3.1. + 3.2. Determinants

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We know if $\det A = ad - bc = 0$, then a is not invertible. But why?

Assume $a \neq 0$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{r_2 \to a r_2}{\sim} \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \overset{r_2 \to r_2 - c \cdot r_1}{\sim} \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

If $ad - bc \neq 0$, we have pivots in all rows

↳ Invertible

Submatrix:

Let A_{ij} denote the submatrix obtained by deleting the i'th row and j'th column.

EX

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

Determinants:

For $n \geq 2$, the determinant of $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij} \cdot \det A_{ij}$ with the signs alternating:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

OR

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

Cofactor:

Given $A=[a_{ij}]$, we will call $(-1)^{i+j}\cdot \det A_{ij}$ the (i,j)-cofactor of A:

$$C_{ij} = (-1)^{i+j}\cdot \det A_{ij}$$

The determinant can thus be stated as:

$$\det A = \sum_{i=1}^{n} a_{ij}\cdot C_{ij}$$

OR

$$\det A = \sum_{j=1}^{n} a_{ij}\cdot C_{ij}$$

Ex: Find det A. Use $i=1$:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (-1)^{1+1} a_{11}\cdot \det A_{11} + (-1)^{1+2} a_{12}\cdot \det A_{12} + (-1)^{1+3} a_{13}\cdot \det A_{13}$$

$$= +1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= -3 + 12 - 9 = 0$$

Ex: Find det A. Use $j=3$

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 0\cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -2$$

Cofactor Expansion:

The method above is called cofactor expansion and can be done on any row or any column. We restate the results from above:

Cofactor Expansion along any row:

$$\det A = (-1)^{i+1} \cdot a_{i1} \cdot |A_{i1}| + (-1)^{i+2} \cdot a_{i2} \cdot |A_{i2}| + \dots + (-1)^{i+n} \cdot a_{in} \cdot A_{in}$$

$$= a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \dots + a_{in} \cdot C_{in}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot A_{ij} \quad = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$$

Cofactor Expansion along any column:

$$\det A = (-1)^{1+j} \cdot a_{1j} \cdot A_{1j} + (-1)^{2+j} \cdot a_{2j} \cdot A_{2j} + \dots + (-1)^{2+j} \cdot a_{nj} \cdot A_{nj}$$

$$= a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \dots + a_{nj} \cdot C_{nj}$$

$$= \sum_{=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot A_{ij} \quad = \sum_{=1}^{n} a_{ij} \cdot C_{ij}$$

Ex: Expand along row 3:

$$\begin{vmatrix} 2 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -4$$

Note: Always expand along row/column with most zeros. Use the algorithm recursively.

Ex

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} = 2 \cdot 5 \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$

$$= 10 \cdot (-18 + 20) = \underline{\underline{20}}$$

Exercise

ⓐ

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} = 3 \cdot 2 \cdot 1 \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = -12$$

ⓑ

$$A = \begin{bmatrix} 1 & -4 & 2 & -1 \\ 0 & 4 & 0 & 2 \\ 3 & -2 & 0 & 5 \\ 0 & 1 & 0 & 7 \end{bmatrix} = 2 \cdot (-3) \begin{vmatrix} 4 & 2 \\ 1 & 7 \end{vmatrix} = -156$$

Consider the following:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} = 1 \cdot 6 \cdot 1 \cdot 4 \cdot 6 = 144$$

Theorem:

If A is a triangular matrix, $\det A$ is the product of the entries on the main diagonal.

> Note: Echelon form is a triangular matrix
> If Echelon Form has a free variable, the determinant $= 0 \Rightarrow$ Not invertible, etc., etc..

Theorem:

Let A be a square matrix. If B is obtained from A → e.g. echelon Form

i) Replacement : $\det B = \det A$

ii) Swap : $\det B = -\det A$

iii) Scaling : $\det B = K \cdot \det A$

Ex

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 2 \\ -3 & -5 & 2 \end{bmatrix} \underset{\sim}{r_2 \to r_2 + r_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 4 \\ -3 & -5 & 2 \end{bmatrix} \underset{\sim}{r_3 \to r_3 + 3r_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 8 \end{bmatrix}$$

$$\underset{\sim}{r_3 \leftrightarrow r_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 8 \\ 0 & 0 & 4 \end{bmatrix}, \quad \det A = -(1 \cdot 1 \cdot 4) = -4$$

Exercise: Find det A using E.F.

$k = \frac{1}{3}$

$$A = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 0 & 3 & 1 & 7 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \underset{\sim}{r_1 \to \frac{1}{3} r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \underset{\sim}{r_3 \to r_3 - r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\underset{\sim}{r_4 \to r_4 + 2r_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\boxed{\begin{aligned} &, \det A = 3 \cdot (1 \cdot 3 \cdot (-1) \cdot (-5)) = 45 \\ &3 \cdot \det B = \cdot \det A \end{aligned}}$$

It is always possible to obtain Echelon form using only replacement and swap:

$$\det A = (-1)^r \cdot \det U, \quad U = \text{Echelon Form}, \; r = \text{no. swap.}$$

Ex

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 0 \end{bmatrix} \underset{\sim}{r_2 \to r_2 + r_3} \begin{bmatrix} 2 & -8 & 6 & 8 \\ 0 & -9 & 6 & 8 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \to r_1 - 2r_4} \begin{bmatrix} 0 & 0 & 6 & 8 \\ 0 & -9 & 6 & 8 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2 \to r_2 - r_1} \begin{bmatrix} 0 & 0 & 6 & 8 \\ 0 & -9 & 0 & 0 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 0 \end{bmatrix} \underset{\sim}{r_1 \leftrightarrow r_4} \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ -3 & 0 & 1 & -2 \\ 0 & 0 & 6 & 8 \end{bmatrix} \underset{\sim}{r_1 \to r_1 - \frac{4}{9} r_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ -3 & 0 & 1 & -2 \\ 0 & 0 & 6 & 8 \end{bmatrix} \overset{r_3 \to r_3 + 3r_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & 8 \end{bmatrix} \overset{r_4 \to r_4 - 6r_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

one swap: $\det A = -(1 \cdot (-9) \cdot 1 \cdot 20) = 180$

Theorem:

i) A square matrix is invertible iff $\det A \neq 0$

ii) $\det A^T = \det A$

iii) $\det AB = \det A \cdot \det B$

Ex

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$\det A = 9$

$\det B = 5$

$\det(AB) = 9 \cdot 5 = \underline{\underline{45}}$

$$\det(AB) = \begin{vmatrix} 25 & 20 \\ 14 & 13 \end{vmatrix} = 25 \cdot 13 - 20 \cdot 14 = \underline{\underline{45}}$$