7. Orthogonality I

Def: Dot Product (Inner Product):

The dot product of two vectors

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is the sum of the entry-by-entry products:

$$\bar{u} \cdot \bar{v} = \bar{u}^T \cdot \bar{v} = [u_1 \ u_2 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 \cdot v_1 + u_2 v_2 + \cdots u_n v_n$$

The dot product is a **scalar**

Def: Length of vector (norm)

$$\| \bar{v} \| = \sqrt{\bar{v} \cdot \bar{v}} = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$$

A vector whose length is 1 is denoted a **unit vector**:

$$\bar{u} = \frac{1}{\| \bar{v} \|} \cdot \bar{v}$$

This is also called normalising a vector

Ex

$\bar{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. Find a unit vector of $\bar{v}$:

$$\bar{u} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2 + 0^2}} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{9}} \bar{v} = 1/3 \cdot \bar{v} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

Notice

$$\bar{u} = \begin{bmatrix} \frac{v_1}{\| \bar{v} \|} \\ \frac{v_2}{\| \bar{v} \|} \\ \vdots \\ \frac{v_n}{\| \bar{v} \|} \end{bmatrix}$$

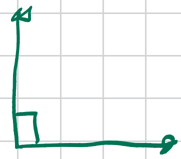
Distance in $\mathbb{R}^n$:

$$\left(\sum |\bar{u}-\bar{v}|^p\right)^{1/p}$$

$$dist(\bar{u},\bar{v}) = \| \bar{u}-\bar{v} \| = \sqrt{(\bar{u}-\bar{v})(\bar{u}-\bar{v})}$$
$$= \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\dots(u_n-v_n)^2}$$

$\hookrightarrow$ Also called Euclidean distance or L_2-norm.

Orthogonal vectors:

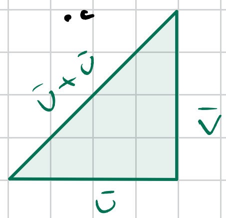


We say that two vectors in $\mathbb{R}^n$ are orthogonal to each other if $\bar{u}\cdot\bar{v}=0$

We can also reformulate this in terms of Pythogorean Theorem:

Two vectors are orthogonal if:

$$\|\bar{u}\|^2+\|\bar{v}\|^2 = \|\bar{u}+\bar{v}\|^2$$



Note the geometry of the sum of two vectors.

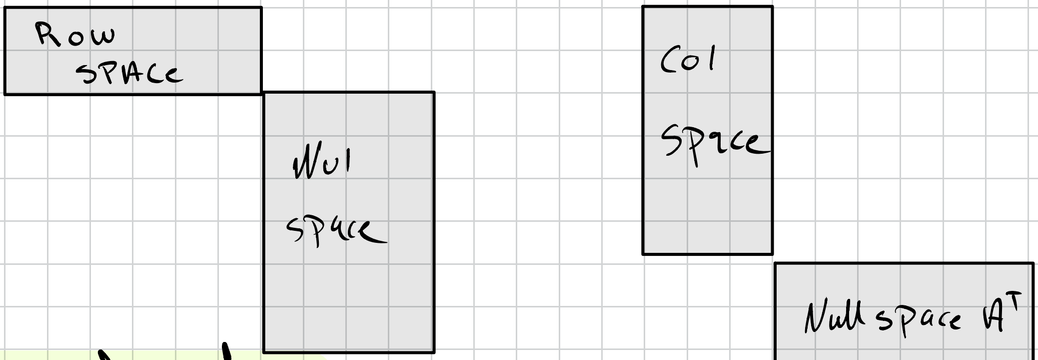
Orthogonal complement:

If W is a subspace in $\mathbb{R}^n$, and $\bar{z}$ is orthogonal to all vectors in W, we say that $\bar{z}$ is orthogonal to W. The set of all orthogonal vectors is called the Orthogonal Complement of W and is denoted $W^\perp$. Note, $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Nul A, Col A, Row A:

Let A be an $m \times n$ matrix:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$



Orthogonal sets:

If $S = \{\bar{u}_1, \bar{u}_2, \dots \bar{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is also linearly independent which means S is a basis for a p-dimensional subspace of $\mathbb{R}^n$.

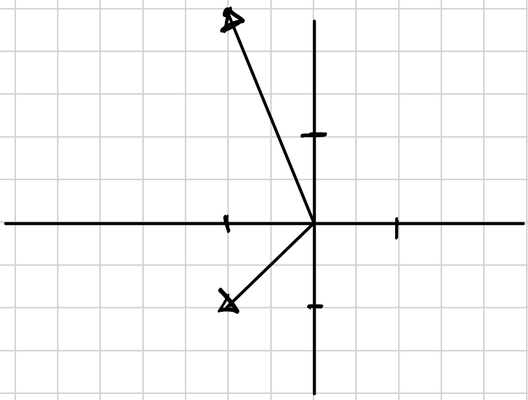
If $p = n$ then S is a basis for $\mathbb{R}^n$

Note: Each distinct pair of vectors are orthogonal

Also note independence does NOT entail orthogonality

$$\bar{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \bar{u} \cdot \bar{v} = (-1)\cdot(-1) + 2(-1) = -1$$

$$\begin{bmatrix} \bar{u} & \bar{v} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

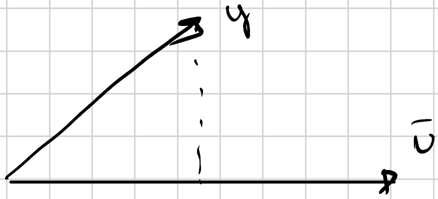


Orthogonality is stricter than independence

$$\bar{y} = c_1 \bar{u}_1 + c_2 \cdot \bar{u}_2 + \dots + c_n \bar{u}_n$$

$$\Updownarrow$$

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j}$$



The c_j's are called weights

Ex

$$S=\left\{ \overset{\bar{v}_1}{\begin{bmatrix}3\\1\\1\end{bmatrix}}, \overset{\bar{v}_2}{\begin{bmatrix}-1\\2\\1\end{bmatrix}}, \overset{\bar{v}_3}{\begin{bmatrix}-1\\-4\\7\end{bmatrix}} \right\}, \quad y=\begin{bmatrix}6\\1\\-8\end{bmatrix}$$

a) Check whether S is an orthogonal set

b) Express $\bar{y}$ as a lin. comp. of S:

a)

$$\left.\begin{aligned} \bar{v}_1\cdot\bar{v}_2 &= 0\\ \bar{v}_1\cdot\bar{v}_3 &= 0\\ \bar{v}_2\cdot\bar{v}_3 &= 0\end{aligned}\right\}$$ They are orthogonal.

What if $\bar{v}_1, \dots, \bar{v}_{100}$?

$$U^T\cdot U = \begin{bmatrix}3 & 1 & 1\\ -1 & 2 & 1\\ -1 & -4 & 7\end{bmatrix}\begin{bmatrix}3 & -1 & -1\\ 1 & 2 & -4\\ 1 & 1 & 7\end{bmatrix}$$

$$\begin{bmatrix}d_{11} & & & 0\\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn}\end{bmatrix}$$

b) $c_1=\dfrac{\bar{y}_1\cdot\bar{v}_1}{\bar{v}_1\cdot\bar{v}_1}=\dfrac{11}{11}$, $c_2=\dfrac{-12}{6}$

$c_3=\dfrac{-66}{66}$, $\bar{y}=\bar{v}_1-2\bar{v}_2-\bar{v}_3$

Exercise:

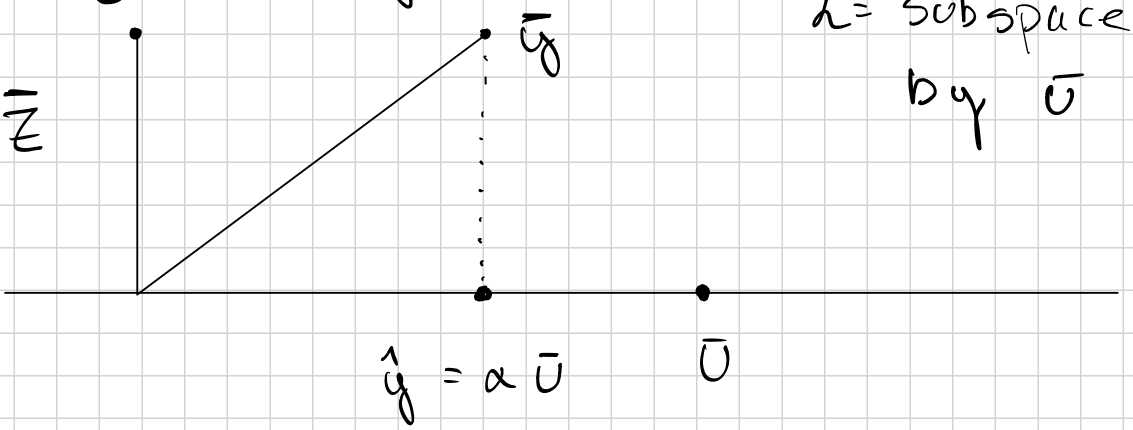
Let S be like above. Express $\bar{y}=\begin{bmatrix}3\\1/2\\-4\end{bmatrix}$ as a linear comb. of S.

$c_1=\frac{1}{2}$, $c_2=-1$, $c_3=-\frac{1}{2}$

$\bar{y}=\frac{1}{2}\bar{v}_1-\bar{v}_2-\frac{1}{2}\bar{v}_3$

Orthogonal Projection:

L = Subspace spanned by $\bar{u}$



$\bar{y} = \hat{y} + \bar{z}$

$\hat{y} = \bar{y} - z$

$\bar{z} = \bar{y} - \hat{y} \quad = \bar{y} - \alpha \bar{u}$, now

$z \perp \bar{u} \rightarrow (\bar{y} - \alpha \bar{u}) \cdot \bar{u} = 0 \Leftrightarrow \bar{y} \cdot \bar{u} - \alpha \cdot \bar{u} \cdot \bar{u} =$

$\Rightarrow \bar{y} \cdot \bar{u} - \alpha (\bar{u} \cdot \bar{u}) = 0 \Rightarrow \bar{y} \cdot \bar{u} = \alpha (\bar{u} \cdot \bar{u}) \Rightarrow \alpha = \dfrac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}}$

$\hat{y} = \alpha \bar{u} = \dfrac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \cdot \bar{u}$

$$\boxed{\hat{y} = \mathrm{Proj}_L \, \bar{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \, \bar{u}}$$

"Projection of $\bar{y}$ on the sub-space spanned by L"

The Orthogonal Decomposition Theorem:

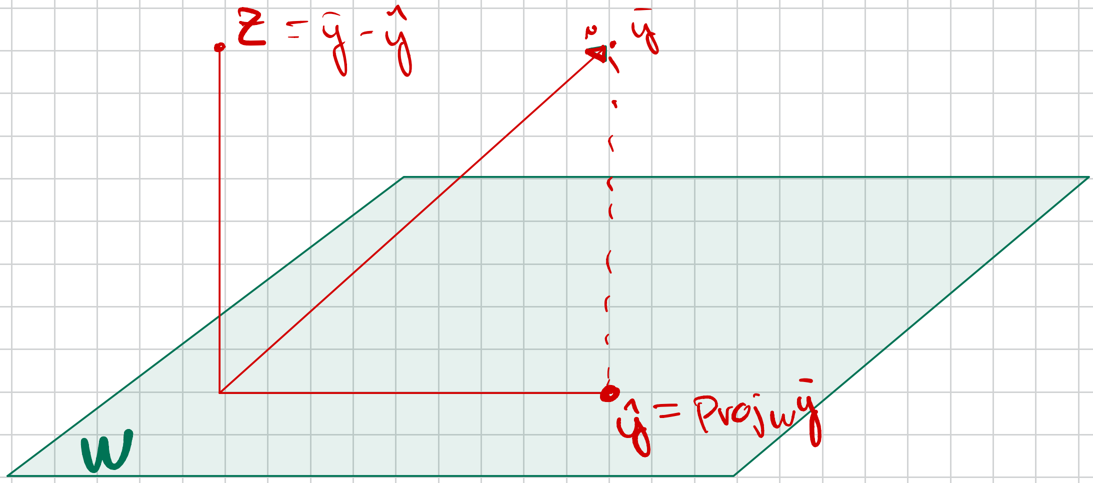
Let W be a subspace of $\mathbb{R}^n$. Then each $\bar{y}$ in $\mathbb{R}^n$ can be written uniquely as

$$\bar{y} = \hat{y} + z$$

Where $\hat{y}$ is in W an $\bar{z}$ is in $W^\perp$. Even further if $\{\bar{u}_1 \ldots \bar{u}_p\}$ is an orthogonal basis:

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \bar{u}_1 + \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} \bar{u}_2 + \ldots + \frac{y \cdot \bar{u}_p}{\bar{u}_p \cdot \bar{u}_p}$$

and $\bar{z} = \bar{y} - \hat{y}$



Ex:

$$\bar{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

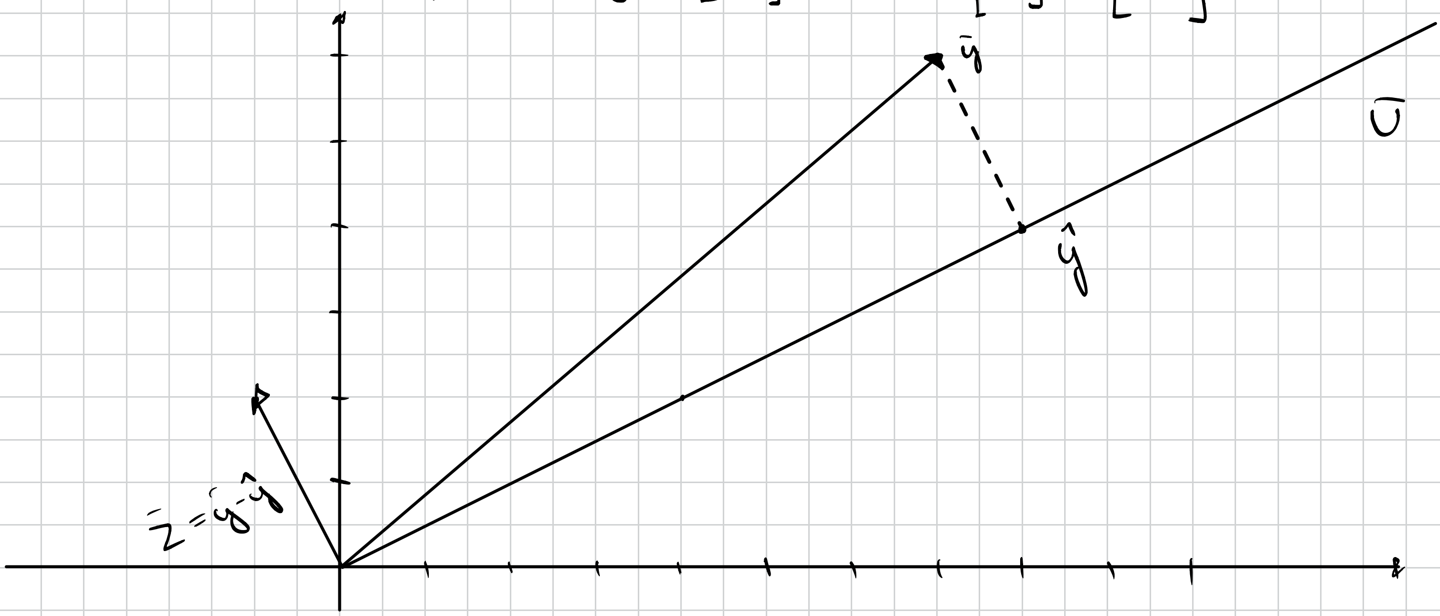
a) Find $\text{proj}_{\bar{u}}\, \bar{y}$

b) Write $\bar{y}$ as the sum of two orthogonal vectors: one in span $\bar{u}$ and one orthogonal to $\bar{u}$

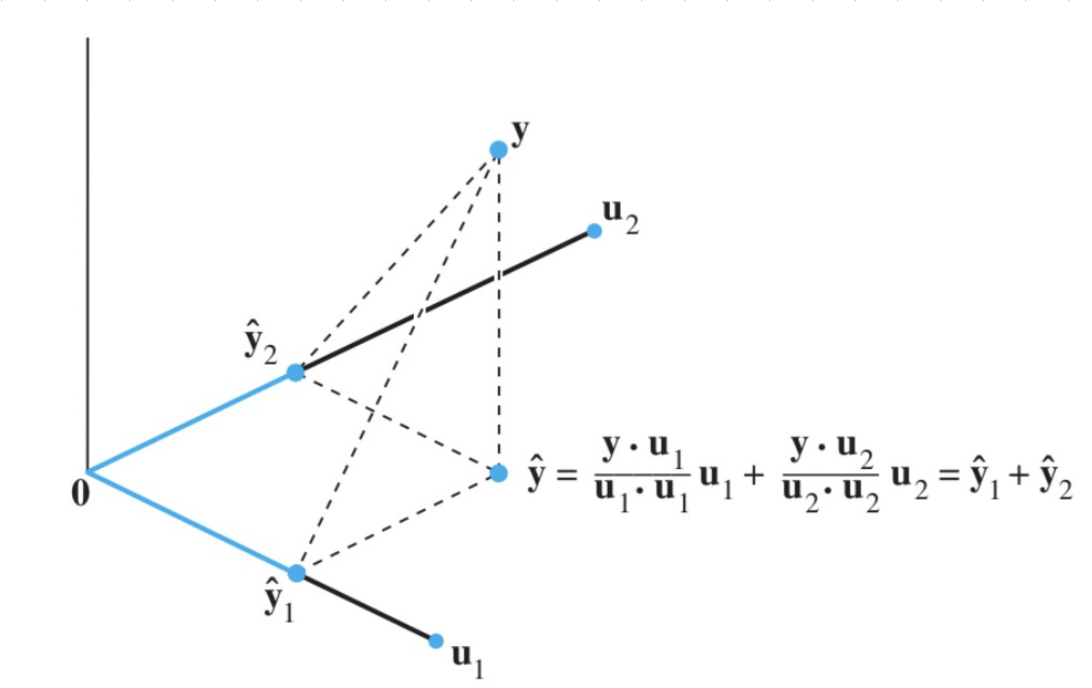
a) $$\hat{y} = \frac{\begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

b) Component of $\bar{y} \perp \bar{u}$

$$\bar{z} = \bar{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



A geometric view



Orthonormal sets:

* An orthogonal set of unit vectors!
* If W is spanned by this set, then the set is an orthonormal basis for W.

An $m \times n$ matrix U has orthonormal columns iff

a) $U^T U = I \rightarrow U^T U$ Gram Matrix

b) $\|U\bar{x}\| = \|\bar{x}\|$ $\qquad G = X^T X$ always symmetric

c) $(U\bar{x}) \cdot (U\bar{y}) = \bar{x} \cdot \bar{y}$

d) $(U\bar{x}) \cdot (U\bar{y}) = 0$ iff. $\bar{x} \cdot \bar{y} = 0 \rightarrow \bar{x} \perp \bar{y}$.

EX:

$$\bar{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

Show that $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$

Step 1: show orthogonal:

a) $\bar{v}_i \cdot \bar{v}_j \; \{\forall\, i,j \in \mathbb{N}_n \mid i \neq j\}$

b) $U^T U = I$

Step 2: show length $= 1$

a) $\|v_i\| = 1$

b) $U^T U = I$

EX:

$$\bar{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Let $W = \{\bar{u}_1, \bar{u}_2\}$. Write $\bar{y}$ as the sum of two orthonormal vectors, one in W and the other in $W^\perp$.

$$\hat{y} = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{10} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} - 1 \\ \frac{3}{2} + \frac{1}{2} \\ -\frac{3}{10} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{105}} \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}$$

$$\bar{z} = \bar{y} - \hat{y}^{*} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix} \longrightarrow \begin{bmatrix} 7 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

so

$$\bar{y} = \frac{1}{\sqrt{105}} \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

* must use non-scaled version of $\hat{y}$ to find $\bar{z}$.