Dynamics of Coaxial Arrays of Vortex Rings

The Periodic N-Vortex Ring problem

Emad Masroor [1] Mark Stremler [2]

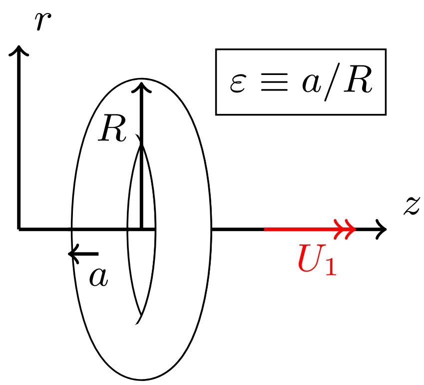
[1]Swarthmore College
Swarthmore, PA

[2]Virginia Tech
Blacksburg, VA

BIRS Workshop:
Vortex Dynamics: the Crossroads of Mathematics, Physics and Applications
Hangzhou, China
December 2023

Vortex rings are solutions to the Euler equations in axisymmetric form

$$\frac{\partial \omega}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \omega}{\partial z} = -\frac{1}{r^2}\frac{\partial \psi}{\partial r}\omega\frac{\partial \omega}{\partial r}$$
$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} = -\omega$$



$B^2 = Ra^2 =$ const.

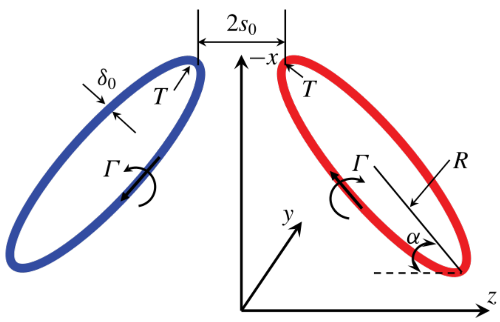
- Scalar-valued ω confined to a torus shape with small ε
- ψ arises from the Green's function for the Laplacian in cylindrical coordinates
- The ring moves forward with speed

$$U_1(\varepsilon) \approx \frac{1}{4\pi}\left(\log\frac{8}{\varepsilon} - \frac{1}{4}\right)$$

Thomson. "The translatory velocity of a circular vortex ring". *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 33 (1867)

Interacting vortex rings lead to many interesting phenomena

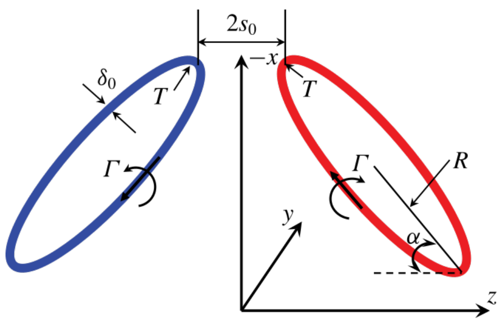
Vortex rings approaching at an angle are a candidate for finite-time singularity in the Euler equations

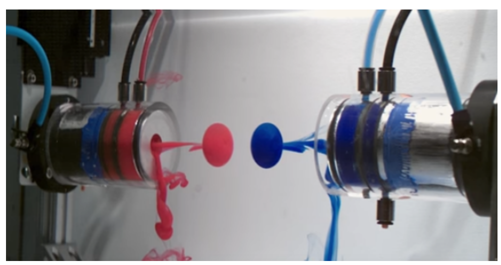


Moffatt and Kimura (2019)

Interacting vortex rings lead to many interesting phenomena

Vortex rings approaching head-on leads to instability & reconnection



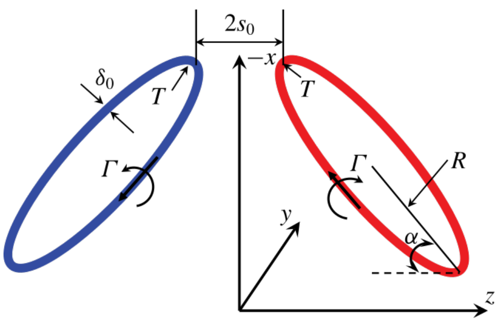


Moffatt and Kimura (2019)

inspired by Lim and Nickels (1992)

Interacting vortex rings lead to many interesting phenomena

Successive vortex rings can be used to feed a 'blob' of finely-tuned turbulence



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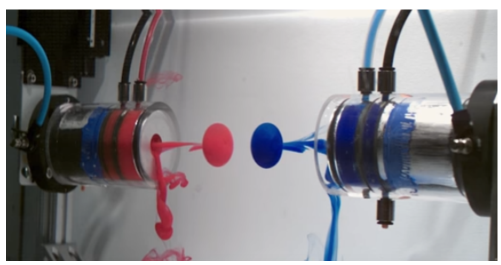

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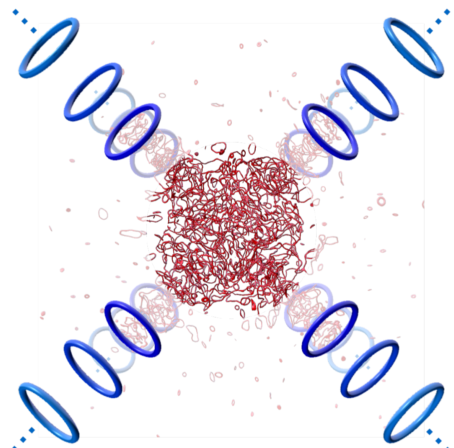


Matsuzawa, Mitchell, Perrard, and Irvine (2022)

Using vortices to model wakes & jets has a long history ...

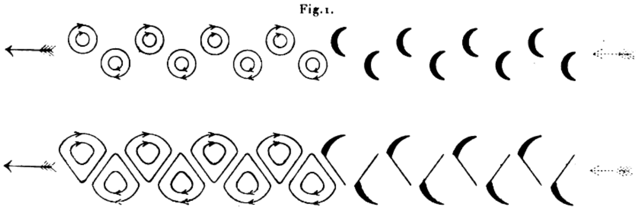
Bénard (1908)

SÉANCE DU 9 NOVEMBRE 1908. 841

30 tourbillons visibles à la fois à l'œil nu, mais les images successives données par le cinématographe y suppléent.

Quand les vitesses de rotation sont faibles (faible vitesse de l'obstacle ou forte viscosité du liquide, et, dans tous les cas, tourbillons près de s'éteindre), les entonnoirs sont sensiblement de révolution. La méthode optique, qui équivaut à dessiner le relief de la surface liquide en l'éclairant sous une incidence presque rasante, les enregistre par des *croissants* demi-circulaires, estompés à l'intérieur (partie supérieure de la figure, à droite).

Fig. 1.



Quand les vitesses de rotation sont plus grandes (grande vitesse de l'obstacle ou faible viscosité du liquide), les entonnoirs des deux rangées se déforment mutuellement; leur contour prend une forme de raquette; les cavités sont plus abruptes du côté intérieur et leur bord y est rectiligne. Si la lumière quasi rasante arrive par exemple dans la direction indiquée par la flèche pointillée, ou à 180°, elle dessine les cavités en forme de *faux* (partie inférieure de la figure, à droite) (1).

Lois de l'équidistance (2), ou intervalle entre deux tourbillons d'une rangée :

Bénard. "Formation de centres de giration à l'arrière d'un obstacle en mouvement". *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences* 147 (1908)

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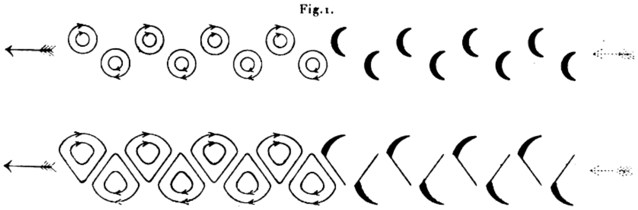

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Lois de l'équidistance [2], ou intervalle entre deux tourbillons d'une rangée :

Kármán (1911)

Die Stabilitätsuntersuchung.

Wir wollen die Frage dahin präzisieren, ob zwei Reihen paralleler, geradliniger Wirbelfäden von gleicher Stärke, aber entgegengesetztem Drehungssinn in der Weise angeordnet werden können, daß das ganze Gebilde ohne Änderung der Anordnung gleichförmig fortschreitet und dabei stabil bestehen kann. Es ist zunächst leicht einzusehen, daß es zwei Anordnungsweisen gibt, bei welchen die zwei parallelen Reihen unverändert fortschreiten können: wir können je zwei Wirbel einander gegenüberstellen (Anordnung *a* in Fig. 1), oder die eine Reihe mit der halben

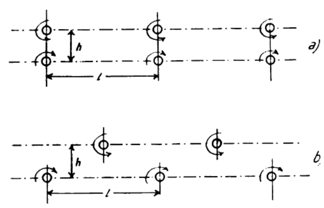


Fig. 1.

Teilung gegen die andere verschieben (Anordnung *b*). Bei gleichmäßiger Teilung der beiden Reihen folgt aus Symmetriegründen für beide Anordnungen, daß jeder Wirbelfaden die gleiche Geschwindigkeit in der x-Richtung erhält, und daß die Geschwindigkeit nach der y-Richtung verschwindet. Es fragt sich nun, welche dieser Anordnungen stabil sein kann?

Geschwindigkeit des q-ten Wirbelfadens

$$\frac{dx_q}{dt} = \frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{y_p - y_q}{(x_p - x_q)^2 + (y_p - y_q)^2}$$

$$\frac{dy_q}{dt} = -\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{x_p - x_q}{(x_p - x_q)^2 + (y_p - y_q)^2},$$

wobei $p = q$ aus der Summation auszuschließen ist. Werden nun die Wirbelfäden von den Gleichgewichtslagen um kleine Beträge ξ_p, η_p verschoben, so können wir die Geschwindigkeiten nach diesen Größen entwickeln, und wir gelangen offenbar zu einem System von Differentialgleichungen für die Störungen ξ_p, η_p, oder für die „kleinen Schwingungen" des Systems.

Wir setzen daher

$$x_p = pl + \xi_p,$$
$$y_p = \eta_p,$$

und man erhält durch Vernachlässigung quadratischer Terme in den ξ, η

$$\frac{d\xi_q}{dt} = \frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{\eta_p - \eta_q}{(p-q)^2 l^2},$$

$$\frac{d\eta_q}{dt} = \frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{\xi_p - \xi_q}{(p-q)^2 l^2}.$$

Die so erhaltenen, unendlich vielen Differentialgleichungen werden auf zwei zurückgeführt durch den Ansatz

$$\xi_p = \xi_0 e^{ip\varphi},$$
$$\eta_p = \eta_0 e^{ip\varphi}.$$

Die beiden Differentialgleichungen lauten alsdann

$$\frac{d\xi_0}{dt} = \eta_0 \frac{\zeta}{2\pi}\sum^{\infty}\frac{e^{ip\varphi} - 1}{p^2 l^2},$$

Bénard. "Formation de centres de giration à l'arrière d'un obstacle en mouvement". *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences* 147 (1908)

Kármán. "Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt". *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 12 (1911)

… and vortex rings figure prominently in this role

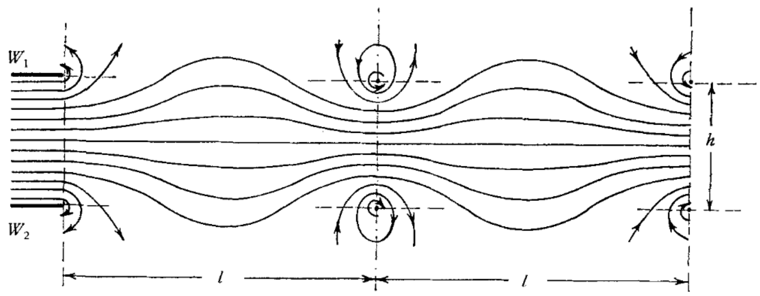
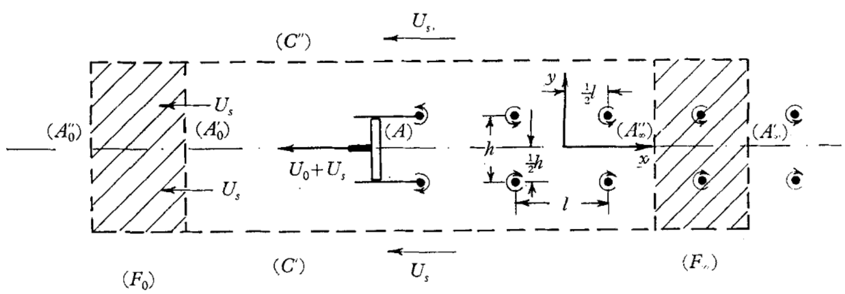


FIGURE 7. Symmetric vortex trail behind a jet propeller.



Siekmann. "On a pulsating jet from the end of a tube, with application to the propulsion of certain aquatic animals". *Journal of Fluid Mechanics* 15.3 (1963). Publisher: Cambridge University Press

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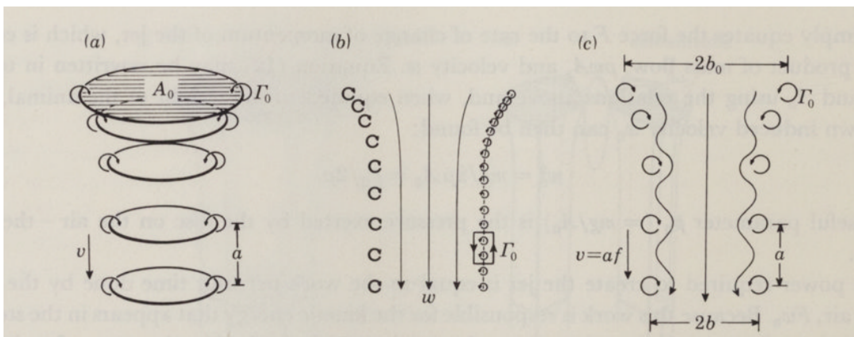

Ellington. "The aerodynamics of hovering insect flight. V. A vortex theory". *Philosophical Transactions of the Royal Society of London. B, Biological Sciences* 305.1122 (24, 1984). Publisher: The Royal Society

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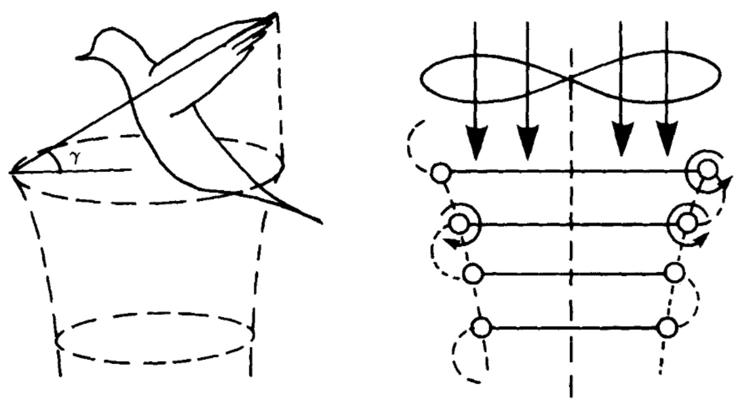
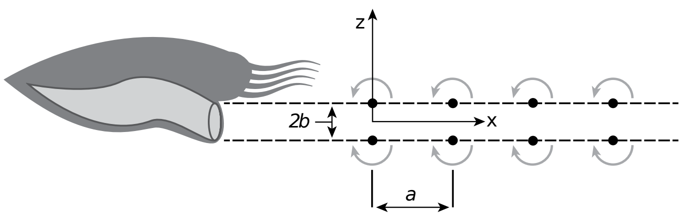
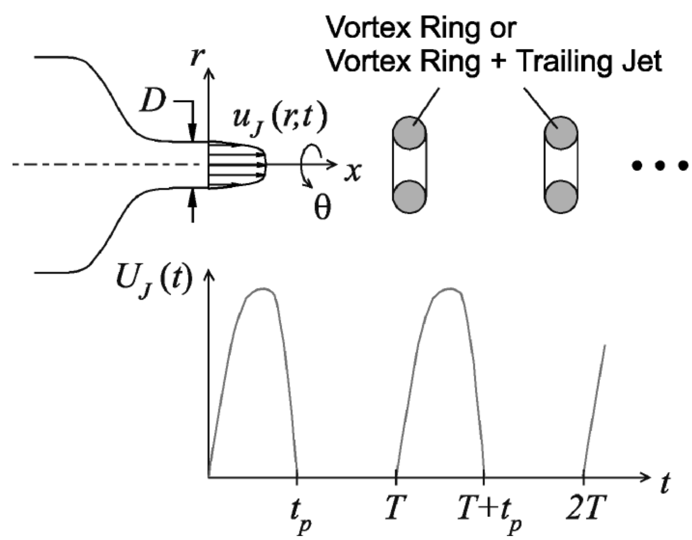


Rayner. "A vortex theory of animal flight. Part 1. The vortex wake of a hovering animal". *Journal of Fluid Mechanics* 91.4 (1979)

Periodic arrays of vortex rings



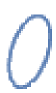
Gordon, Blickhan, Dabiri, and Videler 2017



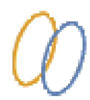
Krueger and Gharib. "Thrust augmentation and vortex ring evolution in a fully pulsed jet". *AIAA Journal* 43.4 (2005)

It is well known how two vortex rings interact

‘Pass through’

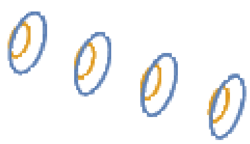


‘Leapfrogging’



delineation of these regimes is due to Lord Kelvin.

How do two *arrays* of vortex rings interact?



what other regimes might exist?

A vortex ring produces two types of streamlines in a **co-moving frame**

$$\psi(z,r;\varepsilon) = \overbrace{\tilde{\psi}(z,r)}^{\text{time-dependent}} - \frac{1}{2}\overbrace{U_1(\varepsilon)}^{\text{self-induced speed}} r^2$$

- Thin annulus-shaped cloud at low ε
- Thick biconcave or elliptical cloud at high ε
- Critical $\varepsilon_c \approx 0.0116$, due to Hicks. “LIX. The mass carried forward by a vortex”. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 38.227 (1919)

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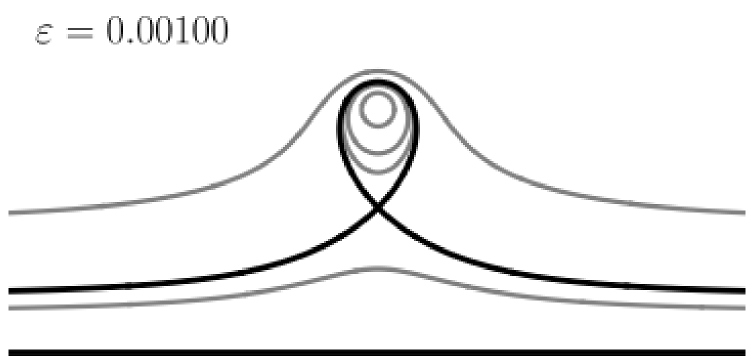

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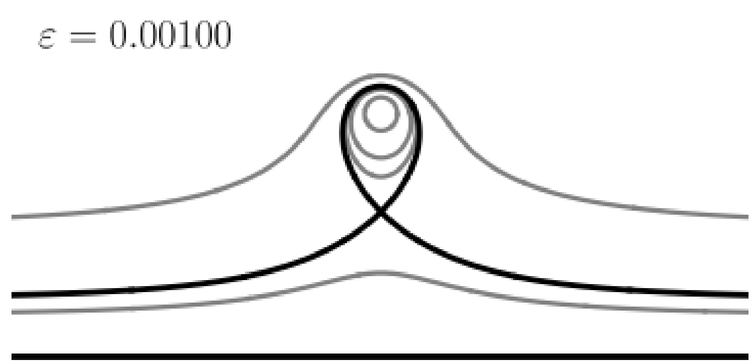
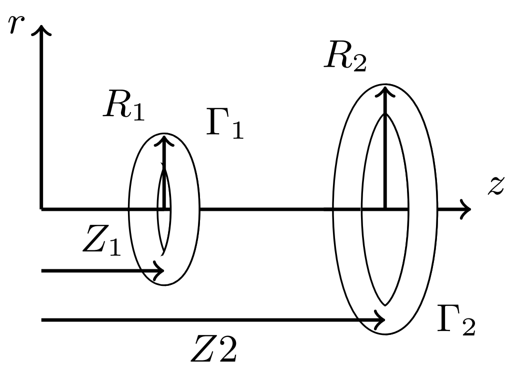


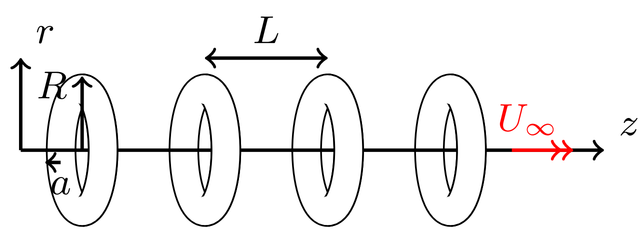
Adding additional vortex rings $\implies$ no privileged co-moving frame ...



$Z_1(t), Z_2(t), R_1(t), R_2(t)$

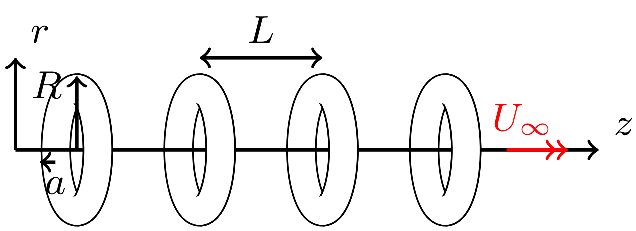
- Different regimes of inter-vortex motion are well-known (Helmholtz 1858)
- A recent comprehensive treatment is Borisov, Kilin, and Mamaev. "The dynamics of vortex rings: Leapfrogging, choreographies and the stability problem". *Regular and Chaotic Dynamics* 18.1 (11, 2013). Publisher: Springer

… except for the special case: an infinite array of identical rings



- *Also* move without change of shape, with speed U_∞

... except for the special case: an infinite array of identical rings



- *Also* move without change of shape, with speed U_∞
- Independently discovered by Vasilev 1916 and Levy and Forsdyke 1927.

- Vasilev. "On the motion of an infinite series of circular vortex rings of the same radius having a common axis". *Zap. Fiz.-Mat. Fak. Imp. Novoross. Univ (Notes of the Physics and Mathematics Department of the Imperial Novorossiysk University)* 10 (1916)
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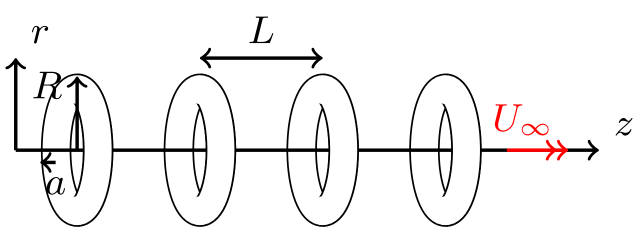

- *Also* move without change of shape, with speed U_∞
- Independently discovered by Vasilev 1916 and Levy and Forsdyke 1927.
- Parameterized by two non-dimensional numbers

$$\varepsilon \equiv a/R, \quad \lambda \equiv L/R$$

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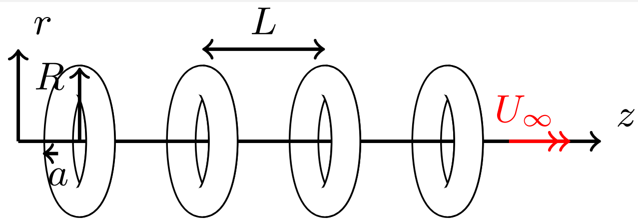
This solution to the Euler equation uses a periodic Green's function and a modified self-induced speed U_∞



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$$\psi_\infty(z, r; \varepsilon, \lambda) = \overbrace{\tilde{\psi}_\infty(z, r; \lambda)}^{\text{time-dependent}} - \frac{1}{2} \overbrace{U_\infty(\varepsilon, \lambda)}^{\text{self-induced speed}} r^2$$

$$\tilde{\psi}_\infty(z, r; \lambda) = \int_\Omega G_\infty(z, r; \bar{z}, \bar{r}, \lambda)\, \omega(\bar{z}, \bar{r})\, d\bar{z}\, d\bar{r} \qquad U_\infty(\varepsilon, \lambda) = U_1(\varepsilon) + U_\infty^*(\lambda)$$

$$G_\infty(z, r; \bar{z}, \bar{r}, \lambda) = \sum_{j=-\infty}^{+\infty} G(z, r; \bar{z} + j\lambda, \bar{r})$$

$$G(z, r; \zeta, \bar{r}) = \frac{\sqrt{r\,\bar{r}}}{2\pi} \left(\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right) \quad k = \left(\frac{4r\,\bar{r}}{(z-\zeta)^2 + (r+\bar{r})^2} \right)^{1/2}$$

and K, E complete elliptic integrals of the first and second kinds

given by Borisov, Kilin, and Mamaev 2013,
originally due to Maxwell 1873

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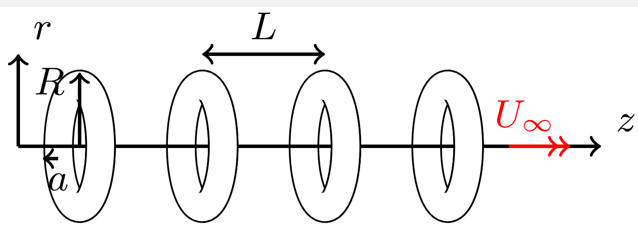

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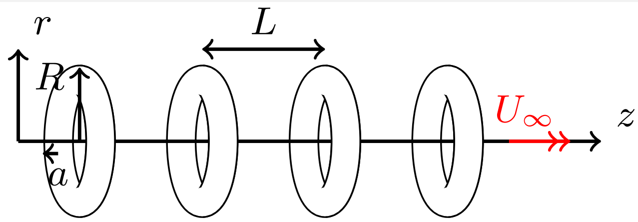

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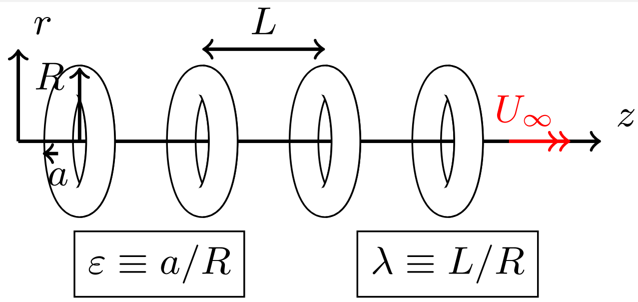
and K, E complete elliptic integrals of the first and second kinds

given by Borisov, Kilin, and Mamaev 2013,
originally due to Maxwell 1873

This solution to the Euler equation uses a periodic Green's function and a modified self-induced speed U_∞



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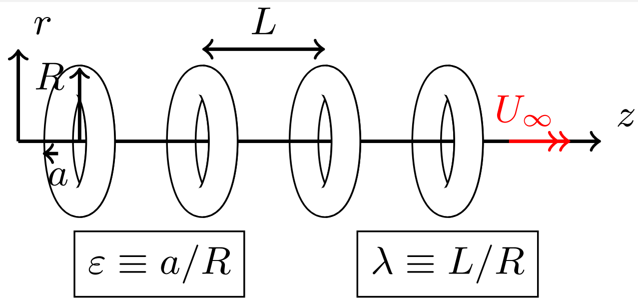

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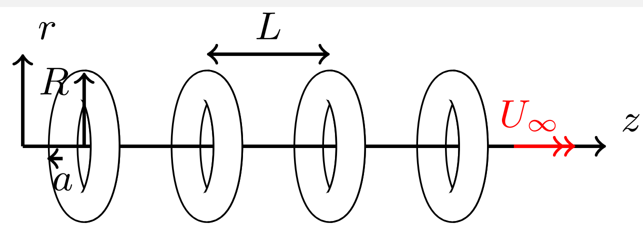

$\boxed{\varepsilon \equiv a/R}$ $\boxed{\lambda \equiv L/R}$

$$\psi_\infty(z, r; \varepsilon, \lambda) = \overbrace{\tilde{\psi}_\infty(z, r; \lambda)}^{\text{time-dependent}} - \frac{1}{2} \overbrace{U_\infty(\varepsilon, \lambda)}^{\text{self-induced speed}} r^2$$

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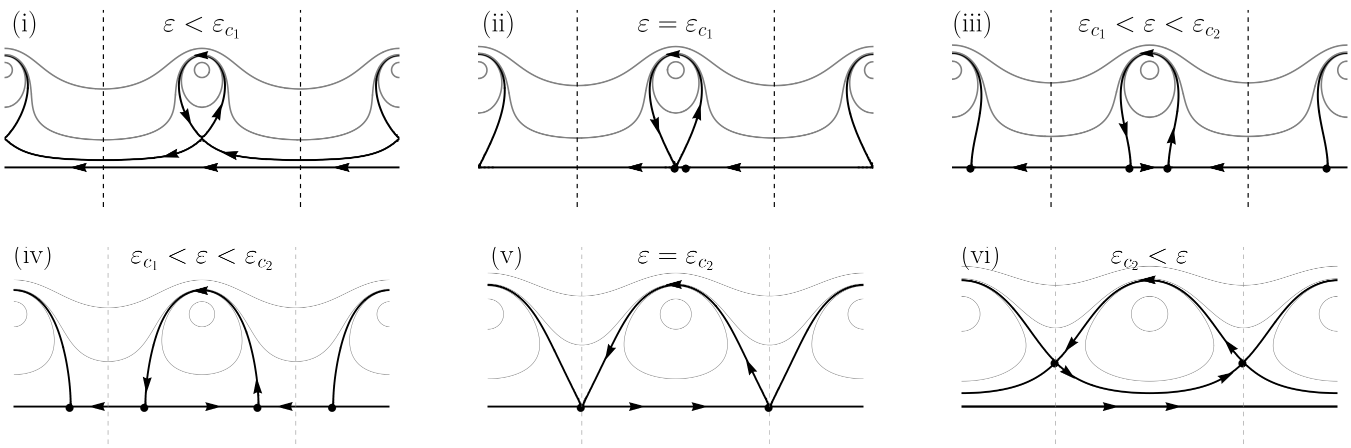
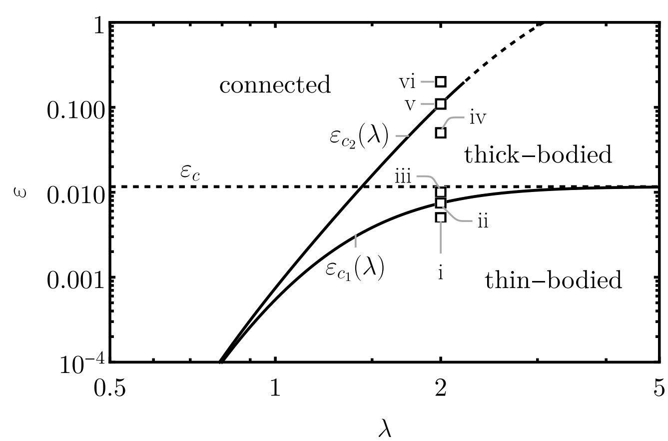
An array of vortex rings exhibits three distinct streamline topologies



- two bifurcations — ε_{c_1} & ε_{c_2}
- a new topology: 'connected'
- 'Interaction length' $\lambda c_2(\varepsilon)$



Masroor and Stremler 2022

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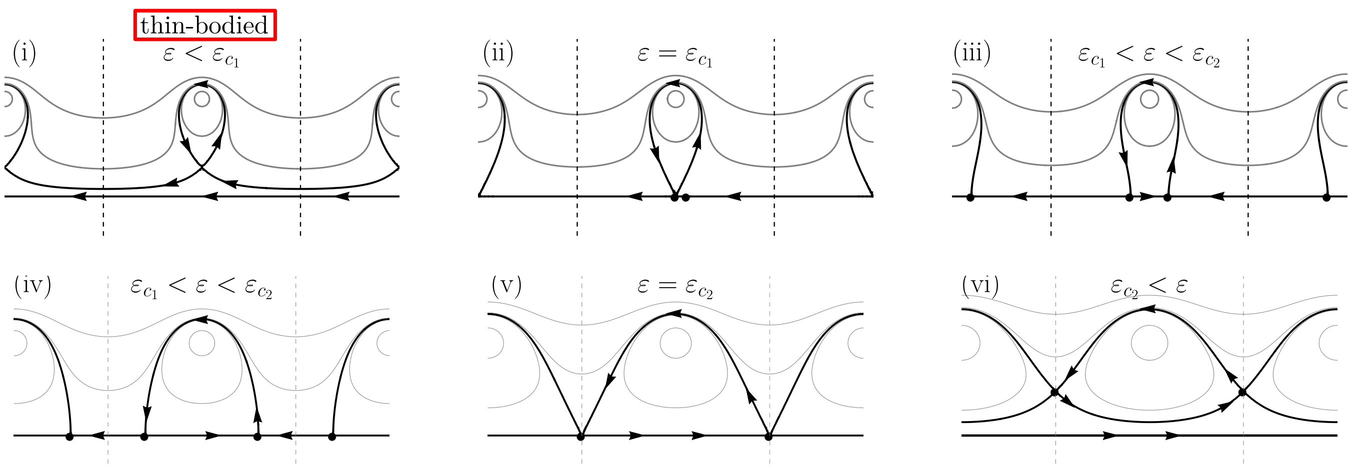

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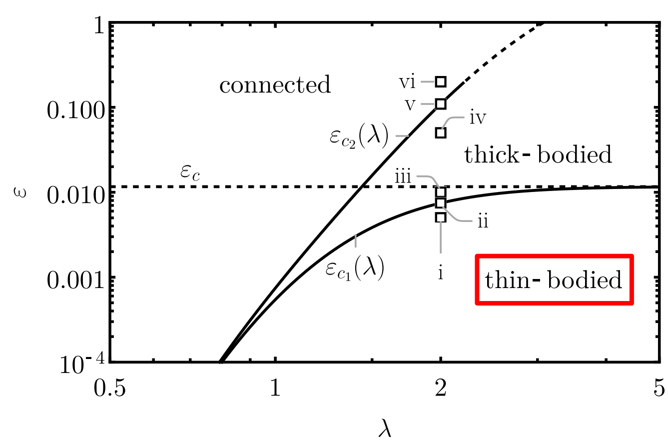

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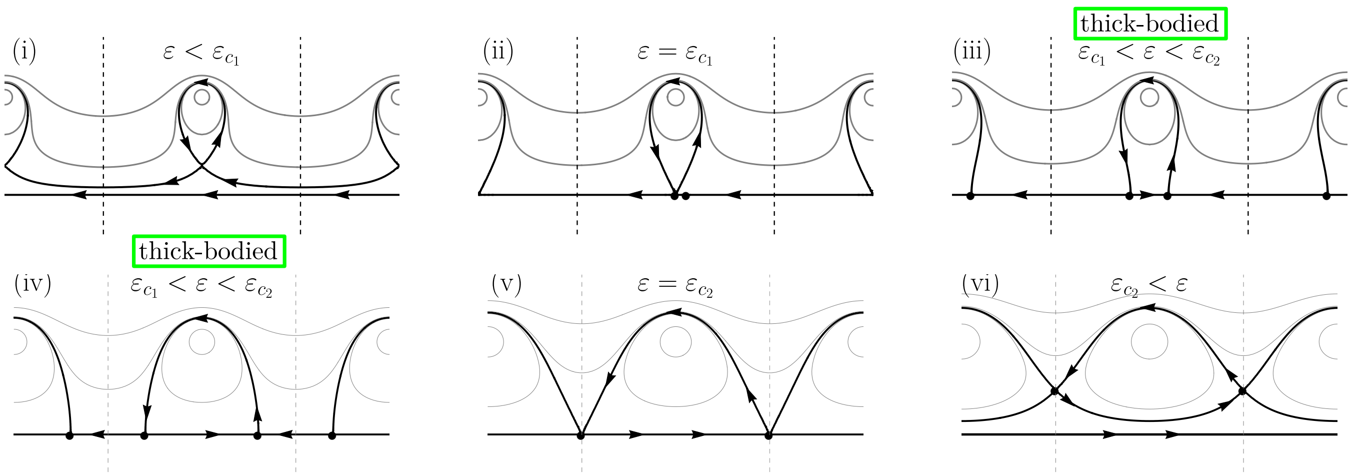

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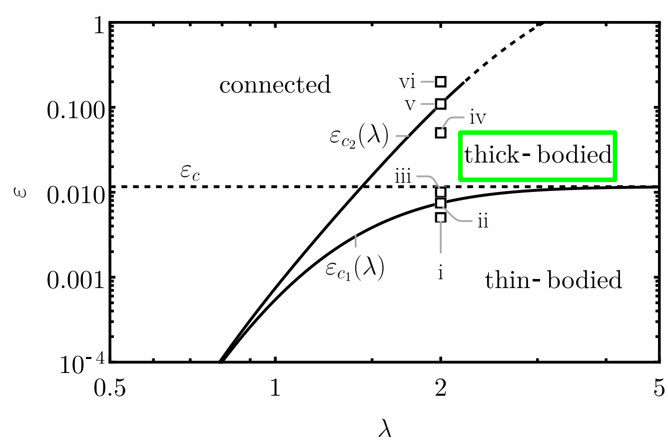

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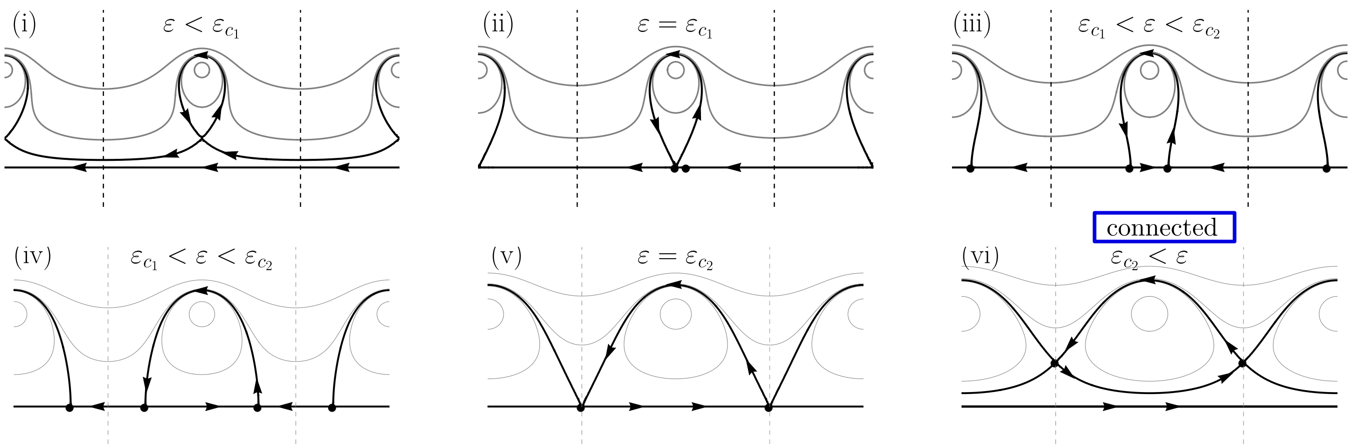

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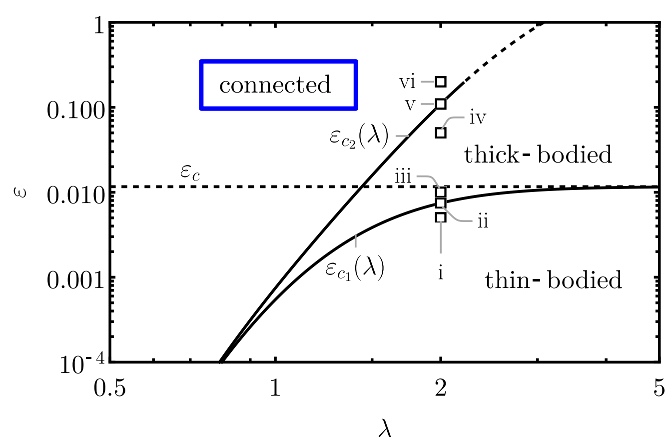

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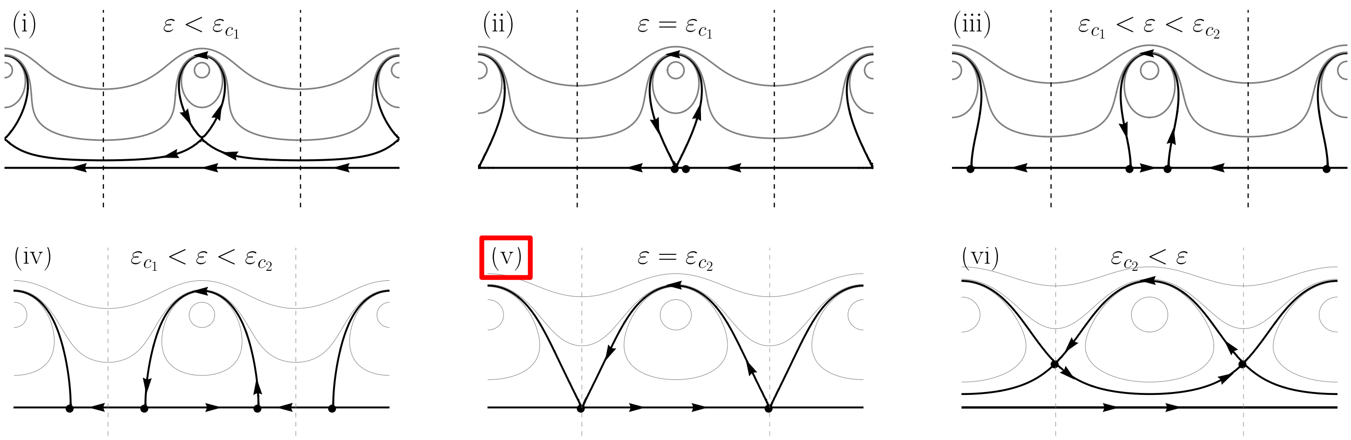

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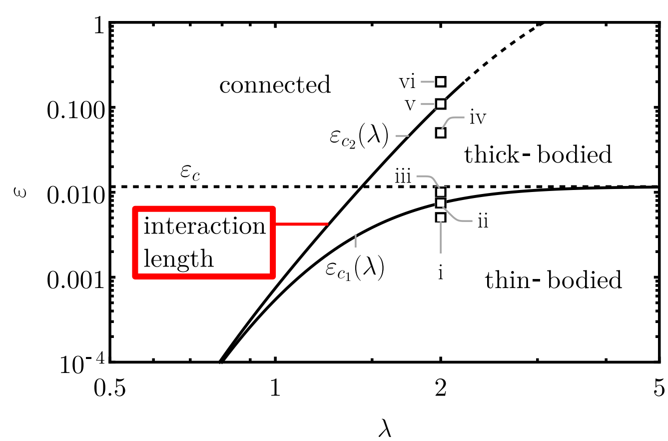
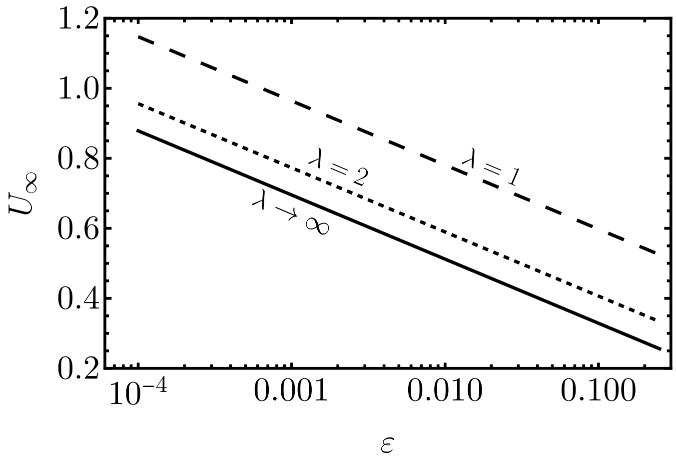


Masroor and Stremler 2022

The self-induced speed of an array of vortex rings varies with both ε and λ

$$U_\infty(\varepsilon,\lambda) = U_1(\varepsilon) + U_\infty^*(\lambda)$$



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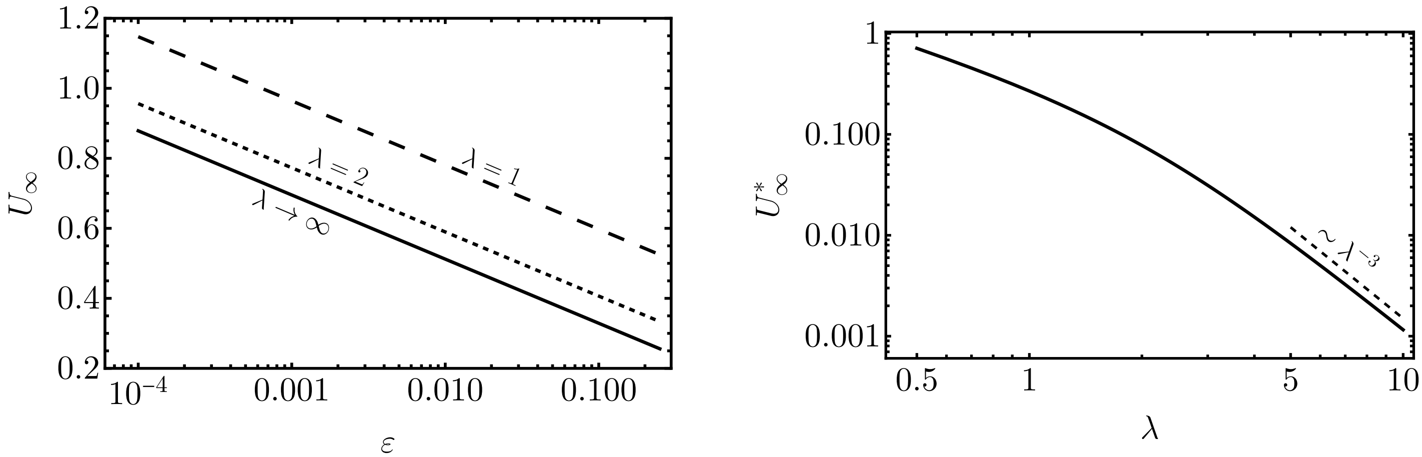
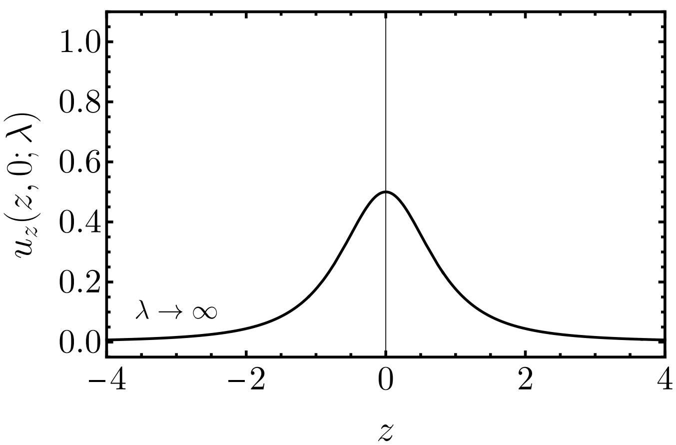
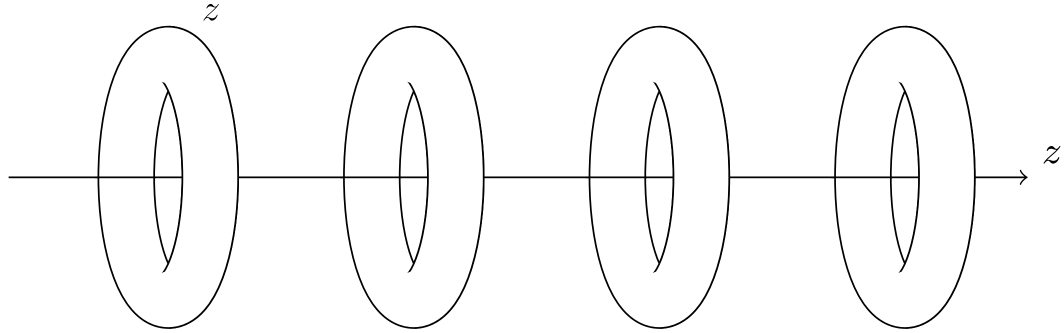


The speed induced along the axis varies with λ





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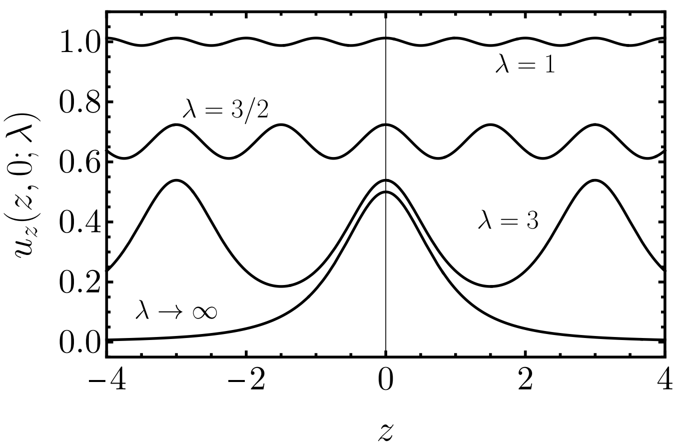

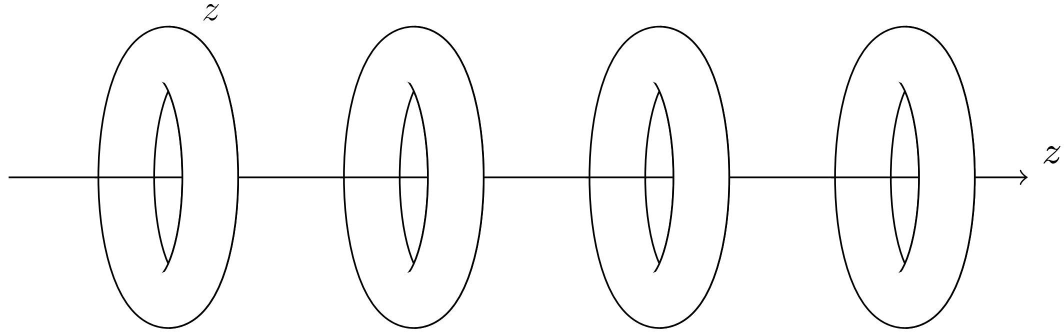

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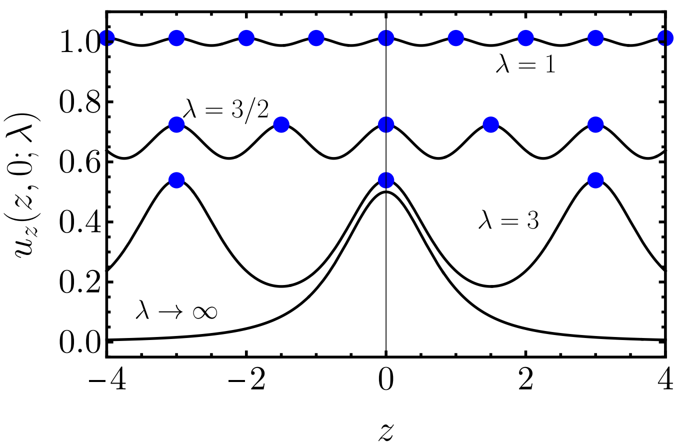

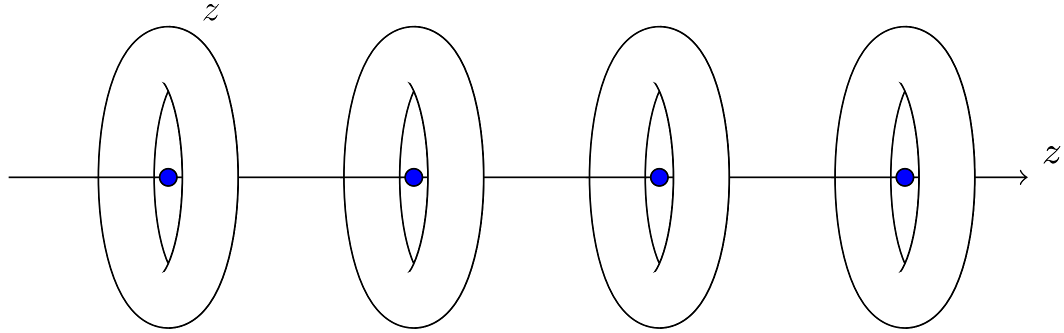

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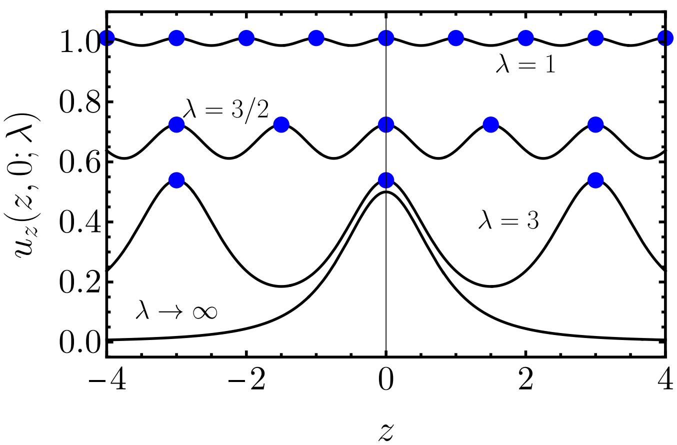

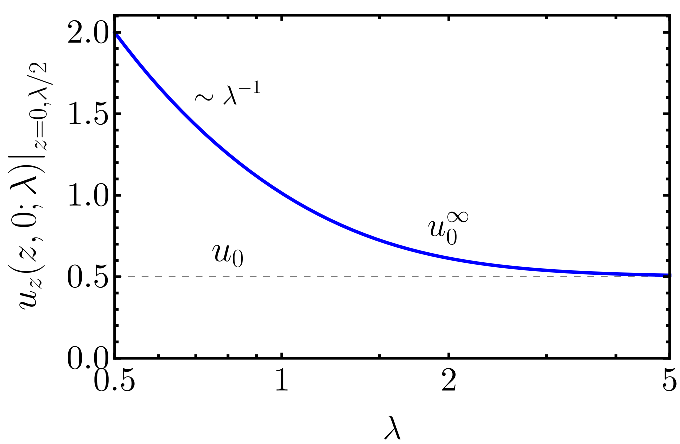

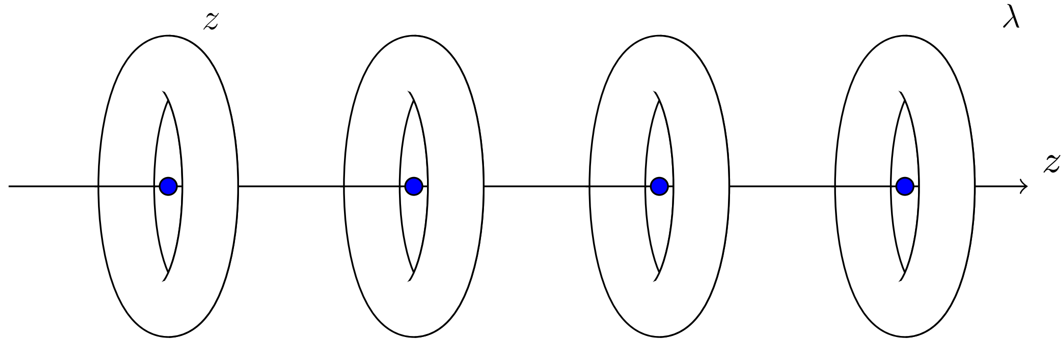

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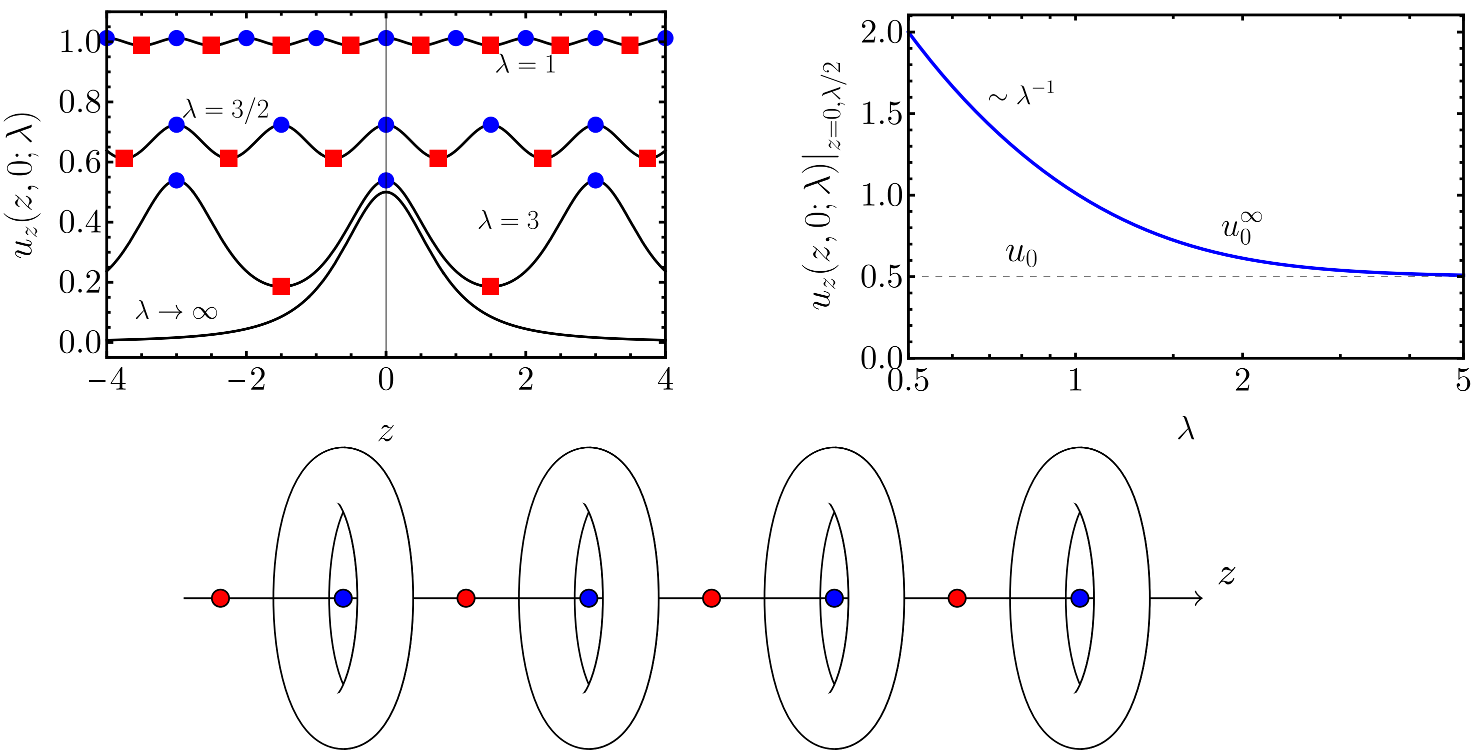

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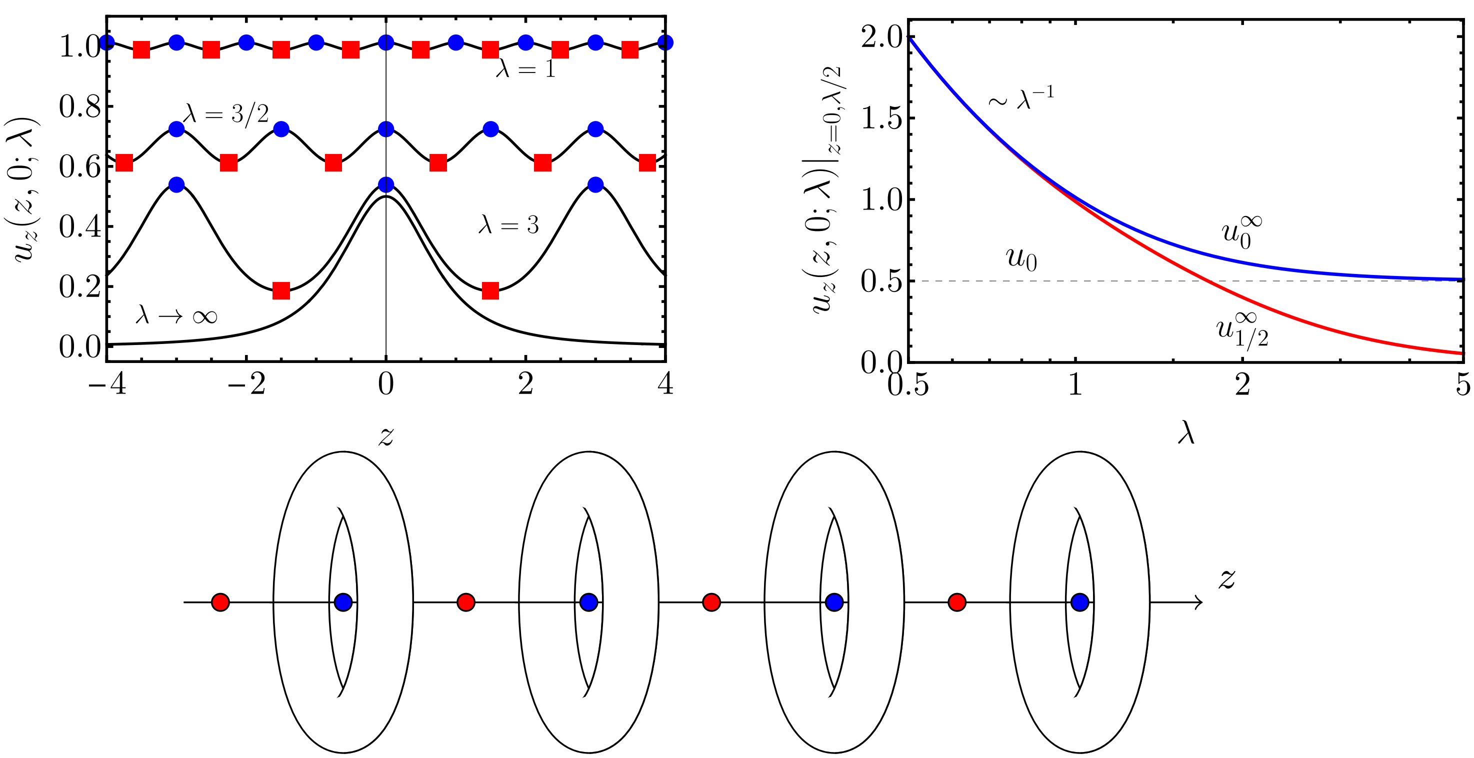
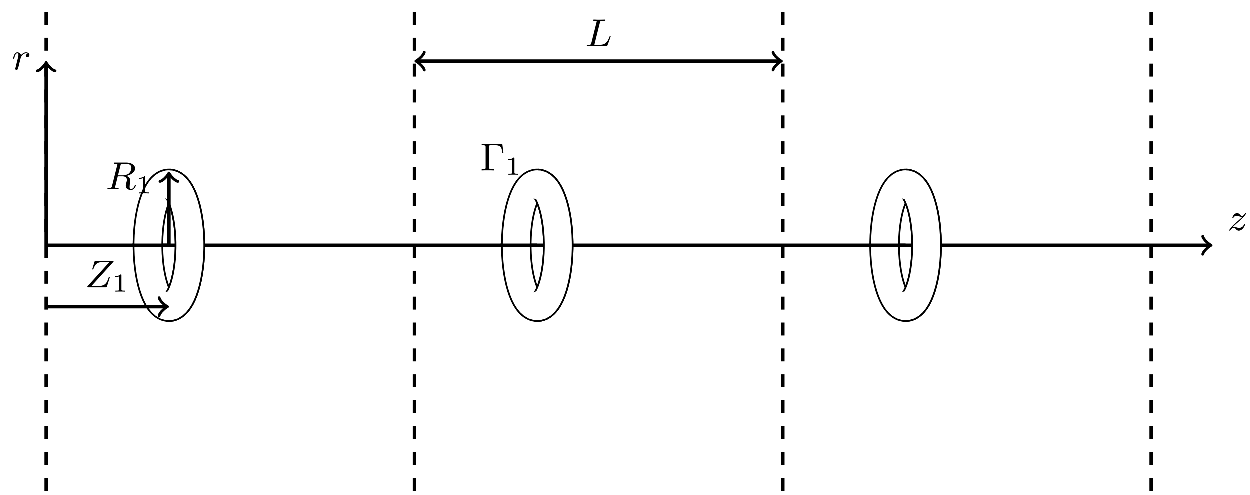


Interacting Vortex Ring Arrays



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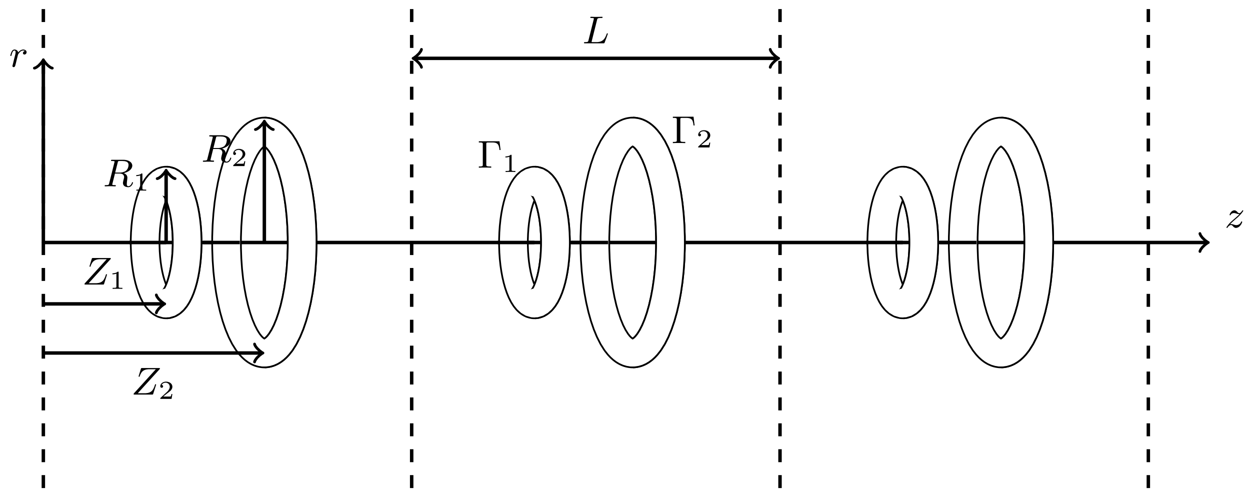

Interaction of N coaxial vortex rings — Borisov et.al (2013)

The interaction of N coaxial vortex rings can be modeled as a dynamical system with N degrees of freedom in $2N$-dimensional phase space:

$$\dot{Z}_i = \frac{1}{\Gamma_i R_i}\frac{\partial H}{\partial R_i}, \quad \dot{R}_i = -\frac{1}{\Gamma_i R_i}\frac{\partial H}{\partial Z_i}$$

whose Hamiltonian H has a self-interaction term and a Green's function:

$$H = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i^2 R_i\left(\log\frac{8R_i^{3/2}}{B_i} - \frac{7}{4}\right) + \sum_{i\neq j}^{N}\Gamma_i\Gamma_j \underbrace{G(R_i, Z_i; R_j, Z_j)}_{\psi \text{ due to ring } j \text{ evaluated at ring } i}$$

The existence of another integral in involution

$$P \equiv \sum_{j}^{N}\Gamma_j R_j^2; \quad \frac{dP}{dt} = 0$$

guarantees a canonical transformation to conjugate variables

$$(Z_1, ...Z_N, R_1, ..., R_N) \to (\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ...\eta_{N-1})$$

$$\dot{\xi}_j = \{\xi_j, H\}$$
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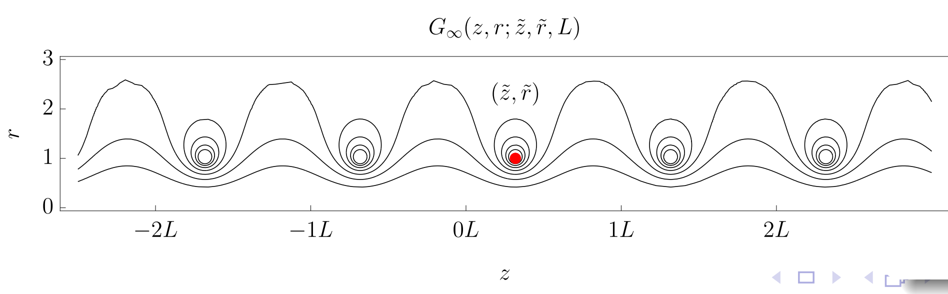

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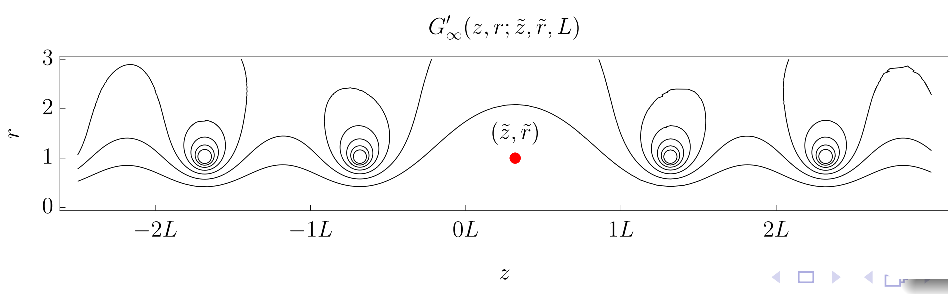

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$$H = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i^2 R_i\left(\log\frac{8R_i^{3/2}}{B_i} - \frac{7}{4}\right) + \sum_{i\neq j}\Gamma_i\Gamma_j G_\infty(R_i, Z_i; R_j, Z_j, L) + \sum_i \Gamma_i^2 G'_\infty(R_i, Z_i; R_i, Z_i, L)$$

The existence of another integral in involution

$$P \equiv \sum_j^N \Gamma_j R_j^2; \quad \frac{dP}{dt} = 0$$

guarantees a canonical transformation to conjugate variables

$$(Z_1, ...Z_N, R_1, ..., R_N) \to (\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ...\eta_{N-1})$$

$$\begin{aligned} \dot{\xi}_j &= \{\xi_j, H\} \\ \dot{\eta}_j &= \{\eta_j, H\} \\ 0 &= \{P, H\} \end{aligned}$$

Interpreting the Hamiltonian for 2 interacting vortex ring arrays

$$H = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i^2 R_i\left(\log\frac{8R_i^{3/2}}{B_i} - \frac{7}{4}\right) + \sum_{i\neq j}\Gamma_i\Gamma_j G_\infty(R_i, Z_i; R_j, Z_j, L) + \sum_i \Gamma_i^2 G'_\infty(R_i, Z_i; R_i, Z_i, L)$$

Let $N = 2$

$H = \frac{1}{2\pi}\Gamma_1^2 R_i\left(\log\frac{8R_1^{3/2}}{B_i} - \frac{7}{4}\right)$	Self-induction of ring 1
$+\frac{1}{2\pi}\Gamma_2^2 R_1\left(\log\frac{8R_2^{3/2}}{B_i} - \frac{7}{4}\right)$	Self-induction of ring 2
$+\Gamma_1\Gamma_2 G_\infty(R_1, Z_1; R_2, Z_2, L)$	Streamfunction at ring 1 induced by ring 2 and its images
$+\Gamma_2\Gamma_1 G_\infty(R_2, Z_2; R_1, Z_1, L)$	Streamfunction at ring 2 induced by ring 1 and its images
$+\Gamma_1^2 G'_\infty(R_1, Z_1; R_1, Z_1, L)$	Streamfunction at ring 1 induced by its own images
$+\Gamma_2^2 G'_\infty(R_2, Z_2; R_2, Z_2, L)$	Streamfunction at ring 2 induced by its own images

Interpreting the reduced variables ξ and η

For N coaxial vortex ring (array)s

- 'Real space' $H(\boldsymbol{Z}, \boldsymbol{R}) \quad (Z_1, ...Z_N, R_1, ..., R_N)$
- 'Phase space' $H(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad (\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ...\eta_{N-1})$

$$\xi_0 = \sum_{i=1}^{N} \Gamma_i Z_i \Bigg/ \sum_{i=1}^{N} \Gamma_i = \frac{\Gamma_1 Z_1 + \Gamma_2 Z_2}{\Gamma_1 + \Gamma_2}$$

$$\xi_i = Z_{i+1} - \sum_{j=1}^{i} \Gamma_j Z_j \Bigg/ \sum_{j=1}^{i} \Gamma_j = Z_2 - Z_1$$

$$\eta_i = \left(R_{i+1}^2 - \sum_{j=1}^{i} \Gamma_j R_j^2 \Bigg/ \sum_{j=1}^{i} \Gamma_j \right) \Gamma_{i+1} \sum_{j=1}^{i} \Gamma_j \Bigg/ \sum_{j=1}^{i+1} \Gamma_j = (R_2^2 - R_1^2) \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}$$

Additionally,

$$B_1 = 1, \quad B_2 = \beta \in (0, \infty),$$
$$\Gamma_1 = 1, \quad \Gamma_2 = \gamma \in (-1, 1]$$

so the reduced Hamiltonian becomes (Borisov, Kilin, and Mamaev 2013)

$$H = H(\xi, \eta; \beta, P, \gamma, L)$$

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Interpreting the reduced variables ξ and η for 2 vortex ring arrays

For 2 coaxial vortex ring (array)s

- 'Real space' $H(\boldsymbol{Z}, \boldsymbol{R}) \quad (Z_1, ...Z_N, R_1, ..., R_N)$
- 'Phase space' $H(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad (\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ...\eta_{N-1})$

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$$H = H(\xi, \eta; \beta, P, \gamma, L)$$

The periodic 2-vortex ring problem in canonical form

$$H(Z_1, R_1, Z_2, R_2)$$

Equations of motion:

$$\dot{Z}_1 = \{Z_1, H\} = \frac{1}{\Gamma_1 R_1} \frac{\partial H}{\partial R_1}$$
$$\dot{R}_1 = \{R_1, H\} = -\frac{1}{\Gamma_1 R_1} \frac{\partial H}{\partial R_1}$$
$$\dot{Z}_2 = \{Z_2, H\} = \frac{1}{\Gamma_2 R_2} \frac{\partial H}{\partial R_2}$$
$$\dot{R}_2 = \{R_2, H\} = -\frac{1}{\Gamma_2 R_2} \frac{\partial H}{\partial R_2}$$

with the modified Poisson bracket

$$\{f, g\} = \sum_{j=1}^{2} \frac{1}{\Gamma_j R_j} \left(\frac{\partial f}{\partial Z_j} \frac{\partial g}{\partial R_j} - \frac{\partial f}{\partial R_j} \frac{\partial g}{\partial Z_j} \right)$$

$$H(\xi, \eta)$$

Equations of motion:

$$\dot{\xi} = \{\xi, H\} = \frac{\partial H}{\partial \eta}$$
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with the usual Poisson bracket

$$\{f, g\} = \left(\frac{\partial f}{\partial \xi} \frac{\partial g}{\partial \eta} - \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \xi} \right).$$

We recover the full system by way of the conjugate variables ξ_0 and P,

$$\dot{\xi}_0 = \frac{\partial H}{\partial P}$$

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$$H(\xi, \eta)$$

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$$\{f, g\} = \left(\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} - \frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \xi}\right).$$

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$$\dot{\xi}_0 = \frac{\partial H}{\partial P}$$

Integrability of the N-vortex problem

N	Point vortices on the plane	Coaxial vortex rings	Coaxial vortex ring arrays
1	stationary	cyclic/trivial	cyclic/trivial
2	cyclic/trivial	integrable	integrable
3	integrable	chaotic	chaotic
4	chaotic	chaotic	chaotic

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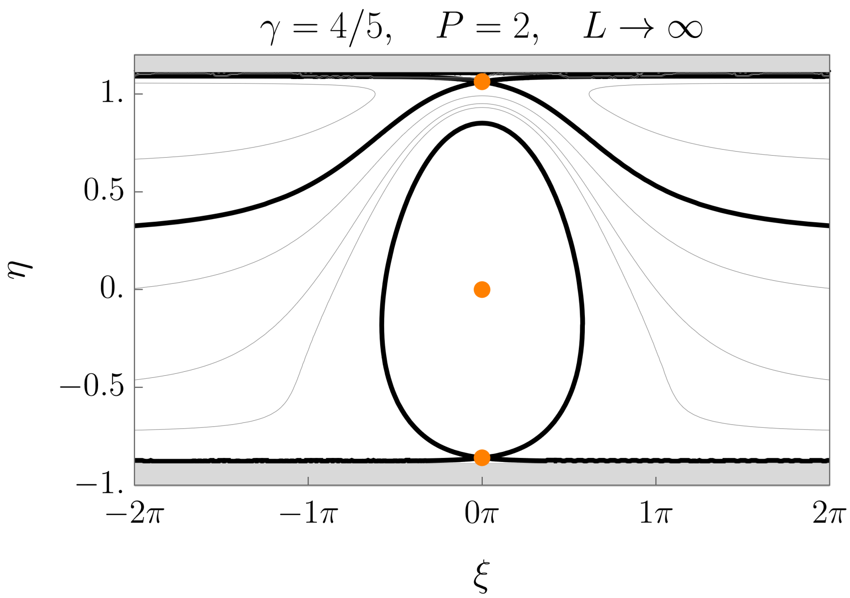
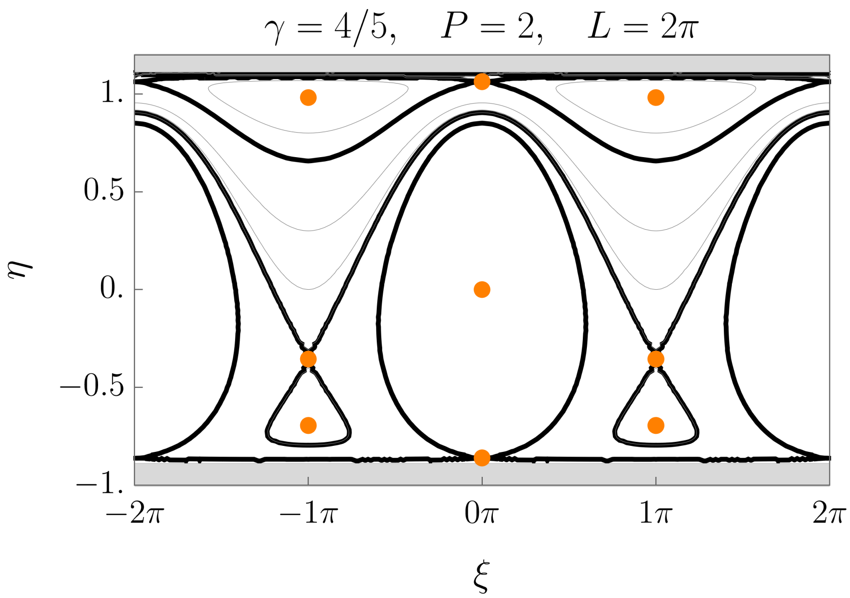
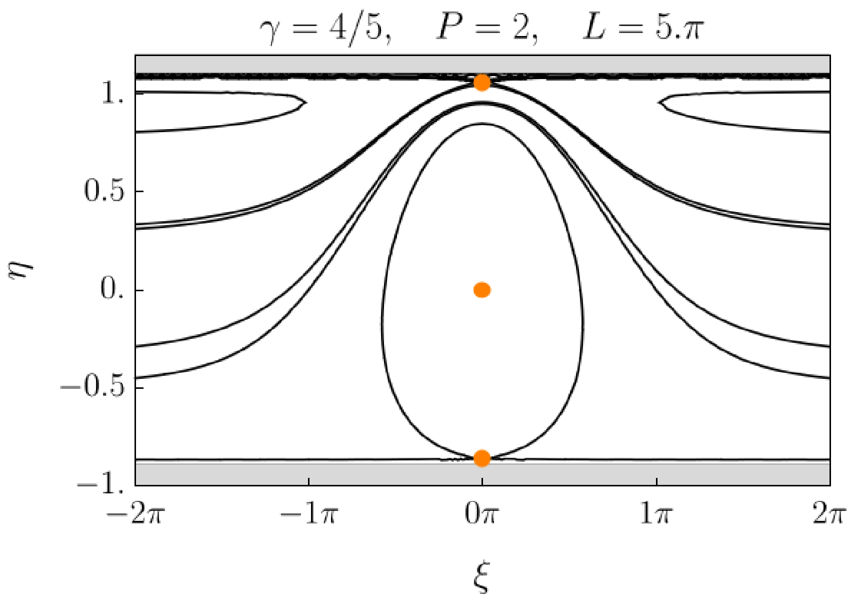
The reduced Hamiltonian for two coaxial vortex rings



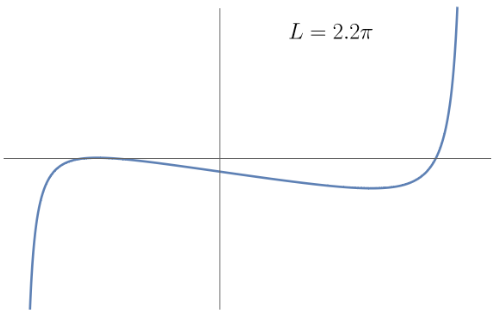
The reduced Hamiltonian for two coaxial vortex ring arrays

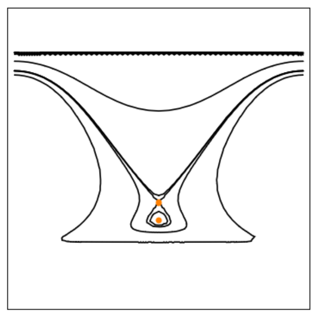


Evolution of the phase space with decreasing L

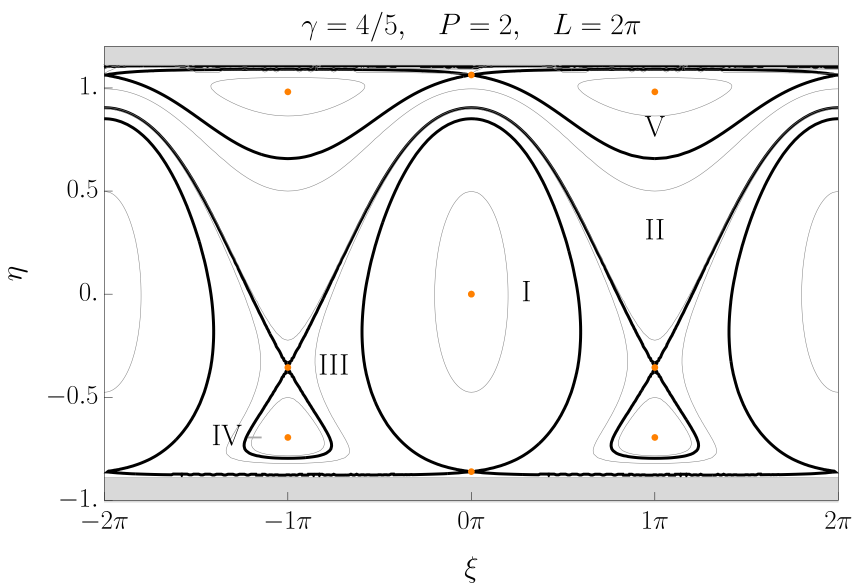


A ‘saddle-center’ bifurcation occurs as the spatial period decreases

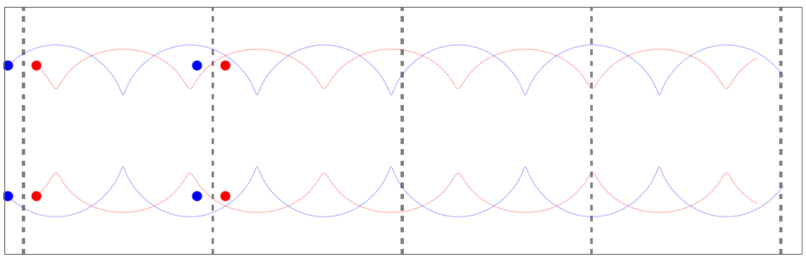


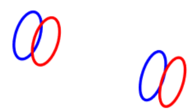


Regimes of inter-vortex motion

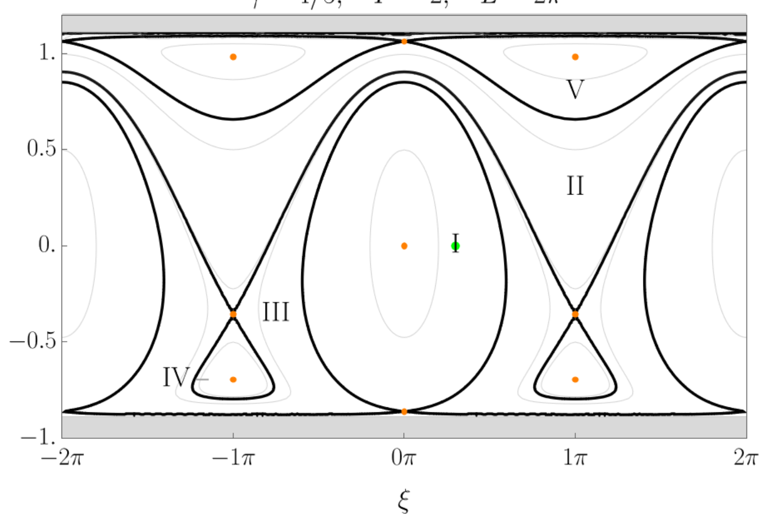


Regime I: Leapfrogging

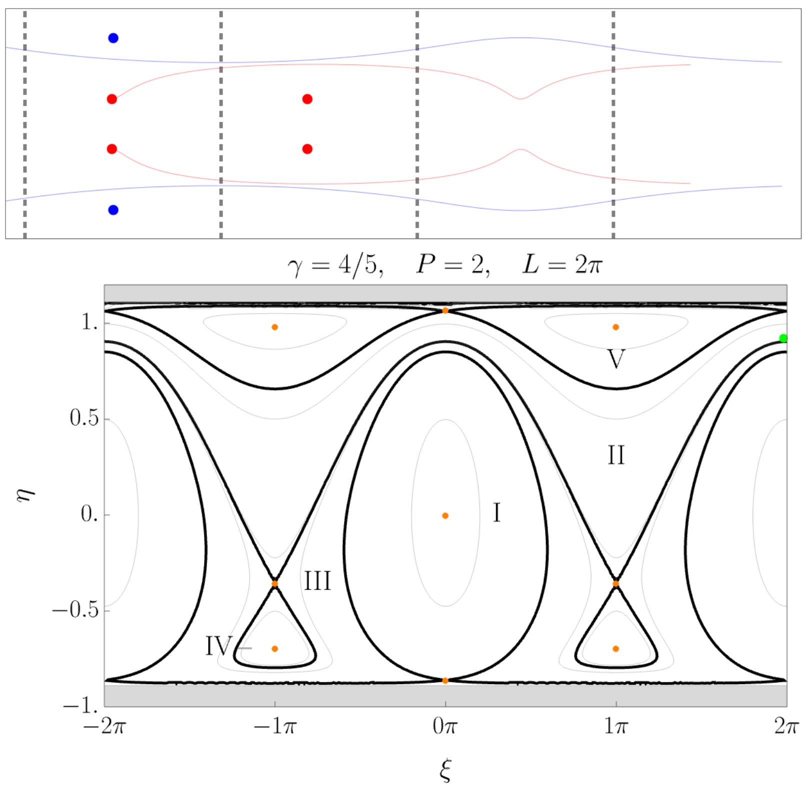




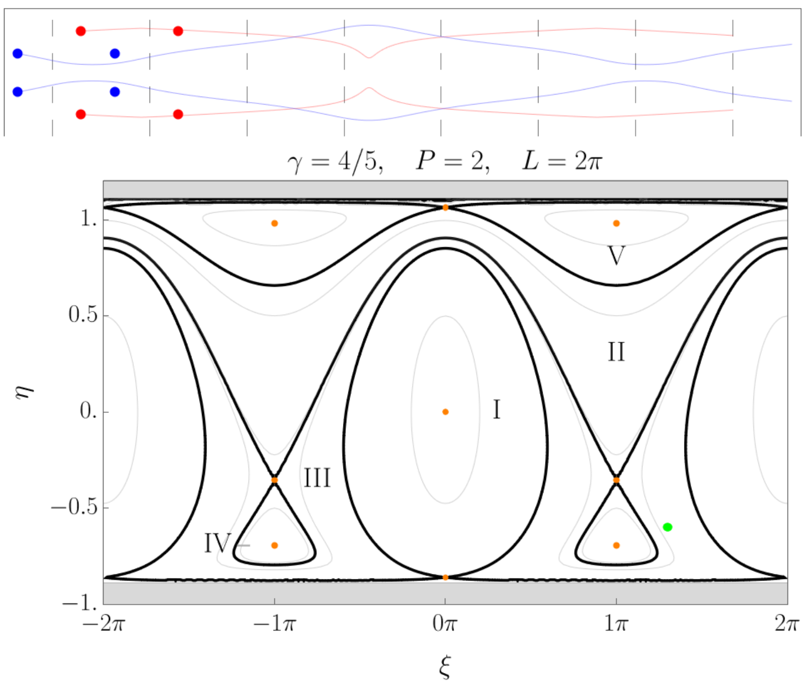
$\gamma = 4/5, \quad P = 2, \quad L = 2\pi$

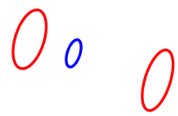


Regime II: Secular Pass-Through

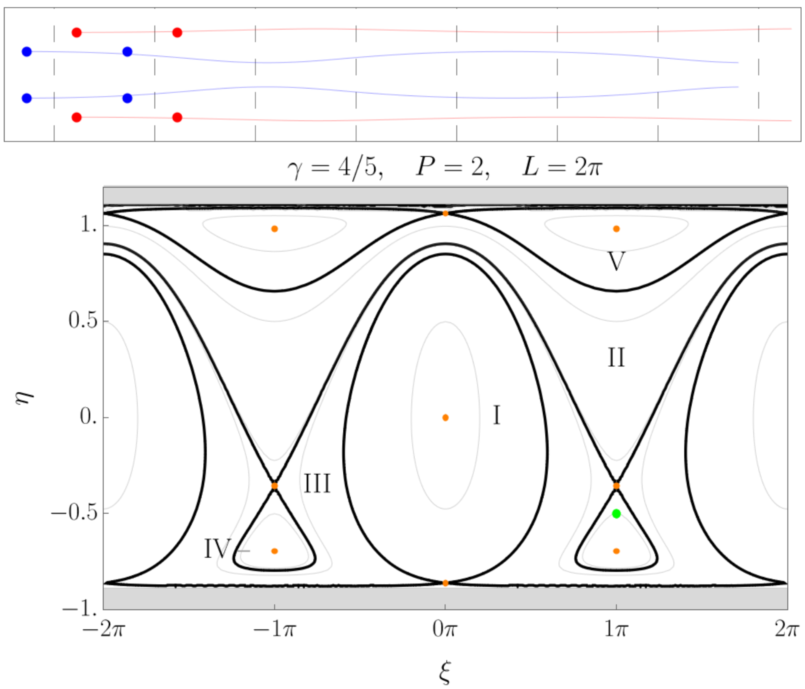


Regime III: Retrograde Pass-Through

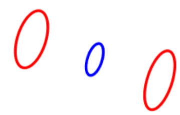




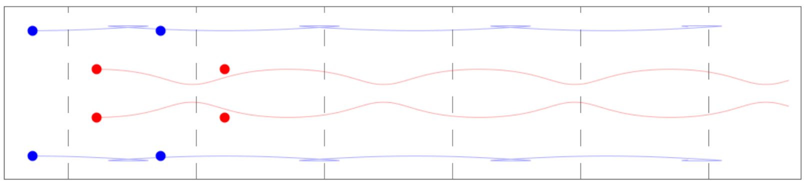
Regime IV

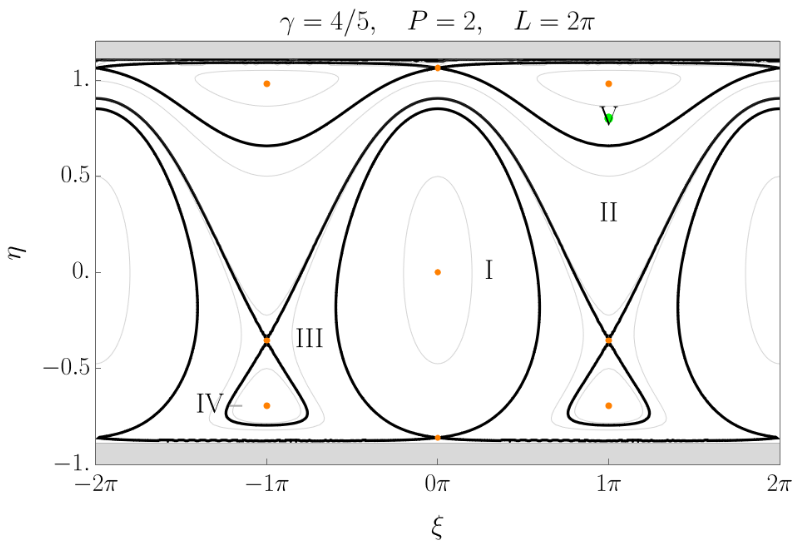


$\gamma = 4/5, \quad P = 2, \quad L = 2\pi$
1.
0.5
0.
−0.5
−1.
η
−2π
−1π
0π
1π
2π
ξ
I
II
III
IV
V



Regime V





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1.
0.5
0.
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−2π
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0π
1π
2π
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I
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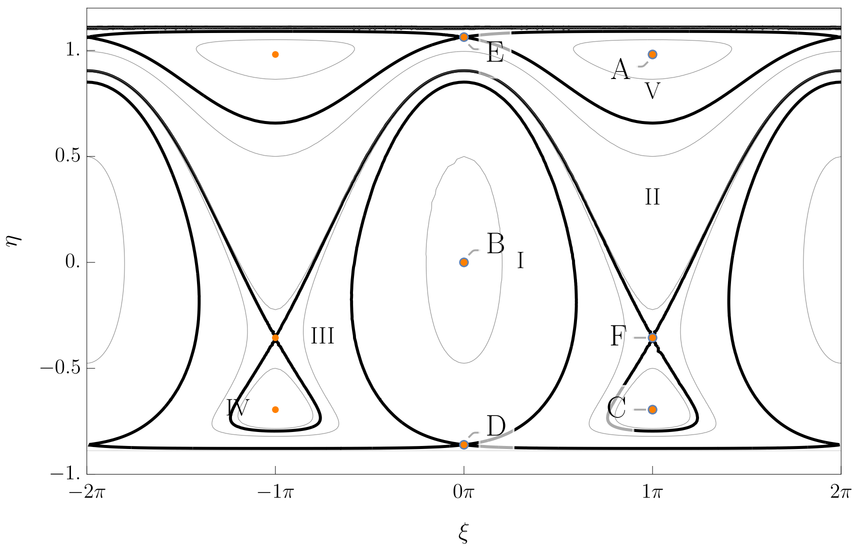
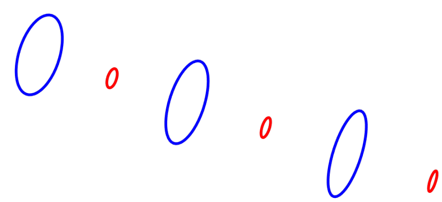
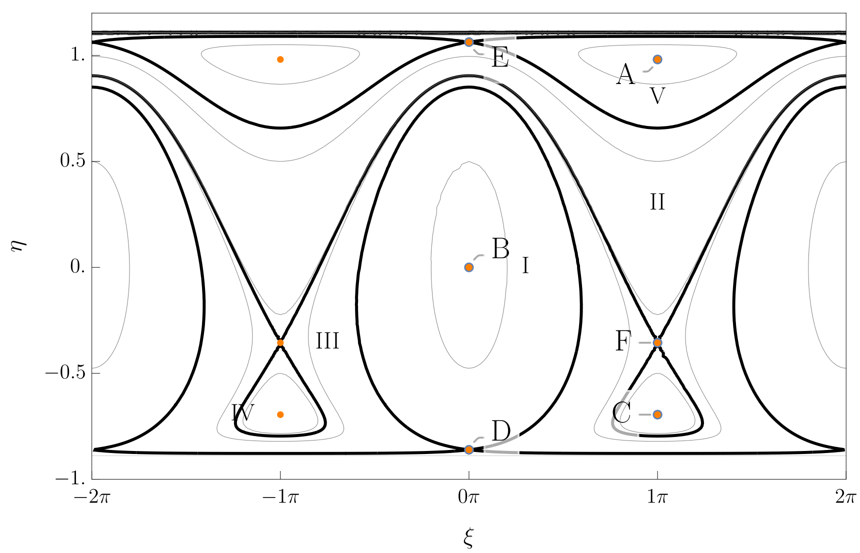
Relative Equilibria



Equilibrium A

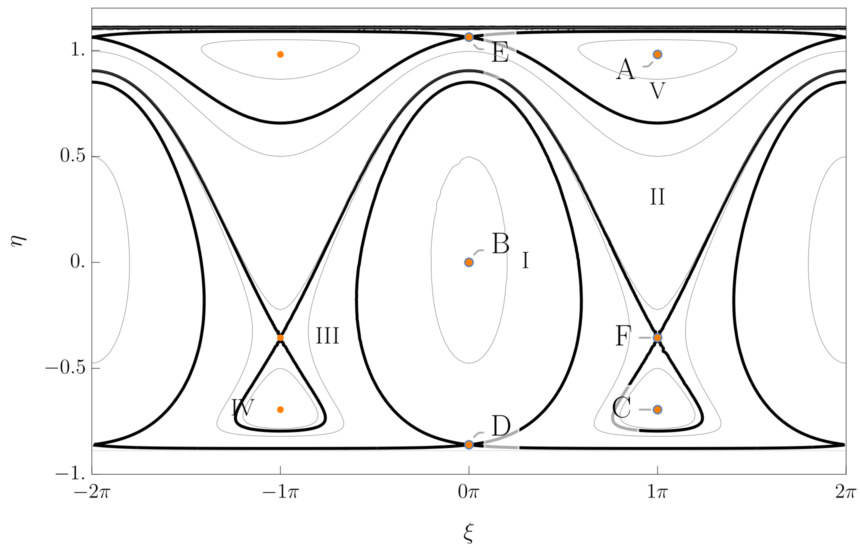


Relative Equilibria

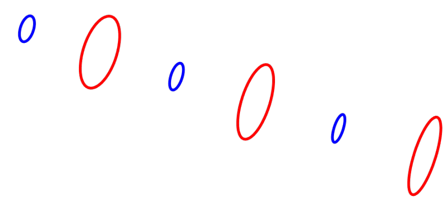


Equilibrium B

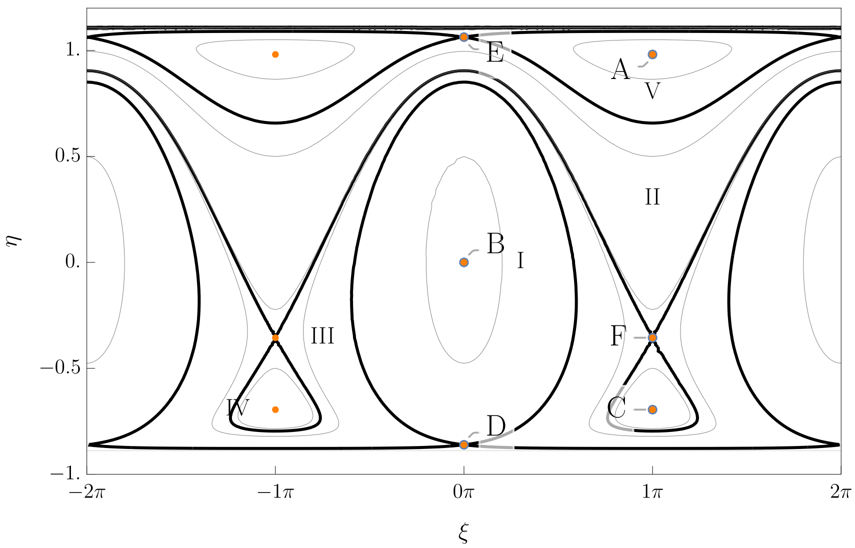
Relative Equilibria



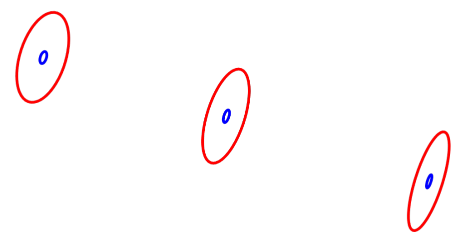
Equilibrium C



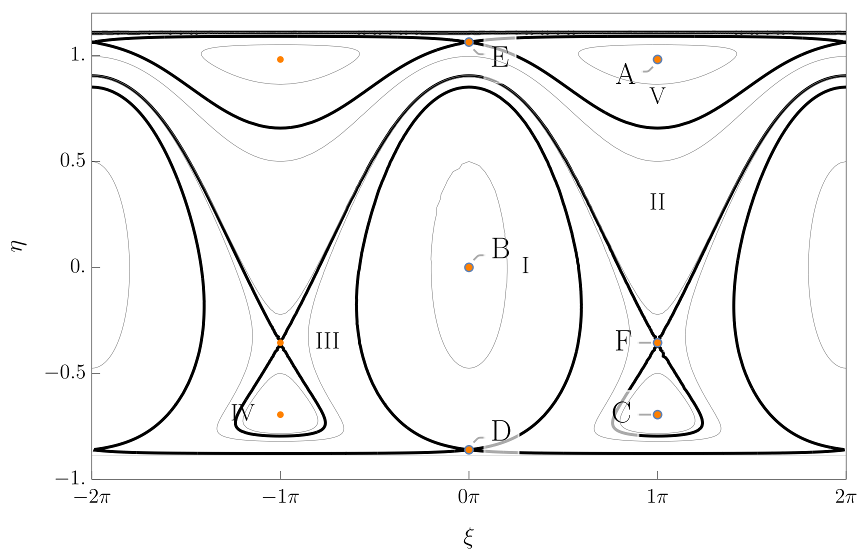
Relative Equilibria



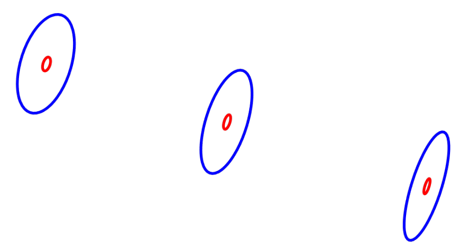
Equilibrium D



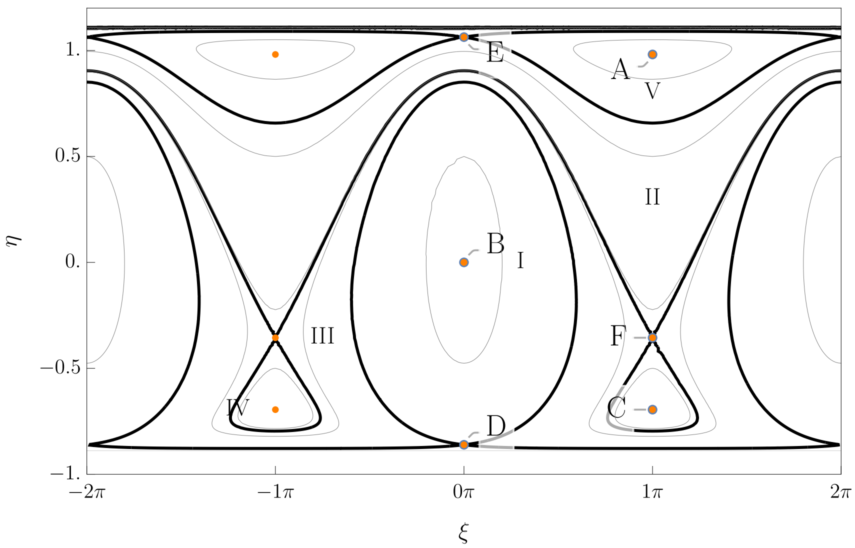
Relative Equilibria



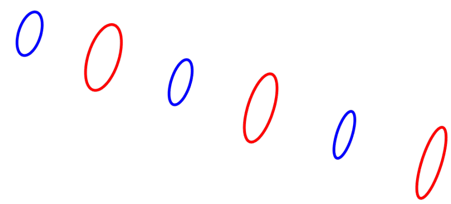
Equilibrium E



Relative Equilibria



Equilibrium F



Summary

- Arrays of coaxial vortex rings – the periodic N-vortex ring problem
- Streamfunction in a co-moving reference frame for a single array
- Hamiltonian dynamics of multiple arrays of vortex rings
- Regimes of vortex motion for 2 vortex ring arrays with $\beta = 1, \gamma = 4/5, P = 2, L = 2\pi$
- Relative Equilibria

Thank you!

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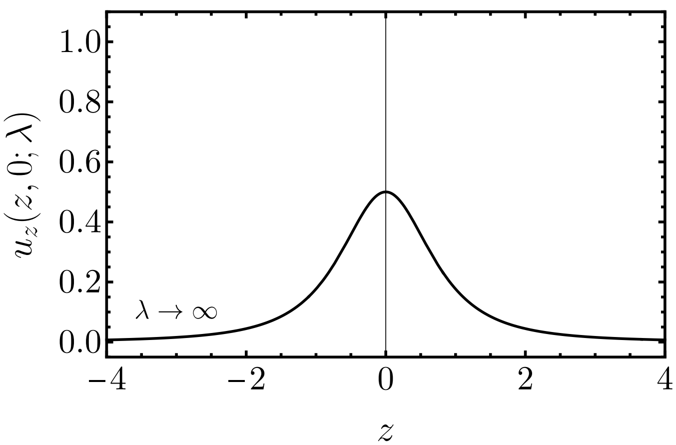
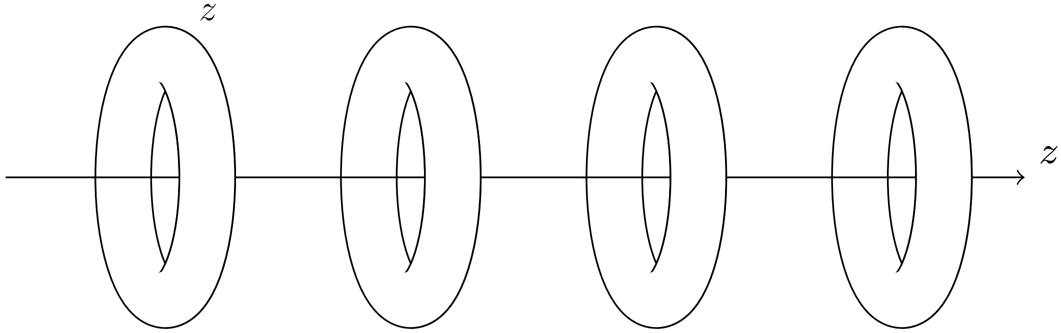
Extra slides

The speed induced along the axis varies with λ





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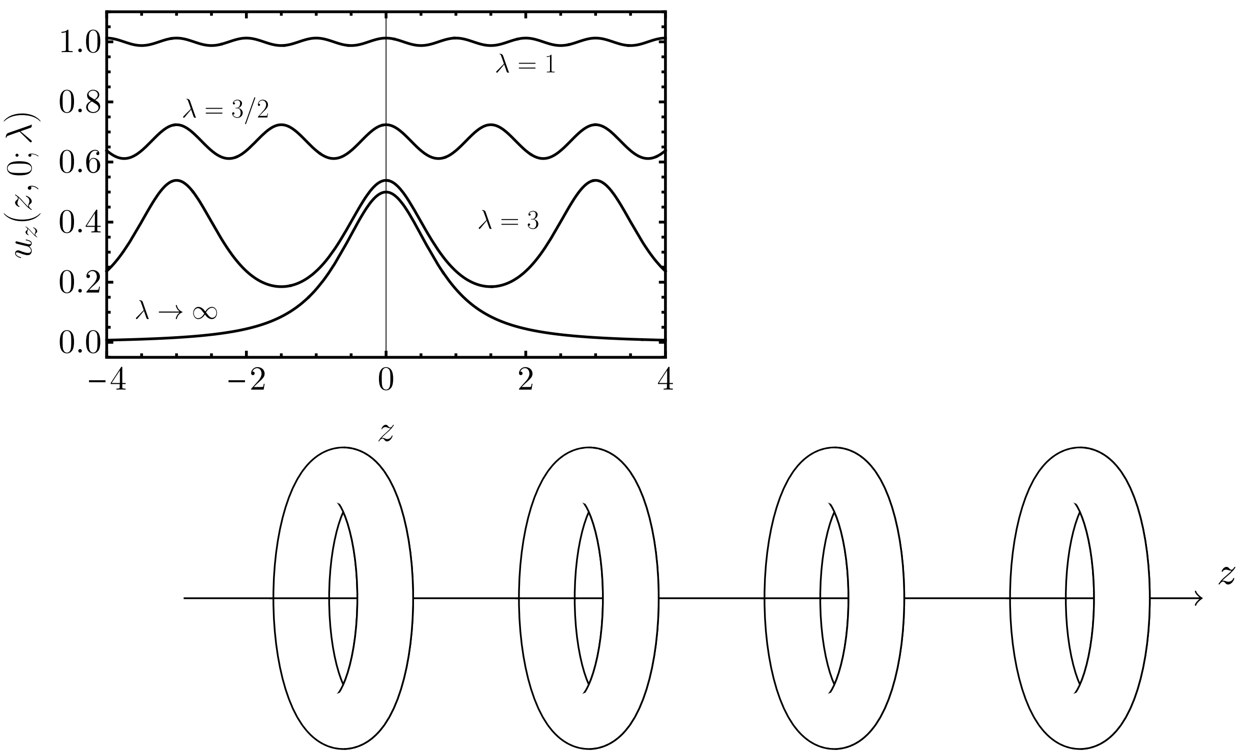

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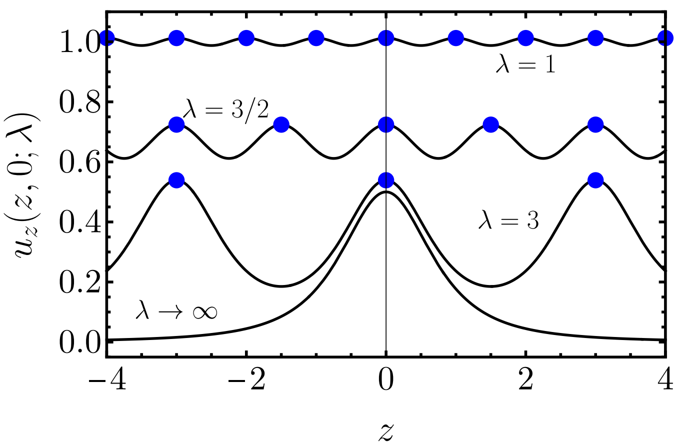
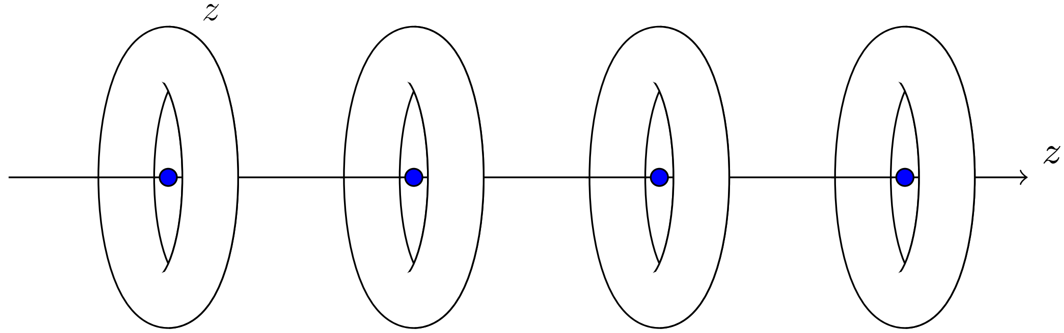
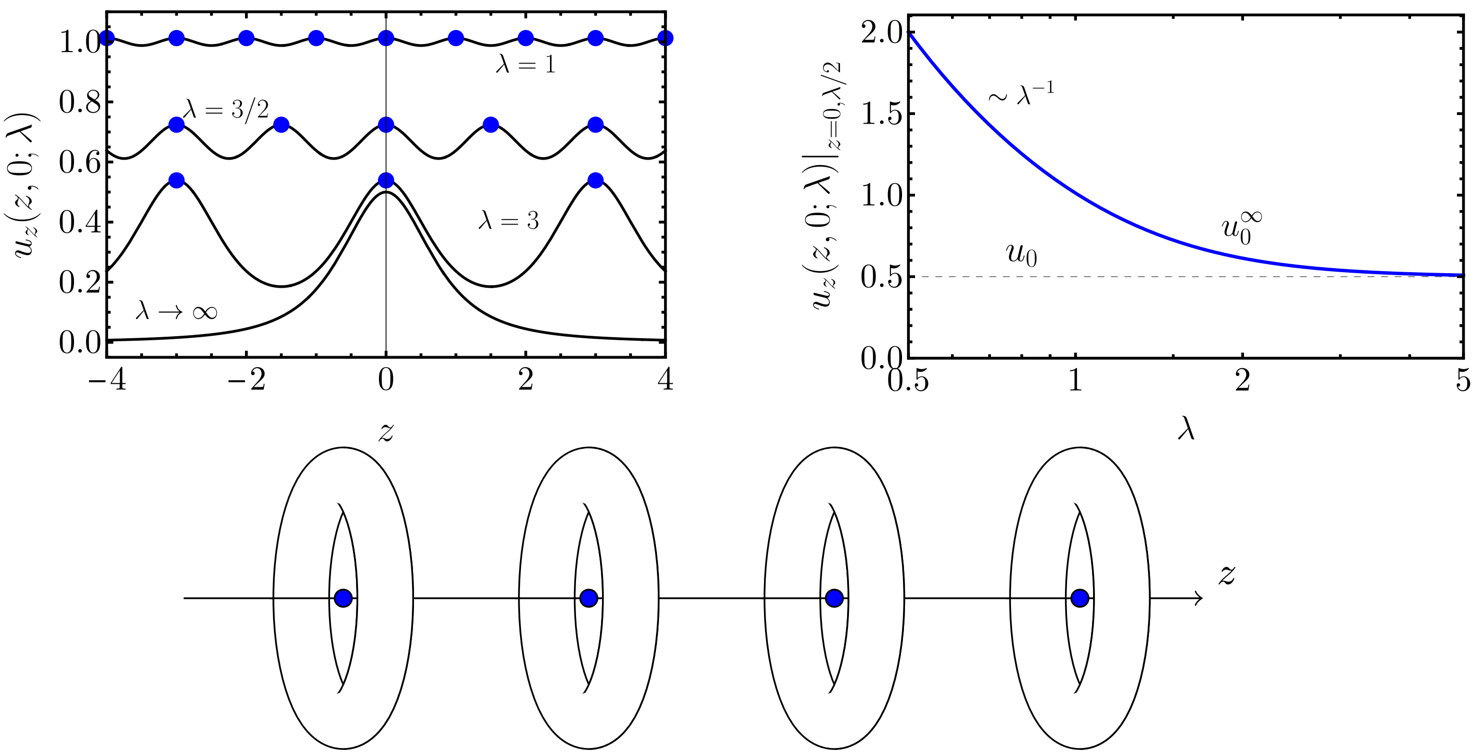
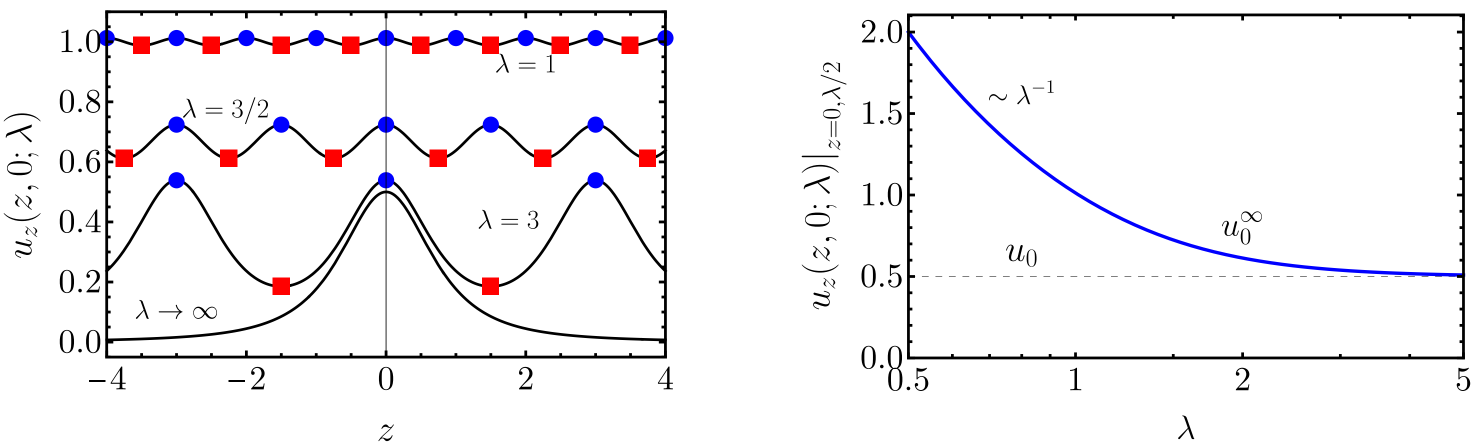


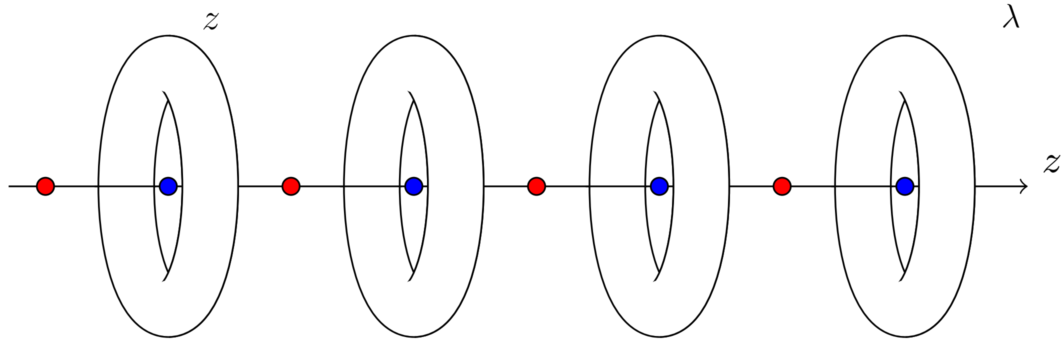


The speed induced along the axis varies with λ

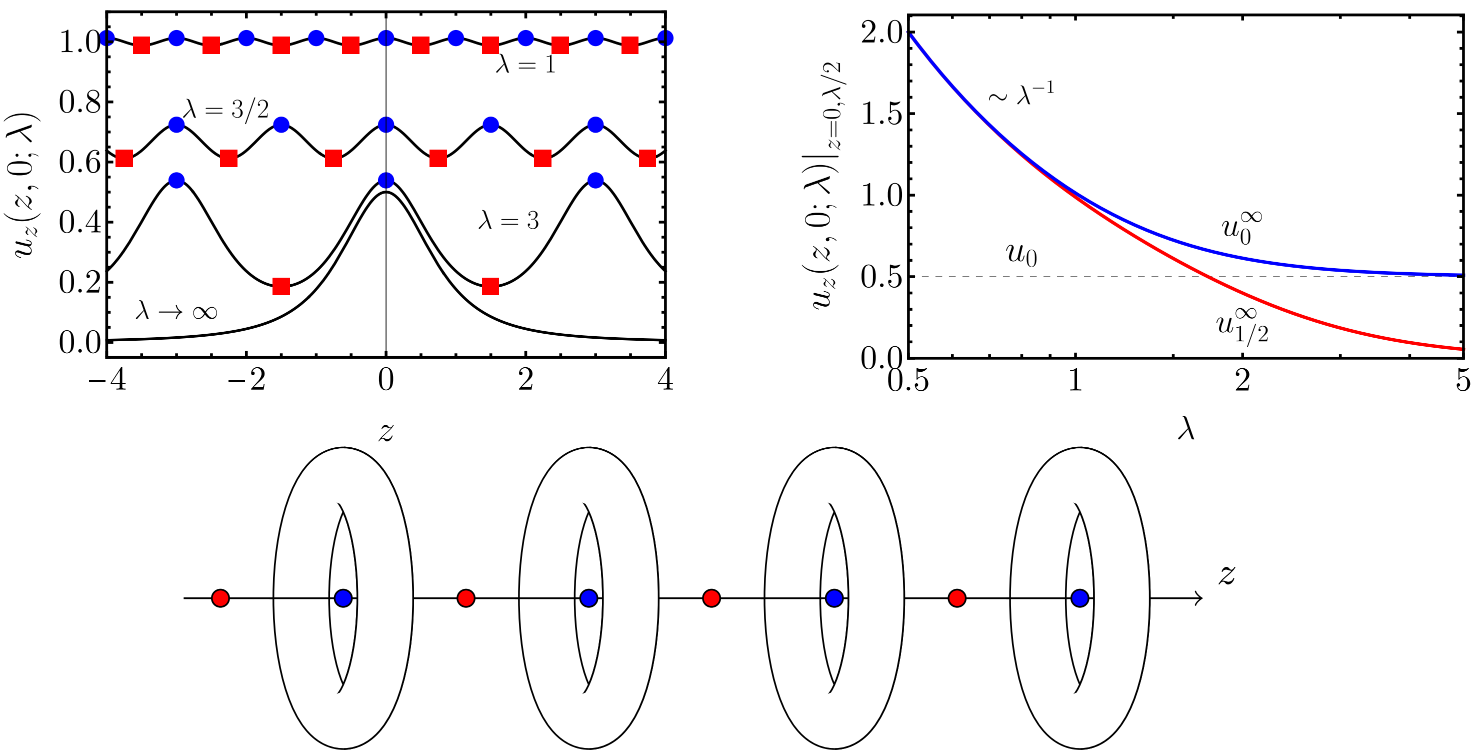


The speed induced along the axis varies with λ

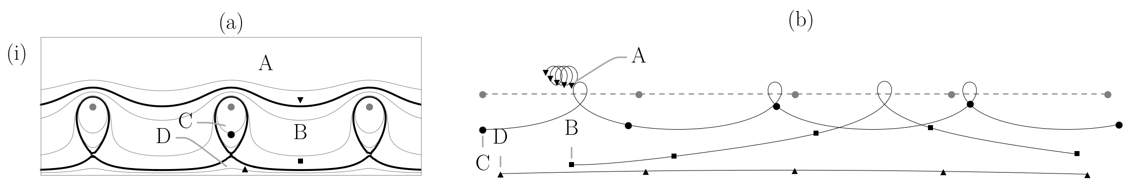




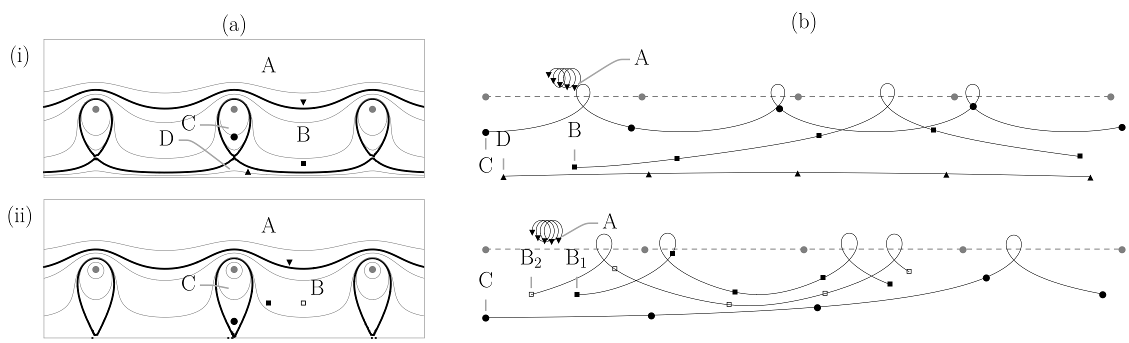
The speed induced along the axis varies with λ



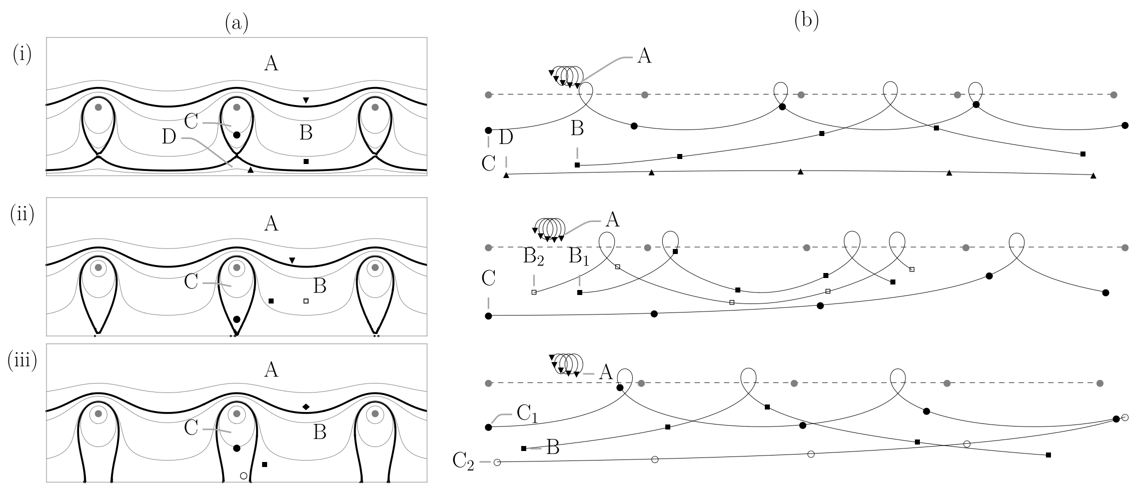
Possible trajectories of a passive particle (without swirl) near an array of vortex rings



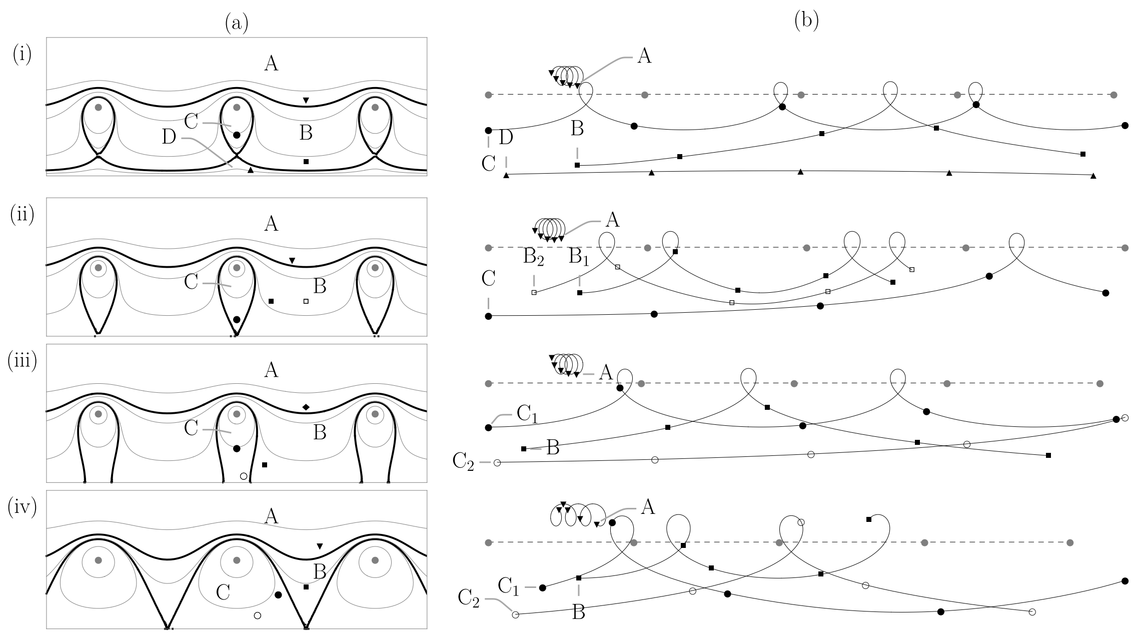
Possible trajectories of a passive particle (without swirl) near an array of vortex rings



Possible trajectories of a passive particle (without swirl) near an array of vortex rings



Possible trajectories of a passive particle (without swirl) near an array of vortex rings



Possible trajectories of a passive particle (without swirl) near an array of vortex rings