6. Differential Equation

Disclaimer: This topic requires pre-existing Calculus skills, more specifically knowledge about differential equations. I recap a bit here.

Definition:

An equation that relates one or more functions and their derivatives is called a **Differential Equation**. The "unknowns" are thus functions:

$$y'(t) = a \cdot y(t) \quad , \ a \text{ is a constant.}$$

This means that solving the equation boils down to finding $y(t)$:

1. $y(t) = e^{at}$, so $y'(t) = (e^{at})' = (at)' \cdot e^{at} = a \cdot e^{at}$
2. $y(t) = c \cdot e^{at}$, so $y'(t) = (c\,e^{at})' = c \cdot (at)' \cdot e^{at} = c \cdot a\, e^{at}$

In summary:

The general solution to

$$y'(t) = a \cdot y(t)$$

is given by

$$y(t) = c \cdot e^{at} \, ,$$

Solutions are uniquely determined by an initial value $y(t_0) = \bar{r}_0$:

$$y(t_0) = \bar{r}_0$$

$$\overset{!}{\Rightarrow} \ c \cdot e^{at} = \bar{r}_0$$

$$c = r_0 \cdot e^{-at_0}$$

Ex Given $y'(t) = 2y(t)$ and $y(0) = -3$ find $y(t)$

$$c = -3 \cdot e^{-2 \cdot 0} = -3 \text{ , so}$$

$$y(t) = -3 e^{-at}$$

Systems of first order differential equations:

$$y_1'(t) = a_{11} y_1(t) + a_{12} y_2(t) + \dots + a_{1n} y_n(t)$$

$$y_2'(t) = a_{21} y_1(t) + a_{22} y_2(t) + \dots + a_{2n} y_n(t)$$

$$\vdots$$

$$y_n'(t) = a_{n1} y_1(t) + a_{n2} y_2(t) + \dots + a_{nn} y_n(t)$$

In vector notation:

$$\bar{b} = A\bar{x}$$

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n'(t) \end{bmatrix}$$

Or $y'(t) = A\, y(t)$, $A = [a_{ij}] \in \mathbb{R}^{n \times n}$

Decoupling:

When A is diagonal, $A = \mathrm{diag}(d_1, d_2, d_3 \dots d_n)$

$$y_1'(t) = d_1 \cdot y_1(t) + 0 + 0$$

$$y_2'(t) = 0 \quad d_2 y_2(t)$$

$$\vdots \quad d_3 y_3(t)$$

$$y_n'(t) = - - - - - - - d_n y_n(t)$$

each derivative only depends on it self!

Ex

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$y_1(t) = 3\, y_1(t)$$
$$y_2(t) = -5\, y_2(t)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot e^{3t} \\ c_2 \cdot e^{-5t} \end{bmatrix} = c_1 \overset{\bar{v}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} e^{3t} + c_2 \overset{\bar{v}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} e^{-5t}$$

Ex

Find a solution to $y'(t) = A\,y(t)$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and $y(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot e^{2t} \\ c_2 \cdot e^{-3t} \end{bmatrix} = \begin{bmatrix} -3 \cdot e^{2t} \\ 2 \cdot e^{-3t} \end{bmatrix}$$

So we may just hope to always get diagonal matrices? Well No!

$$A\bar{v} = \lambda \bar{v}$$

This will give us:

$$y(t) = c \cdot \bar{v} \cdot e^{\lambda t}$$

as a solution to

$$y'(t) = A\,y(t)$$

Proof:

$$y'(t) = \frac{d(c\,\bar{v} \cdot e^{\lambda t})}{dt} = \underbrace{c \cdot \bar{v}}_{\text{Constant}} \frac{d(e^{\lambda t})}{dt} = c \cdot \overbrace{\lambda \bar{v}}^{A\bar{v}} \cdot e^{\lambda t}$$

$$= c\, A\bar{v} \cdot e^{\lambda t} = A\, c \cdot \bar{v}\, e^{\lambda t} = A\,y(t)$$

Theorem:

If A is diagonalisable with $\bar{v}_1, \bar{v}_2 \dots \bar{v}_n$ of eigenvectors and eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$, the solution to $y'(t) = Ay(t)$ is

$$y(t) = c_1 \cdot \bar{v}_1 \cdot e^{\lambda_1 t} + c_2 \cdot \bar{v}_2 \cdot e^{\lambda_2 t} + \dots c_n \cdot \bar{v}_n \cdot e^{\lambda_n t}$$

Given an initial vector $y(t_0) = \bar{r} \in \mathbb{R}^n$, a unique solution can be found using

$$c_1 \bar{v}_1 \cdot e^{\lambda_1 t_0} + c_2 \bar{v}_2 \cdot e^{\lambda_2 t_0} + \dots + c_n \bar{v}_n \cdot e^{\lambda_n t_0} = \bar{r}$$

And it will have a unique solution for $c_1 \dots c_n$ since $\{\bar{v}_1 \dots \bar{v}_n\}$ is a basis.

Ex:

$$\left.\begin{aligned} y_1'(t) &= 0 y_1(t) + 0 y_2(t) \\ y_2'(t) &= 4 y_1(t) + 0 y_2(t) \end{aligned}\right\} \rightarrow y'(t) = A y(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

so we first need to find λ_1 and λ_2

$$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \Longrightarrow \lambda = \begin{cases} 3 \\ -1 \end{cases}$$

For $\lambda = 3$:

$$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = -1$:

$$\begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

General solution:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \\ c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot e^{3t}$$

$$y_1(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{3t}$$

$$y_2(t) = -2c_1 e^{-t} + 2c_2 e^{3t}$$

Assume $g(1)=\begin{bmatrix}3\\2\end{bmatrix}$, find unique solution

$$3 = c_1 \cdot e^{-1} + c_2 \cdot e^{3} = e^{-1} c_1 + e^{3} c_2 = 3$$

$$2 = -2c_1 \cdot e^{-1} + 2c_2 \cdot e^{3} = -2e^{-1} c_1 + 2e^{3} c_2 = 2$$

$$\begin{bmatrix} e^{-1} & e^{3} & 3 \\ -2e^{-1} & 2e^{3} & 2 \end{bmatrix} \overset{r_2 \to -\frac{1}{2}r_2}{\sim} \begin{bmatrix} e^{-1} & e^{3} & 3 \\ e^{-1} & \underline{-e^{3}} & -1 \end{bmatrix} \quad r_2 \to r_2 - r_1$$

$$\begin{bmatrix} e^{-1} & e^{3} & 3 \\ 0 & -2e^{3} & -4 \end{bmatrix} \underset{r_1 \to r_1 - r_2}{\overset{r_2 \to -\frac{1}{2}r_2}{\sim}} \begin{bmatrix} e^{-1} & 0 & 1 \\ 0 & e^{3} & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & e^{1} \\ 0 & 1 & 2 \cdot e^{-3} \end{bmatrix}$$

So

$$g_1(t) = e \cdot e^{-t} + 2 \cdot e^{3} \cdot e^{3t} = e^{1-t} + 2e^{3(1+t)}$$

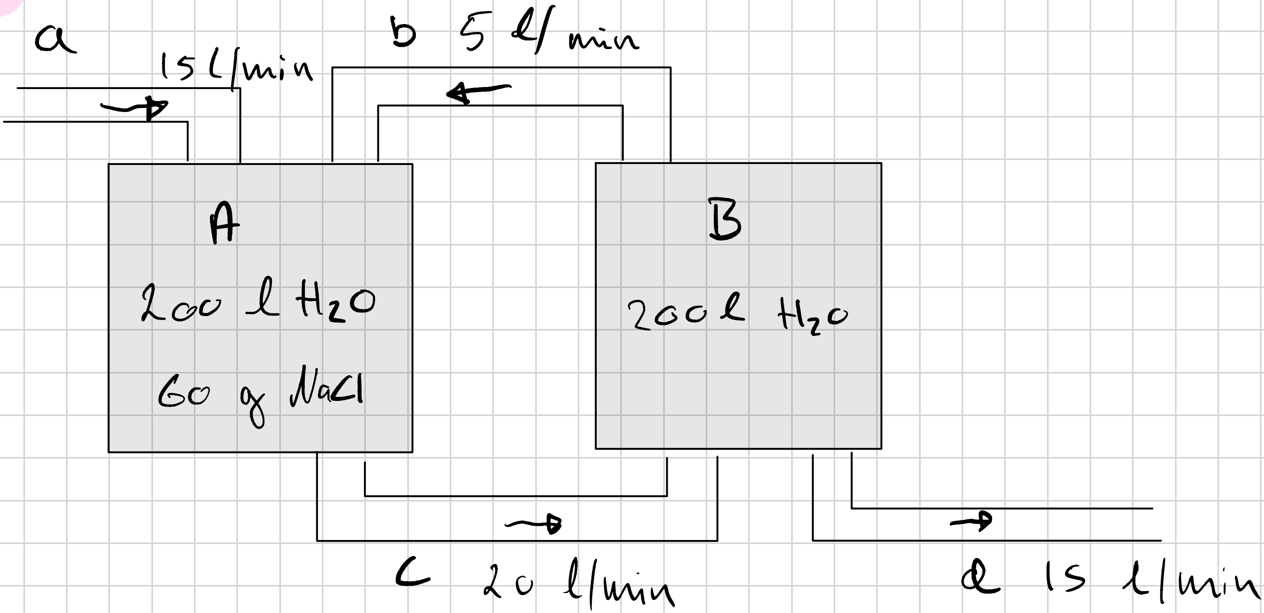
$$g_1(t) = -2e^{1} \cdot e^{-t} + 2 \cdot 2e^{-3} \cdot e^{3t} = -2e^{1-t} + 4e^{3(t-1)}$$

Usually the solution is stated as (for 2d-problem):

$$\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot e^{\lambda_1 t} + c_2 \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{\lambda_2 t}$$

$$\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = e \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + 2 \cdot e^{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

Ex



Let $y_1(t)$ be amount of salt in A at time t
Let $y_2(t)$ be amount of salt in B at time t

$$y_1'(t) = -\frac{20}{200} y_1(t) + \frac{5}{200} y_2(t) = -\frac{1}{10} y_1(t) + \frac{1}{40} y_2(t)$$
$$y_2'(t) = \frac{20}{200} y_1(t) - \frac{5+15}{200} y_2(t) = \frac{1}{10} y_1(t) - \frac{1}{10} y_2(t)$$

$$y'(t) = A y(t), \quad A = \begin{bmatrix} -\frac{1}{10} & \frac{1}{40} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}, \quad y(0) = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

Charpoly A:

$$\lambda^2 - \left(-\frac{1}{10} - \frac{1}{10}\right)\lambda + \left(\frac{1}{100} - \frac{1}{400}\right) = \lambda^2 + \frac{1}{5}\lambda + \frac{3}{400}$$

$$\lambda = \frac{-\frac{1}{5} \pm \sqrt{\left(\frac{1}{5}\right)^2 - 4 \cdot 1 \cdot \frac{3}{400}}}{2 \cdot 1} = \begin{cases} -\frac{1}{20} \\ -\frac{3}{20} \end{cases}$$

For $\lambda = -\frac{1}{20}$:

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{20} & 1/40 \\ 1/10 & -\frac{1}{10} + \frac{1}{20} \end{bmatrix} = \begin{bmatrix} -1/20 & 1/40 \\ 1/10 & -1/20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = -\frac{3}{20}$

$$\begin{bmatrix} -\frac{1}{10} + \frac{3}{20} & 1/40 \\ 1/10 & -\frac{1}{10} + \frac{3}{20} \end{bmatrix} = \begin{bmatrix} 1/20 & 1/40 \\ 1/10 & 1/20 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

General solution:

$$\bar{y}(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-\frac{t}{20}} + C_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3t}{20}}$$

for $t = 0$: $y(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 60 \\ 2 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 60 \\ 0 & -4 & -120 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 30 \\ 0 & 1 & 30 \end{bmatrix}$$

$c_1 = c_2 = 30$

Unique solution:

$$\bar{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 30\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-\frac{t}{20}} + 30\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3t}{20}}$$

You will often see the notation $\bar{x}(t)$ instead of $\bar{y}(t)$ to emphasise that $\bar{x}(t)$ is a vector function.

e.g. $\bar{x}(t) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t/2} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$

Trajectory

$$\mathbf{x}(t) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3e^{-.5t} + 2e^{-2t} \\ 6e^{-.5t} - 2e^{-2t} \end{bmatrix}$$

Figure 2 shows the graph, or *trajectory*, of $\mathbf{x}(t)$, for $t \geq 0$, along with trajectories for some other initial points. The trajectories of the two eigenfunctions $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in the eigenspaces of A.

The functions $\mathbf{x}_1$ and $\mathbf{x}_2$ both decay to zero as $t \to \infty$, but the values of $\mathbf{x}_2$ decay faster because its exponent is more negative. The entries in the corresponding eigenvector $\mathbf{v}_2$ show that the voltages across the capacitors will decay to zero as rapidly as possible if the initial voltages are equal in magnitude but opposite in sign. ■

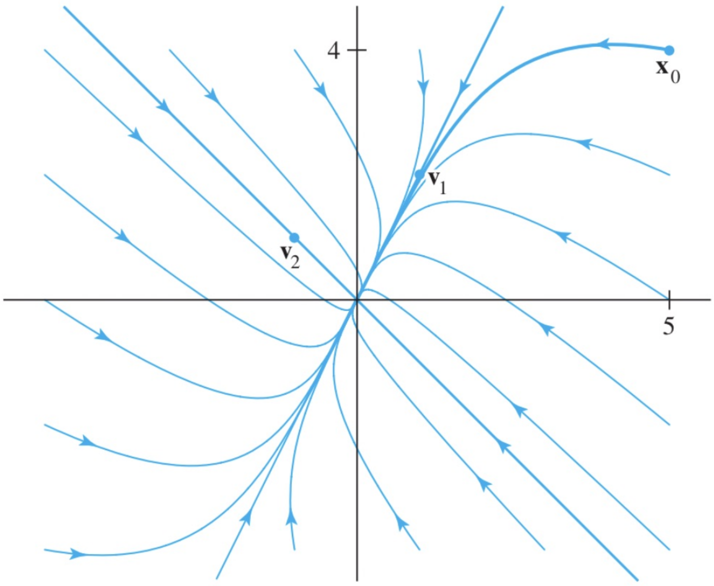


FIGURE 2 The origin as an attractor.

Negative eig. value.
↳ attractor or sink

Greatest attraction is along function with greatest eig. value (numerically)

Here x_2 since $\lambda_2 = -2 > -1/2 = \lambda_1$

If the eig. values of positive, the origin would be a repeller (or source)
↳ greatest repulsion a long line of greatest eig. value.

$$\mathbf{x}(t) = \frac{-3}{70}\begin{bmatrix} -5 \\ 2 \end{bmatrix} e^{6t} + \frac{188}{70}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Trajectories of $\mathbf{x}$ and other solutions are shown in Fig. 3. ■

In Fig. 3, the origin is called a **saddle point** of the dynamical system because some trajectories approach the origin at first and then change direction and move away from the origin. A saddle point arises whenever the matrix A has both positive and negative eigenvalues. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and $\mathbf{0}$, corresponding to the positive eigenvalue. The direction of greatest attraction is the line through $\mathbf{v}_2$ and $\mathbf{0}$, corresponding to the negative eigenvalue.

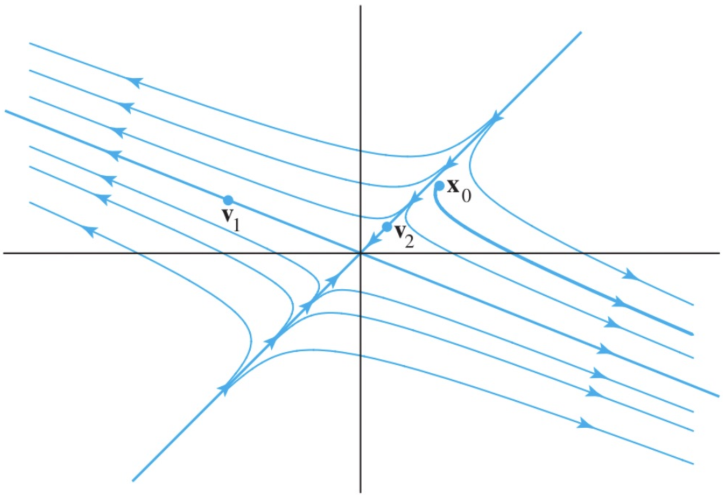


FIGURE 3 The origin as a saddle point.