4. Recap and Exercises

Linear Transformation:

Another way of "solving" $A\bar{x}=\bar{b}$

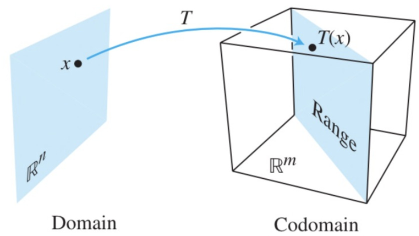
Going from $\bar{x}$ to $A\bar{x}$ is a function (or mapping or transformation) T from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$T: \underbrace{\mathbb{R}^n}_{\text{Domain}} \rightarrow \underbrace{\mathbb{R}^m}_{\text{Codomain}}$$

$$\begin{array}{cc} & \begin{array}{cccc} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \end{array} \\ m & \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \\ & n \end{array} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}$$

The vector $T(\bar{x})$ in $\mathbb{R}^n$ is the image of $\bar{x}$

The set of all images $T(\bar{x})$ is called the range of T.



EX

EXAMPLE 1 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a) find $f(x)$ for $x=2$

b) $2x+7=14$

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T. Insert $\bar{u}$ in $\bar{x}$'s place

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$. ① $A\bar{x}=b \Rightarrow \bar{x}=A^{-1}b$ ② $[A\ b]$ ~~ref~~

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$? → unique solution?

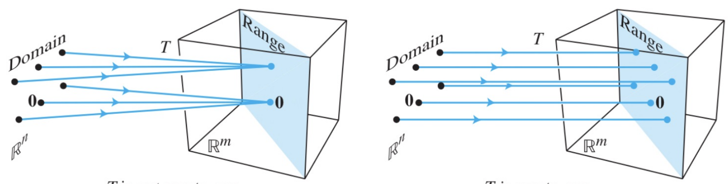
d. Determine if $\mathbf{c}$ is in the range of the transformation T. Is it consistent.

Consistent

$[A\ \bar{c}]$

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.



T is *not* one-to-one   *T* is one-to-one

FIGURE 4 Is every **b** the image of at most one vector?

EXAMPLE 4 Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \underset{r_2 \to \frac{1}{2} r_2,\ r_3 \to \frac{1}{5} r_3}{\overset{r_1 \to r_1 + 2r_2}{\sim}} \begin{bmatrix} 1 & 0 & 6 & 7 \\ 0 & 1 & -1/2 & 3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad x_3 \begin{bmatrix} -6 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? (yes!) Is T a one-to-one mapping? (No!) → Is it unique

$T : \mathbb{R}^n \to \mathbb{R}^m$

$: \mathbb{R}^4 \to \mathbb{R}^3$

→ Is it consistent

In Exercises 3–6, with T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

6. $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$ → $T : \mathbb{R}^3 \to \mathbb{R}^4$

$$\bar{x} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Determinants

Special case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \det A = ad - bc$$

$$\hookrightarrow \sim \begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix}$$

General Formula:

On Row i:

$$\det A = \sum_{j=1}^{n} a_{ij} \cdot \underbrace{(-1)^{i+j} \det A_{ij}}_{\text{Cofactor}}$$

On Col j:

$$\det A = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j} \det A_{ij}$$

When doing cofactor expansion, always choose col or row with the most zeros

The determinant of a triangular Matrix is the product of the main diagonal.

↳ Echelon Form

It is always possible to reach E.F. by swap and replacement only:

$\det A = (-1)^r \cdot \det U$, $U = E.F$, $r = \text{no. of swaps}$.

Useful Theorems:

i) A square matrix is invertible iff $\det A \neq 0$

ii) $\det A^T = \det A$

iii) $\det AB = \det A \cdot \det B$

3.1.1+2

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

$(-1)^{i+j}$

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$ $= -5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix}$

$= -20 + 6 + 16 = 2$

3.1.9

$3\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -39 + 40 = 1$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

9. $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$ $2 \cdot \det A = 2 \cdot 5\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10$

3.1.43

43. **[M]** Is it true that $\det(A + B) = \det A + \det B$? To find out, generate random 5×5 matrices A and B, and compute $\det(A + B) - \det A - \det B$. (Refer to Exercise 37 in Sec-

3.2.1+2

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$

3.2.15–20

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 35$

16. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 21$

17. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -7$

18. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = 7$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} =$ $r_2 \to r_2 - r_1$, $r_2 \to \frac{1}{2} r_2$

$= 14$

20. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

3.2.25

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

~~**24.** $\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$~~

25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

$\det A = -1 \rightarrow$ lin. independent.