5. Eigenvectors and Eigenvalues

Eigenvector:

A nonzero vector $\bar{x}$ s.t. $A\bar{x} = \lambda \bar{x}$

Eigenvalue:

A scalar λ is an eigenvalue of A if there is a non-trivial solution $\bar{x}$ of $A\bar{x} = \lambda \bar{x}$

↳ In such cases we say that $\bar{x}_i$ is an eigenvector corresponding to λ_i.

Ex

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} -12 \\ 10 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Are $\bar{u}$ and $\bar{v}$ eigenvectors of A?

$$A\bar{u} = \overset{A}{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}} \overset{\bar{u}}{\begin{bmatrix} -12 \\ 10 \end{bmatrix}} = \begin{bmatrix} -12+60 \\ -60+20 \end{bmatrix} = \begin{bmatrix} 48 \\ -40 \end{bmatrix} = \overset{\lambda}{-4} \overset{\bar{u}}{\begin{bmatrix} -12 \\ 10 \end{bmatrix}}$$

so $\bar{u}$ is an eigenvector of A with corresponding eigenvalue $\lambda = -4$.

$$A\bar{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 - 12 \\ 15 - 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix} \rightarrow No!$$

$\bar{v}$ is not an eigenvector of A.

Ex: Determine if $\lambda = 7$ is eig. val. of A:

Note $A\bar{x} = 7\bar{x} \Rightarrow A\bar{x} - 7I\bar{x} = 0 \Rightarrow (A - 7I)\bar{x} = 0$

so we need to check if $A - 7I$ has a non-trivial solution:

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1-7 & 6 \\ 5 & 2-7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

so

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow$$ free variable! → non-trivial so $\lambda = 7$ is an eigenvalue of A

What about the corresponding eigenvector?

↳ The vectors in P.V.F. of $[A - \lambda I]$

$[A - 7I]$

$$\bar{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

All multiples of $\bar{x}$ is an eigenvector of A corresponding to $\lambda = 7$.

Except: $x_2 = 0$

Theorem

λ is an eigenvalue of A iff. $A - \lambda I = 0$ has a non trivial solution

Eigenspace:

Is the nullspace of $A - \lambda_i I_n$, denoted $\mathcal{E}_A(\lambda)$

Ex: Find a basis for the eigenspace of $\lambda = 2$

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \overset{\text{RRef}}{\sim} \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: If $\dim \mathrm{Nul}(A - \lambda_i I) > 1$, we are not guaranteed that the vectors are independent.

Theorem:

The eigenvalues of a triangular matrix are the entries on the main diagonal

Ex:

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$$

$\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$

0 is an eigenvalue iff. A is not invertible:

$\underline{\lambda = 0:}$

$A\bar{x} = \lambda x = 0\bar{x} = 0$ so only if $A\bar{x} = 0$ has non-trivial

Theorem:

If $\bar{v}_1 \dots \bar{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1 \dots \lambda_r$ of an $n \times n$ matrix, then $\{\bar{v}_1 \dots \bar{v}_r\}$ is linearly independent.

Note: The eigenspace at $\lambda = 0$ is the same as the nullspace of A: $A\bar{x} = 0\bar{x} = \bar{0}$

The Characteristic Equation:

Ex: Find eigenvalues of

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

We have $(A - \lambda I)\bar{x} = \bar{0}$

We need non-trivial solution

↳ free variables

↳ non-Invertible

↳ $\det(A - \lambda I) = 0$

so $A-\lambda I=\bar{0}$ will have a non-trivial solution when $\det A-\lambda I=0$

$$\det(A-\lambda I)=\det\left(\begin{bmatrix}2&3\\3&-6\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right)$$

$$=\begin{vmatrix}2-\lambda&3\\3&-6-\lambda\end{vmatrix}$$

$$=(2-\lambda)\cdot(-6-\lambda)-9$$

$$=-12-2\lambda+6\lambda+\lambda^2-9$$

$$=\lambda^2+4\lambda-21$$

$$=(\lambda+7)(\lambda-3) \quad =0$$

$$\lambda=\begin{cases}-7\\3\end{cases}$$

Theorem:

A scalar λ is an eigenvalue of an $n\times n$ matrix iff. λ satisfies

$$\det(A-\lambda I)=0$$

This equation is called the characteristic equation (or polynomial).

Ex: Find all the eigenvalues:

$$A=\begin{bmatrix}2&1&0&0&2\\0&3&1&7&-1\\0&0&2&6&5\\0&0&0&4&3\\0&0&0&0&1\end{bmatrix}, \det(A-\lambda I_5)=\begin{vmatrix}2-\lambda&1&0&0&2\\0&3-\lambda&1&7&-1\\0&0&2-\lambda&6&5\\0&0&0&4-\lambda&3\\0&0&0&0&1-\lambda\end{vmatrix}$$

$$(2-\lambda)(3-\lambda)\cdot(2-\lambda)\cdot(4-\lambda)\cdot(1-\lambda)$$

$$\lambda_1=1,\ \lambda_2=2,\ \lambda_3=2,\ \lambda_4=3,\ \lambda_5=4$$

Algebraic and Geometric Multiplicities:

The geometric multiplicity of an eigenvalue λ of A is the dimension of the eigenspace.

↳ $\dim \operatorname{Nul}(A-\lambda I)$

The algebraic multiplicity $\mu(\lambda)$ of an eigenvalue λ of A is the number of times λ appears as a root of the characteristic equation.

↳ In the diagonal matrix above $\mu(2)=2$

algebraic multiplicity $\geq$ geometric multiplicity.

Ex

Let $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} \rightarrow \begin{vmatrix}1-\lambda & 2\\0 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 - 0\cdot 2 = (1-\lambda)^2$

$\Rightarrow \lambda = 1 \quad , \mu(1)=2$

eig. vec:

$$\begin{bmatrix}1-1 & 2\\0 & 1-1\end{bmatrix} = \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} \rightarrow \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \bar{0} \rightarrow 2x_2 = 0$$

$x_2 = 0$

so $\begin{bmatrix}x_1\\0\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix}$ $\quad \dim \operatorname{Nul} A-\lambda I = 1$

Diagonalisation:

An $n\times n$ matrix A is said to be diagonalisable if A is similar to a diagonal matrix D:

$$A = PDP^{-1},$$

Where the diagonals of D are the eigenvalues of A and the columns of P are the corresponding Eigenvectors

When is A diagonalisable?

A is diagonalisable iff. A has n linearly independent eigenvectors.

- Guaranteed if for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity
  - n distinct eigenvalues is a special case of this scenario.

Method: Diagonalisation

1. Find eigenvalues of A:
   a) If triangular → diagonal
   b) roots of $\det(A-\lambda I_n)=0$
2. For each λ, find a basis for the eigenspace: $(A-\lambda I_n)\bar{x}=\bar{0}$ in P.V.F.
3. Construct D by placing the eigenvalues on the diagonal of an $n \times n$ zero matrix. Note it is good practice to place them in descending order: $d_{11} \geq d_{22} \geq d_{33} \ldots \geq d_{nn}$
4. Construct P from the vectors found in step 2. Note the eigenvector must be placed in the same column as its corresponding eigenvalue
5. Find P^{-1}
6. Test $A == PDP^{-1}$
7. Enjoy a job well-done!! :)

Ex

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Step 1: Find λ

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\Big(((-5-\lambda)(1-\lambda)+9\Big) - 3\Big(-3(1-\lambda)+9\Big) + 3\Big(-9-(3(-5-\lambda))\Big)$$

$$= (1-\lambda)(-5+5\lambda-\lambda+\lambda^2+9) - 3(-3+3\lambda+9) + 3(-9+15+3\lambda)$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) - 3(3\lambda+6) + 3(3\lambda+6)$$

$$= (1-\lambda)(\lambda+2)^2, \quad \lambda_1 = 1, \ \lambda_2 = -2, \ \lambda_3 = -2$$

Step 2: Find eig. vecs

$\lambda = 1$:

$$\begin{bmatrix} 1-1 & 3 & 3 \\ -3 & -5-1 & -3 \\ 3 & 3 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \underset{\sim}{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = -2$:

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step 3: Setup D

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Step 4: Setup P

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 5: Find P^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right], \quad P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Step 6: Test $A == P \cdot D \cdot P^{-1}$

↳ $A == PDP^{-1}$ TRUE

Step 7: Enjoy

A question of Power:

$$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$$

$$A^3 = PD\cancel{P^{-1} \cdot P}D\cancel{P^{-1}P}DP^{-1} = PD^3P^{-1}$$

$$=$$

In general:

$$A^k = PD^kP^{-1}$$

Ex

$$A^{1000} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \times \cdots\cdots \times \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

vs.

$$A^{1000} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{1000} & 0 & 0 \\ 0 & (-2)^{1000} & 0 \\ 0 & 0 & (-2)^{1000} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$