# 6. Recap and Exercise

## Eigenvalus and eigenvectors

$$A\bar{x} = \lambda \bar{x}$$

eigenvalue eigenvector

**To check if $\lambda$ is an eig.val. of A**

$[A-\lambda I] \overset{RREF}{\sim}$ check if consistent

**To check if $\bar{v}$ is an eig. vec. of A**

$A \cdot \bar{v}$ and check if result is multiple of $\bar{v}$ $(\lambda \bar{v})$.

**To find eigenvalue, solve:**

$\det(A-\lambda I) = 0$ (Characteristic Equation)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\det(A-\lambda I_2) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - a\lambda - d\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

sum of diagonal of A

det (A)

## Definition:

The sum of the main diagonal of a square matrix is called the trace:

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots a_{nn}$$

so:

$$\det(A-\lambda I) = \lambda^2 - tr(A)\lambda + \det(A)$$

Some Properties of trace:

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

$$tr(A+B) = tr(A) + tr(B)$$

$$tr(AB) = tr(BA)$$

$$tr(cA) = c \cdot tr(A)$$

To find eig. vecs:

Find nullspace of $A - \lambda_i I$

This is also called the eigen space corresponding to $\lambda_i$.

Multiplicities:

When the characteristic polynomial is factorised into linear factors, the number of times $(\lambda - \lambda_i)$ appears is called the algebraic multiplicity of $\lambda_i$, i.e the number of times $\lambda_i$ appears as a root in the characteristic equation.

The dimension of the eigenspace corresponding to $\lambda_i$ is called the geometric multiplicity of $\lambda_i$.

If sum of alg. mult = sum of geo. mult.

↳ A is Diagonalisable.

alg. mult $\geq$ geo. mult.

# Diagonalisation:

An $n \times n$ matrix $A$ is said to be diagonalisable if $A$ is similar to a diagonal matrix $D$:

$$A = PDP^{-1},$$

where the diagonals of $D$ are the eigenvalues of $A$ and the columns of $P$ are the corresponding Eigenvectors

## Method: Diagonalisation

1. Find eigenvalues of $A$:
   a) If triangular $\rightarrow$ diagonal
   b) roots of $\det(A - \lambda I_n) = 0$
2. For each $\lambda$, find a basis for the eigenspace:
   $(A - \lambda I_n)\bar{x} = \bar{0}$ in P.V.F.
3. Construct $D$ by placing the eigenvalues on the diagonal of an $n \times n$ zero matrix. Note it is good practice to place them in descending order: $d_{11} \geq d_{22} \geq d_{33} \ldots \geq d_{nn}$
4. Construct $P$ from the vectors found in step 2. Note the eigenvector must be placed in the same column as its corresponding eigenvalue
5. Find $P^{-1}$

1.5.1+3+5+7 ✓ $\begin{bmatrix} 3-2 & 2 \\ 3 & 8-2 \end{bmatrix}$ RREF Check if consistent

**1.** Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

~~**2.** Is $\lambda = -3$ an eigenvalue of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?~~

**3.** Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? If so, find the eigenvalue. ✓

$\hookrightarrow \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \stackrel{?}{=} \lambda \vec{v} = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

~~**4.** Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$? If so, find the eigenvalue.~~

**5.** Is $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}$? If so, find the eigenvalue. ✓ $\begin{bmatrix} \ \end{bmatrix}$

~~**6.** Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.~~

**7.** Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3-4 & 0 & -1 \\ 2 & 3-4 & 1 \\ -3 & 4 & 5-4 \end{bmatrix}$? If so, find one corresponding eigenvector. ✓

~~**8.** Is $\lambda = 1$ an eigenvalue of $\begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$? If so, find one corresponding eigenvector.~~

~~In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.~~

~~**9.** $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1, 3$~~

# 5.1.13 Find bases

**13.** $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $\lambda = 1, 2, 3$

$\lambda = 1$

$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$, $\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \boxed{\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}}$

$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \boxed{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$

**16.** $A = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}$, $\lambda = 4$

# 5.1.40

[**M**] In Exercises 37–40, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

$$\begin{bmatrix} -23 & 57 & -9 & -15 & -59 \\ -10 & 12 & -10 & 2 & -22 \\ 11 & 5 & -3 & -19 & -15 \\ -27 & 31 & -27 & 25 & -37 \\ -5 & -15 & -5 & 1 & 31 \end{bmatrix}$$

5.2. 1+3+6

Find the characteristic polynomial and the real eigenvalues of the matrices in Exercises 1–8.

**1.** $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$ **6.** $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$

**3.** $\begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$ **4.** $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$ **7.** $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ **8.** $\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$

$\lambda^2 - 4\lambda - 45 \rightarrow \underline{(\lambda-9)(\lambda+5)}$ $\lambda = 9$, $\lambda = -5$

5.2.10+11

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for $3 \times 3$ determinants described prior to Exercises 15–18 in Section 3.1. [*Note*: Finding the characteristic polynomial of a $3 \times 3$ matrix is not easy to do with just row operations, because the variable $\lambda$ is involved.]

**9.** $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$

$\rightarrow (5-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{vmatrix} = (5-\lambda)\big((3-\lambda)(7-\lambda)+2\big)$

$= (5-\lambda)(23 - 10\lambda + \lambda^2)$

$= 115 - 50\lambda + 5\lambda^2 - 23\lambda + 10\lambda^2 - \lambda^3$

$= -\lambda^3 + 15\lambda^2 - 73\lambda + \underline{115}$

**11.** $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 4 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

5.3.1

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute $A^4$.

**1.** $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

5.3.6

In Exercises 5 and 6, the matrix $A$ is factored in the form $PDP^{-1}$. Use the Diagonalization Theorem to find the eigenvalues of $A$ and a basis for each eigenspace.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}$$

# 5.3.10

Diagonalize the matrices in Exercises 7–20, if possible. The real eigenvalues for Exercises 11–16 and 18 are included below the matrix.

~~9.~~ $\cancel{\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}}$

10. $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$$\overset{5}{\phantom{x}}\quad\overset{-2}{\phantom{x}}$$

$$\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

$$\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3/4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4/7 & 1 \\ -1 & 3/7 \end{bmatrix}$$

11. $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$

$\lambda = -1,\ 5$

12. $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

$\lambda = 2,\ 5$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

$\lambda = 1,\ 5$

14. $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

$\lambda = 2,\ 3$

# 5.3.35+36

**[M]** Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

**35.** $\begin{bmatrix} 13 & -12 & 9 & -15 & 9 \\ 6 & -5 & 9 & -15 & 9 \\ 6 & -12 & -5 & 6 & 9 \\ 6 & -12 & 9 & -8 & 9 \\ -6 & 12 & 12 & -6 & -2 \end{bmatrix}$

**36.** $\begin{bmatrix} 24 & -6 & 2 & 6 & 2 \\ 72 & 51 & 9 & -99 & 9 \\ 0 & -63 & 15 & 63 & 63 \\ 72 & 15 & 9 & -63 & 9 \\ 0 & 63 & 21 & -63 & -27 \end{bmatrix}$