5. Recap and Exercises

Key concepts:

- Vector spaces
- subspaces
- Col space
- Nullspace*

$\Big\}$ Bases for col A and Nul A

- Row space
- Dim Nul A aka Nullity
- Rank = Dim Col A

$\Big\}$ concept of dimension

* Null space also known as Kernel → used for Lin. trans.

1. Find Basis for Nul A:

Find P.V.F. These vectors make up a basis for Nul A.

2. Find Basis for Col A:

Find E.F. or RREF and the pivot columns will make up a basis for Col A.

3. Check if $\bar{p}$ is in Nul A:

Test $A\bar{p} == \bar{0}$

4. Check if $\bar{y}$ is in Col A:

$[A\ \bar{y}] \overset{RREF}{\sim}$ and check for consistency.

5. Find dim Nul A:

Find E.F. Count # of free variables.

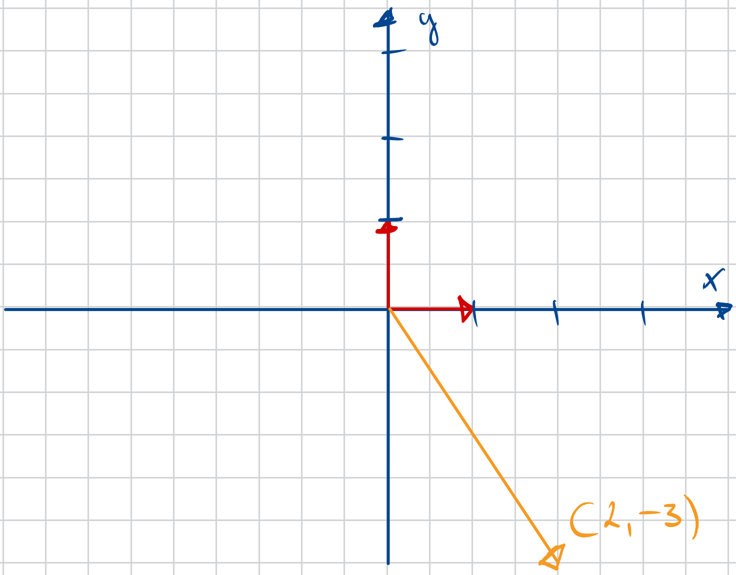
6. Find rank

Find E.F. Count # of pivots

rank + nullity = n

Coordinate systems

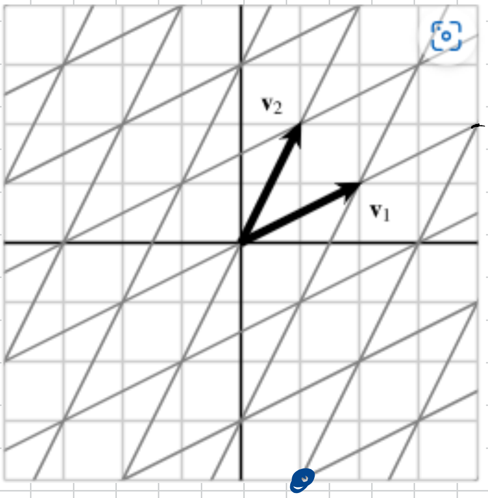
Standard coordinate system



$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\{\bar{e}_1, \bar{e}_2\}$ is a basis for $\mathbb{R}^2$

Alternative coordinate system:



In standard coordinates:

$\bar{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\mathcal{B} = \{\bar{v}_1 , \bar{v}_2\}$

Where is $(2,-3)$ according to $\mathcal{B}$?

We see: $\bar{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} = 2\bar{v}_1 - 3 \cdot \bar{v}_2 = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

The coordinates of $\bar{x}$ in the new coordinate system are the weights that we use to create $\bar{x}$ as a lin. comb of $\bar{v}_1$ and $\bar{v}_2$.

Theorem: The Unique Representation Theorem:

Let $\mathcal{B} = \{\bar{v}_1, \dots \bar{v}_n\}$ be a basis for V. Then for each $\bar{x} \in V$, there is a unique set of scalars $c_1, \dots c_n$ s.t.

$$\bar{x} = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_n \bar{v}_n$$

Definition:

We say that $c_1, \dots c_n$ are the $\mathcal{B}$ - coordinates of $\bar{x}$:

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We call the map

$$\bar{x} \rightarrow [\bar{x}]_{\mathcal{B}}$$

the coordinate mapping of $\mathcal{B}$.

Ex:

$$\{\underset{\substack{\text{O.G.}\\ \downarrow}}{}\begin{bmatrix} 1 \\ -4 \end{bmatrix}\}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Ex:

Let $\bar{v}_1$ and $\bar{v}_2$ be as before and $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

What is $\bar{x}$?

$$\bar{x} = 5 \cdot \bar{v}_1 - 2 \bar{v}_2 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Ex Suppose $\bar{x} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$. What is $[\bar{x}]_B$?

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 = \bar{x}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -8 \\ 1 & 2 & 2 \end{bmatrix} \underset{\sim}{r_2 \to r_2 - \tfrac{1}{2} r_1} \begin{bmatrix} 2 & 1 & -8 \\ 0 & 3/2 & 6 \end{bmatrix} \xrightarrow{r_2 \to \frac{2}{3} r_2} \begin{bmatrix} 2 & 1 & -8 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{r_1 \to r_1 - r_2} \begin{bmatrix} 2 & 0 & -12 \\ 0 & 1 & 4 \end{bmatrix} \underset{\sim}{r_1 \to \tfrac{1}{2} r_1} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 4 \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} 8 \\ -2 \end{bmatrix}_B = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is a change-of-coordinates matrix.

In general for $B = \{\bar{v}_1, \dots \bar{v}_n\}$, the matrix

$$P_B = [\bar{v}_1 \ \ \bar{v}_2 \ \dots \ \bar{v}_n]$$

is the matrix that turns a vector written in terms of B into a vector written in terms of the standard basis $\mathcal{E}$ via

$$P_B^{-1} \bar{x} = [\bar{x}]_B$$

Change of Basis see sec. 4.7, e.g from one basis to another!!

$$[\bar{v}_1 \ \ \bar{v}_2 \ \vdots \ \bar{b}_1 \ \ \bar{b}_2] \sim$$

$$\left[\begin{array}{cc:cc} 1 & 0 & a & c \\ 0 & 1 & b & d \end{array}\right] \quad \begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} c \\ d \end{bmatrix}$$

4.2.2

2. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}.$$

$A \cdot \bar{w} = = \bar{0}$

4.2.3-6

In Exercises 3–6, find an explicit description of Nul A, by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$ RREF $\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ RREF

5. $A = \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ RREF

6. $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ RREF

4.2.17-20

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

$\mathbb{R}^2 \to \mathbb{R}^4$

17. $A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$

$\mathbb{R}^3 \to \mathbb{R}^4$

18. $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$

$\mathbb{R}$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$ $\mathbb{R}^5 \to \mathbb{R}^2$

20. $A = \begin{bmatrix} 1 & -3 & 2 & 0 & -5 \end{bmatrix}$ $\mathbb{R}^5 \to \mathbb{R}$

4.2.24

$\mathbb{R}^4 \to \mathbb{R}^4$

Let $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

Col A : $[A \ \bar{w}]$ RREF

T ↳ check if consistent

Nul A :

T $A \cdot \bar{w} \overset{?}{=} \bar{0}$

4.3.9-10

Find bases for the null spaces of the matrices given in Exercises 9 and 10. Refer to the remarks that follow Example 3 in Section 4.2.

9. $\begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 1 & -2 & 1 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -8 & 0 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

4.3.13

In Exercises 13 and 14, assume that A is row equivalent to B. Find bases for Nul A and Col A.

13. $A = \begin{array}{c} \begin{matrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 \end{matrix} \\ \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \end{array}$, $B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2/2 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Col A = $\{\bar{v}_1, \bar{v}_2\}$

Nul A = $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

4.3.17+18

In Exercises 15–18, find a basis for the space spanned by the given vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

17. [M] $\begin{bmatrix} 2 \\ 0 \\ -4 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 8 \\ -3 \\ 15 \end{bmatrix}, \begin{bmatrix} -8 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

18. [M] $\begin{bmatrix} -3 \\ 2 \\ 6 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -9 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -14 \\ 0 \\ 13 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

4.3.36

[M] Let $H = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $K = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -4 \end{bmatrix},$$

$H = \{\bar{u}_1, \bar{u}_2\}$

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 6 \\ -2 \end{bmatrix}$$

$K = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

Find bases for H, K, and $H + K$. (See Exercises 33 and 34 in Section 4.1.)

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$H+K = \{\bar{u}_1, \bar{u}_2, \bar{v}_2, \bar{v}_3\}$

4.5.6

For each subspace in Exercises 1–8, (a) find a basis for the subspace, and (b) state the dimension.

6. $\left\{ \begin{bmatrix} 3a - c \\ -b - 3c \\ -7a + 6b + 5c \\ -3a + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

$$\begin{bmatrix} 3 \\ 0 \\ -7 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 1 \end{bmatrix}$$

4.5.

In Exercises 11 and 12, find the dimension of the subspace spanned by the given vectors.

11. $$\begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 \\ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, & \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} \\ x & x & & x \end{array}$$

$\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}$

4.5.14 + 17

3 4

Determine the dimensions of Nul A and Col A for the matrices shown in Exercises 13–18.

14. $$A = \begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

17. $$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

4 6.2+4

In Exercises 1–4, assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

2. $A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. $A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 & \bar{v}_5 & \bar{v}_6 \end{array} \\ \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \end{array}$,

$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Rank A = 5
dim nul A = 1

4.6.

[**M**] Let $A = \begin{bmatrix} 7 & -9 & -4 & 5 & 3 & -3 & -7 \\ -4 & 6 & 7 & -2 & -6 & -5 & 5 \\ 5 & -7 & -6 & 5 & -6 & 2 & 8 \\ -3 & 5 & 8 & -1 & -7 & -4 & 8 \\ 6 & -8 & -5 & 4 & 4 & 9 & 3 \end{bmatrix}$.

a. Construct matrices C and N whose columns are bases for Col A and Nul A, respectively, and construct a matrix R whose *rows* form a basis for Row A.

b. Construct a matrix M whose columns form a basis for Nul A^T, form the matrices $S = [\, R^T \quad N \,]$ and $T = [\, C \quad M \,]$, and explain why S and T should be square. Verify that both S and T are invertible.