THEORETICAL PREDICTIONS FOR THE DRAG FORCE DUE TO EXOTIC WAKES

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Summary We present a generalized formulation of von Kármán's analysis of the drag on a bluff body with a vortex-street wake, allowing for the existence of exotic wakes with more than 2 vortices per period. This formulation recovers von Kármán's formula as a special case. We apply the method to exotic wakes by considering systems of N point vortices and their periodic images in relative equilibrium, allowing us to make quantitative comparisons of the drag due to exotic wakes with von Kármán's canonical drag formula.

INTRODUCTION

In 1911, von Kármán considered the drag (or thrust) on a two-dimensional body in uniform flow whose wake takes the form of what is now known as the Bénard - von Kármán vortex street: two rows of counter-rotating vortices [5]. Using the momentum flux through a contour surrounding the cylinder and cutting through the wake, and representing the wake as a series of point vortices in an ideal fluid, he developed the famous formula known by his name that relates the drag on a bluff body to the characteristics of its wake. In dimensionless form, this formula was given by [2] as

$$C_D = \frac{4}{\pi}\frac{L}{d}\frac{U_s^2}{U_\infty^2}\left[\coth^2\frac{\pi h}{L} + \left(\frac{U_\infty}{U_s} - 2\right)\frac{\pi h}{L}\coth\frac{\pi h}{L}\right], \tag{1}$$

where U_s is the self-induced velocity of the double row of vortices, U_∞ is the velocity of the free stream, L is the wavelength of the system, d is the length scale of the bluff body, e.g. the diameter of a cylinder, h is the transverse distance between the two rows, and C_D is the predicted coefficient of drag.

There is a wealth of recent work showing that the Bénard - von Kármán street is not the only arrangement of vortices that persists in the wakes of immersed bodies. The wakes of several moving and/or stationary bluff bodies are known to take on other, repeatable vortex patterns, which were first extensively classified by [8] according to the number and pairings of the vortices in each period of motion. Thus, the '2S' mode was assigned to wakes with two single vortices shed per period, the 'P+S' mode for a pair of vortices together with a single vortex per period, and the '2P' mode for two pairs of vortices per period. These modes are the most extensively documented ones, although more recent work has even found wakes with up to sixteen vortices per period [6]. Theoretical predictions for the forces exerted on a body whose wake takes these complicated forms, analogous to (1) for a Bénard - von Kármán street, are not currently available. Our work aims to fill this gap by generalizing (1) for more complicated vortex-street wakes.

The well-documented experimental observation of discrete, almost-spatially-periodic, temporally persistent vortical structures in bluff-body wakes, such as the 'P+S' and '2P' modes, is mirrored in ideal hydrodynamics by the existence of relative equilibrium solutions to the dynamical system comprised of $N = 3$ and 4 point vortices, respectively, in a singly-periodic vertical strip. The vortex trajectories associated with these equilibrium and near-equilibrium solutions have been shown [7] to approximate the behavior of real vortices in the wakes of bluff bodies reasonably well. Often, the time scale of relative motion of vortices (in a co-moving frame) is of approximately the same order as the convective time scale. Hence, many real vortex-street wakes can be fruitfully analyzed using equilibrium configurations of point vortices in a periodic strip, even when the vortices in the wake are exhibiting relative motion.

EXAMPLE: A PERIODIC VORTEX-STREET WAKE WITH N VORTICES PER PERIOD

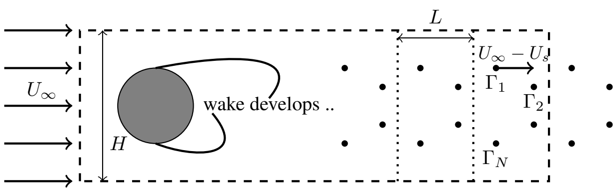

Figure 1: A periodic vortex-street wake with $N = 4$

We consider a wake that, in the well-developed region, can be approximated by a system of N point vortices (together with their periodic images to the left and right), as indicated in figure 1. The strengths $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, ...\Gamma_N)$ and locations in the complex plane $\mathbf{z} = (z_1, z_2, ...z_N)$ of the vortices are assumed to be such that the vortices are in *relative equilibrium*; these equilibria can be found using methods given by, e.g., [1]. The vortices tend to move to the left under their mutual induction at a speed U_s, but they are swept to the right with a velocity $U_\infty - U_s$ under the influence of the free stream.

We then write Newton's second law for the control volume defined by the dashed rectangle, supposing that the width H of the rectangle is made large enough that all disturbances die down at the horizontal boundaries; then, [3] gives the average drag force on the body as the sum of two terms,

$$F_D = \rho\oint y\zeta(\vec{v}\cdot\vec{n})\mathrm{d}s - \frac{1}{2}\rho\mathrm{Re}\int_{-H/2}^{+H/2}(u - iv)^2\mathrm{d}y = F_{D_1} + F_{D_2}, \tag{2}$$

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where ζ is the vorticity, ρ is the fluid density, and $u - iv$ is the complex velocity induced by the system of vortices.

The first term, F_{D_1}, represents the flux of '$y \times \zeta$' out of the control volume, averaged over one period. For a system of point vortices, this is identical to the imaginary part of the linear impulse of the N vortices that leave the contour every period, i.e. the flux of $P = \sum_i^N \Gamma_i y_i$ averaged over one period.

The second term, F_{D_2}, requires the evaluation of the velocity induced by the point vortices at the right-side boundary of the control volume. Although the value of the velocity at any point will change with time, by Cauchy's residue theorem the integral in (2) will not change under certain symmetry conditions; thus, it is sufficient to calculate for any one instant the value of the integral in F_{D_2}.

These two terms can then be non-dimensionalized and added to give the expression for the coefficient of drag as

$$C_D = \frac{4}{\pi}\frac{L}{d}\frac{U_s^2}{U_\infty^2} \cdot f_1 \left[f_1 f_2 + \mathcal{P}\left(\frac{U_\infty}{U_s} - 1\right)\right], \tag{3}$$

where $\mathcal{P}$ is a non-dimensionalized form of P, and the non-dimensional quantities f_1 and f_2 are defined by the relations

$$U_s = \Gamma/(2L) \cdot f_1(\mathbf{z}, \mathbf{\Gamma}),$$
$$f_2(\mathbf{z}, \mathbf{\Gamma}) = \pi L/(\Gamma) \int (u - iv)^2 \mathrm{d}y.$$

Scalar Γ is the sum of the strengths of all vortices with positive circulations (or, equivalently, the sum of the strengths of all vortices with negative circulations). Thus f_1 expresses the relationship of $\mathbf{z}$ and $\mathbf{\Gamma}$ with U_s, and f_2 expresses the relationship of $\mathbf{z}$ and $\mathbf{\Gamma}$ with the integral in F_{D_2}.

Equation (3) is a generalized form of von Kármán's drag formula, and it can be applied to any spatially-periodic wake with the vortices in relative equilibrium by determining f_1, f_2, and $\mathcal{P}$ as a function of the vortices' positions and strengths. For the Bénard - von Kármán street, the three terms have straightforward closed forms: $f_1 = -\coth \mathcal{P}$, $f_2 = 1 - \mathcal{P}\tanh\mathcal{P}$ and $\mathcal{P} = -\pi h/L$. By substituting these expressions, we verify that (1) is indeed a special case of (3).

CONCLUSIONS

We use this method to give theoretical predictions for the drag force exerted on a body whose wake takes the often-observed 'P+S' or '2P' forms, using relative equilibrium configurations for $N = 3$ and 4 respectively. A representative example for a system of 4 vortices and their periodic images in relative equilibrium, representing a '2P' wake, is given in figure 2, where we show the predicted coefficient of drag as a function of Strouhal number $St \equiv f \cdot d/U_\infty$, with f the frequency of vortex shedding. The Strouhal number is itself an empirically-known, increasing function of the Reynolds number — as documented by, e.g. [9] — that varies between about 0.18 and 0.25 in the Reynolds number range $100 < Re < 1000$.

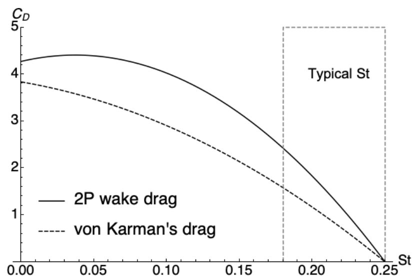

Figure 2: Predicted drag as a function of Strouhal number

It has long been known that abrupt changes in the mode of vortex shedding are related to changes in the forces experienced by the shedding body [4]. This work allows us to classify exotic wakes such as the '2P' and 'P+S' modes according to whether they lead to an increase or decrease in the drag force they cause on the body compared to the standard Bénard - von Kármán -type wake. Our model therefore suggests a theoretical basis for possible drag reduction or thrust-enhancement due to a 'switch' from one vortex-shedding mode to another.

References

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