The Lorenz equations

$\sigma, r, b > 0$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - \underline{xz}$$
$$\dot{z} = \underline{xy} - bz$$

Simplified model of atmospheric convection.

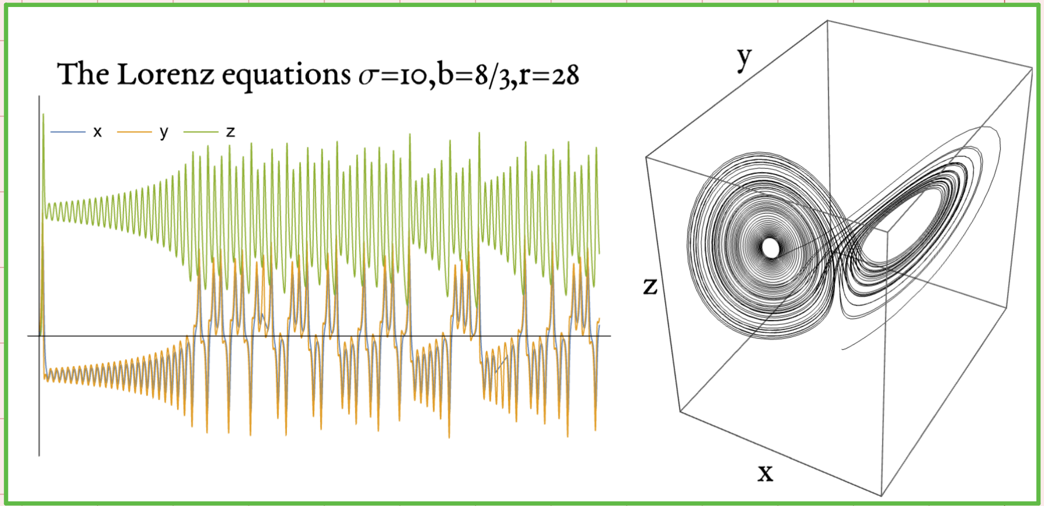
→ Nonlinear

→ 3-dimensional phase space

→ Symmetric : $(x,y,z) \longmapsto (-x,-y,z)$

tinyurl.com/E91lorenz1
tinyurl.com/E91lorenz2

Wed, Apr 2 Lecture 17



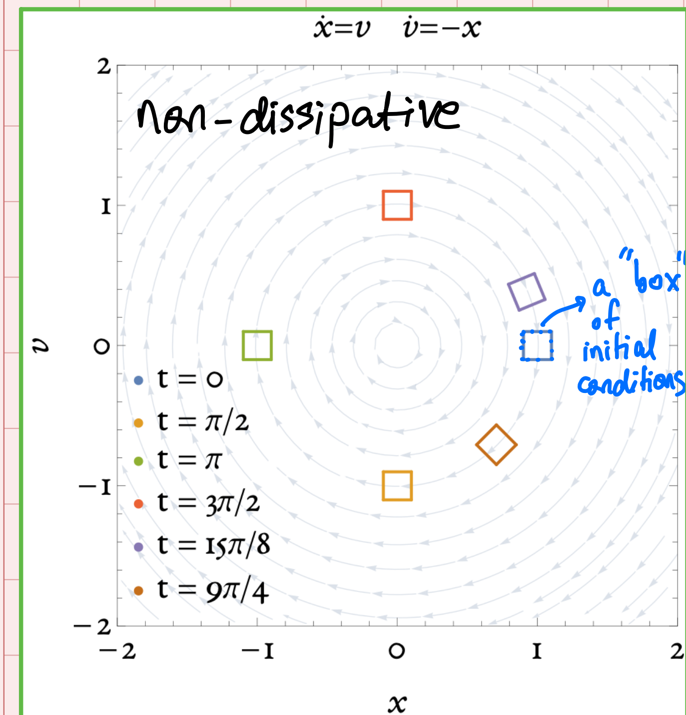
→ Dissipative : "Volumes in phase space" shrink exponentially with time

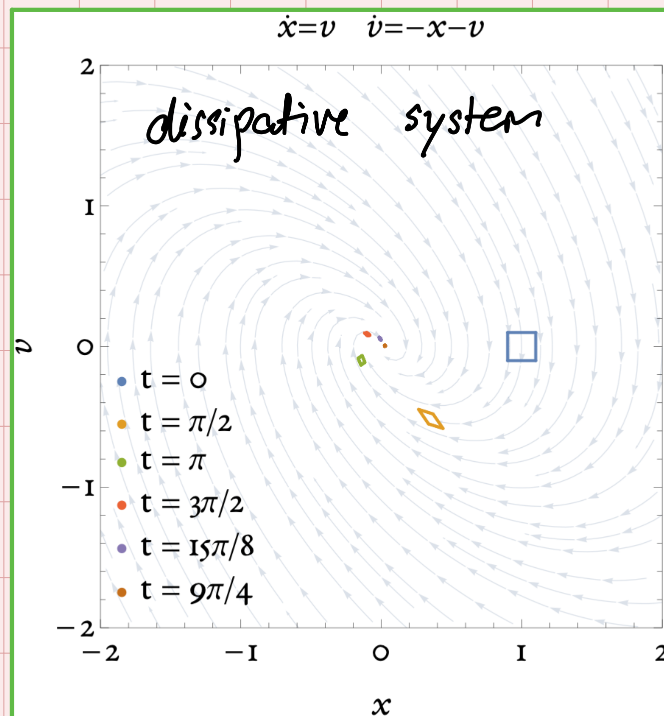
i.e. $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, $\nabla \cdot \underset{\sim}{f} < 0$ for dissipative systems.

$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$

If $\nabla \cdot \underset{\sim}{f} > 0$ in some region, volumes expand

If $\nabla \cdot \underset{\sim}{f} < 0$ " " " , volumes shrink





The rate of change of volumes is $\dot{V} = \int_V (\nabla \cdot \underset{\sim}{f})\, dV$

For Lorenz : $\nabla \cdot f = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} \sigma(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix}$

$\rightsquigarrow \nabla \cdot \underset{\sim}{f} = -\sigma - 1 - b$. It happens to not be a function of (x, y, z). Const. everywhere.

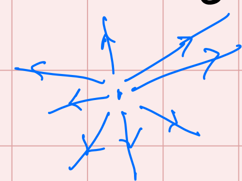
$\dot{V} = -(\sigma+1+b)V$

$V(t) = V(0)\, e^{-(\sigma+1+b)t}$

Because of dissipation

→ Quasiperiodicity is not allowed because q.p. flow occurs on the surface of a fixed torus in phase space. But if volumes in phase space are always shrinking, you can't have such an invariant torus.

→ No repelling fixed points or repelling closed orbits are allowed.



Fixed Points of Lorenz system

$(x,y,z) = \underset{\sim}{0}$ is always a fixed pt.

$(x,y,z) = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$ (c^+, c^-) are fixed pts if $r > 1$

Linear stability of origin:

linearized Lorenz equations:

$\dot{x} = \sigma(y-x)$

$\dot{y} = rx - y$

$\dot{z} = -bz$ — contracting direction

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-\sigma & \sigma\\ r & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

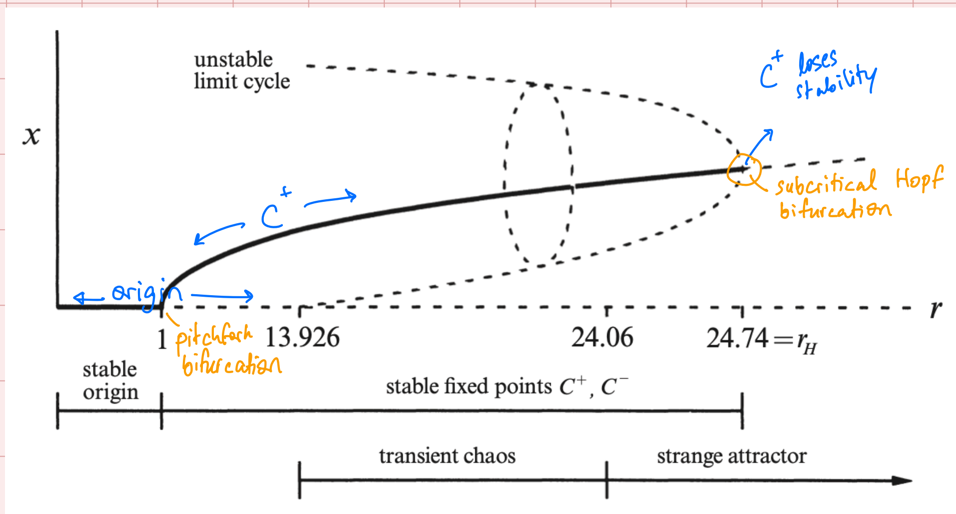
$\tau = -(\sigma+1)$

$\Delta = \sigma - \sigma r = \sigma(1-r)$

$\left.\begin{matrix}\\ \\ \end{matrix}\right\}$ $r > 1$: saddle pt

$r < 1$: stable node

C^+ and C^- are stable for $1 < r < r_H = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$



so what exists after $r = r_H$?

- trajectories don't go out to infinity
- (it can be shown) there are no stable limit cycles
- no attracting fixed pts.
- but phase space is dissipative

<E91lorenz3>

$\rightsquigarrow$ A strange attractor

Attractor :

if "A" is an attractor, it has these properties →

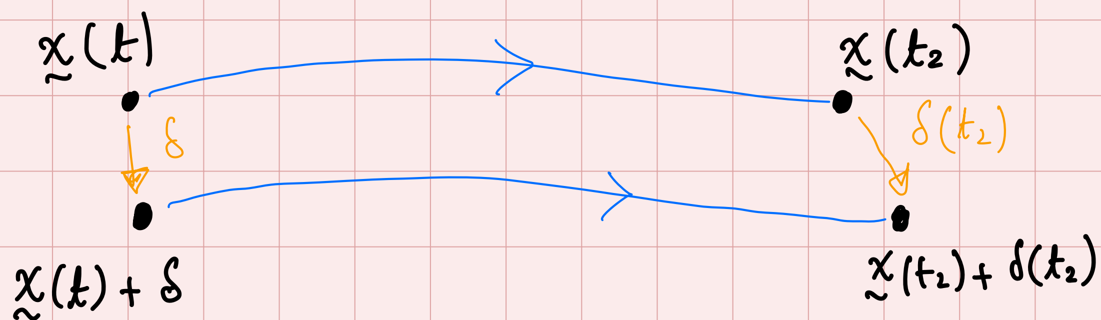
1. *A* is an *invariant set:* any trajectory $\mathbf{x}(t)$ that starts in *A* stays in *A* for all time.
2. *A attracts an open set of initial conditions*: there is an open set *U* containing *A* such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to *A* tends to zero as $t \to \infty$. This means that *A* attracts all trajectories that start sufficiently close to it. The largest such *U* is called the *basin of attraction of A*.
3. *A* is *minimal:* there is no proper subset of *A* that satisfies conditions 1 and 2.

An attractor has a certain shape — what is the shape of the Lorenz attractor? — dimension "2.05"

tinyurl.com/E91lorenz3

For Lorenz's parameters $\sigma = 10$, $b = 8/3$, $r = 28$, the system exhibits chaos. Strange attractors are chaotic ones.

Strogatz: Chaos is aperiodic long-term behaviour in a deterministic system that shows sensitive dependence on initial conditions.



$\delta(t)$: how far apart are the two trajectories

use both as independent as initial conditions. ⟶ See how δ evolves