Item 1

You can obtain 10 points in this assignment, equally distributed between the problems. You must document how you obtained the result.

Let

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & 1 \\ 3 & 3 & a \end{bmatrix}$$

and consider the following matrix equation involving A:

$A\vec{x} = \begin{bmatrix} -8 \\ 2 \\ b \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. In all problems, state all inputs as integers between 0 and 99.

a. Find an expression for the determinant of A using cofactor expansion on the second row.

$$\det\ A = -\square \det\begin{bmatrix} \square & \square \\ \square & a \end{bmatrix} + \square \det\begin{bmatrix} -1 & 2 \\ 3 & a \end{bmatrix} - \square \det\begin{bmatrix} -1 & \square \\ \square & \square \end{bmatrix} = \square -$$

Correct answers:

$$\det\ A = -1\det\begin{bmatrix} 3 & 2 \\ 3 & a \end{bmatrix} + 0\det\begin{bmatrix} -1 & 2 \\ 3 & a \end{bmatrix} - 1\det\begin{bmatrix} -1 & 3 \\ 3 & 3 \end{bmatrix} = 18 - 3a$$

b. For which values of *a, b* does the matrix equation have exactly one solution?

$a \neq$ 1 $\square$

Correct answers:

1 6

c. For which values of *a, b* does the matrix equation have no solution?

$a =$ 1 ☐ and $b \neq$ 2 ☐

Correct answers:

1 6 2 0

For which values of *a, b* does the matrix equation have an infinite number of solutions?

$a =$ 1 ☐ and $b =$ 2 ☐

Correct answers:

1 6 2 0

Item 2

It is possible to obtain 5 points in this assignment. You must document how you obtained the result.

Let *k* be a constant and let

$$B = \begin{bmatrix} 1 & -3 \\ -1 & 3 \\ 2 & k \end{bmatrix}$$

For what values of *k* will $B^T B$ only have positive eigenvalues? State your answer as an integer between 0 and 99.

$k \neq -$ 1 [ ]

Correct answers:

1 6

Item 3

It is possible to obtain 10 points in this assignment. You must document how you obtained the result.

Consider

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & r \\ 0 & r & 1 \end{bmatrix}$$, where r is a positive constant. It is known that -1 is an eigenvalue of A.

Find an orthogonal matrix *P* and a diagonal matrix *D* such that $P^{-1}AP = D$, where the entries on the diagonal of *D* appear in descending order. State all inputs as integers between 0 and 99.

$$P = \begin{bmatrix} \frac{\square}{\sqrt{35}} & -\frac{3}{\sqrt{\square}} & \frac{\square}{\sqrt{14}} \\ \frac{\square}{\sqrt{35}} & \square & -\frac{2}{\sqrt{14}} \\ \frac{\square}{\sqrt{35}} & \frac{1}{\sqrt{\square}} & \frac{\square}{\sqrt{14}} \end{bmatrix}$$

Correct answers:

$$P = \begin{bmatrix} \frac{1}{\sqrt{35}} & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & -\frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$D = \begin{bmatrix} \square & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Correct answers:

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Item 4

It is possible to obtain 5 points in this assignment. You must document how you obtained the result.

Find the dimension of the intersection of *U* and *V*:

$$U = span\left\{\begin{bmatrix}1\\-1\\-1\\-2\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\-2\\0\\-3\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\\-2\\1\end{bmatrix}\right\}$$

$$V = span\left\{\begin{bmatrix}1\\-2\\-3\\0\\-2\end{bmatrix}, \begin{bmatrix}1\\-1\\-3\\2\\-4\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\\2\\-5\end{bmatrix}\right\}$$

$Dim(U \cap V) =$ 1

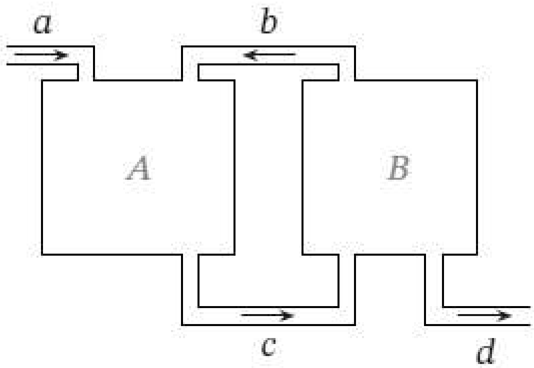
Correct answers:

1 2

Item 5

It is possible to obtain 10 points in this assignment, equally divided between the two problems. You must document how you obtained the result.

Two containers are connected by pipes, as in the figure below. Container A holds 50 l and container B 50 l. Initially, there is 25 g of salt in container A and no salt in container B. Through pipe b 1 l/min of a salt-water mixture is added to A and through pipe c 4 l/min of a salt-water mixture is added to container B. Assume only clean water is added through pipe a, and assume that the water level is constant.



a. How many grams of salt will there be in both tanks after 10 minutes? Round down to the nearest integer.

1

Correct answers:

1 21

b. At what time will the amount of salt in both tanks be half of what it was initially? State your answer as a positive integer. Always **round up** to the nearest integer in this type of problem. Note, the answer is less than t = 100.

1

Correct answers:

1 27

Item 6

It is possible to obtain 10 points in this assignment, six in the first and four in the second problem. You must document how you obtained the result.

Assume the following is known with regard to a matrix *A*:

$$A^T A = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

a. Find the eigenvalues of $A^T A$, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and the corresponding eigenvectors of $A^T A$, denoted x_1, x_2, x_3, respectively, below.

$\lambda_1 =$ 1 ☐

Correct answers:

1 12

And the corresponding eigenvector for λ_1 is

$$x_1 = \begin{bmatrix} 1 \\ \square \\ \square \end{bmatrix}$$

Correct answers:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\lambda_2 =$ 1 ☐

Correct answers:

1 10

And the corresponding eigenvector for λ_2 is

$$x_2 = \begin{bmatrix} -2 \\ \square \\ 0 \end{bmatrix}$$

Correct answers:

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 =$ 1 $\square$

Correct answers:

1 0

And the corresponding eigenvector for λ_3 is

$$x_3 = \begin{bmatrix} -1 \\ -2 \\ \square \end{bmatrix}$$

Correct answers:

$$x_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

b. It is known that the columns of the matrix below are the left singular vectors of A corresponding to the eigenvectors and eigenvalues found in (a).

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Find matrix A and state the sum of all its entries as an integer between 0 and 99.

Correct answers:

8