2.1 Linear Algebra

Problem 1: $x \in \mathbb{R}^M, y \in \mathbb{R}^N \quad Z \in \mathbb{R}^{P\times Q} \qquad f: \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}^{P\times Q} \to \mathbb{R}$

$$f(x,y,Z) = \underbrace{x^T \cdot Ay}_{①} + \underbrace{Bx}_{②} - \underbrace{y^T CZD}_{③} - \underbrace{y^T E^T y}_{④} + F$$

①: $x^T \cdot Ay$

$\hookrightarrow x^T \in \mathbb{R}^{1\times M}$

$\hookrightarrow y \in \mathbb{R}^N$

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ M \end{pmatrix} \in \mathbb{R}^{M\times 1}$$

$\hookrightarrow \mathbb{R}^{1\times M} \cdot \mathbb{R}^{M\times N} \cdot \mathbb{R}^{N\times 1}$ is a valid expression

$\Rightarrow A \in \mathbb{R}^{M\times N}$

$\Rightarrow ① \in \mathbb{R}^{1\times 1} \Leftrightarrow ① \in \mathbb{R}$

$$\underset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{pmatrix}} = \underset{1\times 2}{\begin{pmatrix} 14 & 32 \end{pmatrix}}$$

②: $B \cdot x$

$\hookrightarrow \mathbb{R}^{1\times M} \cdot \mathbb{R}^M$

$\Rightarrow B \in \mathbb{R}^{1\times M}$

$\Rightarrow ② \in \mathbb{R}^{1\times 1} \Leftrightarrow ② \in \mathbb{R}$

③: $y^T \cdot CZ \cdot D$

$\hookrightarrow \mathbb{R}^{1\times N} \cdot \mathbb{R}^{N\times P} \cdot \mathbb{R}^{P\times Q} \cdot \mathbb{R}^{Q\times 1}$

$\Rightarrow C \in \mathbb{R}^{N\times P}, D \in \mathbb{R}^{Q\times 1} \Leftrightarrow D \in \mathbb{R}^Q$

$\Rightarrow ③ \in \mathbb{R}$

④: $y^T \cdot E^T \cdot y$

$\hookrightarrow \mathbb{R}^{1\times N} \cdot \mathbb{R}^{N\times N} \cdot \mathbb{R}^{N\times 1}$

$\hookrightarrow E^T \in \mathbb{R}^{N\times N} \to E \in \mathbb{R}^{N\times N}$

⑤: F

$\hookrightarrow F \in \mathbb{R}$

$\Rightarrow A \in \mathbb{R}^{M\times N}, B \in \mathbb{R}^{1\times M}, C \in \mathbb{R}^{N\times P}, D \in \mathbb{R}^Q, E \in \mathbb{R}^{N\times N}, F \in \mathbb{R}$

Problem 2: $x \in \mathbb{R}^N, M \in \mathbb{R}^{N \times N}$ $f(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j M_{ij}$

example: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $f(x) = \begin{matrix} 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 7 + \\ 2 \cdot 1 \cdot 2 + 2 \cdot 2 \cdot 5 + 2 \cdot 3 \cdot 8 + \\ 3 \cdot 1 \cdot 3 + 3 \cdot 2 \cdot 6 + 3 \cdot 3 \cdot 9 \end{matrix}$

$$
\begin{aligned}
f(x) = {} & x_1 \cdot x_1 \cdot M_{11} + x_1 \cdot x_2 \cdot M_{12} + \dots + x_1 \cdot x_N \cdot M_{1N} + \\
& x_2 \cdot x_1 \cdot M_{21} + x_2 \cdot x_2 \cdot M_{22} + \dots + x_2 \cdot x_N \cdot M_{2N} + \\
& \dots + \\
& x_N \cdot x_1 \cdot M_{N1} + x_N \cdot x_2 \cdot M_{N2} + \dots + x_N \cdot x_N \cdot M_{NN}
\end{aligned}
$$

Problem 3: $A \in \mathbb{R}^{M \times N}$ $x \in \mathbb{R}^N$ $b \in \mathbb{R}^M$

$Ax = b$

a) 1. x and b need the same dimension

2. All rows of A have to be independent.

Problem 4: $A, B \in \mathbb{R}^{N \times N}$ $AB = BA = I$

$A \cdot A^{-1} = I = AB \Leftrightarrow B = A^{-1}$

$B \cdot B^{-1} = I = BA \Leftrightarrow A = B^{-1}$

$\Rightarrow$ The eigenvalues of A are the reciprocals of the eigenvalues of B.

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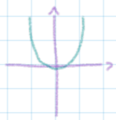
Problem 5: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}ax^2 + bx + c$

$$\min_{x \in \mathbb{R}} f(x)$$

a) $f'(x) = ax + b$, $f''(x) = a$

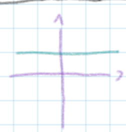
$f'(x) = 0 \Rightarrow ax + b = 0 \Leftrightarrow x = -\frac{b}{a}$

i. one solution: $a > 0$, $b \neq 0$ and $c \in \mathbb{R}$ any value.



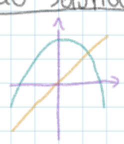
a has to be bigger than 0, otherwise with $a = 0$ or $a < 0$ the resulting extreme point isn't a minima. Also, b has to be not 0 in order to obtain a minima. With this parameters we would obtain a parabola which is opened upwards.

ii. ∞ solutions: $a = 0$, $b = 0$, $c \in \mathbb{R}$ any value



In this case, we need the same restriction for b as before. Additionally, a has to be equal to 0. In this way we achieve a constant straight line resulting in an infinite amount of possible solutions.

iii. no solution: 1) $a = 0$, $b \neq 0$ and $c \in \mathbb{R}$ can take any value.



With $a = 0$, $b \neq 0$ we obtain a straight line which has no minima.

2) $a < 0$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$ can take any value.

With $a < 0$, $b \neq 0$ we obtain a parabola which is opened downwards (no minima).

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Problem 7:

$$p(a|b,c) = p(a|c) \Rightarrow p(a|b) = p(a)$$

	a	$\bar{a}$	
b	0,05	0,05	0,1
c	0,1	0,1	0,2
$\bar{b}$	0,05	0,05	0,1
c	0,05	0,05	0,1
b	0,05	0,05	0,1
$\bar{c}$	0,05	0,05	0,1
$\bar{b}$	0,15	0,05	0,2
$\bar{c}$	0,05	0,05	0,1
	0,55	0,45	1

$$p(a \cap b \cap c) = 0{,}15$$

$$p(b \cap c) = 0{,}3$$

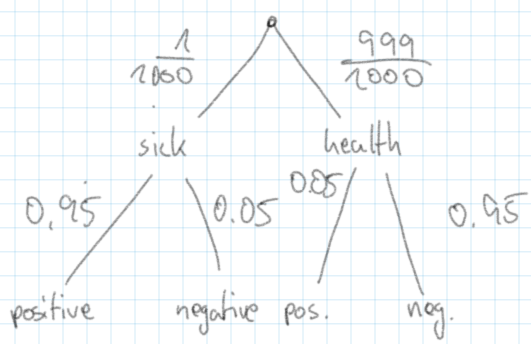
$$p(a|b,c) = \frac{0{,}15}{0{,}3} = 0{,}5$$

$$p(a|c) = \frac{p(a \cap c)}{p(c)} = \frac{0{,}15}{0{,}3} = 0{,}5$$

$$p(a|b) = \frac{p(a \cap b)}{p(b)} = \frac{0{,}1}{0{,}2} = 0{,}5$$

$$p(a) = 0{,}55$$

Problem 10:



$$p(\text{sick} \mid \text{positive}) = \frac{p(\text{sick} \cap \text{positive})}{p(\text{positive})} = \frac{0{,}00095}{0{,}0509} \approx 0{,}0187$$

Problem 11:

$$\mathbb{E}_{x \sim N(\mu,\sigma)}[f(x)] = \int p(x) \cdot f(x)\, dx$$

$$= \int \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \cdot f(x)\, dx$$