

2.1. Matrix Operations

Adding and Subtracting

- * Matrices must be same size
- * Entry by Entry

Ex

$$\begin{bmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 5 \end{bmatrix}$$

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$A + (-A) = 0$$

$$A - B = A + (-B)$$

$$A + B = A + C \Leftrightarrow B = C$$

Scaling:

- * Multiply each entry by scalar

Ex

$$3 \begin{bmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 \\ 9 & 0 & 6 \end{bmatrix}$$

$$s(A + B) = sA + sB$$

$$(A + B)s = sA + sB, s \in \mathbb{R}$$

$$(r+s)A = rA + sA, r, s \in \mathbb{R}$$

$$r(sA) = (rs)A$$

Multiplying Matrices

$$A: m \times n \quad 3 \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \quad 3 \times 5 \quad m \times n \quad 3 \text{ rows} \\ 5 \text{ columns}$$

$$B: k \times l \quad 5 \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \quad 5 \times 2 \quad k \times l \quad 5 \text{ rows} \\ 2 \text{ columns}$$

AB is only defined if $n=k$: Matrix must match row of second

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \quad \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix}$$

$$3 \times 5 \quad 5 \times 2$$

Match

size of AB

Note: BA is not defined

$$5 \times 2 \quad 3 \times 5$$

No match!

$$\begin{bmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \\ 5 & 10 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

$$AB = A \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \dots & \bar{b}_k \end{bmatrix} = \begin{bmatrix} A\bar{b}_1 & A\bar{b}_2 & \dots & A\bar{b}_k \end{bmatrix}$$

$$= \left[A \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k1} \end{bmatrix} \quad A \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{k2} \end{bmatrix} \quad \dots \quad A \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{kk} \end{bmatrix} \right]$$

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} A \begin{bmatrix} 2 \\ 3 \end{bmatrix} & A \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 2 + 2 \cdot 3 & -1 + 2 \\ 2 \cdot 2 + 3 \cdot 3 & -2 + 3 \\ 2 \cdot 3 + 3 \cdot 4 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 13 & 1 \\ 18 & 1 \end{bmatrix}$$

Row-Column rule:

If AB is defined, then the ij entry of AB is the dot product of the i 'th row of A with the j 'th column of B . $A:m \times n$ and $B:n \times l$

$$AB_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{in} \cdot b_{nj}$$

Remember that we are indexing from 1.

Ex

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 2 \\ 9 & 7 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 3 & 4 & 5 & 4 \end{bmatrix}$$

$$AB_{23} = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 5 = 20$$

$$\begin{bmatrix} x & x & x & x \\ x & x & 20 & x \\ x & x & x & x \end{bmatrix}$$

Given B and C are defined appropriately, where A is $m \times n$:

$$A(BC) = (AB)C$$

$$A(B+C) = AB+AC$$

$$(B+C)A = BA+CA$$

$$rAB = (rA)B = A(rB)$$

$$I_m A = A = A I_n, \text{ where } I \text{ is the identity matrix}$$

If $AB = BA$ we say that A and B commute with one another.

Power

If A is square, i.e an $n \times n$ matrix:

$$A^k = \underbrace{A \cdot A \cdot A \cdots A}_{K \text{ times}}$$

Transpose

If A is an $m \times n$ matrix, then the transpose of A , A^T , is the $n \times m$ matrix whose columns are formed from the rows of A :

Ex

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \\ 7 & 8 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 8 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(rA)^T = rA^T, \quad r \in \mathbb{R}$$

$$(AB)^T = B^T A^T$$

Ex

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$(A\bar{x})^T = \left(\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right)^T = \left(\begin{bmatrix} 5 & -9 \\ -10 + 12 \end{bmatrix} \right)^T = \begin{bmatrix} 5 & -9 \\ -10 + 12 \end{bmatrix}^T = \begin{bmatrix} 5 & -9 \\ -10 + 12 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}$$

$$\bar{x}^T \cdot A^T = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 5 - 9 & -10 + 12 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}$$

$$(A\bar{x})^T = \bar{x}^T \cdot A^T$$

$$\bar{x}^T \cdot \bar{x}^T = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$\bar{x}^T \cdot x = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$A^T \cdot x^T = 2 \times 2 \quad 1 \times 2 \quad \%_c$$

$$(ABC)^T$$

$$(DC)^T = C^T D^T = C^T (AB)^T$$

$$= C^T B^T A^T$$

Inverse

We say that a square matrix is invertible iff.

$$A^{-1} \cdot A = I \quad \text{and} \quad A \cdot A^{-1} = I$$

Where A^{-1} is the inverse of A .

2×2 Matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, A is invertible:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Where $ad - bc$ is called the determinant of A :

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det A = 1 \cdot 4 - 2 \cdot 3 = -2$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Confirm:

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 + 3 & 1 - 1 \\ -6 + 6 & 3 - \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$n \times n$ Matrices

Note: If A is invertible, $A\bar{x} = \bar{b}$ has a unique solution:

$$A\bar{x} = \bar{b}$$

$$A^{-1} \cdot A\bar{x} = A^{-1} \cdot \bar{b}$$

$$\bar{x} = A^{-1} \bar{b}$$

~~$$A^{-1} \cdot A\bar{x} = \bar{b} \cdot A^{-1}$$~~

Ex Use the inverse to solve

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned} \rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 + 6 \\ 15/2 - 6/2 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

Let A and B be invertible $n \times n$ matrices

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Algorithm for finding A^{-1} :

An $n \times n$ matrix is invertible iff. it is row equivalent with I , i.e. we get the identity matrix when we reduce to reduced echelon form.

1. Set up $[A \ I]$, i.e. A concatenated with I
2. Row reduce until A reaches reduced echelon form. The matrix I will now have changed into A^{-1} :

$$[A \ I] \xrightarrow{\text{REF}} [I \ A^{-1}]$$

$$Ex \quad A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & 5 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

A I

$$\begin{array}{|c|c|c|} \hline & \text{A} & \\ \hline \boxed{\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix}} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{I} \\ \hline \end{array}$$

$$r_2 \rightarrow r_2 - 4r_1 \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$r_3 \rightarrow \frac{1}{2}r_3 \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$r_1 \rightarrow r_1 - r_3 \sim \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & -3 & 0 & -4 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$r_2 \rightarrow -\frac{1}{3}r_2 \sim \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$r_1 \rightarrow r_1 - 2r_2 \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} & \frac{2}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Theorem (The Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

- (a) A is invertible.
- (b) A is row equivalent to I_n .
- (c) A has n pivot positions (i.e. one for each row and column).
- (d) The equation $A\bar{x} = \bar{0}$ has *only* the trivial solution.
- (e) The columns of A are linearly independent.
- (f) The linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$ is one-to-one.
- (g) The equation $A\bar{x} = \bar{b}$ has at least one solution for each \bar{b} in \mathbb{R}^n .
- (h) The columns of A span \mathbb{R}^n .
- (i) The linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$ is onto.
- (j) There is an $n \times n$ matrix C such that $CA = I$.
- (k) There is an $n \times n$ matrix D such that $AD = I$.
- (l) A^T is invertible.