

Introduction

Book: (Mandatory for exercises)

- Important to retrieve right copy.
- Lecture Notes vs. Class Notes

Prerequisites: Some recap today

Differential Equations

Exam: 4 hour ; two parts

Documentation must be uploaded

Python part must be .ipynb format

Tools : Python 3

Jupyter Notebook

↳ VS code

↳ Jupyter lab

↳ Data Spell (jetbrains)

↳ Google Colab

Itslearning not used. Go to

rbrooksdk.github.io/ACI1_25

Workflow : You will receive multiple flows with assignments. Code is always

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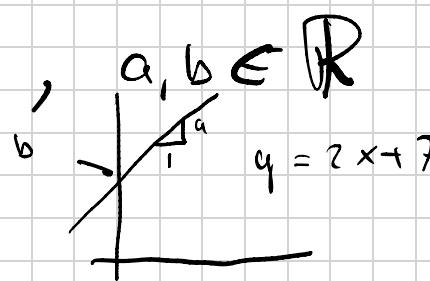
1.1. Systems of linear equations

Linear equations:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

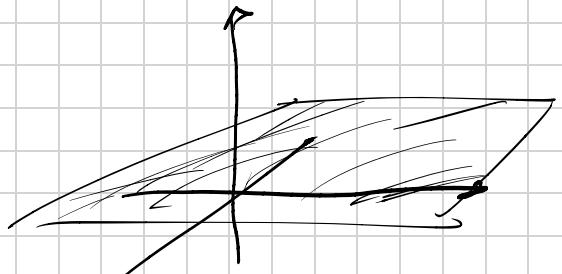
Ex:

$$y = ax + b \rightarrow y = 2x + 7$$



Ex: (Plane equation)

$$ax + by + cz = d$$



A system of Linear Equations

A collection of one or more linear equations involving the same variables:

Ex:

$$2x_1 + 3x_2 + x_3 = 3$$

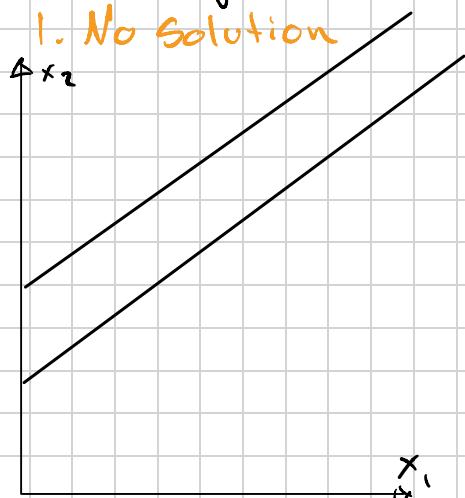
$$7x_2 - 4x_3 = 10$$

$$x_3 = 1$$

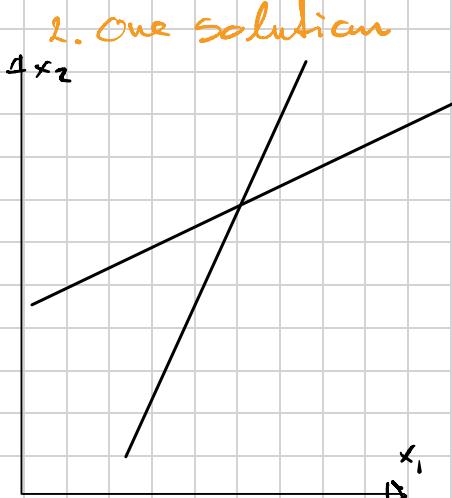
Solution set:

A solution of a linear system is a list of numbers $s_1, s_2, s_3 \dots$ that **satisfies** the system, i.e. makes the system "true".

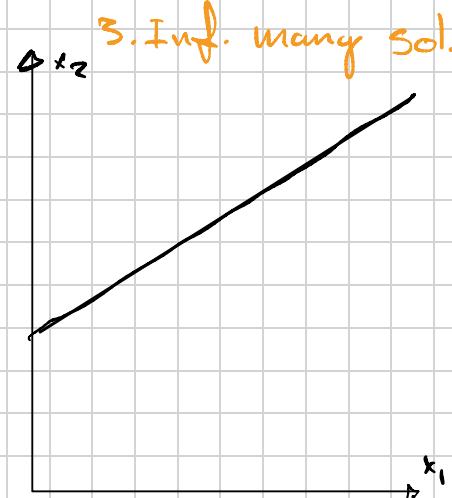
1. No Solution



2. One Solution



3. Inf. many sol.



1. No solution \rightarrow Inconsistent

2. Exactly one solution (Unique) }
3. Infinitely many solutions }

Existence question: Does a solution exist?
↳ If yes, is it unique.

The Matrix:

Consider the system:

$$2x_1 + 3x_2 + x_3 = 3$$

$$7x_2 - 4x_3 = 10$$

$$x_3 = 1$$

We can "code" this system into two types of matrices:

Coefficient Matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 7 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 & b \\ 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

3 rows
4 columns

Solving a System:

We can solve a system by working on the augmented matrix. The objective is to get all ones on the diagonal at the coefficient and then we will have the solution on the augmented part (the last column)

Ex:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & b \\ \left[\begin{array}{cccc} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{array} \right] & r_1 \rightarrow r_1 - r_3 \end{array}$$

$$\left[\begin{array}{cccc} 2 & 3 & 0 & 2 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$r_2 \rightarrow r_2 + 4r_3 \quad \begin{array}{cccc|c} 2 & 3 & 0 & 2 \\ 0 & 7 & 0 & 14 \\ 0 & 0 & 1 & 1 \end{array} \quad r_2 \rightarrow \frac{1}{2}r_2$$

$$\left[\begin{array}{cccc} 2 & 3 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$r_1 \rightarrow r_1 - 3r_2 \quad \begin{array}{cccc|c} 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \quad r_1 \rightarrow \frac{1}{2}r_1$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Before we went from equations to matrix.
Let us now go from matrix to equations:

$$\begin{cases} 1x_1 + 0x_2 + 0x_3 = -2 \\ 0x_1 + 1x_2 + 0x_3 = 2 \\ 0x_1 + 0x_2 + 1x_3 = 1 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} x_1 = -2 \\ x_2 = 2 \\ x_3 = 1 \end{array}$$

$$\rightarrow \underline{(x_1, x_2, x_3) = (-2, 2, 1)}$$

Elementary Row Operations:

- 1) Replacement (one row by self + multiple of another)
- 2) Swap (swap two rows)
- 3) Scaling (multiply all entries in a row with a non-zero constant)

Consistency and Matrices:

1. If a system has no solution (i.e. is inconsistent), then the matrix will also have an inconsistency when reduced:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{aligned} x_1 &= 2 \\ x_2 &= 3 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

2. If a system has a unique solution, the reduced matrix will be "nice" like in our example, i.e. only ones on the diagonal on the coefficient part and real numbers in the last column.

3. If a system has infinitely many solutions, the reduced matrix will have a row of all zeros.

Ex: Determine if consistent

$$x_2 + 4x_3 = 2$$

$$x_1 - 3x_2 + 2x_3 = 6$$

$$x_1 - 2x_2 + 6x_3 = 9$$

Solution:

$$\left[\begin{array}{cccc} 0 & 1 & 4 & 2 \\ 1 & -3 & 2 & 6 \\ 1 & -2 & 6 & 9 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cccc} 1 & -3 & 2 & 6 \\ 0 & 1 & 4 & 2 \\ 1 & -2 & 6 & 9 \end{array} \right] \xrightarrow{r_3 \rightarrow r_3 - r_1}$$

$$\left[\begin{array}{cccc} 1 & -3 & 2 & 6 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 3 \end{array} \right] \xrightarrow{r_3 \rightarrow r_3 - r_2} \left[\begin{array}{cccc} 1 & -3 & 2 & 6 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

No solution!

Ex Give Solution

$$x_1 + 2x_2 + 3x_3 = 4$$

$$3x_1 + 6x_2 + 9x_3 = 12$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

Solution:

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \end{array} \right] \xrightarrow{r_2 \rightarrow r_2 - 3r_1} \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Infinitely Many Solutions

Row Reduction and Echelon Forms

Echelon Form:

- i) All nonzero rows are above all rows of zeros
- ii) Each leading non-zero entry is to the right of the above leading non-zero entry
- iii) All entries in a column below a leading non-zero entry are zero

$$\left[\begin{array}{cccc|c} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 7 & 4 & 9 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array} \right]$$

Reduced Echelon Form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

- iv) the leading non-zero entry in each row is 1
- v) Each leading 1 is the only non-zero entry in its column.

Each matrix is row equivalent to one and only one matrix in reduced echelon form

Pivots

A leading non-zero entry in echelon form is called a **pivot** and its column a **pivot column**:

$$\left[\begin{array}{ccccc} 0 & -3 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right] \sim \left[\begin{array}{ccccc} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Pivot columns: x_1, x_2 and x_4 : Basic variables

non-pivot columns: x_3 and x_5 free variables

Ex:

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \text{Basic : } x_1, x_2, x_4 \\ \text{free : } x_3 \text{ and } x_5 \end{array}$$

If a system is consistent:

- a) Unique \rightarrow no free variables
- b) at least one free variable

Exercise:

- ① i) Find reduced echelon
 ii) state solution

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right] \quad x_1 + 2x_2 + 3x_3 = 4$$

Solution:

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right] \quad r_2 \rightarrow r_2 - 5r_1 \quad \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 9 & 10 & 11 & 12 \end{array} \right] \quad r_3 \rightarrow r_3 - 9r_1$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array} \right] \quad r_3 \rightarrow r_3 - 2r_2 \quad \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad r_2 \rightarrow -\frac{1}{4}r_2$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad r_1 \rightarrow r_1 - 2r_2 \quad \left[\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 - x_3 = -2$$

$$x_2 + 2x_3 = 3$$

$$0 = 0$$

$$x_1 = x_3 - 2$$

$$x_2 = -2x_3 + 3$$

$$x_3 = x_3$$

Ex: Find h and k s.t. the system has i) no solution; ii) unique solution; iii) infinitely many.

$$x_1 - 3x_2 = 1$$

$$2x_1 + hx_2 = k$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 \\ 2 & h & k \end{array} \right] \xrightarrow{r_2 \rightarrow r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{array} \right]$$

i) $h+6 = 0 \quad \wedge \quad k-2 \neq 0$

$$h = -6 \quad \wedge \quad k \neq 2$$

(ii) $h+6 \neq 0 \quad \wedge \quad k \in \mathbb{R}$

(iii) $h+6 = 0 \quad \wedge \quad k-2 = 0$

$$h = -6 \quad \wedge \quad k = 2$$

1. 3. Vector Equations

A matrix with only one column is called a column vector.

$$\bar{v} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad \bar{v}, \bar{v} \in \mathbb{R}^2$$
$$\bar{v} \neq \bar{v}$$

Same rules apply for vectors as for numbers (see p. 27)

Linear Combinations:

Given a set of vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_p \in \mathbb{R}$, the vector \bar{y} given by:

$$\bar{y} = c_1 \cdot \bar{v}_1 + c_2 \cdot \bar{v}_2 + c_3 \cdot \bar{v}_3 + \dots + c_p \cdot \bar{v}_p$$

is called a linear combination of $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p$ with weights c_1, c_2, \dots, c_p .

Ex: $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\bar{v}_1 + \bar{v}_2 = \begin{bmatrix} ? \\ 1 \end{bmatrix}$$

$$\bar{v}_1 + \bar{v}_2 = \bar{y}$$

$$\bar{v}_1 + 2\bar{v}_2 = \begin{bmatrix} ? \\ 0 \end{bmatrix}$$

$$-2\bar{v}_1 - 3\bar{v}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$2\bar{v}_1 + 3\bar{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Often we want to know if another vector \bar{b} can be formed as a lin. comb. of some other vectors $\bar{a}_1, \bar{a}_2 \dots \bar{a}_n$.

A **vector equation**:

$$x_1 \cdot \bar{a}_1 + x_2 \cdot \bar{a}_2 + \dots + x_n \cdot \bar{a}_n = \bar{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3 \ \dots \ \bar{a}_n \ \bar{b}]$$

More specifically:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array} \right] \quad m \times n$$

Ex:

$$\bar{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \bar{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}, \bar{a}_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

Vector Equation:

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix}$$

Linear System of Equations

$$2x_1 + 0x_2 + 6x_3 = 10$$

$$-1x_1 + 8x_2 + 5x_3 = 3$$

$$1x_1 - 2x_2 + 1x_3 = 7$$

Solution:

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & -2 & 1 & 7 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 2 & 0 & 6 & 10 \end{bmatrix}$$

$$r_2 \rightarrow r_2 + r_1$$

$$r_3 \rightarrow r_3 - 2r_1$$

$$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$r_3 \rightarrow r_3 - \frac{4}{6}r_2$$

$$\left\{ \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 0 & 0 & -4 - \frac{40}{6} \end{bmatrix} \right\}$$

Inconsistent!
b is not a lin comb
of $\bar{a}_1, \bar{a}_2, \bar{a}_3$

Span { \bar{v}_3 :

If $\bar{v}_1, \dots, \bar{v}_p$ are in \mathbb{R}^n , then the set of all lin. comb. of $\bar{v}_1, \dots, \bar{v}_p$ is denoted $\text{span}\{\bar{v}_1, \dots, \bar{v}_p\}$ and is called the subset of \mathbb{R}^n spanned by $\bar{v}_1, \dots, \bar{v}_p$.

Ex:

$$\bar{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \bar{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, \bar{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

$\text{span}\{\bar{a}_1, \bar{a}_2\}$ is a plane through the origin in \mathbb{R}^3 . Is \bar{b} in that plane?

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

1.4. Matrix Equation

A column vector \bar{x} and a matrix A can be combined as the product of a matrix and a vector:

$$A\bar{x} = [\bar{a}_1 \ \bar{a}_2 \ \dots \ \bar{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{a}_1 + x_2\bar{a}_2 + \dots + x_n\bar{a}_n$$

If A is an $m \times n$ matrix and if $\bar{b} \in \mathbb{R}^m$ the matrix equation $A\bar{x} = \bar{b}$ has the same solution as the corresponding vector equation.

Ex:

linear eq

$$x_1 + x_2 + x_3 = 6$$

$$2x_1 + x_2 + 3x_3 = 11$$

$$x_1 + 2x_2 + x_3 = 8$$

Vector eq

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

Matrix Eq.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 11 \\ 1 & 2 & 1 & 8 \end{array} \right] \xrightarrow{\substack{r_2 \rightarrow r_2 - 2r_1 \\ r_3 \rightarrow r_3 - r_1}} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{array} \right]$$

$$\xrightarrow{r_3 \rightarrow r_3 - r_2} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{r_2 \rightarrow r_2 - r_3} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{r_1 \rightarrow r_1 - r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{r_2 \rightarrow -1 \cdot r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{r_1 \rightarrow r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\text{So } A \bar{x} = \bar{b}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

Please Note:

If A is a $m \times n$ matrix, then all of the following are equivalent

- For each \bar{b} in \mathbb{R}^m , $A \bar{x} = \bar{b}$ has a solution
- Each \bar{b} in \mathbb{R}^m is a lin. comb. of the columns of A .
- The columns of A span \mathbb{R}^m
- A has a pivot in every row.

1.5. Solution sets of linear Systems

A linear system is said to be **homogeneous** if $A\bar{x} = 0$

We call $\bar{x} = \bar{0}$ the **trivial solution**
 ↳ looking for **non-trivial**.

$A\bar{x} = \bar{0}$ has a non-trivial solution iff. the equation has at least one free variable.

Ex:

$$\begin{array}{rcl} 2x_1 + 3x_2 - 3x_3 & = 0 \\ 1x_1 - 3x_2 + 2x_3 & = 0 \\ -2x_1 + 5x_2 - 7x_3 & = 0 \end{array} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 1 & -1 & 1 & 0 \\ -2 & 5 & -7 & 0 \end{array} \right] \sim$$

$$\left[\begin{array}{cccc} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

We can write this as:

$$x_1 - \frac{2}{3}x_3 = 0 \Rightarrow x_1 = \frac{2}{3}x_3$$

$$x_2 - \frac{5}{3}x_3 = 0 \Rightarrow x_2 = \frac{5}{3}x_3$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_3 \\ \frac{5}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Parametric
Vector
Form P.V.F

$$Ex: \quad x_1 - 2x_2 - 5x_3 = 0$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Ex (non-homogeneous)

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 2 \\ x_1 - x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 - 7x_3 &= -2 \end{aligned} \rightarrow \begin{bmatrix} 2 & 1 & -3 & 2 \\ 1 & -1 & 1 & 1 \\ -2 & 5 & -7 & -2 \end{bmatrix} \sim$$

$$\begin{aligned} x_1 - \frac{2}{3}x_3 &= 1 & x_1 &= 1 + \frac{2}{3}x_3 \\ x_2 - \frac{5}{3}x_3 &= 0 & x_2 &= \frac{5}{3}x_3 \end{aligned} \quad \begin{bmatrix} 1 & 0 & -\frac{2}{3}x_3 & 1 \\ 0 & 1 & -\frac{5}{3}x_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{2}{3}x_3 \\ \frac{5}{3}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \quad PVF$$

$$Ex: \quad x_1 - 2x_2 - 5x_3 = 3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 + 2x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} ? \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \quad DVF$$

Exercise:

(a) $x_1 + 4x_2 - 5x_3 = 0$ $2x_1 - x_2 + 8x_3 = 9$ $\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array} \right]$

$$\bar{x} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

(b)

$$\left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

RREF of augmented

$$\bar{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

1.7. Linear Dependence

If the vector equation

$$x_1 \bar{v}_1 + x_2 \bar{v}_2 + \dots + x_p \bar{v}_p = 0$$

has only the trivial solution, then the vectors are lin. independent.

Also two vectors are independent if one is not a multiple of the other.

In general a vector is independent of a set of vectors if it is NOT a lin. comb. of the set.

ex: $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{array} \right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

\bar{v}_3 is a lin. Comb. of \bar{v}_1 and \bar{v}_2 :

$$-2 \bar{v}_1 + \bar{v}_2 = \bar{v}_3$$

Important Theorems:

- a) If a set $S = \{\bar{v}_1, \dots, \bar{v}_p\}$ in \mathbb{R}^n contains the zero vector, then the set is lin. dep.
- b) A set of two vectors in \mathbb{R}^n is lin. independent iff. neither is a multiple of the other
- c) Any set $\{\bar{v}_1, \dots, \bar{v}_p\}$ in \mathbb{R}^n is lin. dep. if $p > n$, i.e. more columns than rows.
(augmented)
- d) $S = \{\bar{v}_1, \dots, \bar{v}_p\}$ is lin. dep. iff. at least one vector in S is a linear comb. of the others, assuming $\bar{v}_1 \neq \bar{0}$. So \bar{v}_j ($1 < j \leq p$) is a linear comb. of the preceding vectors $\bar{v}_1, \dots, \bar{v}_{j-1}$

Exercise:

$$\bar{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \bar{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \bar{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \bar{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$$

① Is any pair lin. dep.? (No!)

② Is $\{\bar{u}, \bar{v}, \bar{w}, \bar{z}\}$ lin. dep.?

$$\begin{bmatrix} 3 & -6 & 0 & 3 \\ 2 & 1 & -5 & 7 \\ -4 & 7 & 2 & -5 \end{bmatrix} \quad \begin{array}{l} \begin{array}{rcl} 9 & -6 & 3 \\ -6 & 1 & -5 \\ \hline 3 & 7 & -5 \end{array} \\ 3\bar{u} + \bar{v} = \bar{z} \end{array}$$