Note: contact the instructor if you would like the code behind any of the results.

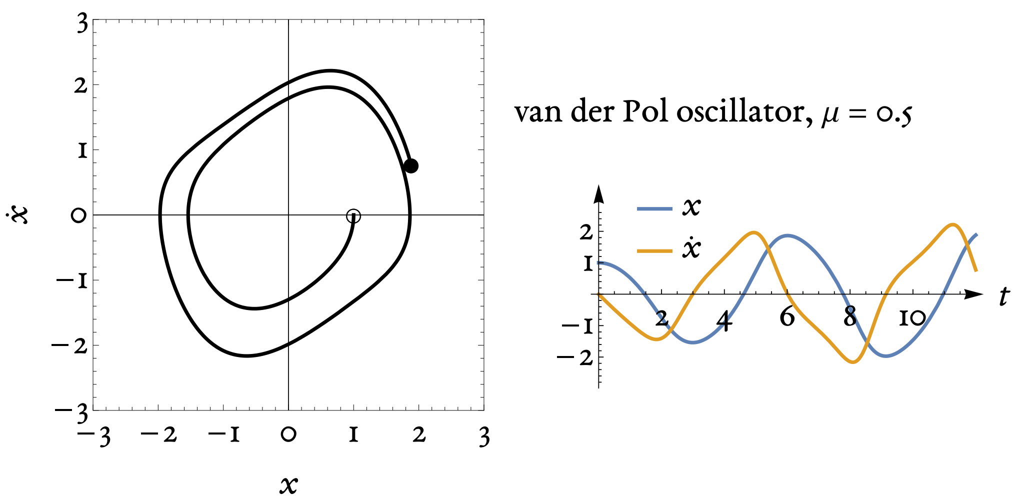
1. The van der Pol oscillator. Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

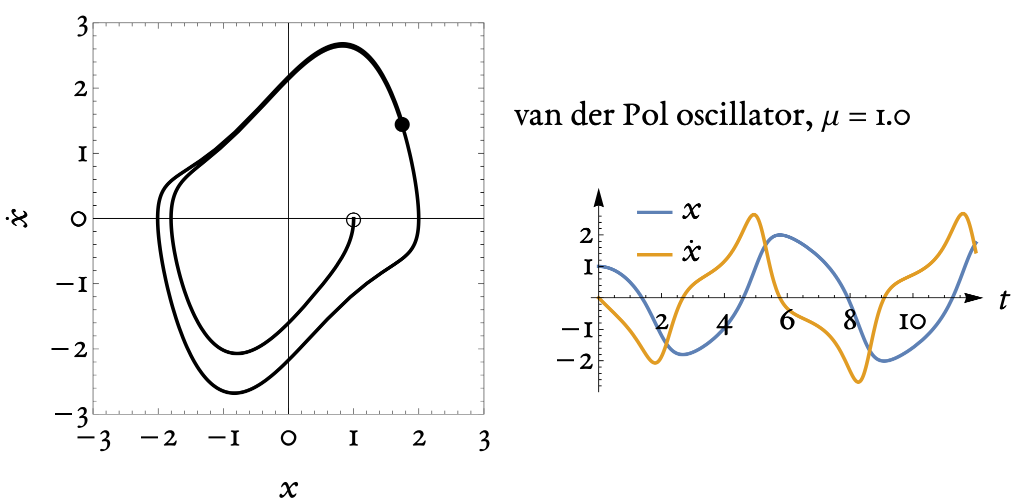
(a) Numerically solve (1) using a programming language of your choice for the initial condition

$$x(0) = 1, \quad \dot{x}(0) = 0$$

with $\mu = 1$. Plot your results in two different forms: (1) parametrically with $x$ on the horizontal axis and $\dot{x}$ on the vertical axis, and (2) both $x$ and $\dot{x}$ against time. Integrate the differential equation(s) from $t = 0$ to $t = 12$. Your figures should look similar to the following. Turn in your code together with the figures for a complete solution.
Note that the sample has a different value of $\mu$.

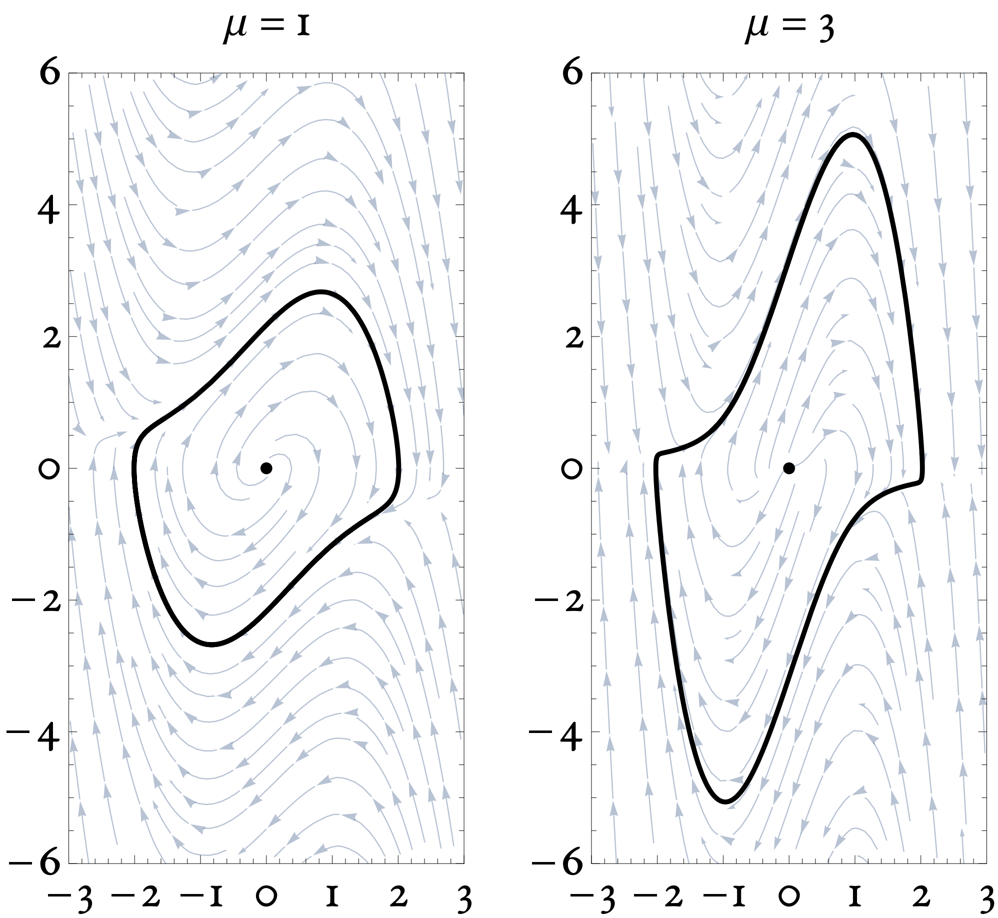



*Ans.* The solution is as follows.



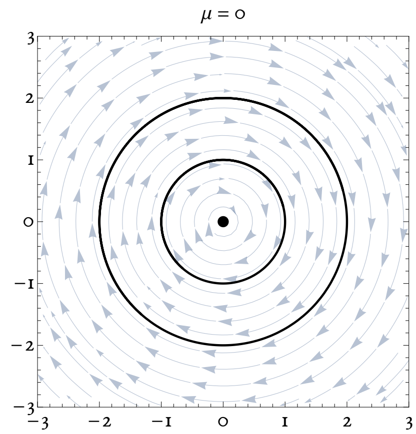
(b) Make a computer-generated phase portrait for this system using a program of your choice; `pplane` is perfectly acceptable. As usual, the horizontal axis should be $x$ and the vertical axis should be $\dot{x}$. Do this for two values of the parameter, $\mu = \{1, 3\}$. The range of the axes should be the same for both cases. Your phase portrait should be legible, and should highlight all the interesting dynamical features of the system, including fixed points and limit cycles, if any.

*Ans.* The solution is given in the figures below.



(c) Consider the case when $\mu = 0$. Does this system have a limit cycle when $\mu = 0$? Elaborate on your answer using the corresponding phase portrait.

*Ans.* No, this system does not have a limit cycle at $\mu = 0$. Visually, this can be seen in the phase portrait below, where it is clear that infinitely many closed orbits are available; there are no *isolated* closed orbits. The black curves show examples of closed orbits that are not limit cycles.



(d) Show, mathematically and with the help of computer visualization of the phase portrait, that the origin ($x = \dot{x} = 0$) changes its stability when $\mu$ changes sign. Recall that, typically, a fixed point is stable if its eigenvalues have negative real part, and unstable if its eigenvalues have positive real part.

*Ans.* Let us use $x$ to refer to $x$ and $y$ to refer to its derivative, $\dot{x}$. Then, (1) becomes

$$\dot{x} = y, \tag{2a}$$
$$\dot{y} = -\left(\mu\left(x^2 - 1\right)y + x\right). \tag{2b}$$

To determine the stability of the origin, we must first linearize this system of equations, which will give us the Jacobian matrix. The Jacobian of this system is

$$A = \begin{bmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu\left(x^2 - 1\right) \end{bmatrix},$$

which at the origin $x = y = 0$ evaluates to

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}. \tag{3}$$

The eigenvalues of this matrix can be found using the standard techniques of linear algebra; here we find the eigenvalues to be

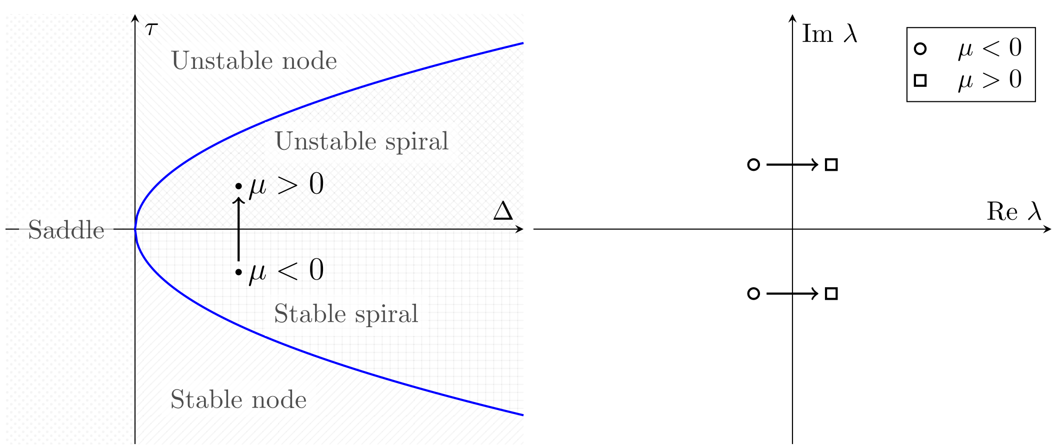
$$\lambda = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right).$$

Near $\mu = 0$, $\lambda$ is guaranteed to be complex since $\mu^2 - 4 < 0$ for small $\mu$, which makes square-root term imaginary. The real part of this complex number is simply

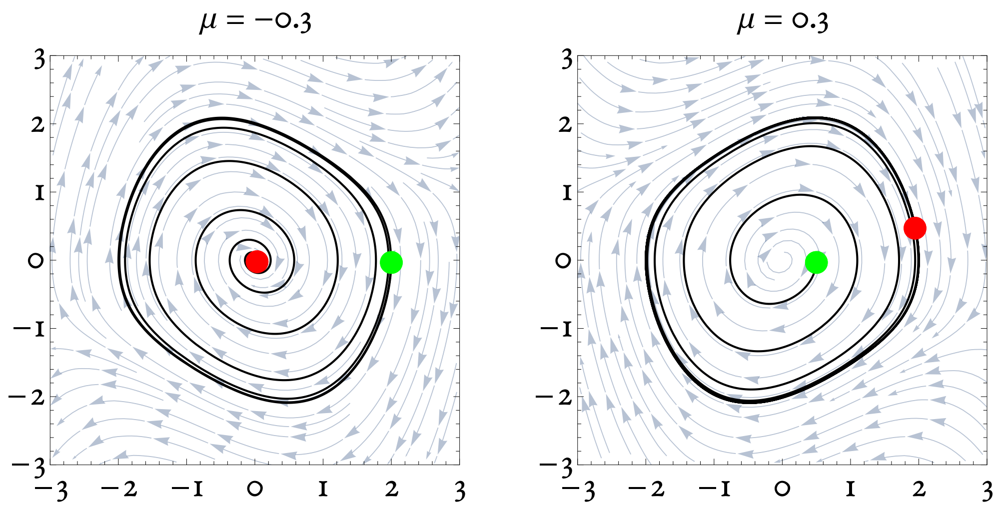
$$\mathrm{Re}\lambda = \frac{1}{2}\mu,$$

which means that the real part of the eigenvalues changes sign when $\mu$ changes sign. Thus, the origin becomes unstable when $\mu > 0$ and is stable when $\mu < 0$.

Alternatively, we can also note that the trace and determinant of (3) are $\tau = \mu$ and $\Delta = 1$ respectively. Thus, when $\mu$ increases from negative to positive, the origin goes from the 'stable spiral' region to the 'unstable spiral' region.



Now, the phase portrait shows the same thing: when $\mu < 0$, points around the origin 'fall into' the origin; when $\mu > 0$, points near the origin escape away from the origin.



2. Consider the system

$$\dot{x} = \mu x + y + \sin x \tag{4a}$$
$$\dot{y} = x - y \tag{4b}$$

(a) Write down the equations that must be satisfied by the coordinates of the fixed points of this system. You need not solve it at this stage.

*Ans.* Fixed points of this system occur when both $\dot{x} = 0$ and $\dot{y} = 0$. For the second condition, it is sufficient to require that $y = x$. For the first condition, we have

$$0 = \dot{x} = \mu x + y + \sin x,$$

and substituting $y = x$, we get

$$x(\mu + 1) + \sin x = 0 \tag{5}$$
$$y = x \tag{6}$$

(b) Write down, in symbolic form, the Jacobian for this system in terms of $\mu$ and $x$ and/or $y$.

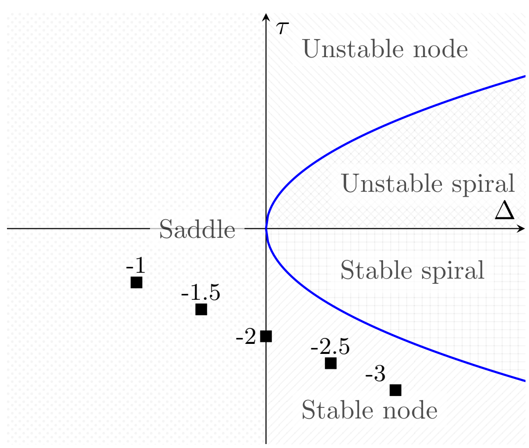
*Ans.*

$$A = \begin{bmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{bmatrix} \tag{7}$$

(c) Classify the fixed point $(0, 0)$ based on the value taken by $\mu$. What type is it, and for what value of $\mu$ does the answer change?

*Ans.* For the fixed point at the origin, $\cos 0 = 1$ and the Jacobian matrix becomes $A = \begin{bmatrix} \mu + 1 & 1 \\ 1 & -1 \end{bmatrix}$. This matrix has trace $\tau = \mu$ and determinant $\Delta = -2 - \mu$. If we plot this parametrically on our usual classification diagram, we find:

Thus, we see that the fixed point is a saddle for $\mu > -2$ and a stable node for $\mu < -2$.




(d) For small distances away from $(0, 0)$, some other fixed points may also exist. Determine an approximate expression for the coordinates of these other fixed points in terms of $\mu$. You may need to use the series expansion of $\sin x$ or $\cos x$, as appropriate. Recall that, near $x = 0$,

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$
$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

*Ans.* As suggested, let's replace trigonometric expressions with their series representation. Note that **all** fixed points must lie on $y = x$, which should be clear by inspection of (4a). Thus, it is sufficient to deal only with equation (5), i.e.,

$$
\begin{aligned}
x(\mu+1) + \sin x &= 0 \\
\implies x(\mu+1) + x - \frac{x^3}{3!} &\approx 0 \\
\mu + 1 + 1 - \frac{x^2}{6} &\approx 0 \\
\implies \mu + 2 - \frac{x^2}{6} &\approx 0 \\
x^2 - 6(\mu+2) &\approx 0 \\
&\implies x \approx \sqrt{6(\mu+2)} \quad \text{if } \mu > -2
\end{aligned}
$$

Thus, the other fixed points are located approximately at $\left(\sqrt{6(\mu+2)}, \sqrt{6(\mu+2)}\right)$.

(e) What type of bifurcation occurs in this system?

*Ans.* Supercritical pitchfork bifurcation.

3. Applying the Poincare-Bendixson Theorem. Consider the system

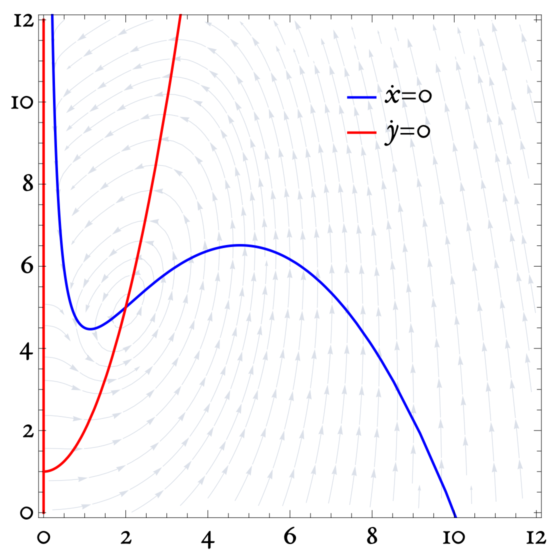
$$\dot{x} = a - x - \frac{4xy}{1+x^2} \tag{8a}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right) \tag{8b}$$

where the parameters are set to $a = 10, b = 2$.

(a) Plot the **nullclines** of this system in the range $x, y > 0$. Use this information to determine the location of a fixed point in this quadrant.

*Ans.* The nullclines are shown below.




The equations for the nullclines are, respectively

$$a - x - \frac{4xy}{1+x^2} = 0, \text{ and}$$
$$bx\left(1 - \frac{y}{1+x^2}\right) = 0.$$

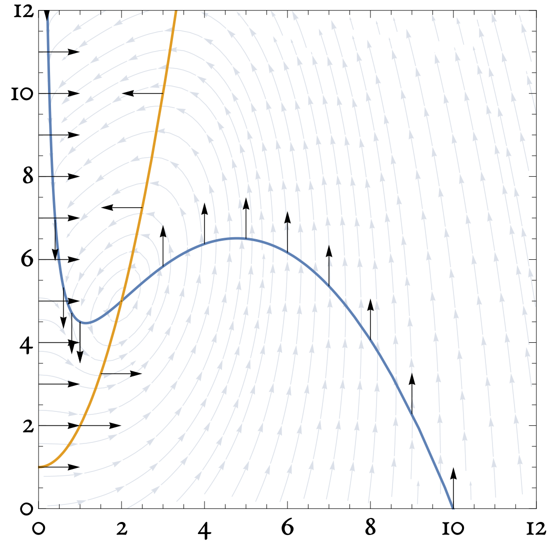
After some algebra, we find that the curves can be written explicitly as

$$y = \frac{(a-x)(1+x)^2}{4x}, \text{ and}$$
$$y = 1 + x^2.$$

Solving for the intersection of these curves, we find the point

$$\frac{(a-x)(1+x^2)}{4x} = 1 + x^2$$
$$\implies \frac{a-x}{4x} = 1$$
$$\implies x = \frac{a}{5}, \text{ and correspondingly } y = 1 + \frac{a^2}{25}$$

(b) Sketch some arrows on the nullclines to indicate the direction of flow on the nullclines. Hint: these will simply be horizontal or vertical arrows.



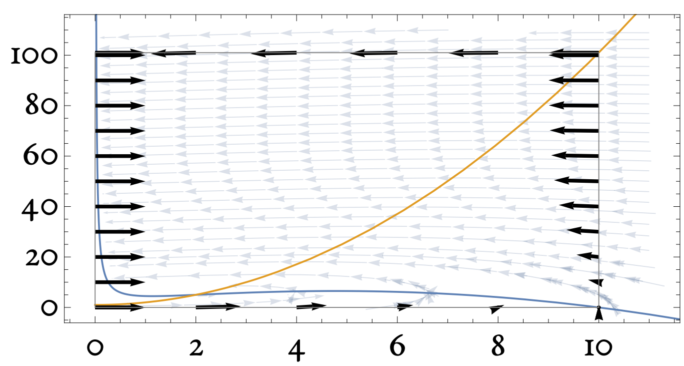

*Ans.* The required diagram is shown here.

(c) On the same axes, draw the rectangle defined by the edges of the region

$$\{x, y\} \text{ such that } 0 < x < 10,\ 0 < y < 101.$$

Sketch a few arrows of the vector field along the edges of this rectangle.

*Ans.* The diagram is shown here.




(d) Using linearization, determine the type of fixed point that occurs at the point you determined in part 3a. Use your result to show that any trajectory that starts ‘near the fixed point will move away from the fixed point. A mathematical proof is not necessary; words and a sketch is enough.

*Ans.* A fixed point occurs at $x = a/5, y = 1 + a^2/25$. Let us now write the Jacobian for this system. Using Mathematica, we find

$$A = \begin{bmatrix} \dfrac{8x^2y}{(x^2+1)^2} - \dfrac{4y}{x^2+1} - 1 & -\dfrac{4x}{x^2+1} \\ \dfrac{2bx^2y}{(x^2+1)^2} + b\left(1 - \dfrac{y}{x^2+1}\right) & -\dfrac{bx}{x^2+1} \end{bmatrix}.$$

When we substitute the fixed point's coordinates ($x = a/5, y = 1 + a^2/25$), we find that the Jacobian is

$$\begin{bmatrix} \dfrac{8a^2}{25\left(\dfrac{a^2}{25}+1\right)} - 5 & -\dfrac{4a}{5\left(\dfrac{a^2}{25}+1\right)} \\ \dfrac{2a^2 b}{25\left(\dfrac{a^2}{25}+1\right)} & -\dfrac{ab}{5\left(\dfrac{a^2}{25}+1\right)} \end{bmatrix}$$

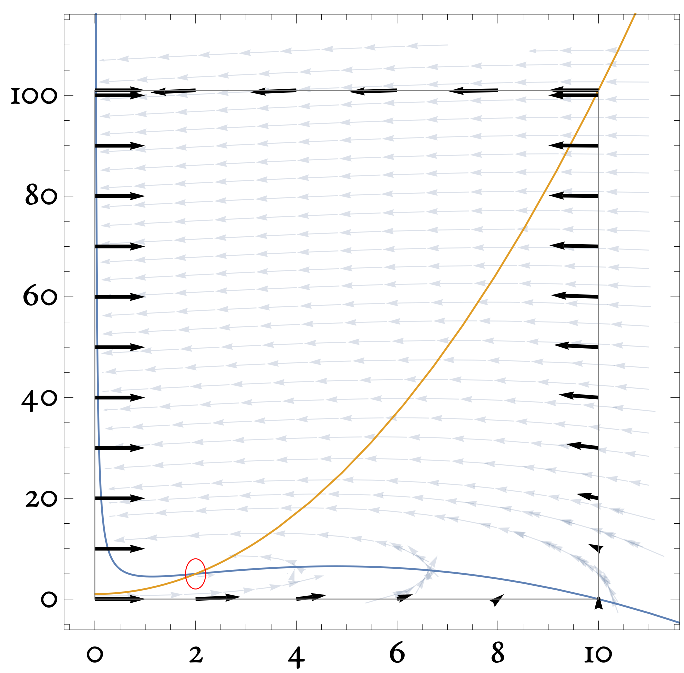
With $a = 10$ and $b = 2$, we find that the Jacobian is

$$\begin{bmatrix} 7/5 & -8/5 \\ 16/5 & -4/5 \end{bmatrix},$$

a matrix whose determinant is 4 and trace is $5/3$. This means that the fixed point at $x = a/5, y = 1+a^2/25$ is an unstable spiral.

(e) Use the Poincare-Bendixson Theorem to argue why it must be the case that a limit cycle exists for this system. Indicate, using a sketch or words, the region $\mathcal{D}$ of phase space in which the theorem applies.

*Ans.* The theorem applies in a region given by the rectangle from part (c), EXCLUDING a small area around the fixed point. If we draw a tiny circle around the repelling fixed point, shown in red below, we can be sure that the flow is always leaving this circle. Separately, we have shown that the flow is always entering the black rectangle. Thus, we know that the flow must eventually settle into a limit cycle somewhere inside the region enclosed by the black rectangle on the outside and the red circle on the inside.




4. Complete the in-class exercise from lecture 9.2. You can access the exercise at the following URL: `https://emadmasroor.github.io/classes/E91_S25/Exercises/Exercise5.pdf`