Institute of
Pervasive Computing

Trees (Height Balanced)

Algorithms and Data Structures 2, 340300
Lecture – 2022W
Univ.-Prof. Dr. Alois Ferscha, teaching@pervasive.jku.at

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

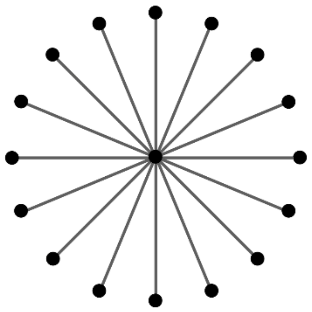
Definition & Terminology

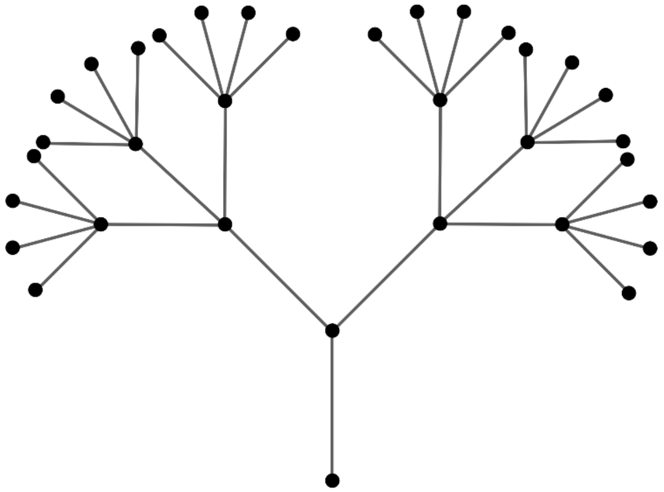
A **tree** is

- an acyclic, simple, coherent graph
- i.e. does not contain loops and cycles:
  between each pair of nodes there is at maximum one edge

Generalization of lists:

- Element (node) may have **multiple successors**.
- Exactly 1 node has no predecessor: **root**
- Nodes without successors: **leaves**

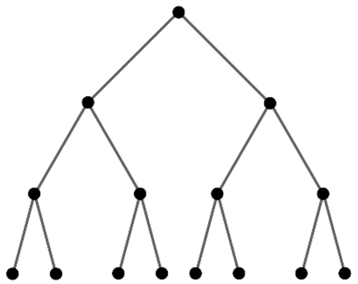




Frequently used data structure: decision trees, syntax trees, derivation trees, search trees, ...

Frequently used representation form: quantity, bracket, recursive indentation, graph, ...

Here: Use of trees to **store keys** and realization of dictionary operations (**search**, **insert**, **remove**) in e.g. binary trees



Definition & Terminology

Tree *B* is **ordered**, if successors of each node are ordered (1., 2., 3. etc.; left, right).

In an ordered tree, the **subtrees B_i** of each node form an **ordered set** (e.g.: arithmetic expression)

Order of B: maximum number of successors of a node

Path of length *k*: Follow $p_0, ..., p_k$ of nodes, such that p is successor of p_{i-1}

- **Height** of a **tree**:
  maximum distance of a leaf to the root.
- **Depth** of a **node**:
  distance to the root (the **number of edges** on a path from this node to the root)
  The nodes at level *i* are all nodes with depth *i*.

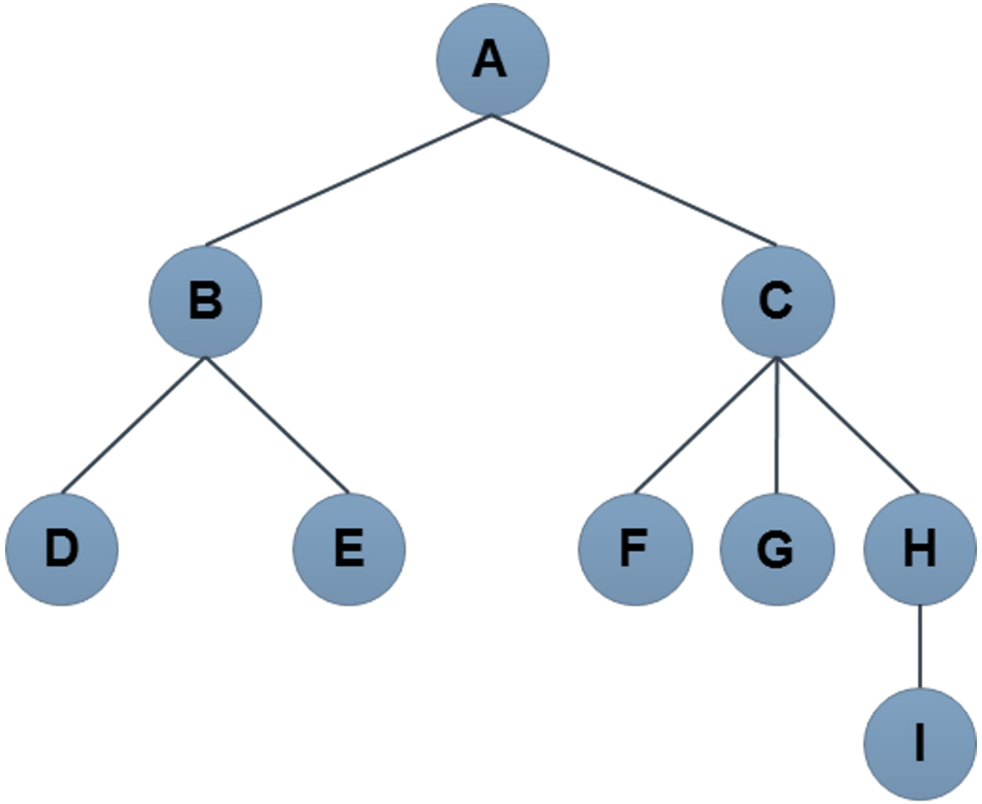
A tree of order *n* is called **complete** if all leaves have the **same depth** and the maximum number of nodes is present at each level.

Definition & Terminology

A	**root**
B	is **parent** of D and E
C	is **sibling** of B
D and E	are **children** of B
D, E, F, G, I	are **external nodes** or **leaves**
A, B, C, H	are **internal nodes**

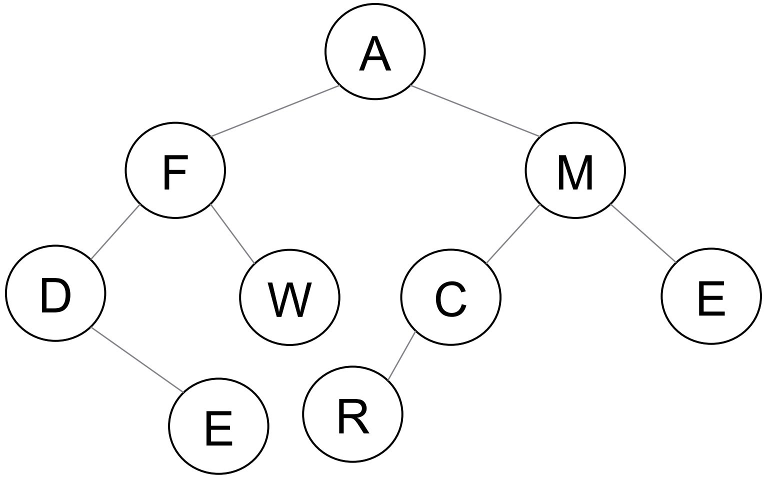
The **depth** of E is 2.
The **height** of the tree is 3.
The **order** of B is 2, the order of C is 3.

We always have: number of edges = number of nodes - 1



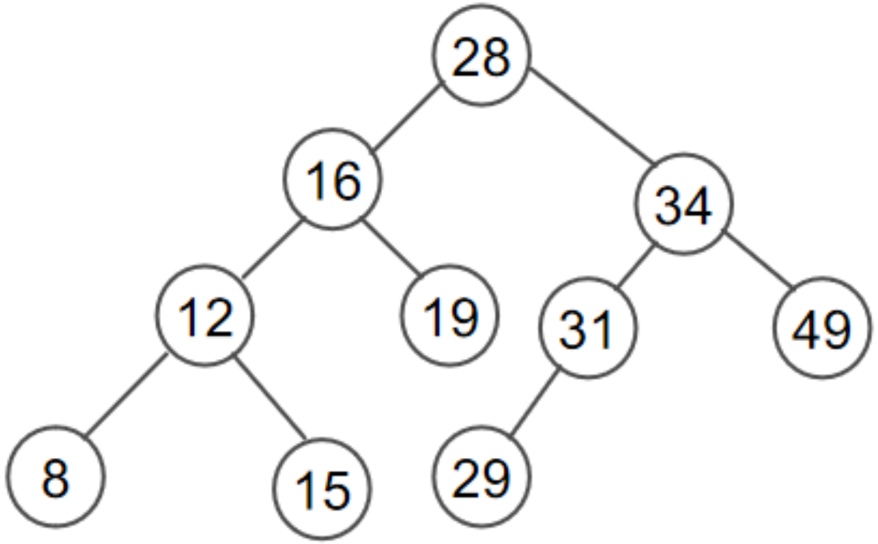
Binary Trees

If *T* is a binary tree **(order of all nodes <= 2)** with **n nodes** and a **height *h***, then *T* has the following properties



- Number of **external nodes** is at least ***h+1*** and at maximum ***2^h***
- Number of **internal nodes** is at least ***h*** and at maximum ***2^h –1***
- Number of **nodes** is at least ***2h+1*** and at maximum ***2^{h+1} –1***
- The **height** of *T* is at least ***log(n+1) –1*** and at maximum ***(n-1)/2***

Traversing Binary Trees



- **Preorder**:
  28, 16, 12, 8, 15, 19, 34, 31, 29, 49
- **Postorder**:
  8, 15, 12, 19, 16, 29, 31, 49, 34, **28**
- **Inorder**:
  8, 12, 15, 16, 19, **28**, 29, 31, 34, 49

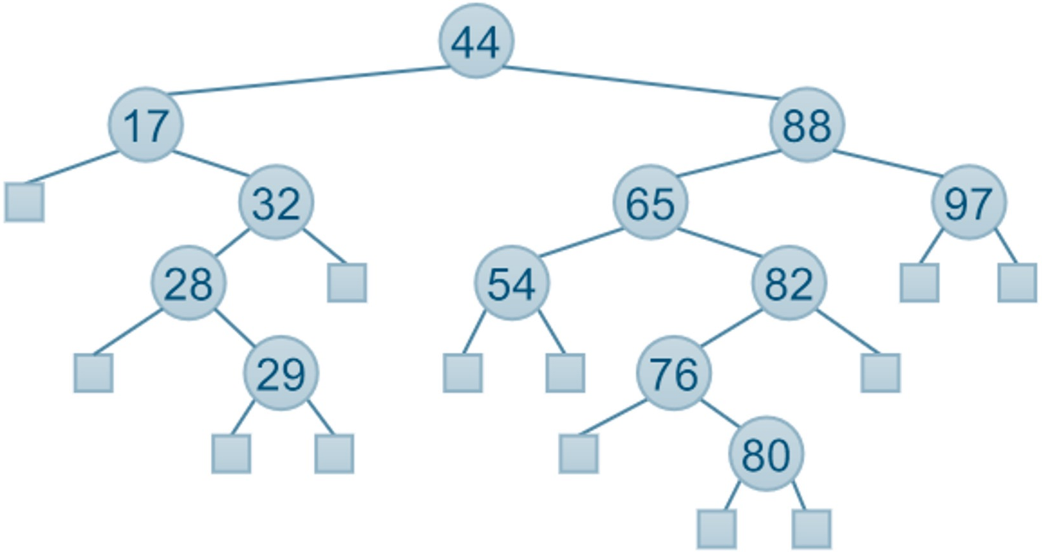
Hints: If you traverse the tree starting from the root, you have

- **Preorder** by visiting all nodes on the left side of which you pass by.
- **Postorder** by visiting all nodes on the right side of which you pass by.
- **Inorder** by visiting all nodes on the bottom side of which you pass by.

Binary Search Trees

A **binary search tree** is a **binary tree** ***T*** where:

- Each internal node stores a **key-value pair** of a dictionary
- Keys which are stored in nodes of the **left** subtree of a node *v*, are **less than** or **equal** to the key stored in *v*
- Keys which are stored in nodes of the **right** subtree of a node *v*, are **greater** than the key stored in *v*
- **External nodes** serve only as **placeholders** and do not store elements



Searching in Binary Search Trees

Assumption: For each node **n** with children $\mathbf{c_l}$, $\mathbf{c_r}$ and key **k** we have (search tree condition):

- All keys stored in subtree of $\mathbf{c_l}$ are **smaller** than **k**
- All keys stored in subtree of $\mathbf{c_r}$ are **larger** than **k**
- All keys stored in subtree of $\mathbf{c_l}$ are **smaller than all** keys stored in subtree of $\mathbf{c_r}$

Search for key x:

1. Compare key **k** of inspected node **n** with **x**.
2. If **n == null**, **x** is **not in** the binary tree.
3. If **k < x**, set **n =** $\mathbf{c_r}$**(n)** and jump to 1.
4. If **k > x**, set **n =** $\mathbf{c_l}$**(n)** and jump to 1.
5. If **k == x**, found.

Maximum number of inspected nodes: depth of tree

Search within the nodes, e.g. by linear or binary search. As l ≤ m, the complexity is constant.

Searching in Binary Search Trees

Binary search tree is a **decision tree**:
in each **internal node v** the key to be searched for is compared with the **key stored in v**

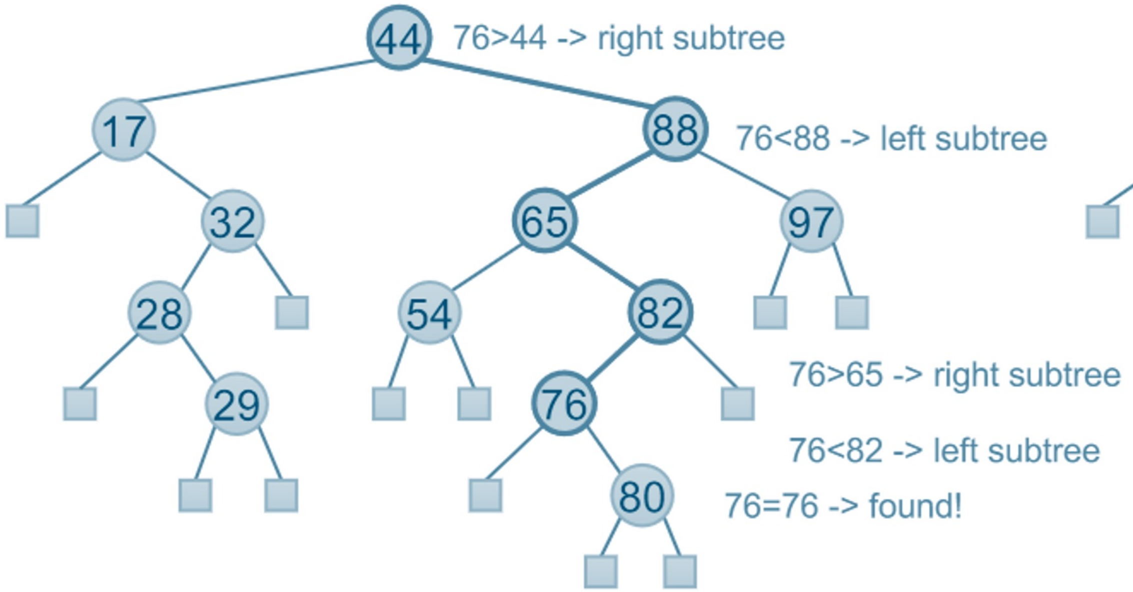
Pseudocode

```
Algorithm TreeSearch(k, v)
  Input: key k to be searched, node v of a binary search tree
  Output: node w in subtree of v, for successful search internal node with key k, for unsuccessful
search
    external node after all internal nodes smaller than k and before all internal nodes greater than k
  if v is an external node then
    return v
  if k = key(v) then
    return v
  else if k < key(v) then
    return TreeSearch(k, T.leftChild(v))
  else
    return Treesearch(k, T.rightChild(v))
```

Examples for Search in Binary Search Trees

TreeSearch(76, root))

(**sucessful** search)



76>44 -> right subtree

76<88 -> left subtree

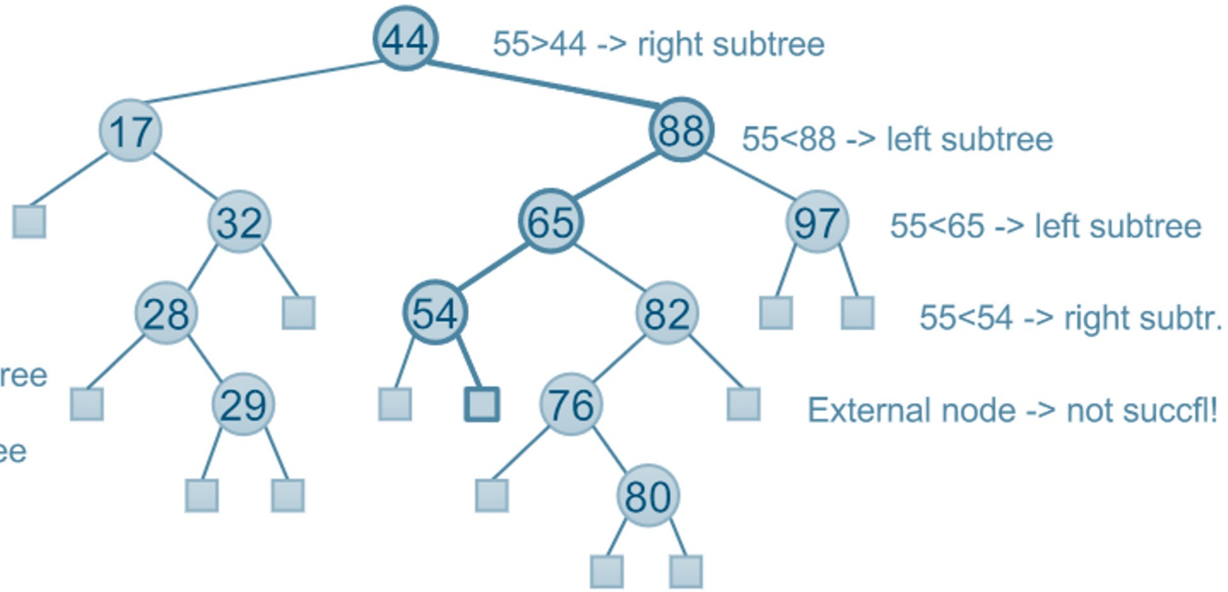
76>65 -> right subtree

76<82 -> left subtree

76=76 -> found!

TreeSearch(55, root)

(**unsucessful** search)



55>44 -> right subtree

55<88 -> left subtree

55<65 -> left subtree

55<54 -> right subtr.

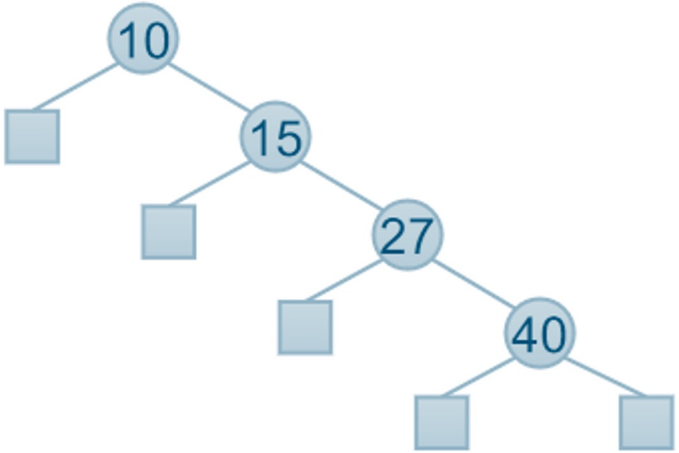
External node -> not succfl!

Complexity of Binary Search Trees

For **searching**, **inserting**, **removal**, nodes along a **root-leaf-path** (plus possibly siblings of such nodes) needs to be traversed.

- Complexity per node: O(1)
- Therefore the runtime is O(h), where h is the **height of the tree**

In worst case the height of the binary search tree is h = N
(tree „degenerates“ to sorted sequence)



- Worst case complexity O(N)
- Therefore, the tree must be kept **balanced** to allow operations in O(log N)

Trees

Problem: trees can degenerate (e.g. become lists)

Removal of a node

- **Very easy:** Removal of a leaf
- **Easy:** Removal of a node **p** with 1 successor:
  replace node with successor
- **More difficult:** Removal of a node **p** with 2 successors:
  Search for the **leftmost node q** in **right subtree** (symmetric successor).

  Replace **p** with **q**, remove **q** from its original position.

Binary Search Tree :: Access Cost

Cost measure: Number of nodes visited or number of search steps or key comparisons required.

Average access cost z of a tree **B** ...

- is obtained by calculating its **total path length** **PL** as the **sum of the lengths of all paths** from the root to each node $\mathbf{K_i}$.

$$PL(B) = \sum_{i=1}^{N} Level\ (K_i)$$

- The mean path length is calculated by $l = \frac{PL}{N}$

Maximum access cost We have the longest search path and thus the maximum access cost, when the binary search tree degenerates into a linear list.

$$height\ h = l_{max} + 1 = N$$

Binary Search Tree :: Access Cost

Maximum average access cost:

$$z_{max} = \frac{1}{N} * \sum_{i=0}^{N-1} (i+1) * 1 = N * \frac{(N+1)}{2N} = \frac{(N+1)}{2} = O(N)$$

Minimum average access cost:

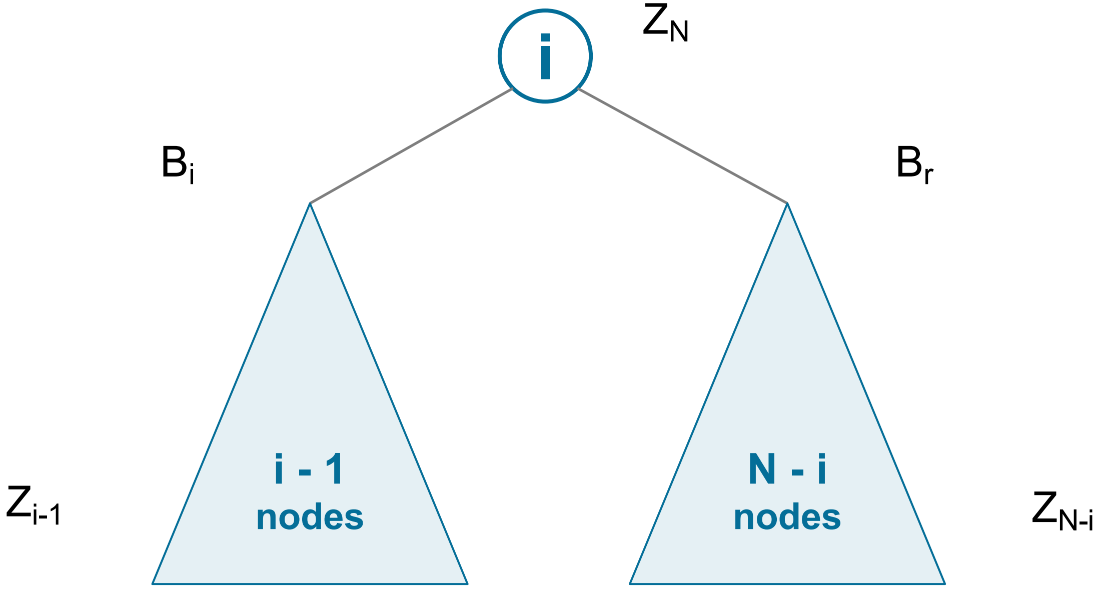
- They can be expected in an **almost complete** or **balanced** tree structure.
- Total number of nodes: $2^{h-1}-1 < N \leq 2^h-1$
- height $h = [\log_2 N] + 1$
- **Minimum average access cost**: **$z_{min} \approx \log_2 N - 1$**

Binary Search Tree :: Access Cost

Average access cost

- Extreme cases of average access costs have only little significance.
- The **difference** between the **average** and the **minimum** access cost is a measure of the **urgency of balancing**.

Determination of the average access cost:



Binary Search Tree :: Access Cost

N different keys with values 1, 2, ..., N are given in random order.

The probability that the first key has the value **i** is $\frac{1}{N}$
(Assumption: same access probability to all nodes)

For the tree with **i** as root we get:

$$Z_N(i) = \frac{1}{N} * \left((Z_{i-1} + 1) * (i - 1) + 1 + (Z_{N-i} + 1) * (N - i)\right)$$

The recursive equation can be represented in non-recursive, closed form by means of the harmonic function:

$$H_N = \sum_{i=1}^{N} \frac{1}{i}$$

Binary Search Tree :: Access Cost

We get:

$$z_N = 2 * \frac{(N+1)}{N} * H_N - 3 = 2ln(N) - c$$

Relative additional cost:

$$\frac{z_N}{z_{min}} = \frac{2ln(N)-c}{\log_2(N)-1} \sim \frac{2\ln(N)-c}{\log_2(N)} = 2\ln(2) = 1{,}386\ldots$$

The **balanced binary search tree** causes the **least cost** for all basic operations.

However, **perfect balancing** at any time is very **expensive**.

Balanced Trees

Efficiency of dictionary operations (insert, search, remove) on trees depends directly on the tree height.

Tree with N nodes:

- Minimum height: $\lfloor \mathbf{\log_2 N} \rfloor$, maximum height: ***N-1***
- Access on average ***O(log₂ N),*** but in worst case linear.

Aim:

- Fast access with $z_{max} \sim O(\log_2 N)$
- Insert and remove operations in logarithmic complexity.

Heuristics:

For each node in the tree, the number of nodes in each of its two subtrees should be kept as constant as possible.

Balanced Trees

Two different approaches

- **Height-balanced trees**:

  The maximum allowed **height difference** of the two subtrees is limited

- **Weight-balanced trees**:

  The **ratio** of the **node weights (number of nodes)** of both **subtrees** meets certain conditions.

Height-Balanced Search Trees

- **B-Trees**
  always **balanced** due to **balance sustaining search**, **insert**, **remove**
  number of children per inner node between **t** (=m/2) and **2t** (=m)

  access in **O(log(n))**

- **AVL Trees**
  The heights of **each internal node's children** only **differ** by a **maximum of 1**

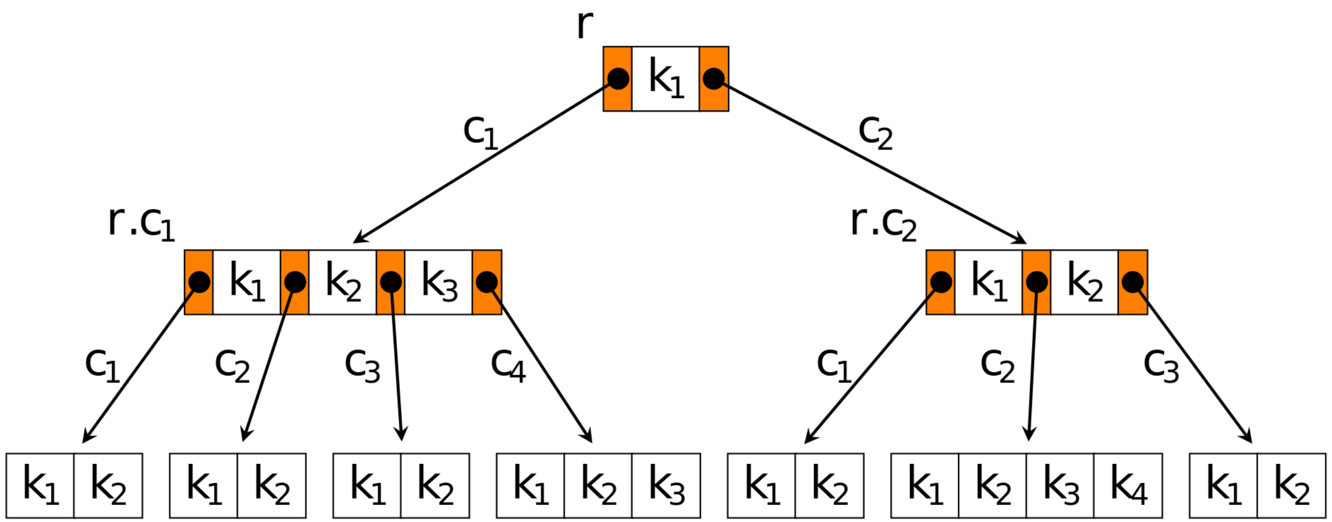
  access in **O(log(n))**

Example from https://de.wikipedia.org/wiki/B-Baum

B-Trees

Tree structure (m-Tree) that is **always balanced**

- **self-balancing** mechanism integrated in access operations (**search**, **insert**, **remove**)
- maintains **sorted** data (**keys**)
- **search**(), **insert**(), **remove**() in **logarithmic time**
- supports "**locality**"-**principle** in memory organization



Graphics: Flying sheep, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=19226668

1. Every node has at most ***m*** children (***m*** = 2t).
2. Every non-leaf node (except root) has at least ⌈t = ***m***/2⌉ child nodes.
3. The root has at least 2 children if it is not a leaf node.
4. A non-leaf node with ***k*** children contains ***k***−1 keys.
5. All leaves appear in the same level (height h)

B-Trees

m = 2t ... maximum number of children per node
n ... number of keys stored,

then the **height h** is:

$$h \leq \log_t \left(\frac{n+1}{2}\right) + 1$$ (therefore access in **O(log(n))**)

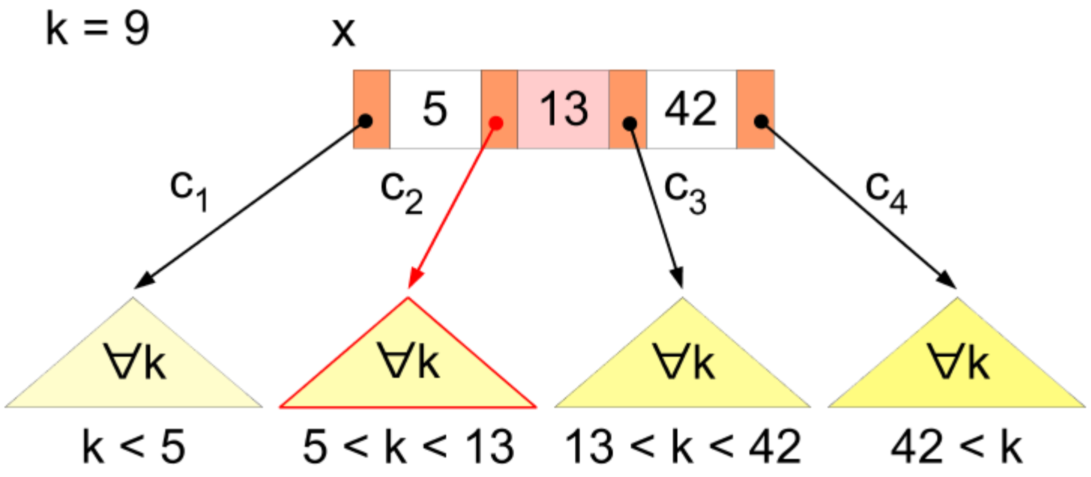
t ... minimum number of children per node
n ... **t^h-1**

let **t** = 1024, **h** = 4 then

$1024^4 – 1 = (2^{10})^4 - 1 = 2^{40} - 1$ = 1.099.511.627.775 keys (i.e. access time 10 times lower)

B-Trees

Search (key = 9)

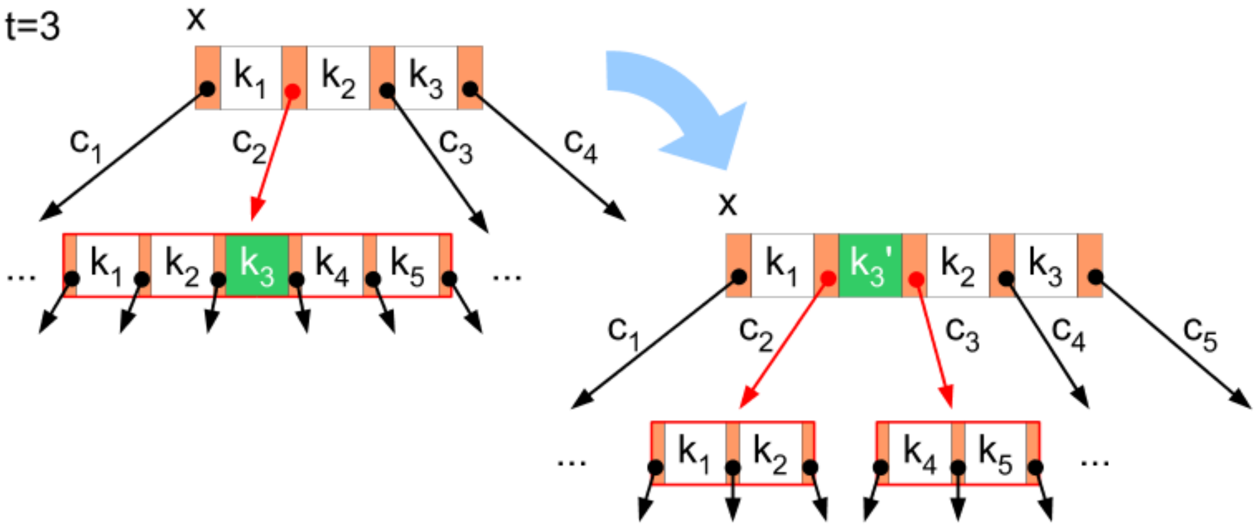


Insert

Search for insert position **(inner node)**

If inner node **full**

(i.e more than **2t-1** keys) then preventively **split** **before stepping down**



Graphics: Haui, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=245051, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=245228

B-Trees

Remove

Removing (**inner node**) could destroy balance

before stepping down

check whether there are **enough** keys in the **subtree**

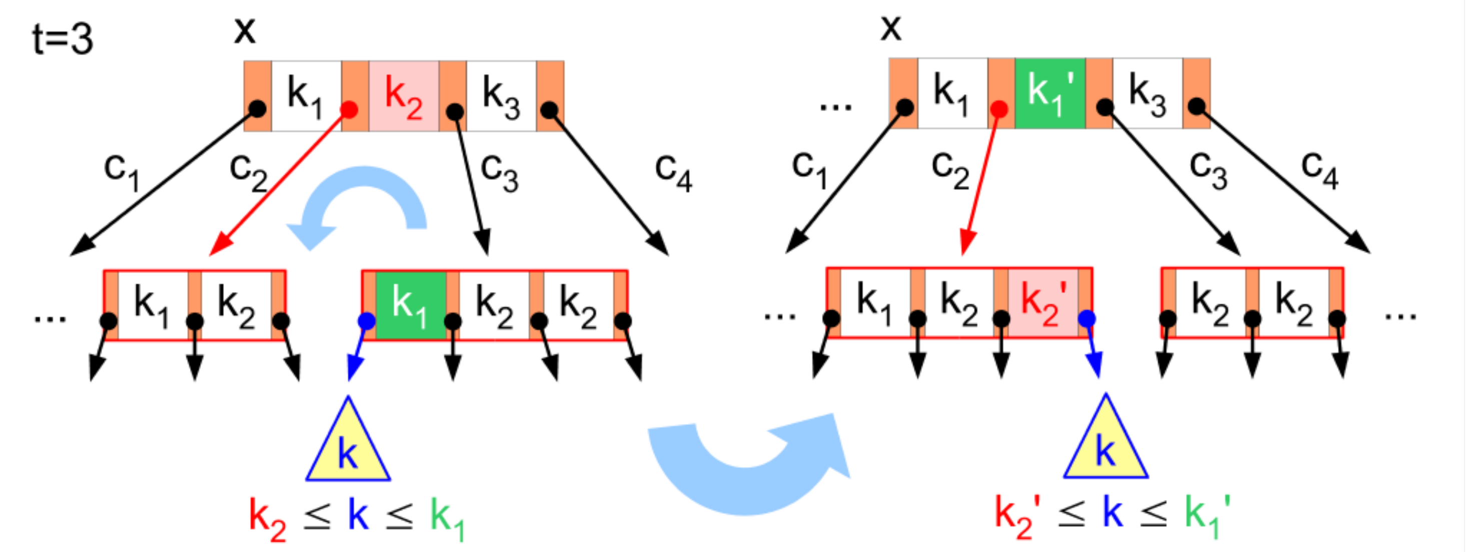
If considered inner node in subtree has just minimum number of keys (**t-1** keys), then

Move / **Merge**

to prevent from imbalance after **Remove**

Move

transfer a sibling to **expand (minimum) inner node**



207

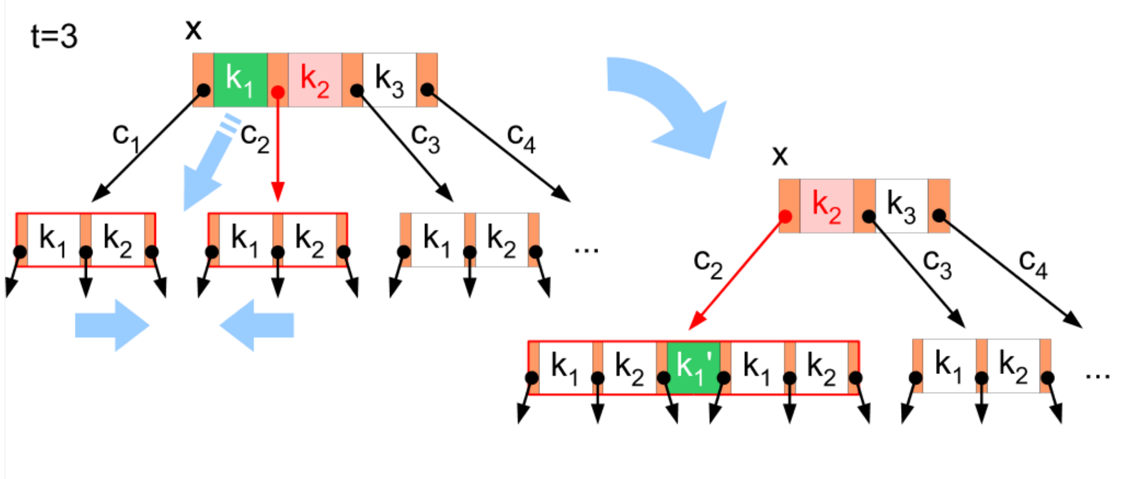
B-Trees

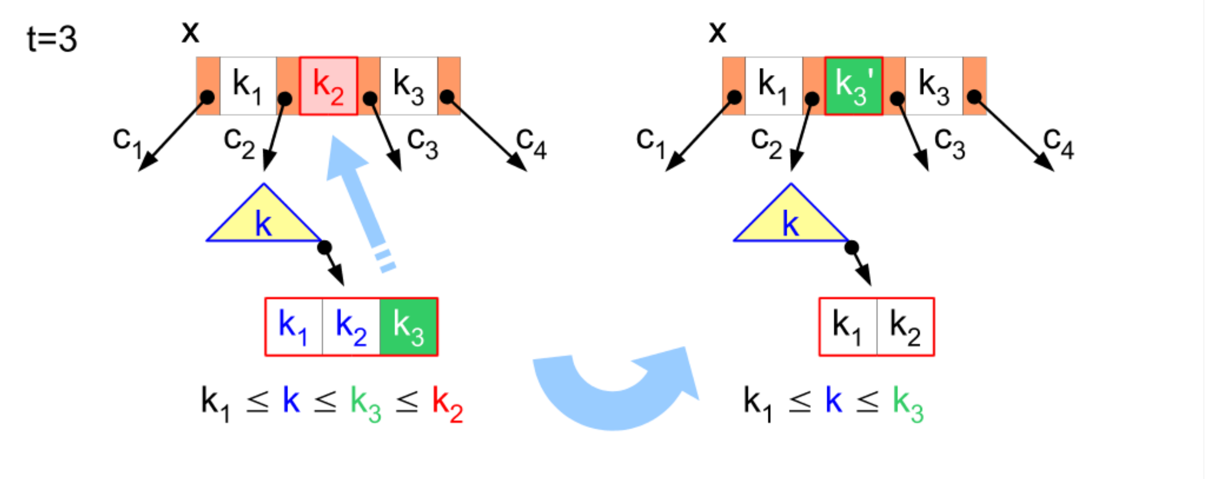
Merge

before stepping down
merge two a siblings
if both (left and right child)
are at **minimum number of inner nodes**

Remove

before deleting key from inner node
separate/adjust the **key ranges**
of both left and right child
(here: inorder transfer,
i.e. rightmost of left child)





Graphics: Haui, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=248208, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=248209

B-Trees

Example (**t=2**) i.e. min 1, max 3 keys per inner node

a–c) **insert** 5, 13, 27

d–e) **insert** 9 – induces **splitting** of root

f) **insert** 7 leaf node

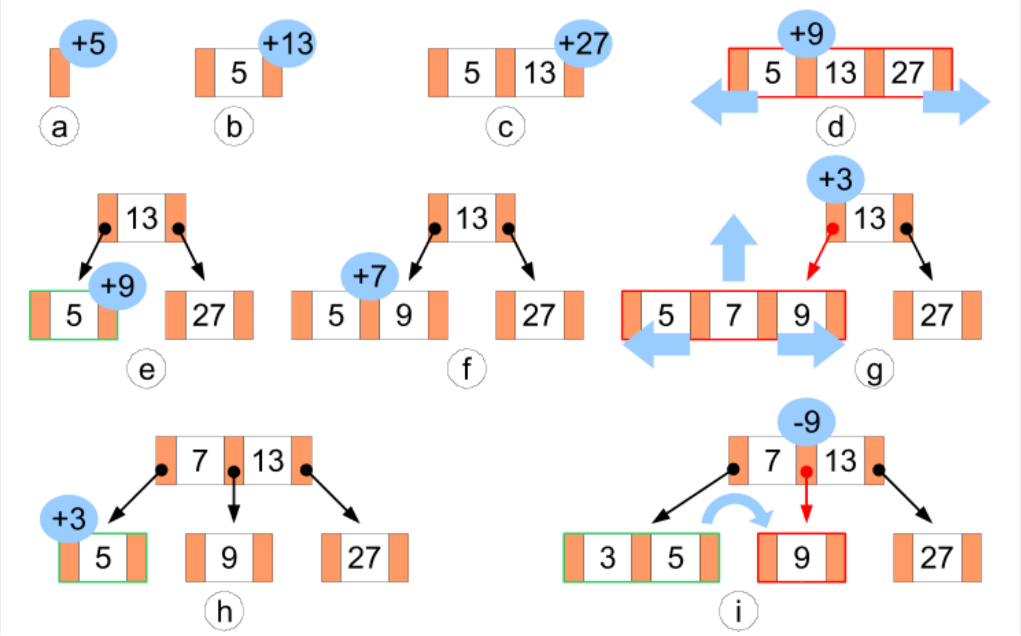
g–h) **insert** 3 needs **splitting**

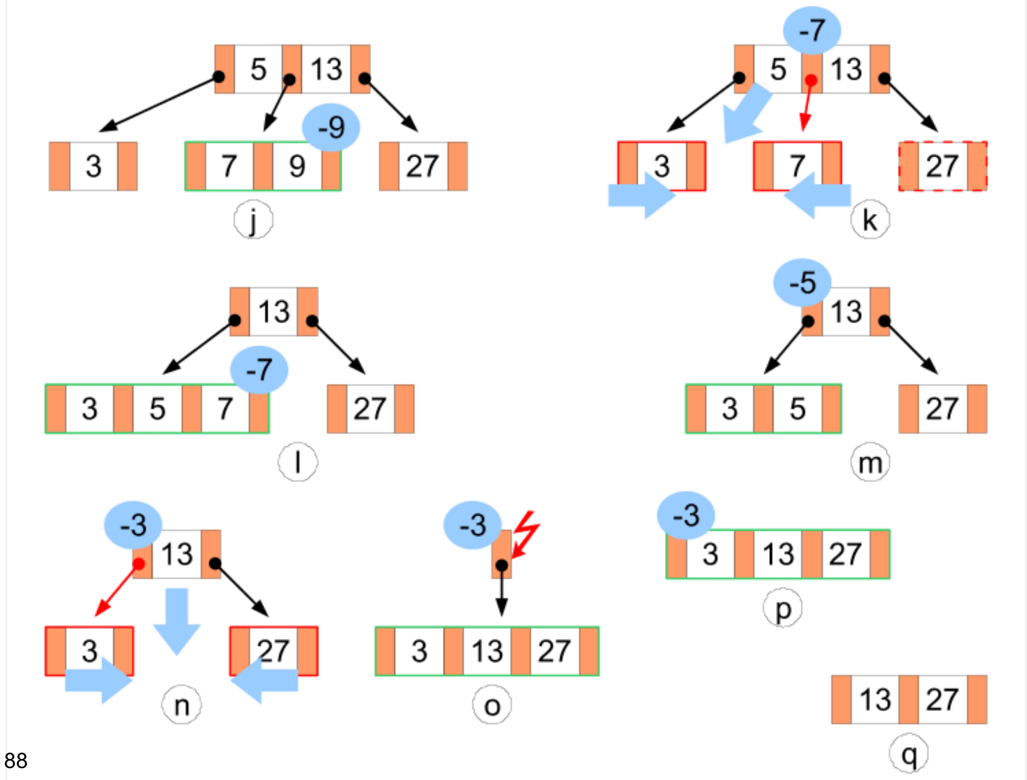
i–j) **remove** 9, needs induces **move** of sibling

k–l) **remove** 7 induces **merge**

m) **remove** 5 (**leaf**)

n–q) **remove** 3 induces **merge** of two children of root
empty root replaced by only child





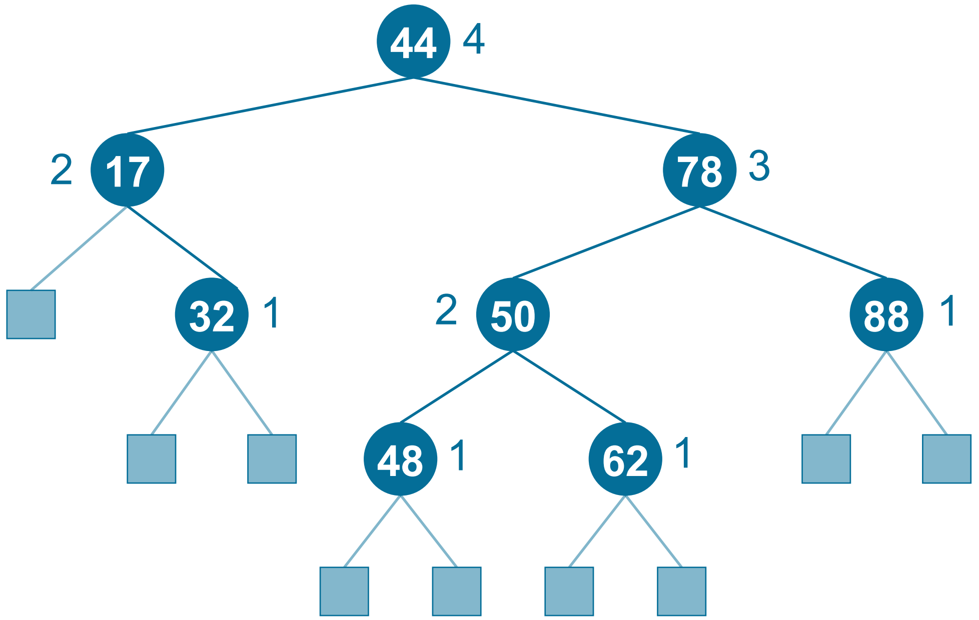
Example from https://de.wikipedia.org/wiki/B-Baum; Graphics: Haui, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=250188

AVL Trees (Adelson-Velskii, Landis 1962)

AVL trees are balanced binary search trees

An AVL tree is a binary search tree, in which the heights of **each internal node's children** only **differ** by a **maximum of 1**.

Example (numbers next to the nodes indicate their height):



Height of an AVL tree

Assertion: The height of an AVL tree T that stores n keys is O(log n)

Sketch of a proof:
Find n(h), the smallest possible number of internal nodes in an AVL tree of height h.

- Trivial: n(1) = 1, n(2) = 2

Height of an AVL tree

Assertion: The height of an AVL tree T that stores n keys is O(log n)

Sketch of a proof:

Find n(h), the smallest possible number of internal nodes in an AVL tree of height h.

- Trivial: $n(1) = 1$, $n(2) = 2$
- $n \geq 3$: The AVL tree of height h with minimum n(h) consists of a root node, an AVL subtree of height h-1 and an AVL subtree of height h-2.

i.e. $n(h) = 1 + n(h-1) + n(h-2)$

Fibonacci progression (exponential!)

as $n(h-1) > n(h-2)$ it follows:

$n(h) > 2\, n(h-2)$

$n(h) > 4\, n(h-4)$

$n(h) > 8\, n(h-6)$

...

$n(h) > 2^i\, n(h-2i)$ then $n(h) \geq 2^{h/2-1}$

choose $i = \lceil h/2 \rceil - 1$

- findElement: O(logn)
- findAllElements O(log n+s) (s=number of findings)

Using the logarithm we get: $h < 2 \log n(h) + 2$

$\log n(h) > h/2 - 1$

Therefore we have: The height of an AVL tree is O(log n)

Insert in AVL trees

By **inserting** a node into an AVL tree, the **height** of some nodes in this tree changes

Insert operation may cause the AVL tree to be **unbalanced**

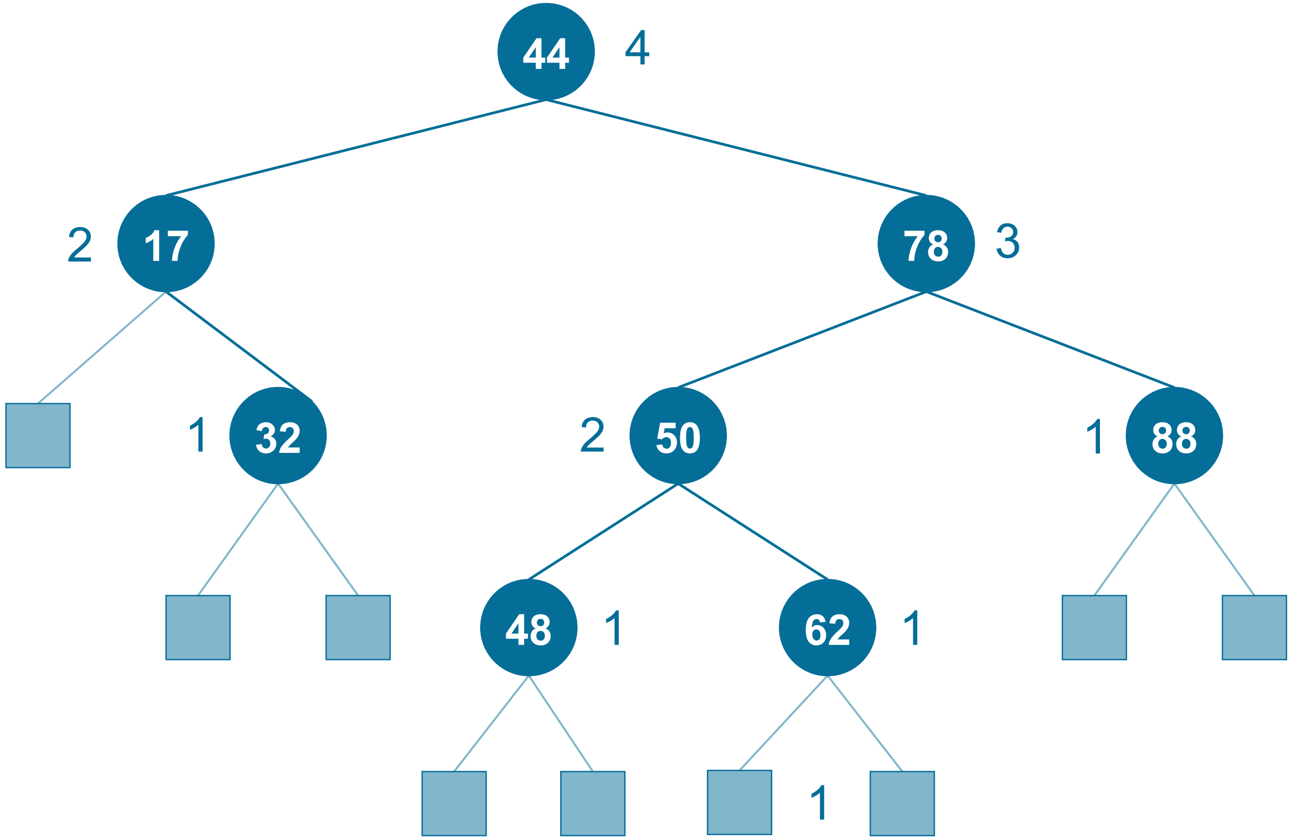
- Go up **from the new node** in the tree until the first node **x** is found, whose "grandfather" **z** is an unbalanced node.
- Since **z** was unbalanced by inserting a node that lies in a sub-tree with root y (where y is a child of **z**), we have height(y) = height(sibling(y)) + 2

Re-balancing of the sub-tree with root z requires **restructuring**:

- **x**, **y** and **z** are renamed to a, b, c (according to **in-order traversing**)
- **z** is replaced by **b**, whose children are now **a** and **c**, whose children are again **four subtrees**, which were formerly the children of x, y and z

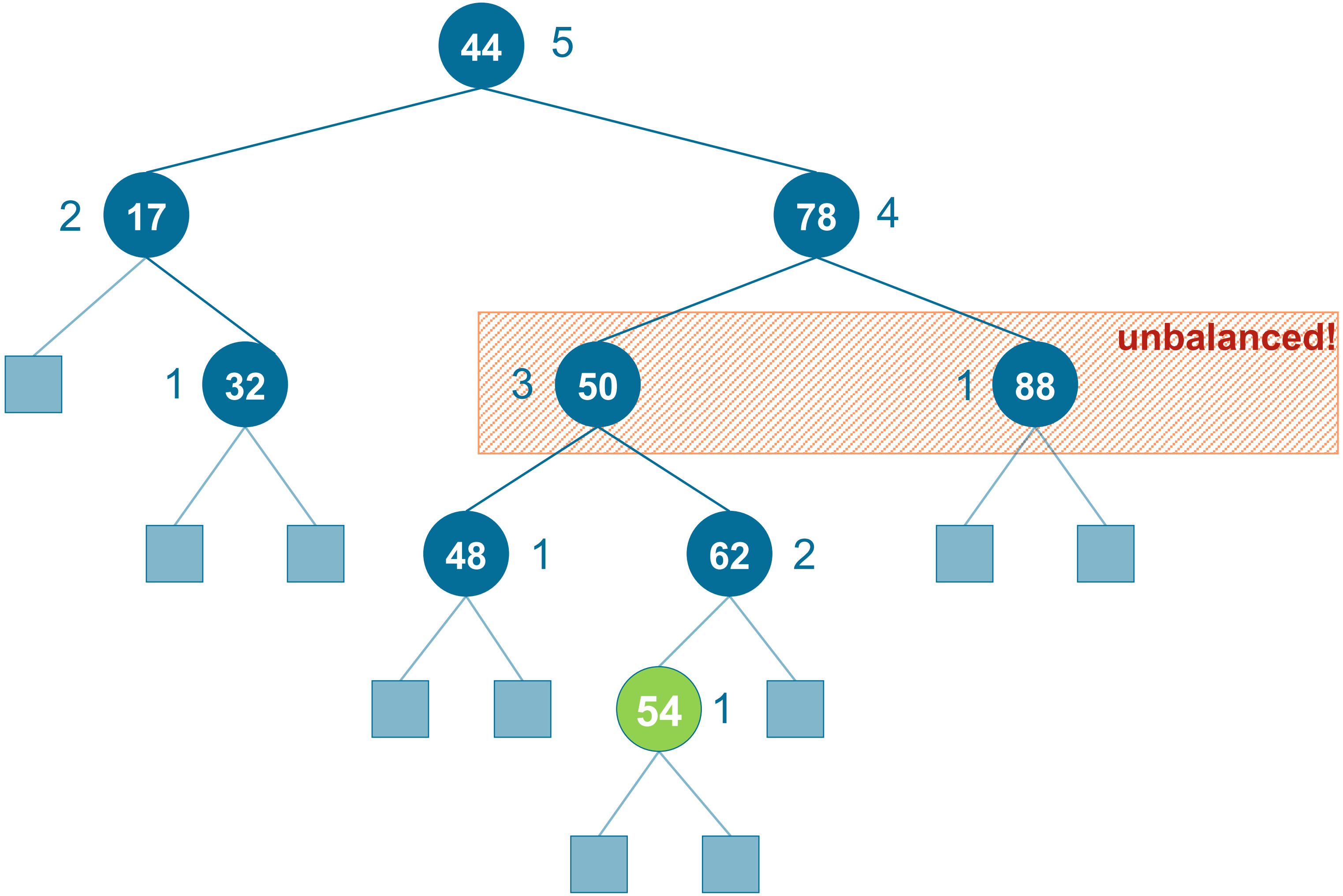
Insert in AVL trees

Insert 54



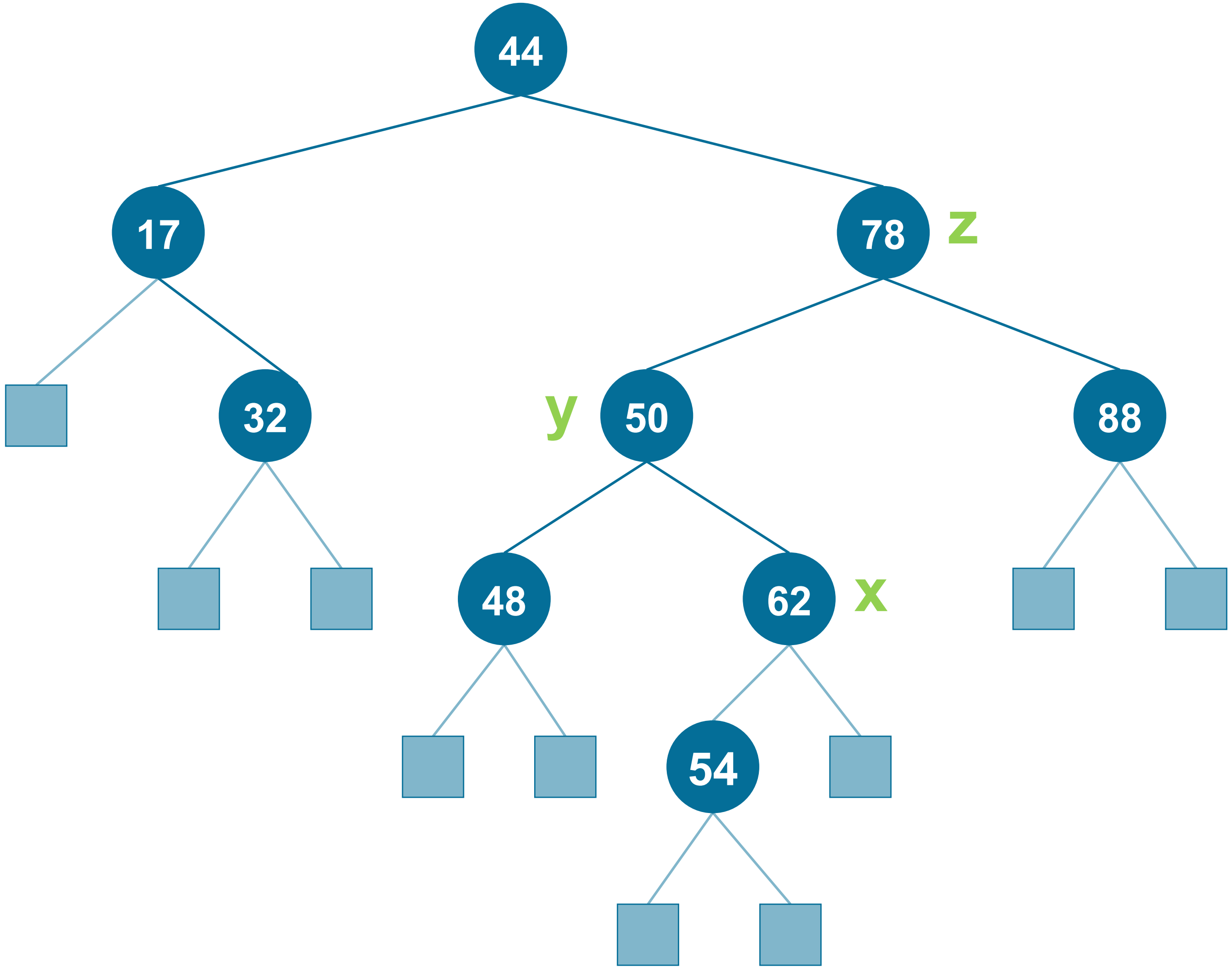
Insert in AVL trees

After inserting 54:



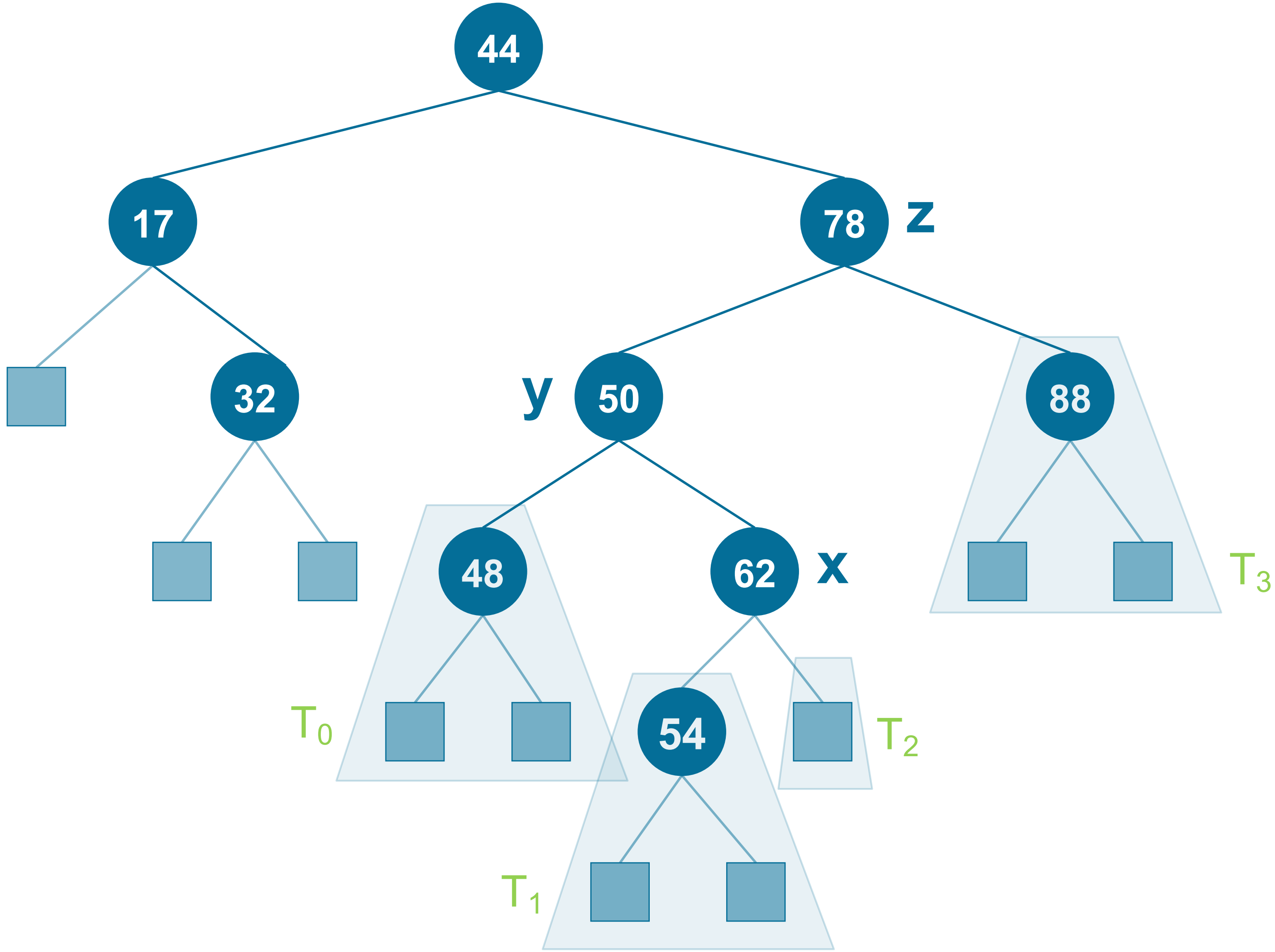
Insert in AVL trees

After inserting 54:



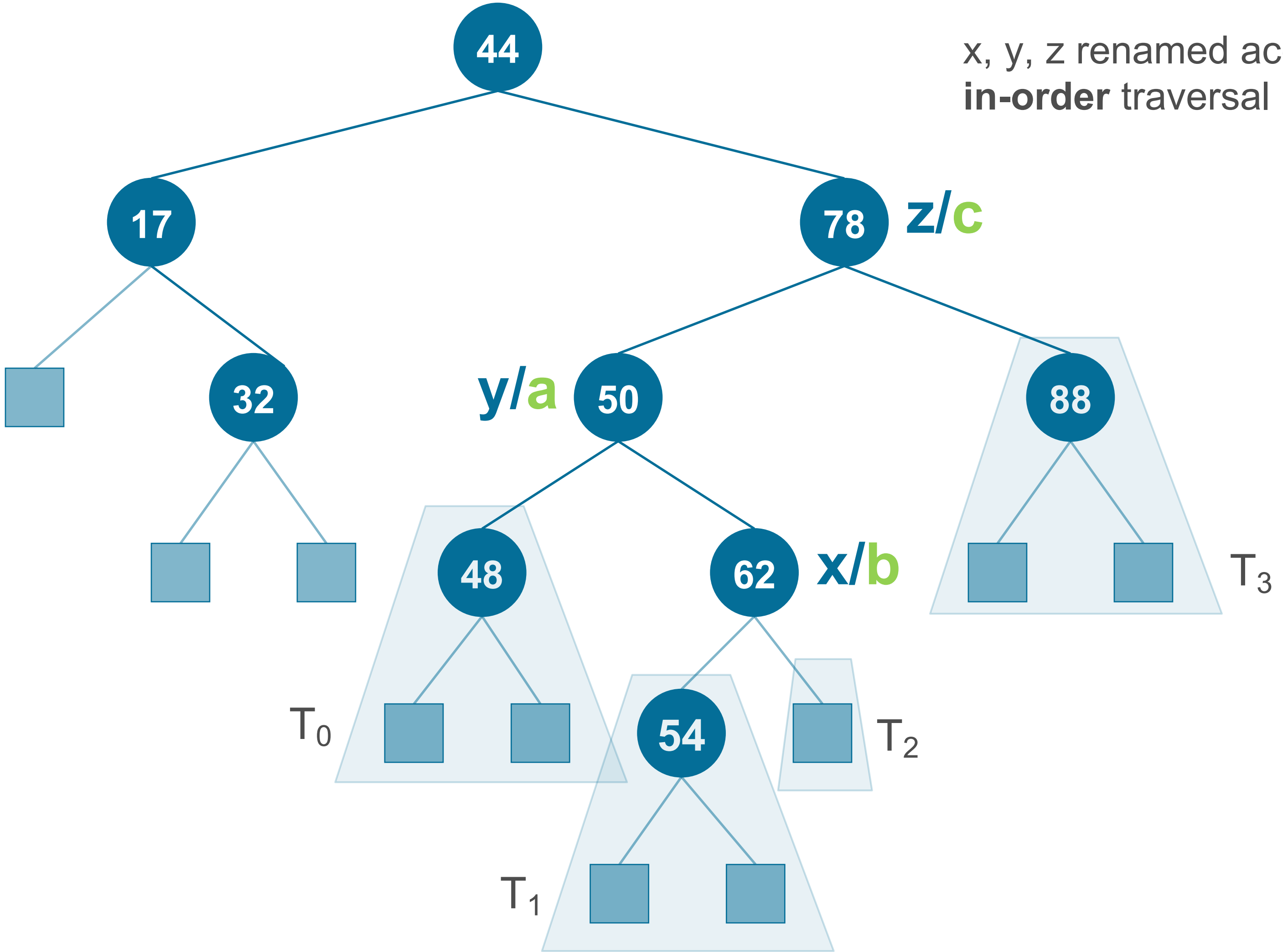
Insert in AVL trees

After inserting 54:



Insert in AVL trees

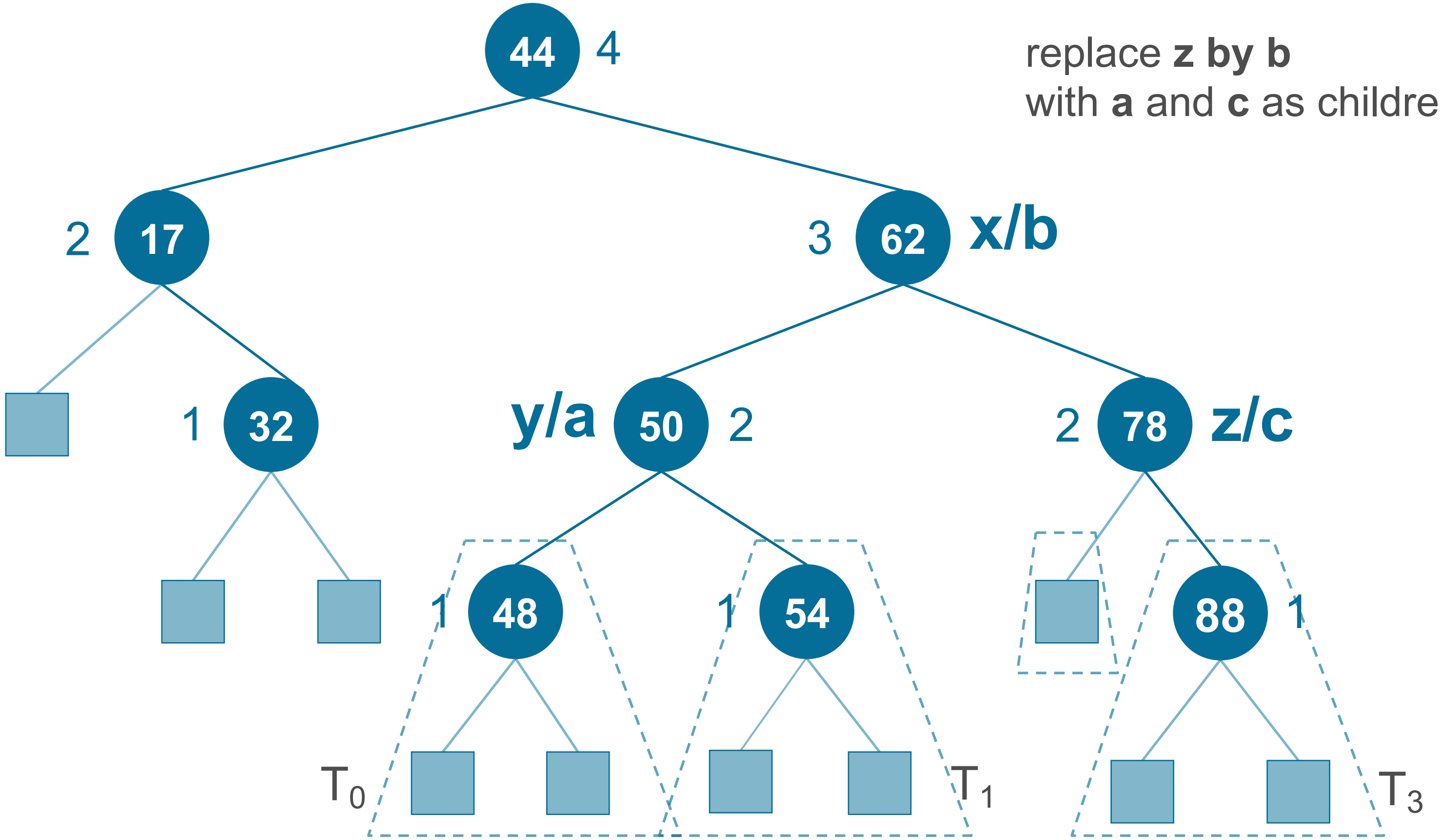
After inserting 54:



x, y, z renamed according to **in-order** traversal

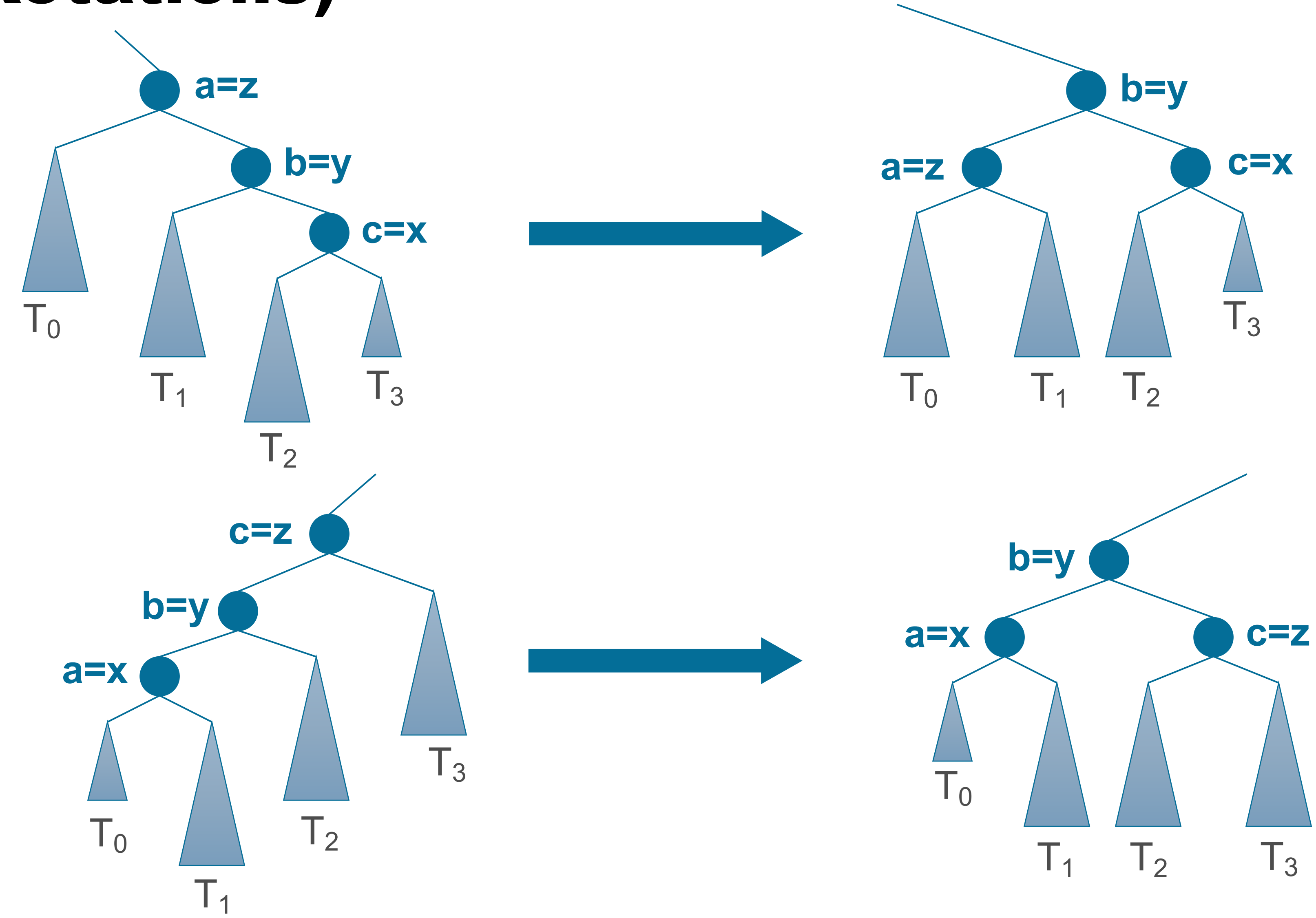
Insert in AVL trees

After inserting 54:

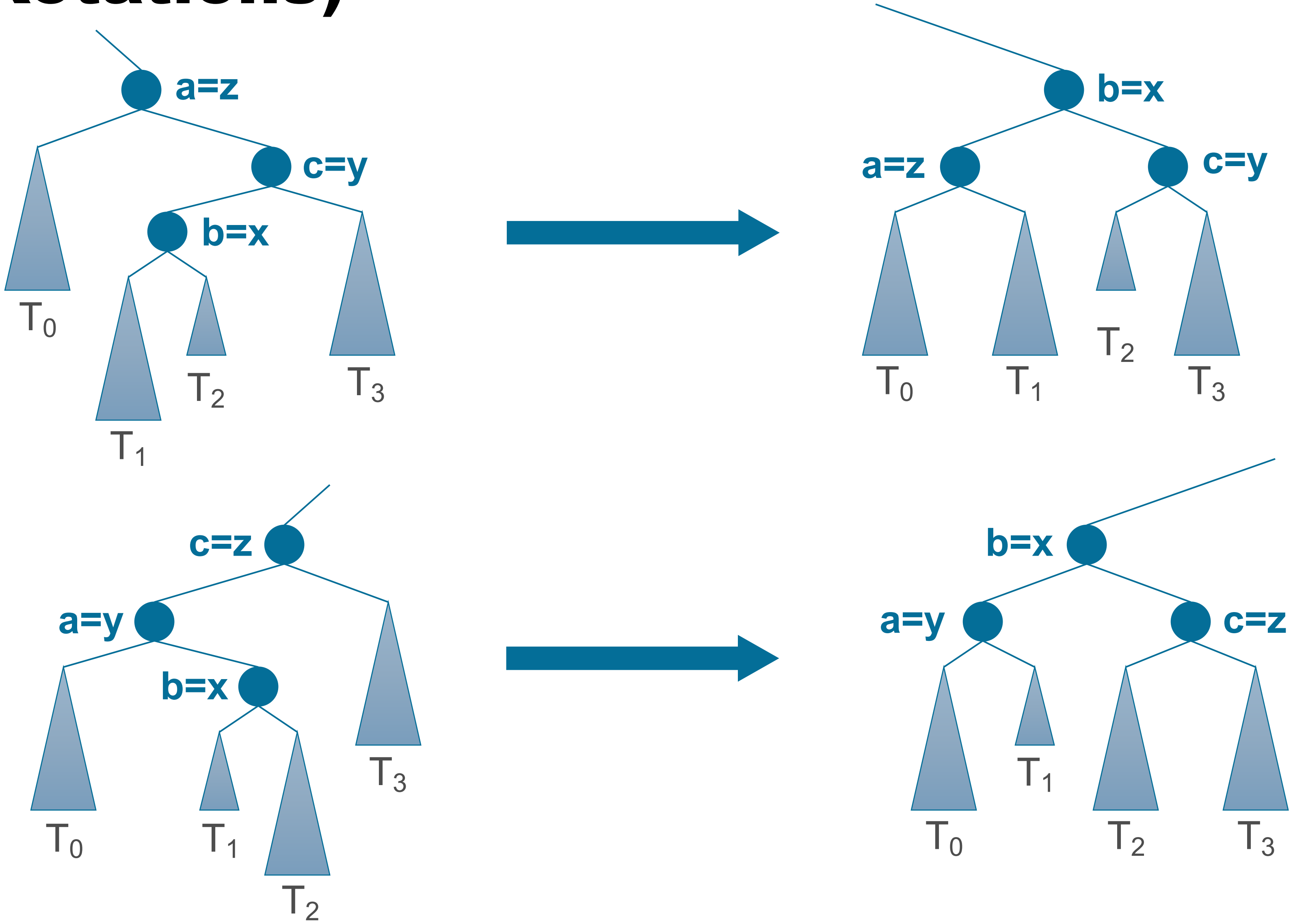


replace **z by b**
with **a** and **c** as children

Restructuring in AVL trees (Single Rotations)



Restructuring in AVL trees (Double Rotations)



Algorithm for Restructuring

Algorithm restructure (x):

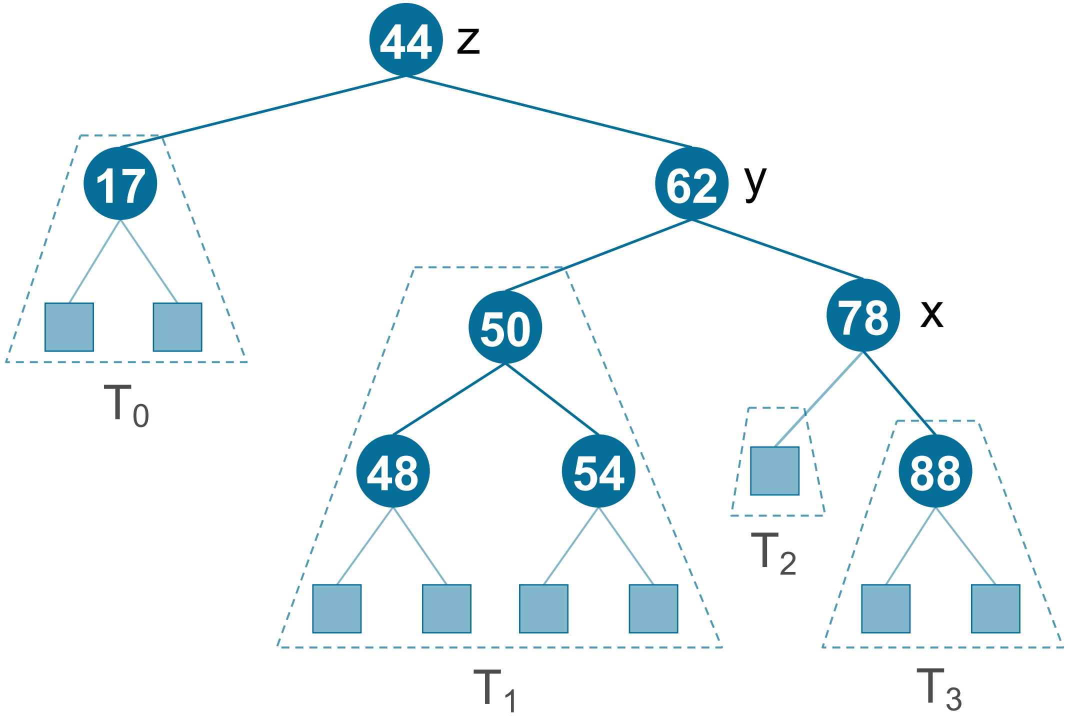
```
Input:
 node x of a binary tree T with parent y and grandparent z
Output:
 tree T restructured by (single or double) rotation of nodes x, y, z


1.Let (a,b,c) be the in-order-sequence of the nodes x, y, z and let
  ($T_0$, $T_1$, $T_2$, $T_3$) be the in-order-sequence of the subtrees of x, y and z
2.Replace subtree with root z by subtree with root b
3.Assign a as left child of b and assign $T_0$ and $T_1$ as left/right subtree of a
4.Assign c as right child of b and $T_2$, $T_3$ as left/right subtree of c
```

Cut/Link Restructuring Algorithm

Each tree to be balanced can be divided into 7 parts:

- nodes **x, y, z** and
- **4 subtrees (T_0, T_1, T_2, T_3)**, **with the children of x, y, z** as root

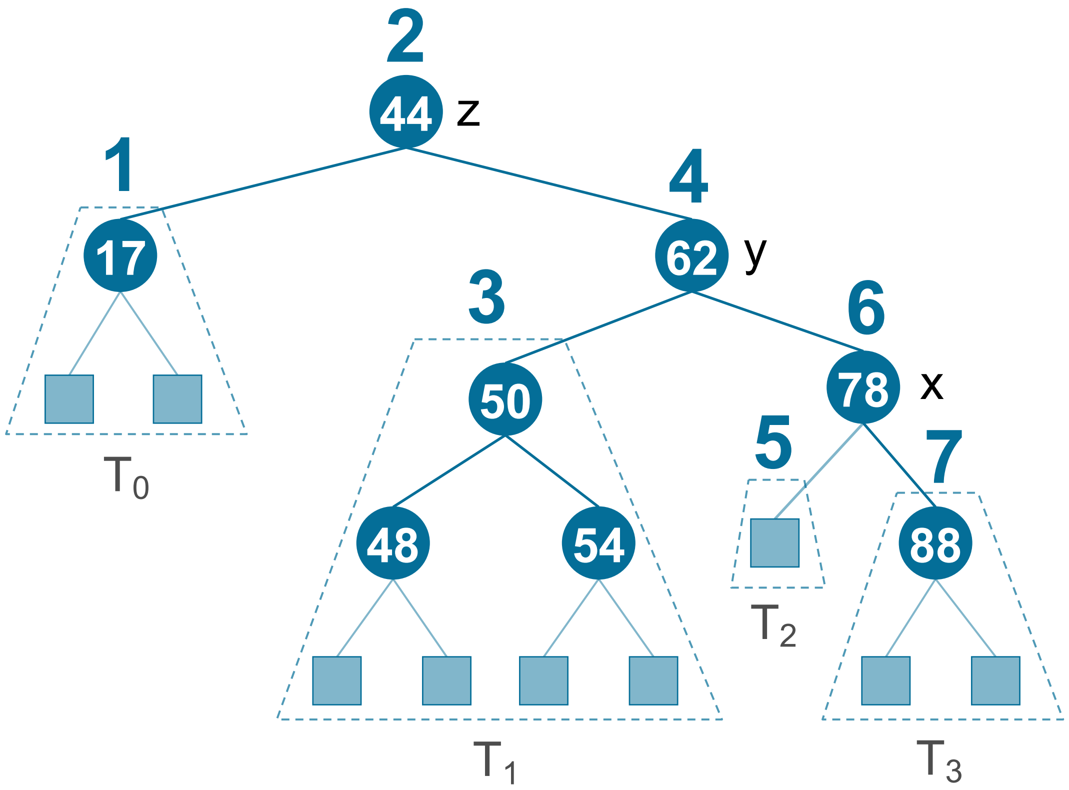


Create a new tree from the 7 parts, which is balanced and in which the **in-order sequence** of the parts is retained.

Cut/Link Restructuring Algorithm

Example

number 7 parts according to in-order traversal



Cut/Link Restructuring Algorithm

Example

Create an array with indices 1..7

1	2	3	4	5	6	7

“Cut“ the 4 subtrees and place them into the array according to their numbering.

T_0		T_1		T_2		T_3
1	2	3	4	5	6	7

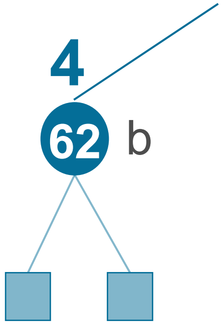
“Cut" x, y and z (in the order child, parent, grandparent) and put them in the array according to their numbering

T_0	z=a	T_1	y=b	T_2	x=c	T_3
1	2	3	4	5	6	7

Cut/Link Restructuring Algorithm

Example

- Reassemble the tree again
- Set element
  at position 4 (b) as root

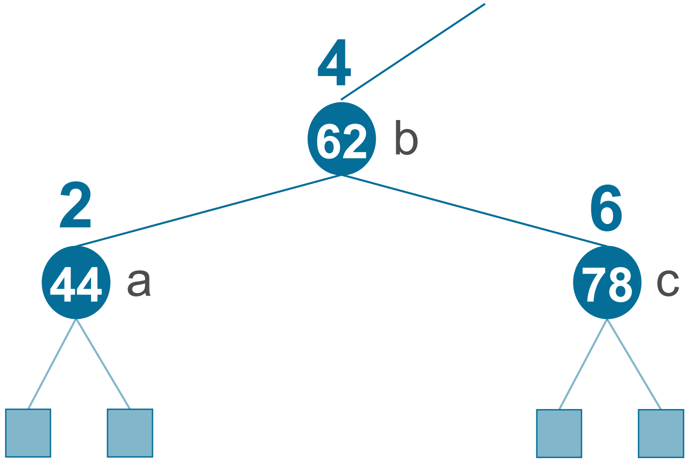


T_0	z=a	T_1	y=b	T_2	x=c	T_3
1	2	3	4	5	6	7

Cut/Link Restructuring Algorithm

Example

- Reassemble the tree again
- Set elements
  at position 2 and 6 as children

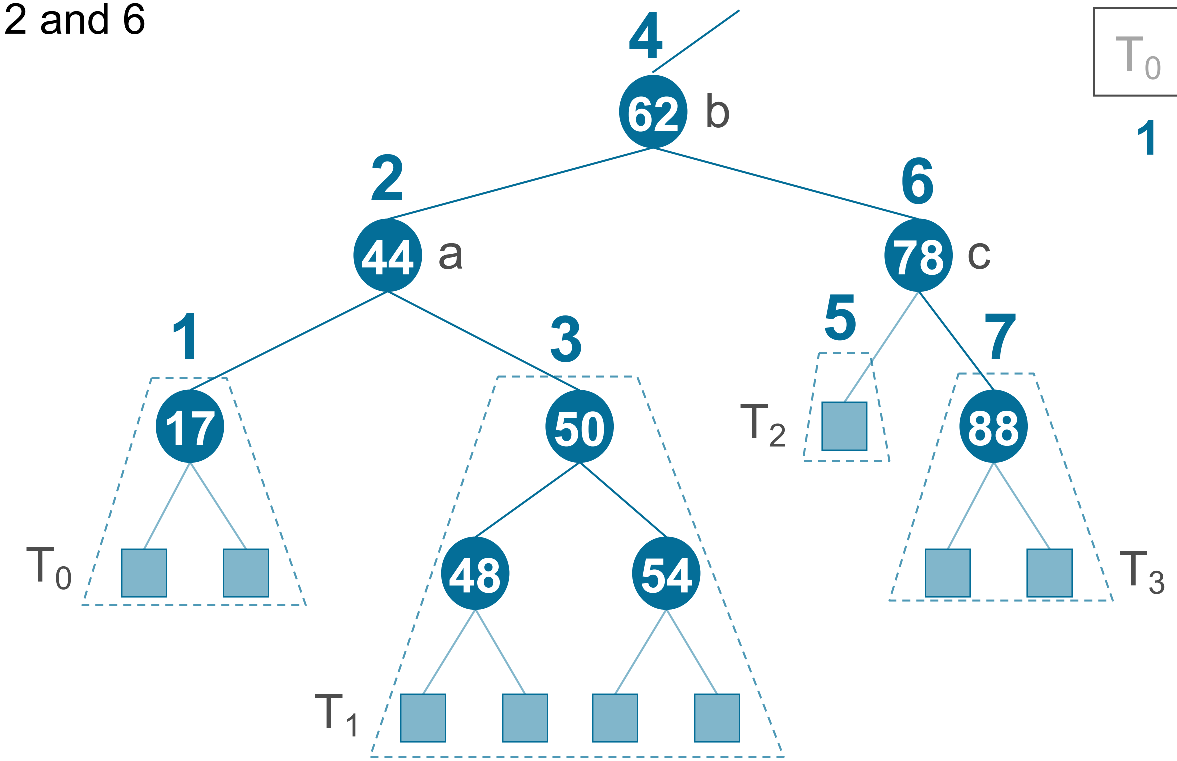


T_0	z=a	T_1	y=b	T_2	x=c	T_3
1	2	3	4	5	6	7

Cut/Link Restructuring Algorithm

Example

- Reassemble the tree again
- Set elements at position
  1, 3 or 5, 7 as child of 2 and 6



T_0	z=a	T_1	y=b	T_2	x=c	T_3
1	2	3	4	5	6	7

Cut/Link Restructuring Algorithm

Cut/Link restructuring algorithm has the **same effects** as the **four rotation cases** previously considered

Advantage:

- **No case distinction** necessary
- More „**elegant**“ solution

Disadvantage:

- May require more code

The two procedures **do not differ** in terms of **complexity** (**runtime**).

Removal of Nodes

Using **removeAboveExternal(w)** can unbalance a tree:

- Let **z be the first unbalanced node** which is visited while traversing **up** in the tree.

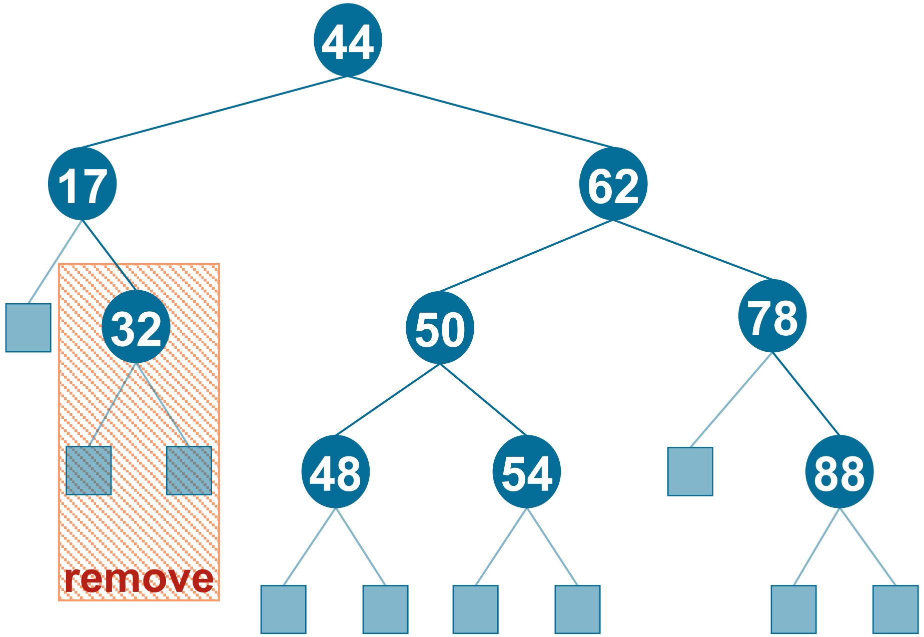
- Let **y** be the **child of z** with the **largest height** and let **x** be the **child of y** with **largest height**.

- The algorithm **restructure(x)** can be used to restructure and balance the subtree with root *z*.

- However, restructuring can **destroy the balance at higher levels**, so that the verification (and restructuring if necessary) must be **continued** until the root node is reached.

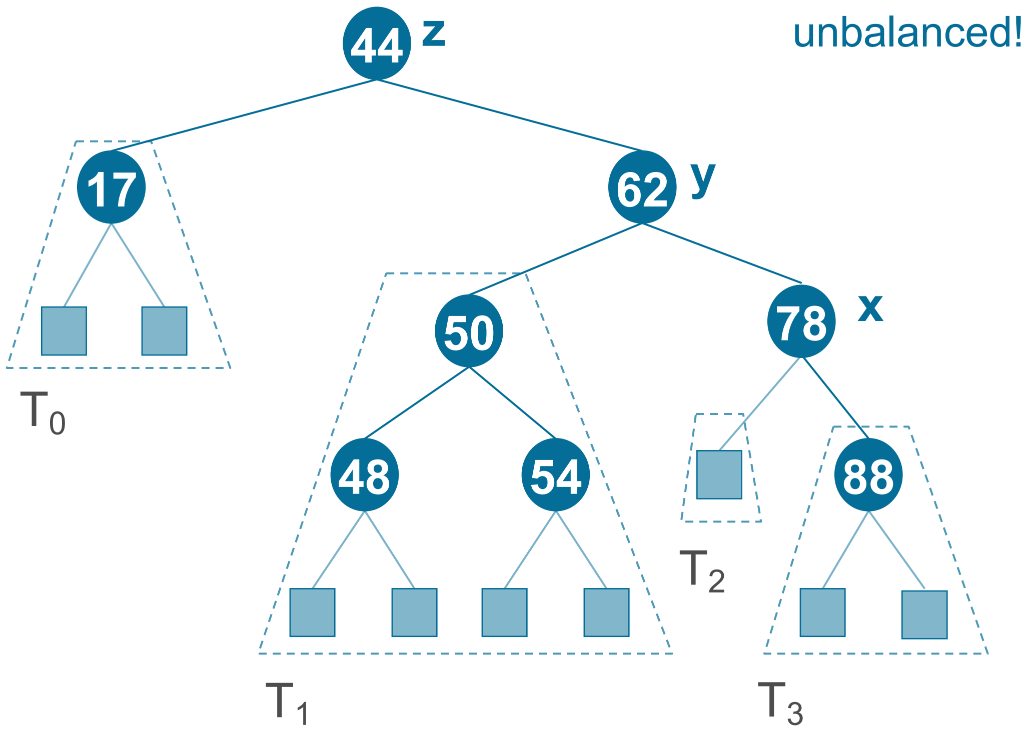
Removal of Nodes

Example



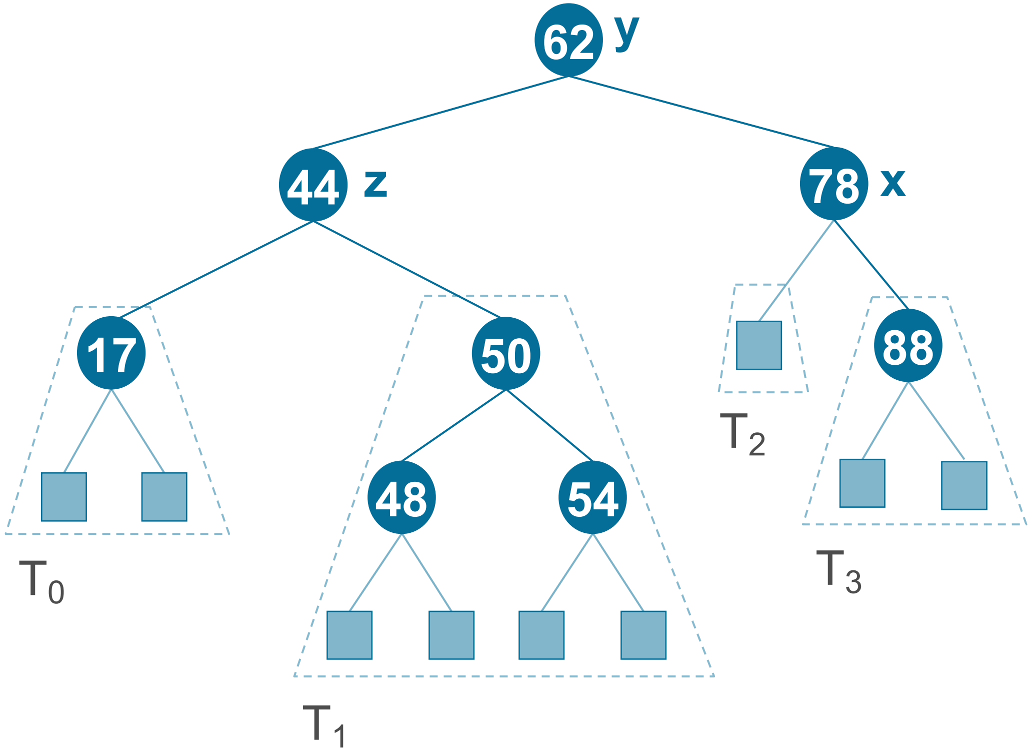
Removal of Nodes

Example



Removal of Nodes

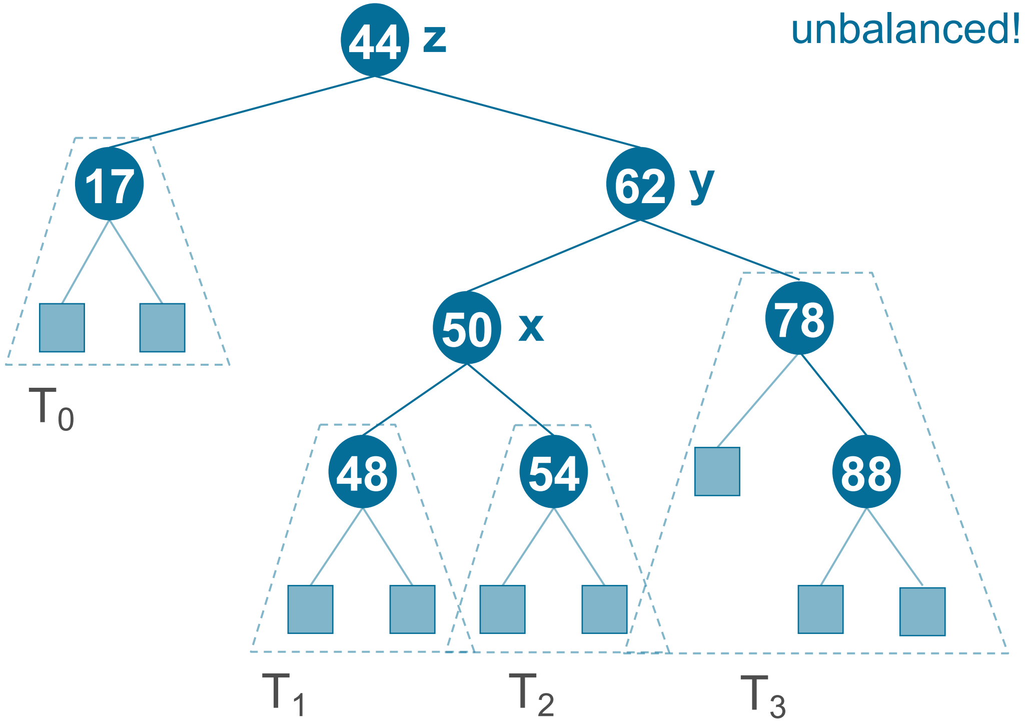
Example



Removal of Nodes

Example

Alternative

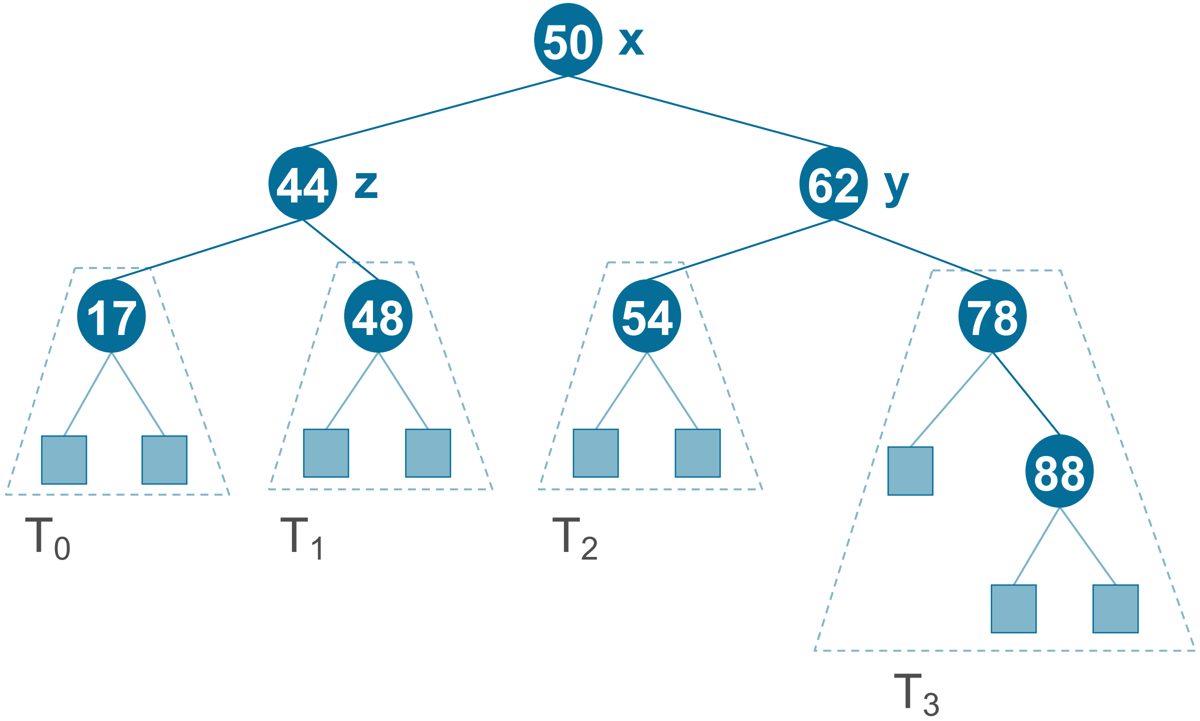


unbalanced!

Removal of Nodes

Example

Alternative



AVL Trees :: Complexity

Possible complexity: N = number of keys

`size(), isEmpty()`	$\Theta(1)$
`get(), insert(), remove()`	$O(\log N)$
`getAll(), removeAll()`	$O(\log N + s)$

(s = Number of elements in respective `enumeration`)

Balanced Trees

Balanced trees are introduced as a compromise between balanced and natural search trees, whereby **logarithmic search complexity** is required in the **worst case**.

For the height h_b of an AVL tree with *N* nodes we have:

$\lfloor \log_2 N \rfloor \le h_b \le 1{,}44 * \log_2 (N+2)$

- The upper limit can be derived from Fibonacci trees, a subclass of the AVL trees.
- Let N(h) be the minimum number of nodes of a height-balanced tree with height h. We have:
  - $N(0)=1, N(1)=2, N(2)=4, N(3)=7, N(4)=12, N(5)=20, \dots$
  - $N(h) = 1 + N(h-1) + N(h-2) = Fib(h+3) – 1$
  - $Fib(h) = 1/\sqrt{5} * (((1 + \sqrt{5})/2)^h - ((1 - \sqrt{5})/2)^h)$
  - for all h we have: $Fib(h) \ge 1/\sqrt{5} * ((1 + \sqrt{5})/2)^h – 1$
  - if $N(h)=Fib(h+3)-1$ we have: $\log_2 (N(h)+2) \ge \log_2 (1/\sqrt{5}) + (h+3) \log_2 ((1 + \sqrt{5})/2))$
- From this the estimation follows: $h \le 1{,}44 \log_2 (N(h)+2)$ → $h =$ **O (log N(h))**

0,1,1,2,3,5,8,13,21, …

Minimum number of nodes grows exponential with height
→ so vice versa: height grows logarithmically with node number

Institute of
Pervasive Computing

Trees (Height Balanced)

Algorithms and Data Structures 2, 340300
Lecture – 2022W
Univ.-Prof. Dr. Alois Ferscha, teaching@pervasive.jku.at

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at