A4)

$$f'(x_1, x_2) = \left(\frac{1}{x_2}, -\frac{x_1}{x_2^2}\right)$$

$$f'(1, 10^{-4}) = (10^4, -10^8)$$

$$K_{a,\infty} \approx \|(10^4, -10^8)\|_\infty = 10^4 + 10^8 \approx 10^8$$

$$K_{r,\infty} \approx \frac{1 + 10^{-4}}{\frac{1}{10^{-4}}} \cdot 10^8 = 10^4 + 1 \approx 10^4$$

$$K_{a,2} = \|f'(1, 10^{-4})\|_2 = \sqrt{\rho(A^T A)}$$

$$A = (10^4, -10^8)$$

$$\Rightarrow A^T = \begin{pmatrix} 10^4 \\ -10^8 \end{pmatrix} \Rightarrow A^T A = 10^8 \begin{pmatrix} 1 & -10^{-4} \\ -10^{-4} & 10^8 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 10^{-4} \\ 10^{-4} & 10^8 - \lambda \end{pmatrix} \overset{!}{=} 0$$

$$\Rightarrow ((1-\lambda) \cdot (10^8 - \lambda)) - 10^8 \overset{!}{=} 0$$

$$\Rightarrow \lambda(-1 - 10^8 + \lambda) \overset{!}{=} 0$$

$$\lambda = 0 \quad \wedge \quad \lambda = 1 + 10^8$$

$$\Rightarrow \quad K_{a,2} = \sqrt{10^8(1+10^8)} \approx 10^8$$

$$\Rightarrow \quad K_{r,2} = \frac{\|x\|_2}{|f(x)|} \cdot 10^8$$

$$= \frac{\sqrt{1+10^{-8}}}{10^4} \cdot 10^8$$

$$= 10^4 \cdot \sqrt{1+10^{-8}}$$

$$\approx 10^4$$

A5)

$$AB = \begin{pmatrix} 1 & 0 & \cdots & \beta \\ -\beta & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & -\beta & 1 & 0 \\ 0 & \cdots & 0 -\beta & 1 \end{pmatrix} \cdot -\frac{1}{\beta^n} \begin{pmatrix} 0 & \beta^{n-1} & \cdots & \beta \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ -\beta^{n-1} & \cdots & \cdots & \beta^0 \end{pmatrix}$$

$$= -\frac{1}{\beta^n} \begin{pmatrix} \beta^{n-1} \cdot -\beta & \cdots & \beta^2 - \beta^2 \beta - \beta^0 \cdot \beta \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta^{n-1} \cdot -\beta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} = E$$

$$\Rightarrow B = A^{-1}$$

$$K_{A,\infty} = \|A^{-1}\|_\infty \cdot \|A\|_\infty$$

$$\|A^{-1}\|_\infty = \frac{1}{\beta^n} \cdot \sum_{i=0}^{n-1} \beta^i$$

geom. Reihe

$$= \frac{1}{\beta^n} \cdot \frac{\beta^n - 1}{\beta - 1}$$

$$= \frac{1 - \frac{1}{\beta^n}}{\beta - 1}$$

$$\|A\|_\infty = 1 + \beta \quad \text{für } \beta > 0$$

$$\Rightarrow \|A\|_\infty = \frac{1 - \frac{1}{\beta^n}}{\beta - 1} \cdot (1 + \beta)$$

$$= \left(1 - \frac{1}{\beta^n}\right) \cdot \left(\frac{\beta + 1}{\beta - 1}\right)$$

für $\beta > 0$

$$\leq \frac{\beta + 1}{\beta - 1}$$