Mon, Apr 7 Lecture 18

Time Horizon of Prediction in systems with sensitive dependence on initial conditions.

$$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$$

for Lorenz system $\lambda \approx 0.9$ when $\sigma = 10, b = \frac{8}{3}, r = 28$

Find time t^* at which two initially nearby (δ_0) trajectories have diverged by more than ϵ.

$$\epsilon \approx \|\delta_0\| e^{\lambda t^*}$$

$$\Rightarrow \frac{\epsilon}{\|\delta_0\|} \approx e^{\lambda t^*} \Rightarrow \boxed{t^* \approx \frac{1}{\lambda} \log\left[\frac{\epsilon}{\|\delta_0\|}\right]}$$

the largest Liapunov Exponent of the system. $= 0.9$ for Lorenz

Example

Two measurements were made to a precision of $\delta_0 = 10^{-7}$

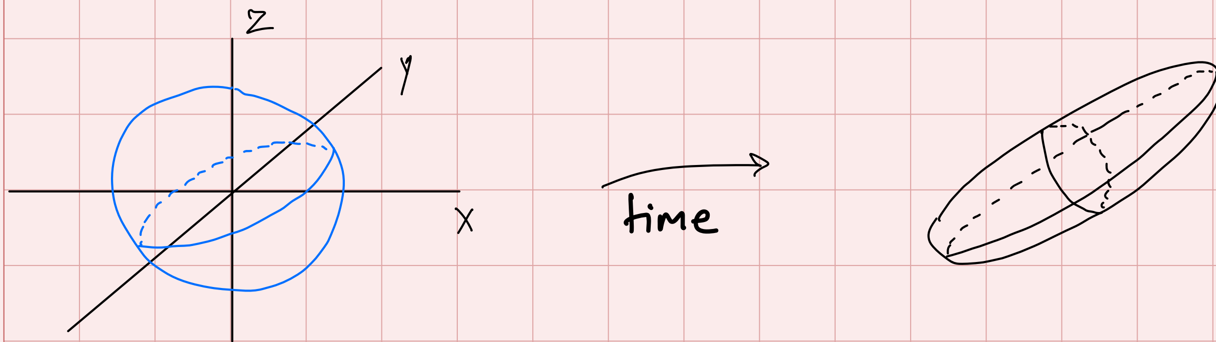
We consider deviations more than $\epsilon = 10^{-3}$ to be unacceptable. $\Rightarrow t^* \approx 10.2$

$3\times$

Increase initial precision. $\Rightarrow t^* \approx 30.7$

Now, $\delta_0 = 10^{-15}$ (100,000,000 × more precise)

What exactly is λ ?



sphere of initial conditions with infinitesimal radius δ

Ellipsoid with principal axes

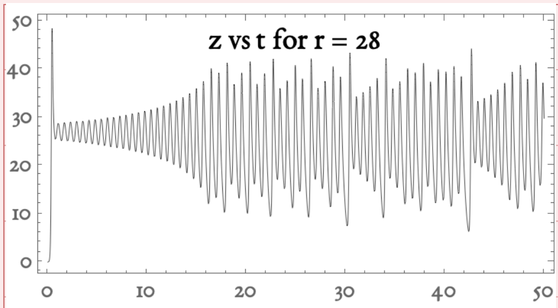
$$\delta_1(0) \longrightarrow \delta_1(0)e^{\lambda_1 t}$$
$$\delta_2(0) \longrightarrow \delta_2(0)e^{\lambda_2 t}$$
$$\delta_3(0) \longrightarrow \delta_3(0)e^{\lambda_3 t}$$

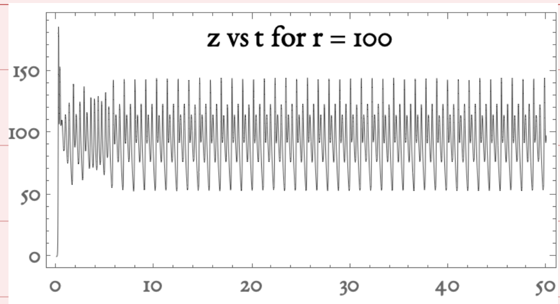
At large times, the $\underbrace{\text{largest } \lambda_k}_{\text{Liapunov Exponent.}}$ controls the size of the ellipsoid.

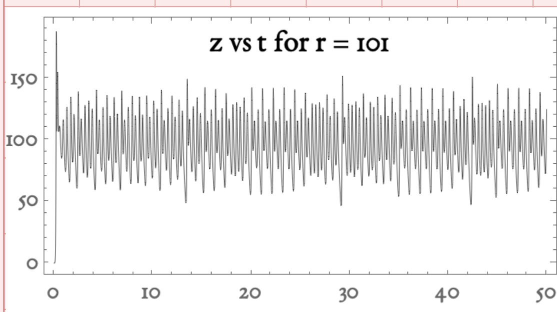
Negative $\lambda_k \Rightarrow$ that direction shrinks
Positive $\lambda_k \Rightarrow$ that direction expands.

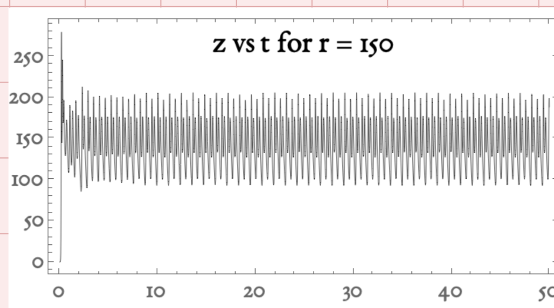
If Liapunov Exponent > 0, $\Rightarrow$ sensitive dependence on initial conditions.

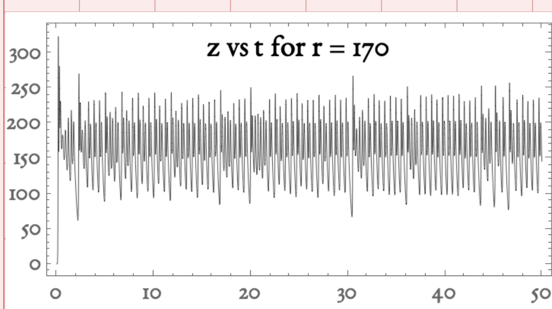
tinyurl.com/E91lorenz4 $\longrightarrow$ remix

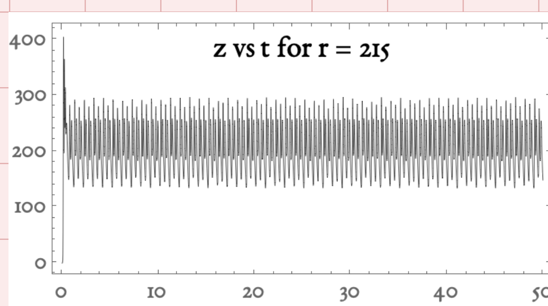




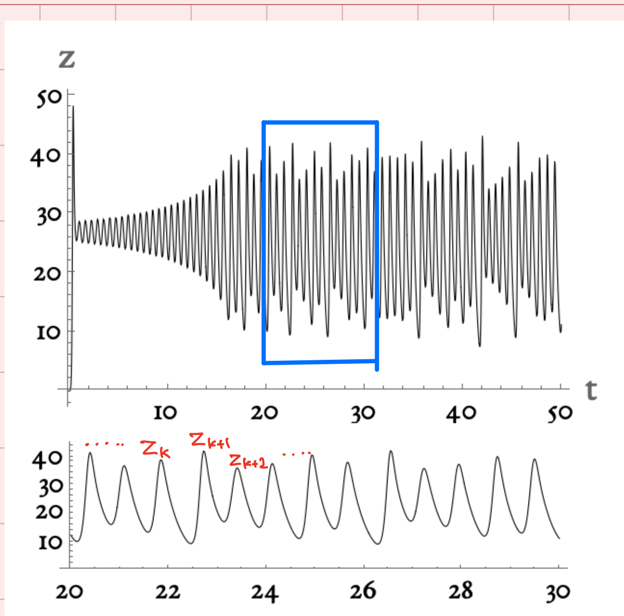








order → chaos → order → chaos → ?



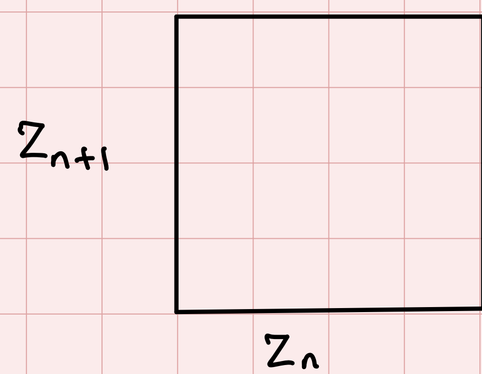
Continuous Time → Discrete Time

Differential Eqs → Maps

$\dot{x} = f(x)$ $\qquad$ $x_{n+1} = f(x_n)$

These two are not the same, even if you have a "map" and a differential eqn. describing the same underlying system.

If you know "f" for a map, you can plot → by graphing the function.



For Lorenz system, you have to do this empirically