Augmented Red-Black Trees

To illustrate how to augment a data structure for a specific purpose, we will focus here on the red-black tree. Based on the four-step process in the following section, we will begin by proving a theorem that, while specific to the red-black tree, applies in principle to any augmentation.

Theorem (Augmentation): Let f be a data field that augments a red-black tree $\mathbf{T}$ of n nodes, and suppose the contents of f for node x can be computed using only the information contained in x and the information in $left[x]$ and $right[x]$ (including f). Then we can maintain the values of f in all nodes in $\mathbf{T}$ during insertion and deletion without asymptotically affecting the $O(\lg n)$ performance of these operations.

Proof: Recall that, for red-black trees, insertion occurs in two phases. During the first phase, x is inserted as a child of an existing node, which we denote $parent[x]$. The value of $f(x)$ can be computed in $O(1)$ time since, by supposition, it only depends on information in the current node and the node's children (of which there are none). Once $f(x)$ is determined, the changes to any other f values propagate up the tree. Thus phase 1 requires $O(\lg n)$ time. For phase 2, structural changes only occur via rotations. Since only two nodes are affected by a rotation, the total time for updating f is $O(1)$ during the rotation. Since there are at most two nodes per rotation and at most $O(\lg n)$ rotations, total time to update f in phase 2 is $O(\lg n)$.

Like insertion, deletion occurs in two phases. In phase 1, a structural change occurs if the deleted node is replaced by its successor or whenever either the deleted node or the successor node is spliced out. Such changes propagate up the tree, causing $O(\lg n)$ updates to f. The repair in phase 2 is $O(1)$, and there are $O(\lg n)$ repairs, so the total time to update f in phase 2 is $O(\lg n)$.

Since all phases for insertion and deletion require $O(\lg n)$ updates to f, the asymptotic running time for insertion and deletion in the augmented red-black tree is still $O(\lg n)$. QED