3. Recap and Exercises

* **Add/subtract** : entry by entry
* **Scaling** : Multiply each entry with a constant
* **Multiplication** : Number of column in first must match number of rows in second. Matrix multiplication is basically taking the dot product between all columns in first and corresponding row in second
* **Transpose A^T**: Swap row with columns

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

* **Inverse A^{-1}** (only square matrices) : special case for 2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \underbrace{\frac{1}{ad-bc}}_{\text{Determinant}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

General Algorithm

$$[A \ I] \xrightarrow{RREF} [I \ A^{-1}]$$

* **Power** : $\underbrace{A^k = A \cdot A \cdot \dots \cdot A}_{k\text{-times}}$
* **Solving Matrix Equations** : $A\bar{x} = \bar{b} \Rightarrow \bar{x} = A^{-1} \cdot \bar{b}$
* **The Invertible Matrix Theorem**

1.7.11

In Exercises 11–14, find the value(s) of h for which the vectors are linearly *dependent*. Justify each answer.

11. $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$

1.7.15-20

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

15. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

16. $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$

17. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$

18. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

1.7.41

[M] In Exercises 41 and 42, use as many columns of A as possible to construct a matrix B with the property that the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. Solve $B\mathbf{x} = \mathbf{0}$ to verify your work.

41. $A = \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ -5 & -3 & -7 & -11 & 15 \\ 4 & 3 & 5 & 2 & 1 \\ 8 & -7 & 23 & 4 & 15 \end{bmatrix}$

1.8.3-6

In Exercises 3–6, with T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$

$x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

1.8.10

For Exercises 9 and 10, find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

$$A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$$

1.9.15

In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

15. $\underbrace{\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}}_{A\bar{x}}$

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

2.1.1.+2

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, AC, CD

2. $A + 3B$, $2C - 3E$, DB, EC

2.1.10
2.1.13

skip

2.1.40+41

40. **[M]** Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute S^k for $k = 2, \dots, 6$.

41. **[M]** Describe in words what happens when A^5, A^{10}, A^{20}, and A^{30} are computed for

$$A = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/3 & 1/6 \\ 1/4 & 1/6 & 7/12 \end{bmatrix}$$

2.2.9

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

9. a. In order for a matrix B to be the inverse of A, the equations $AB = I$ and $BA = I$ must both be true. T

b. If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB. F

c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible. F

d. If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$. T

e. Each elementary matrix is invertible. T

2.2.30-32

Find the inverses of the matrices in Exercises 29–32, if they exist. Use the algorithm introduced in this section.

30. $\begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

2.3.11+12+15+17

In Exercises 11 and 12, the matrices are all $n \times n$. Each part of the exercises is an *implication* of the form "If ⟨ statement 1 ⟩, then ⟨ statement 2 ⟩." Mark an implication as True if the truth of ⟨ statement 2 ⟩ *always* follows whenever ⟨ statement 1 ⟩ happens to be true. An implication is False if there is an instance in which ⟨ statement 2 ⟩ is false but ⟨ statement 1 ⟩ is true. Justify each answer.

11. a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix. T

b. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent. T

c. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. F

d. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions. T

e. If A^T is not invertible, then A is not invertible. T

12. a. If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$. T

b. If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then the row reduced echelon form of A is I. (F)

c. If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$. T

d. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one. F

e. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution is unique. F

~~**13.** An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.~~

~~**14.** An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.~~

15. Is it possible for a 4×4 matrix to be invertible when its columns do not span $\mathbb{R}^4$? Why or why not? No!

~~**16.** If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent. Explain why.~~

17. Can a square matrix with two identical columns be invertible? Why or why not? No!