iint_{S_3} h[N]^T[N][t]\,dS = -\iiint_V [N]^T Q\,dV - \iint_{S_2} [N]^T q^*\,dS - \iint_{S_3} h[N]^T T_\infty\,dS$$

$$\left[\iiint_V [B]^T[D][B]\,dV + \iint_{S_3} h[N]^T[N]\,dS\right][t] = -\iiint_V [N]^T Q\,dV - \iint_{S_2} [N]^T q^*\,dS - \iint_{S_3} h[N]^T T_\infty\,dS \quad \text{Eq. F10}$$

Simplifying the Eqn F10 gives

$$[k][t] = [f_Q] + [f_q] + [f_h] \quad \text{Eq. F11}$$

Where k is the element conduction matrix

$$[k] = \iiint_V [B]^T[D][B]\,dV + \iint_{S_3} h[N]^T[N]\,dS$$

f_Q Heat source (positive)/ sink (negative) matrix

$$[f_Q] = \iiint_V [N]^T Q\,dV$$

f_q Heat flux matrix (into the surface (positive)/ out the surface (negative))

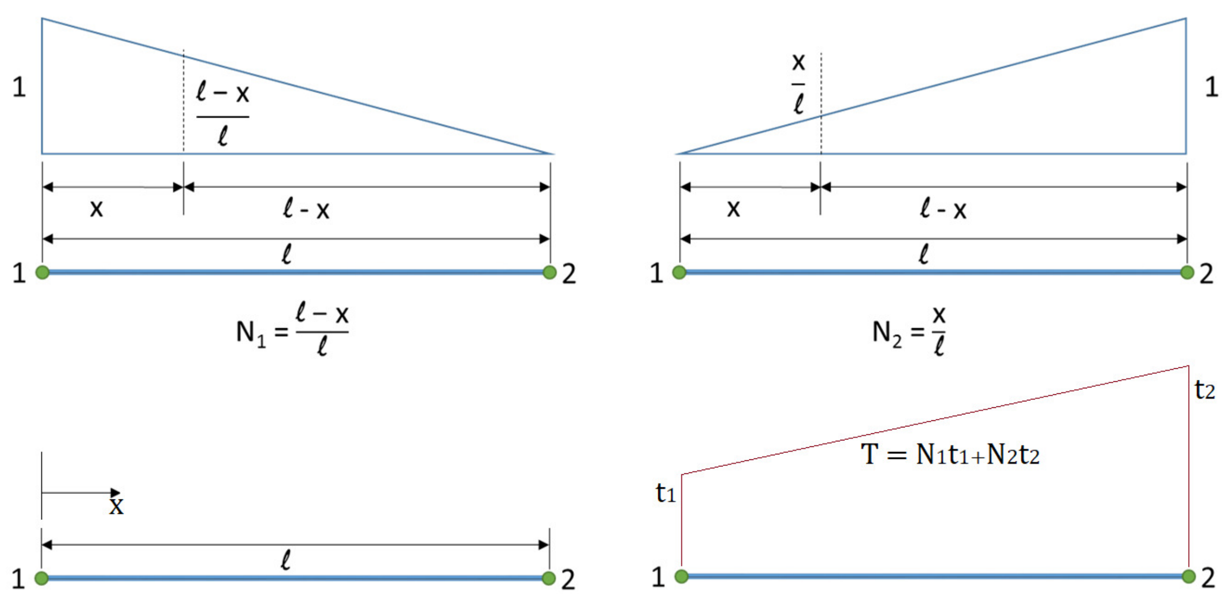
$$[f_q] = \iint_{S_2} [N]^T q^* \, dS$$

f_h Heat convection matrix

$$[f_h] = \iint_{S_3} h[N]^T T_\infty \, dS$$

E. One-dimensional finite element formulation

Shape (interpolation) functions



Knowing the temperatures at node ends, the temperature distribution between the nodes can be interpolated using the shape functions which is the fundamental idea behind finite element formulation.

$$T(x) = \left(\frac{l-x}{l}\right) t_1 + \left(\frac{x}{l}\right) t_2$$

$$T(x) = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \qquad \text{Eq E1}$$

$$T(x) = [N][t]$$

$$[N] = \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix} \qquad \text{Eq E2}$$

$$[B] = \frac{d}{dx}\begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix}$$

$$[B] = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \qquad \text{Eq E3}$$

$$[g] = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

k is the element conduction matrix

$$[k] = \iiint_V [B]^T[D][B]\, dV + \iint_{S_3} h[N]^T[N]\, dS$$

$$[k] = [k_c] + [k_h]$$

$$[k_c] = \iiint_V [B]^T[D][B]\, dV$$

Substituting Eq E3

$$[k_c] = \iiint_V \begin{bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{bmatrix}^T [K_{xx}] \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} dV$$

$$[k_c] = \int_0^L \begin{bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{bmatrix}^T [K_{xx}] \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} A dx$$

$$[k_c] = \frac{AK_{xx}}{L^2} \int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

$$[k_c] = \frac{AK_{xx}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{Eq E4}$$

Substituting Eq E2

$$[k_h] = \iint_{S_3} h \begin{bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix} dS$$

$$[k_h] = hP \int_0^L \begin{bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix} dx$$

$$[k_h] = \frac{hPL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{Eq E5}$$

Now the element conduction matrix of 1D finite element is

$$[k] = \frac{AK_{xx}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{Eq E6}$$

f_Q Heat source (positive)/ sink (negative) matrix

$$[f_Q] = \iiint_V [N]^T Q \, dV$$

Substituting Eq E2

$$[f_Q] = QA\int_0^L \begin{bmatrix} 1-\frac{x}{l} \\ \frac{x}{l} \end{bmatrix} dx$$

$$[f_Q] = \frac{QAL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{Eq E7}$$

f_q Heat flux matrix (into the surface (positive)/ out the surface (negative))

$$[f_q] = \iint_{S_2} [N]^T q^*\, dS$$

$$[f_q] = q^* P\int_0^L \begin{bmatrix} 1-\frac{x}{l} \\ \frac{x}{l} \end{bmatrix} dx$$

$$[f_q] = \frac{q^* PL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{Eq E8}$$

f_h Heat convection matrix

$$[f_h] = \iint_{S_3} h[N]^T T_\infty\, dS$$

$$[f_h] = hPT_\infty\int_0^L \begin{bmatrix} 1-\frac{x}{l} \\ \frac{x}{l} \end{bmatrix} dx$$

$$[f_h] = \frac{hPT_\infty L}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{Eq E9}$$

Finally, we must consider the convection from the free end of an element.

$$[k_h]_{end} = \iint\limits_{S_{end}} h[N]^T[N]\,dS$$

N1 = 0 & N2 = 1 if the right end is exposed (shown below) and N1 = 1 & N2 = 0 if the left end is exposed

$$[k_h]_{end} = \iint\limits_{S_{end}} h\begin{bmatrix}0\\1\end{bmatrix}[0 \quad 1]\,dS$$

$$[k_h]_{end} = hA\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \qquad \text{Eq E10}$$

$f_{h,end}$ Heat convection matrix at end

$$[f_h]_{end} = \iint\limits_{S_3} h[N]^T T_\infty\,dS$$

N1 = 0 & N2 = 1 if the right end is exposed (shown below) and N1 = 1 & N2 = 0 if the left end is exposed

$$[f_h]_{end} = hAT_\infty \iint\limits_{S_3} \begin{bmatrix}0\\1\end{bmatrix} dS$$

$$[f_h]_{end} = hAT_\infty \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{Eq E11}$$

$$[k][t] = \left[f_Q\right] + \left[f_q\right] + [f_h] \qquad \text{Eq. F11}$$

$$\left[[k_c]+[k_h]+[k_{h,end}]\right][t]=[f_Q]+[f_q]+[f_h]+[f_{h,end}]$$

Combining Eq E6,E7,E8,E9,E10,E11

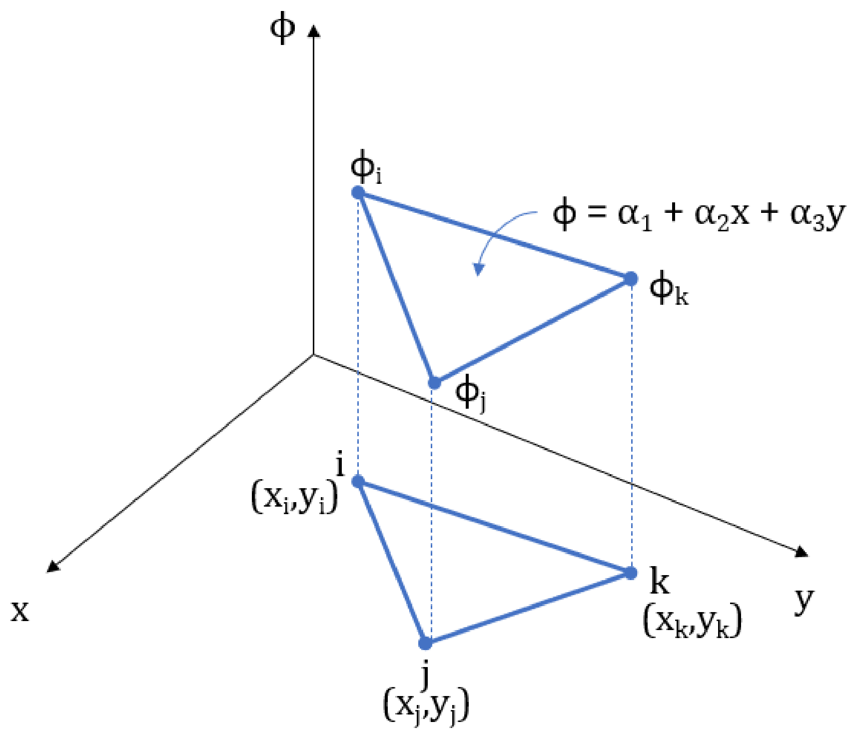
[

$$\left[\frac{AK_{xx}}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{hPL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}+hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right]\begin{bmatrix}t_1\\ t_2\end{bmatrix} = \frac{QAL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}+\frac{q^*PL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}+\frac{hPT_\infty L}{2}\begin{bmatrix}1\\ 1\end{bmatrix}+hAT_\infty\begin{bmatrix}0\\ 1\end{bmatrix} \quad \text{Eq E12}$$

Note that $k_{h,end}$ and $f_{h,end}$ is only applicable to end elements (in this case the end is assumed to be right side)

F. Two-dimensional finite element formulation

Shape (interpolation) functions



The interpolation polynomial for linear triangular element is

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y \qquad \text{Eq F1}$$

With nodal conditions

$$\phi = \phi_i \text{ @ } x = x_i, y = y_i$$

$$\phi = \phi_j \text{ @ } x = x_j, y = y_j$$

$$\phi = \phi_k \text{ @ } x = x_k, y = y_k$$

Substituting these conditions into the Eq F1 produces the system of equations

$$\phi_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$

$$\phi_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j$$

$$\phi_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \qquad \text{Eq F2}$$

Which yields

$$\alpha_1 = \frac{1}{2A}\left[\left(x_j y_k - x_k y_j\right)\phi_i + \left(x_k y_i - x_i y_k\right)\phi_j + \left(x_i y_j - x_j y_i\right)\phi_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[\left(y_j - y_k\right)\phi_i + \left(y_k - y_i\right)\phi_j + \left(y_i - y_j\right)\phi_k\right]$$

$$\alpha_3 = \frac{1}{2A}\left[\left(x_k - x_j\right)\phi_i + \left(x_i - x_k\right)\phi_j + \left(x_j - x_i\right)\phi_k\right]$$

Where A is the area of the triangle given by

$$2A = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \qquad \text{Eq F3}$$

Substituting for α_1, α_2 and α_3 in Eq F1 and re-arranging produces an equation in ϕ in terms of three shape functions and ϕ_i, ϕ_j and ϕ_k that is

$$\phi = \phi_i N_i + \phi_j N_j + \phi_k N_k \qquad \text{Eq F4}$$

Where

$$N_i = \frac{1}{2A}[a_i + b_i x + c_i y]$$

$$N_j = \frac{1}{2A}[a_j + b_j x + c_j y]$$

$$N_k = \frac{1}{2A}[a_k + b_k x + c_k y] \qquad \text{Eq F5}$$

And

$$a_i = x_j y_k - x_k y_j \,, \qquad b_i = y_j - y_k \;\; and \;\; c_i = x_k - x_j$$

$$a_j = x_k y_i - x_i y_k \,, \qquad b_j = y_k - y_i \;\; and \;\; c_j = x_i - x_k$$

$$a_k = x_i y_j - x_j y_i \,, \qquad b_k = y_i - y_j \;\; and \;\; c_k = x_j - x_i \qquad \text{Eq F6}$$

The scalar quantity ϕ is related to nodal values by a set of shape function that are linear in x and y. Since x and y are linear the gradient is constant within linear triangular element

$$\frac{\partial \phi}{\partial x} = \phi_i \frac{\partial N_i}{\partial x} + \phi_j \frac{\partial N_j}{\partial x} + \phi_k \frac{\partial N_k}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{2A} [b_i \phi_i + b_j \phi_j + b_k \phi_k] \qquad \text{Eq F7}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{2A} [c_i \phi_i + c_j \phi_j + c_k \phi_k] \qquad \text{Eq F8}$$

The scalar function is Temperature T in our case so Eq F4 becomes

$$T = t_i N_i + t_j N_j + t_k N_k$$

$$T = [N_i \quad N_j \quad N_k] \begin{bmatrix} t_i \\ t_j \\ t_k \end{bmatrix} \qquad \text{Eq G1}$$

Temperature gradient matrix

$$[g] = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} \qquad \text{Eq G2}$$

$$[g] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} \end{bmatrix} \begin{bmatrix} t_i \\ t_j \\ t_k \end{bmatrix} \qquad \text{Eq G3}$$

$$[g] = [B][t]$$

$$[B] = \frac{d}{dx}[N]$$

Where the [B] matrix is obtained by using Eq F7 & Eq F8

$$[B] = \frac{1}{2A} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \qquad \text{Eq G4}$$

Heat flux/ temperature gradient relationship is

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = -[D][g] \qquad \text{Eq G4}$$

Where the material property matrix is

$$[D] = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix} \qquad \text{Eq G5}$$

k is the element conduction matrix

$$[k] = \iiint_V [B]^T[D][B]\,dV + \iint_{S_3} h[N]^T[N]\,dS$$

$$[k] = [k_c] + [k_h]$$

$$[k_c] = \iiint_V [B]^T[D][B]\,dV$$

Substituting Eq G4

$$[k_c] = \iiint_V \frac{1}{4A^2}\begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix}\begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}\begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} dV \qquad \text{Eq G6}$$

Assuming constant thickness in the element and noting all the terms of the integrand of Eq G6 are constant

$$[k_c] = \iiint_V [B]^T[D][B]\,dV$$

$$[k_c] = tA[B]^T[D][B] \qquad \text{Eq G7}$$

Convection term of the total stiffness matrix is

$$[k_h] = \iint_{S_3} h[N]^T[N]\,dS$$

$$[k_h] = \iint_{S_3} h\begin{bmatrix} N_i & N_i \\ N_j & N_j \\ N_k & N_k \end{bmatrix}\begin{bmatrix} N_i & N_j & N_k \\ N_i & N_j & N_k \end{bmatrix} dS$$

$$[k_h] = h\iint_{S_3} \begin{bmatrix} N_iN_i & N_iN_j & N_iN_k \\ N_jN_i & N_jN_j & N_jN_k \\ N_kN_i & N_kN_j & N_kN_k \end{bmatrix} dS \qquad \text{Eq G8}$$

If the side between node i and j of the triangular element is subjected to convection, then $N_k = 0$

$$[k_h] = \frac{hL_{i-j}t}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If the side between node j and k of the triangular element is subjected to convection, then $N_i = 0$

$$[k_h] = \frac{hL_{j-k}t}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

If the side between node i and k of the triangular element is subjected to convection, then $N_j = 0$

$$[k_h] = \frac{hL_{k-i}t}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

f_Q Heat source (positive)/ sink (negative) matrix

$$[f_Q] = \iiint_V [N]^T Q \, dV$$

$$[f_Q] = \frac{QAt}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{Eq G9}$$

f_q Heat flux matrix (into the surface (positive)/ out the surface (negative))

$$[f_q] = \iint_{S_2} [N]^T q^* \, dS$$

$$[f_q] = \iint_{S_2} q^* \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} dS \qquad \text{Eq G10}$$

If the side between node i and j of the triangular element is subjected to heat flux, then N_k =0

$$[f_q] = \frac{q^* L_{i-j} t}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

If the side between node j and k of the triangular element is subjected to heat flux, then N_i =0

$$[f_q] = \frac{q^* L_{j-k} t}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

If the side between node i and k of the triangular element is subjected to heat flux, then N_j =0

$$[f_q] = \frac{q^* L_{i-m} t}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

f_h Heat convection matrix

$$[f_h] = \iint_{S_3} h[N]^T T_\infty \, dS$$

$$[f_h] = \iint_{S_2} h T_\infty \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} dS \qquad \text{Eq G11}$$

If the side between node i and j of the triangular element is subjected to heat convection, then N_k =0

$$[f_h] = \frac{hT_\infty L_{i-j} t}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

If the side between node j and k of the triangular element is subjected to heat convection, then N_i =0

$$[f_h] = \frac{hT_\infty L_{j-k} t}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

If the side between node i and k of the triangular element is subjected to heat convection, then N_j =0

$$[f_h] = \frac{hT_\infty L_{i-m} t}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$[k][t] = [f_Q] + [f_q] + [f_h] \qquad \text{Eq. F11}$$

$$[[k_c] + [k_h]][t] = [f_Q] + [f_q] + [f_h]$$

Combining Eq G7, G8, G9, G10, G11

$$\left[tA\begin{bmatrix}b_i & c_i\\ b_j & c_j\\ b_k & c_k\end{bmatrix}\begin{bmatrix}K_{xx} & 0\\ 0 & K_{yy}\end{bmatrix}\begin{bmatrix}b_i & b_j & b_k\\ c_i & c_j & c_k\end{bmatrix} + h\iint_{S_3}\begin{bmatrix}N_iN_i & N_iN_j & N_iN_k\\ N_jN_i & N_jN_j & N_jN_k\\ N_kN_i & N_kN_j & N_kN_k\end{bmatrix}dS\right]\begin{bmatrix}t_i\\t_j\\t_k\end{bmatrix} = \frac{QAt}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \iint_{S_2} q^*\begin{bmatrix}N_i\\N_j\\N_k\end{bmatrix}dS + \iint_{S_2} hT_\infty\begin{bmatrix}N_i\\N_j\\N_k\end{bmatrix}dS \qquad \text{Eq G12}$$

Using the above equation, we can solve 2D heat transfer problems.

Suppose the temperature of a region is known. That temperature can be applied as source by multiplying it with global k matrix.

Applying boundary condition, which is the specified temperature

$$[k][T] = [f]$$

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & \cdots & k_{1n} \\ k_{21} & k_{22} & \ddots & \ddots & k_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} T_1 = xx \\ T_2 = xx \\ \vdots \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_n \end{bmatrix}$$

Assume T1 & T2 temperatures are known, In order to maintain the nodes T1, T2 at the known temperature a source has to exist to maintain the temperature. That source can be found by multiplying global k matrix with temperature matrix (with known temperature and other values as zero) and subtracting from f matrix.

$$[k][T] = [f] - [k][T_0]$$

$$\begin{bmatrix} \cancel{k_{11}} & \cancel{k_{12}} & \cancel{\cdots} & \cancel{\cdots} & \cancel{k_{1n}} \\ \cancel{k_{21}} & \cancel{k_{22}} & \cancel{\ddots} & \cancel{\ddots} & \cancel{k_{2n}} \\ \cancel{\vdots} & \cancel{\ddots} & \ddots & \ddots & \vdots \\ \cancel{\vdots} & \cancel{\ddots} & \ddots & \ddots & \vdots \\ \cancel{k_{n1}} & \cancel{k_{n2}} & \cdots & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \cancel{T_1 = xx} \\ \cancel{T_2 = xx} \\ \vdots \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} \cancel{f_1} \\ \cancel{f_2} \\ \vdots \\ \vdots \\ f_n \end{bmatrix}$$

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