2. Recap and Exercises

We can place equations and vectors in Matrices

Ex

$$5x_1 - x_2 + 2x_3 = 7$$
$$-2x_1 + 6x_2 + 9x_3 = 0$$
$$-7x_1 + 5x_2 - 3x_3 = -7$$

$$\Downarrow$$

$$\begin{bmatrix} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{bmatrix}$$

Ex

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \quad \bar{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$

On matrices we can perform elementary row operations (Replace, swap, scale).

1. scaling

Multiplying row 1 by 1/5

$$\begin{bmatrix} 1 & -\frac{1}{5} & \frac{2}{5} & \frac{7}{5} \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & 7 \end{bmatrix}$$

2. Replace

Adding 2 times row 1 to row 2

$$\begin{bmatrix} 1 & -\frac{1}{5} & \frac{2}{5} & \frac{7}{5} \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ -7 & 5 & -3 & 7 \end{bmatrix}$$

3. Replace

Adding 7 times row 1 to row 3

$$\begin{bmatrix} 1 & -\frac{1}{5} & \frac{2}{5} & \frac{7}{5} \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & \frac{18}{5} & -\frac{1}{5} & \frac{84}{5} \end{bmatrix}$$

4. Scaling

Multiplying row 2 by 5/28

$$\begin{bmatrix} 1 & -\frac{1}{5} & \frac{2}{5} & \frac{7}{5} \\ 0 & 1 & \frac{7}{4} & \frac{1}{2} \\ 0 & \frac{18}{5} & -\frac{1}{5} & \frac{84}{5} \end{bmatrix}$$

5. Replace

Adding 1/5 times row 2 to row 1

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} & \frac{3}{2} \\ 0 & 1 & \frac{7}{4} & \frac{1}{2} \\ 0 & \frac{18}{5} & -\frac{1}{5} & \frac{84}{5} \end{bmatrix}$$

6. Replace

Adding -18/5 times row 2 to row 3

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} & \frac{3}{2} \\ 0 & 1 & \frac{7}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{13}{2} & 15 \end{bmatrix}$$

Echelon!

7. Scaling

Multiplying row 3 by -2/13

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} & \frac{3}{2} \\ 0 & 1 & \frac{7}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{30}{13} \end{bmatrix}$$

6. Replace

Adding -3/4 times row 3 to row 1

$$\begin{bmatrix} 1 & 0 & 0 & \frac{42}{13} \\ 0 & 1 & \frac{7}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{30}{13} \end{bmatrix}$$

7. Replace

Adding -7/4 times row 3 to row 2

$$\begin{bmatrix} 1 & 0 & 0 & \frac{42}{13} \\ 0 & 1 & 0 & \frac{59}{13} \\ 0 & 0 & 1 & -\frac{30}{13} \end{bmatrix}$$

Reduced Echelon!

Pivots → Basic variables

Non Pivots → Free variables

Matrix Equation

$A\bar{x} = \bar{b}$

$A\bar{x} = \bar{0}$

$$A\bar{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$

Note: $A\bar{x} = b$ has a solution iff. b is a lin. comb. of the columns of A

$$= x_1 \cdot \bar{a}_1 + x_2 \cdot \bar{a}_2 + \dots + x_n \cdot \bar{a}_n$$

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nm} \end{bmatrix}$$

If A is a $m \times n$ matrix, then all of the following are equivalent

a) For each $\bar{b}$ in $\mathbb{R}^m$, $A\bar{x} = \bar{b}$ has a solution

b) Each $\bar{b}$ in $\mathbb{R}^m$ is a lin. comb. of the columns of A.

c) The columns of A span $\mathbb{R}^m$

d) A has a pivot in every row.

Linear Dependence

If the vector equation

$$x_1 \bar{v}_1 + x_2 \bar{v}_2 + \dots x_p \cdot \bar{v}_p = 0$$

$A\bar{x} = 0$

has only the trivial solution, then the vectors are lin. independent.

Important Theorems:

$\{ \quad \}$

a) If a set $S = \{\bar{v}_1, \dots \bar{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is lin. dep.

b) A set of two vectors in $\mathbb{R}^n$ is lin. independent iff. neither is a multiple of the other

c) Any set $\{\bar{v}_1 \dots \bar{v}_p\}$ in $\mathbb{R}^n$ is lin. dep. if $p > n$, i.e. more columns than rows.

d) $S = \{\bar{v}_1, \dots \bar{v}_p\}$ is lin. dep. iff. at least one vector in S is a linear comb. of the others, assuming $\bar{v}_1 \neq \bar{0}$. So $\bar{v}_j$ $(1 < j \leq p)$ is a linear comb. of the preceeding vectors $\bar{v}_1, \dots \bar{v}_{j-1}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercises

1.2.1

1. a. $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\end{bmatrix}$ b. $\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$ c. $\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$ d. $\begin{bmatrix}1&1&0&1&1\\0&2&0&2&2\\0&0&0&3&3\\0&0&0&0&4\end{bmatrix}$

a. RREF b. RREF c. % d. E.F.

1.2.7, 9, 11

7. $\left[\begin{array}{ccc|c}1&3&4&7\\3&9&7&6\end{array}\right]$

9. $\begin{bmatrix}0&1&-2&3\\1&-3&4&-6\end{bmatrix}$

11. $\left[\begin{array}{ccc|c}3&-2&4&0\\9&-6&12&0\\6&-4&8&0\end{array}\right]$

$\underset{\sim}{r_2 \to r_2 - 3r_1}$ $\begin{bmatrix}1&3&4&7\\0&0&1&3\end{bmatrix} \to \begin{bmatrix}1&3&0&-5\\0&0&1&3\end{bmatrix} -$ $x_1 = -5-3x_2$, $x_2 = x_2$, $x_3 = 3$

$a_1t + a_2t^2 + a_3t^3$

1.2.33

Find the interpolating polynomial $p(t) = a_0 + a_1t + a_2t^2$ for the data (1, 6), (2, 15), (3, 28). That is, find a_0, a_1, and a_2 such that

$$a_0 + a_1(1) + a_2(1)^2 = 6$$
$$a_0 + a_1(2) + a_2(2)^2 = 15$$
$$a_0 + a_1(3) + a_2(3)^2 = 28$$

$$\begin{array}{cccc} a_0 & a_1 & a_2 & p(t) \end{array}$$
$$\begin{bmatrix}1&1&1&6\\1&2&4&15\\1&3&9&28\end{bmatrix} \sim \begin{bmatrix}1&0&0&1\\0&1&0&3\\0&0&1&2\end{bmatrix}$$

$P(t) = 1 + 3t^2 + 2t^2$

1.3.11

$\mathbf{a}_1 = \begin{bmatrix}1\\-2\\0\end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix}5\\-6\\8\end{bmatrix}, \mathbf{b} = \begin{bmatrix}2\\-1\\6\end{bmatrix}$

$$\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix} \underset{\sim}{\text{RREF}} \begin{bmatrix}1&0&5&2\\0&1&4&3\\0&0&0&0\end{bmatrix}$$

1.3.13

$A = \begin{bmatrix}1&-4&2\\0&3&5\\-2&8&-4\end{bmatrix}, \mathbf{b} = \begin{bmatrix}3\\-7\\-3\end{bmatrix}$

$$\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix} \underset{\sim}{\text{RREF}} \begin{bmatrix}1&0&26/3&0\\0&1&5/3&0\\0&0&0&1\end{bmatrix}$$

1.3.28

A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?

$\to 27.6x_1 + 30.2x_2$

b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.

$$x_1 \overset{\bar{A}}{\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix}} + x_2 \overset{\bar{B}}{\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}} = \bar{y}$$

c. [M] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

$$x_1 \overset{\bar{A}}{\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix}} + x_2 \overset{\bar{B}}{\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}} = \begin{bmatrix} 162 \\ 23\,610 \\ 1623 \end{bmatrix}$$

$$\begin{bmatrix} \bar{A} & \bar{B} & \bar{y} \end{bmatrix} \underset{\sim}{\text{RREF}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & 3.9 \\ 0 & 1 & 1.8 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

1.4.6

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

$$-3\begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

1.4.8

$$z_1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2\begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3\begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

1.4.11

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix} \underset{\sim}{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$$

1.4.17+18

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

17. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

18. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B above? Do the columns of B span $\mathbb{R}^3$?

$\mathbb{R}^3$ $\quad$ $\mathbb{R}^4$

1.5.7

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 & -8 \\ 0 & 1 & -4 & 5 \end{bmatrix} \quad \bar{x} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

1.5.15

Describe and compare the solution sets of $x_1 + 5x_2 - 3x_3 = 0$ and $x_1 + 5x_2 - 3x_3 = -2$.

$$x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

1.5.23 + 24

Check solution