8. Recap and Exercises

Dot Product:

$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + \dots a_n b_n$

↳ a.dot(b) sympy

dot(a,b) Numpy.

The length and distance:

$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$ v.norm()

$\|\bar{a} - \bar{b}\| = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + \dots (a_n-b_n)^2}$ (a-b).norm()

Orthogonality:

$\bar{u} \cdot \bar{v} = 0 \Leftrightarrow \bar{u} \perp \bar{v}$

$W^{\perp}$ = orthogonal Complement

↳ set of all vectors orthogonal to W

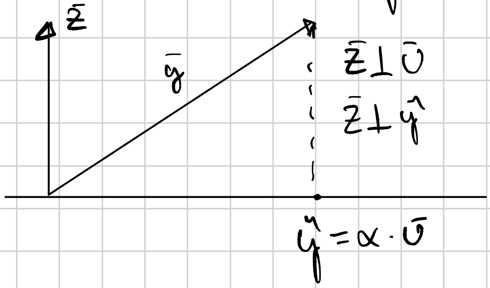
If S is an orthogonal set it is also a lin. ind. set

↳ Check: $U^T U = \begin{bmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & \ddots \end{bmatrix}$

U is called orthonormal if $U^T U = I$

↳ Normalize: $\frac{1}{\|\bar{v}\|} \cdot \bar{v}$ $\bar{v}$.normalized()

Orthogonal Projections



$\bar{z} = \bar{y} - \hat{y}$

$\bar{z} \perp \bar{u} \rightarrow (\bar{y} - \hat{y}) \cdot \bar{u} = 0$

$(\bar{y} - \alpha \bar{u}) \cdot \bar{u} = 0$

$\bar{y} \cdot \bar{u} - \alpha \bar{u} \cdot \bar{u} = 0$

$\bar{y} \bar{u} = \alpha \bar{u} \cdot \bar{u}$

$\hat{y} = \alpha \bar{u} = \frac{\bar{y} \bar{u}}{\bar{u} \bar{u}} \bar{u} \longleftarrow \alpha = \frac{\bar{y} \bar{u}}{\bar{u} \cdot \bar{u}}$

Projection on to subspaces!

Let $W = \{\bar{v}_1, \bar{v}_2 \dots \bar{v}_p\}$ be a subspace of $\mathbb{R}^n$

$$\text{Proj}_W \bar{y} = \frac{\bar{y} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \cdot \bar{v}_1 + \frac{\bar{y} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} \bar{v}_2 + \dots \frac{\bar{y} \cdot \bar{v}_p}{\bar{v}_p \cdot \bar{v}_p} \bar{v}_p$$

$$\boxed{= \sum_{i=1}^{p} \frac{\bar{y} \cdot \bar{v}_i}{\bar{v}_i \bar{v}_i} \bar{v}_i}$$

Exercise

Let

$$\bar{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \bar{u}_2 = \begin{bmatrix}1\\-1\\2\end{bmatrix}, \qquad and\ \bar{y} = \begin{bmatrix}1\\3\\4\end{bmatrix}.$$

And let $H = span\{\bar{u}_1, \bar{u}_2\}$, let $A = [\bar{u}_1 \quad \bar{u}_2]$, and let $H^{\perp}$ denote the subspace that is orthogonal to H

a) Show that $\{u_1, u_2\}$ is an orthogonal basis for H.
b) Find a basis for $H^{\perp}$.
c) Calculate the orthogonal projection of $\bar{y}$ onto H

a) $\bar{u}_1 \cdot \bar{u}_2 = 1\cdot 1 + 1\cdot(-1) + 0\cdot 2 = 0$

↳ yes they are orthogonal

2×3

b) $\bar{u}_1 \cdot \bar{u}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 + x_2 = 0 \quad [1 \quad 1 \quad 0]$

$\bar{u}_2 \cdot \bar{u}_3 = \begin{bmatrix}1\\-1\\2\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x - x_2 + 2x_3 = 0 \quad [1 \quad -1 \quad 2]$

$$\begin{bmatrix}1 & 1 & 0\\1 & -1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\end{bmatrix}$$

$\bar{x} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$

$([\bar{u}_1 \quad \bar{u}_2]^T).rref()$

$\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix} = 0$

$\begin{bmatrix}1\\-1\\2\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix} = 0$

$H^{\perp} = \{\bar{u}_3\}$

$y = \begin{bmatrix}1\\3\\4\end{bmatrix}$

c) $\hat{y} = \dfrac{\bar{y}\cdot\bar{u}_1}{\bar{u}_1\cdot\bar{u}_1}\bar{u}_1 + \dfrac{\bar{y}\cdot\bar{u}_2}{\bar{u}_2\cdot\bar{u}_2}\bar{u}_2$

$= \dfrac{4}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \dfrac{6}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$

$= \begin{bmatrix}2\\2\\0\end{bmatrix} + \begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}3\\1\\2\end{bmatrix}$

$$\frac{V}{V.\text{norm}}$$

$$\begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = -3 \cdot 1 \cancel{+ 7 \cdot (-8)} - - - \ldots$$

$$\bar{a} \cdot \bar{b} = 0$$

$$X = \underbrace{\sum_{i=1}^{3} x.\text{project}(v_i)} + x.\text{project}.\bar{v}_4$$

$$X - X_{v_4} =$$