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Dipartimento di Ingegneria Elettrica e Tecnologie dell'Informazione

Corso di Laurea in Ingegneria dell'Automazione e Robotica

FIELD AND SERVICE ROBOTICS

Homework 1 Report

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Contents

1 Fully actuation and underactuation of ATLAS robot

In this section, considering the ATLAS robot from Boston Dynamics, its underactuation and fully actuation will be analyzed in two different cases: while standing and while doing backflip.

1.0.1 Standing

While standing, ATLAS is fully actuated, thus the **statement a is true**. In fact, in this configuration the ATLAS can achieve **instantaneously** any desired acceleration in any direction. For example, ATLAS can exert a vertical force through its legs, allowing it to accelerate upward (jump) or decelerate downward (e.g. brake in a landing), can apply horizontal forces through its feet, so it can like walk, run, or move laterally, controlling acceleration along the X and Y axis, or can rotate on itself without changing horizontal position.

1.0.2 Doing backflip

While doing backflip, ATLAS is underactuated, thus the **statement b is false**. In fact, in this configuration for every $\ddot{q}$ there is no input that allows the robot to change the direction imposed when it comes off the ground during movement: neither accelerations of the center of mass other than gravity (i.e., it cannot move laterally or change its vertical velocity) nor rotational accelerations independent of the initial angular moment (i.e., it cannot begin to rotate or stop without moving body parts to balance) are allowed.

Thus, it is evident how ATLAS's feet in contact with the ground act as a support and force application point, enabling the robot to move, balance and react to external forces in a controlled manner. The presence of the ground allows ATLAS to maintain stability and control accelerations more precisely than when it is in the air, ensuring its fully actuation.

2 Degrees of freedom and configuration space topology

This section focuses on determining the number of degrees of freedom of two mechanisms, comparing the results obtained with their possible motions and writing, for each mechanism, the configuration space topology. Grubler's formula was used to determine analytically the number of DoFs:

$$DoFs = m(N - 1 - J) + \sum_{i=1}^{J} f_i \tag{1}$$

2.0.1 Experimental surgical manipulator

Consider the spatial mechanism in Fig.1

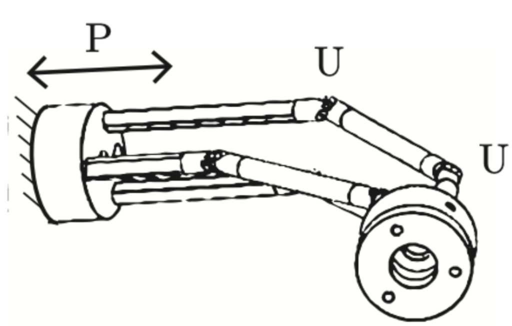

Figure 1: Experimental surgical manipulator

that consists of a fixed platform, a movable platform to which the surgical instrument is attached and 3 arms having identical PUU kinematic structures connecting the fixed platform with the movable one.

Due to the spatial mechanism, $m = 6$. By considering that:

Parameter	**Description**	**Value**
N	**Number of links:** the structure consists of two links for each of the three limbs, a fixed platform and a mobile platform	8
J	**Number of joints:** the mechanism has 1 prismatic joint and 2 universal joints for each of the three limbs	9
$\sum_{i=1}^{J} f_i$	**Number of freedom provided by the i-th joint:** each Prismatic joint provides 1 DoF; the Universal joint gives 2 DoFs	15

Table 1

applying the Grubler's formula 1, it can be determined that the mechanism consists of 3 DoFs. The result obtained is consistent with intuitive expectations in that the mechanism allows rotational motion of the end effector around the x-axis and y-axis and its translational motion along the z-axis. Gruebler's formula, thus, validated that the number of degrees of freedom of this mechanism matches the expected physical behavior. Without joint limits, the overall configuration space topology is given by:

$$Topology : \mathbb{R} \times T^2$$

If, on the other hand, joint limits are considered, the overall configuration space topology is given by:

$$Topology : I^3$$

2.0.2 Spatial mechanism made by 6 identical bars with all spherical joints

Consider the spatial mechanism in Fig.2, characterized by a top plate (moving platform), a base plate (fixed base) and six bars each with two spherical joints connecting the top plate to the bottom plate.

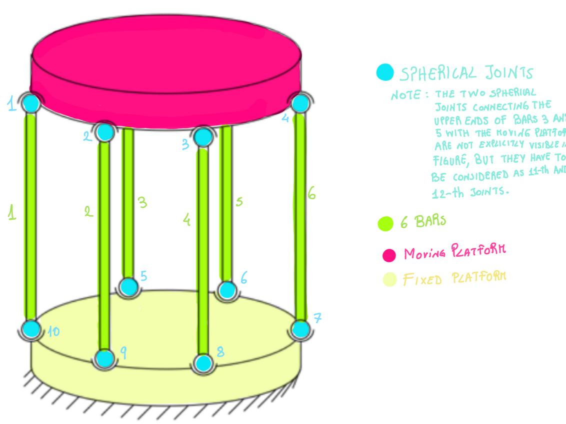

Figure 2: Spatial mechanism made by 6 identical bars with all spherical joints

Due to the spatial mechanism, m = 6. Considering that N = 8 (six bars, ground and moving platform), J = 12 (two spherical joints for each of the six bars) and $\sum_{i=1}^{J} f_i = 36$ (the spherical joint gives 3 DoFs), the application of Grubler's formula 1 leads to a number of DoFs of the entire structure equal to six. Let now analyze the result obtained from the application of Grubler's formula, to check its validity. The result obtained does not match with the expected one. In fact, given the structure of the mechanism under analysis, 8 DoFs in total are expected: 6 DoFs related to the rotation on themselves of the six axes, 1 DoF due to the lateral translation and 1DoF due to vertical translation of the moving platform, which are an effect of the rotations of the individual bars. Regarding translation, it should be recognized that pure translational motions are not possible since the latter are constrained by the geometry of the structure and the fixed length of the bars. However, since each bar is connected by two spherical joints to the two platforms, limited translations can be achieved through combinations of rotations. The overall configuration space topology, following intuition, is given by:

$$Topology : T^6 \times I^2$$

3 Underactuation or fully actuation

3.0.1 Car with inputs the steering angle and the throttle

The car has a total of 7 DoFs. However, for control purposes it is usual to consider only a subset of DoFs to describe its model. In particular, neglecting the wheel positions, only the 3 DOFs given by the planar rigid body chassis are taken into account. Since the car is provided with two inputs, namely the steering angle and the throttle, it turns out to be characterized by a number of DoFs related to the mathematical model that is greater than the number of control inputs. This implies that the car is underactuated and that **statement a is true**. In fact, it cannot achieve instantaneously any desired acceleration in any direction. This result turns out to be easily deduced by observing that, since the two inputs (steering and throttle) do not allow direct lateral forces to be applied, the car can only move along the road where the wheels face and cannot perform movements perpendicular to this direction, e.g. lateral displacement; the car also has no direct control over its vertical height or acceleration without an additional system, such as an active suspension.

3.0.2 The KUKA youBot system

The KUKA youBot system has a total of 12 DoFs, since, without considering the gripper, it consists of four wheels (4 DoFs), a planar rigid body (3 DoFs) and five rotational joints for the arm (5 DoFs). Despite the 12 total DoFs, to describe its model it is possible to consider only the 8 DoFs given by the rigid planar chassis and the revolute joints of the manipulator, neglecting the wheel angles. In view of the presence of nine independent actuators, four for each of the wheels and five for the robotic arm, the KUKA youBot is fully actuated. Hance, **statement b is true**. This result is consistent with the fact that it can instantaneously achieve any desired acceleration in any direction given the presence of omnidirectional wheels.

3.0.3 The hexarotor system with co-planar propellers

The hexarotor system is composed by a rigid body and six co-planar propellers (all lying in the same plane). Neglecting the DoFs given by the propellers, the system is characterized by a number of DoFs equal to six (three for position and three for orientation) and consists of six actuators. Although the number of actuators coincides with the number of generalized variables might suggest that the system is fully actuated, however, it is possible to prove that the hexarotor turns out to be underactuated. Each propeller generates a lift force along the axis perpendicular to the same plane and there is only one possibility to control the magnitude of each force. If all forces are balanced, the net force along the axis passing through the center of mass will be parallel to the others and the drone will have the ability to move up and down along the z axis. When some of the lift forces come to be unbalanced, the hexarotor will tilt (roll, rotation around the x-axis or pitch, rotation around the y-axis) and can move laterally (along the x-axis or y-axis). There is no combination of inputs that allows the drone to move laterally by applying only translational accelerations without tilting it. So, there is no possibility of achieving accelerations along the x and y axes without rotating the body. This shows that the system is underactuated and therefore **statement c is false**. It is worth noting that even if the drone had more propellers, it would still remain underactuated; the only thing that changes is the distribution of forces with respect to the center of mass.

3.0.4 The KUKA iiwa 7-DOF robot

The KUKA iiwa robot can be compared to a human arm that has seven DoFs: three in shoulder, three in wrist and one in elbow. This manipulator is intrinsically redundant since, if the base and the hand position and orientations are fixed, requiring six DoFs, the elbow can be moved, thanks to the additional DoF. The fact that iiwa is redundant does not imply nor is it a consequence of whether or not it is underactuated. It is good to emphasize that there is no relationship between the concepts of redundacy and underactuation, which therefore turn out to be completely independent and distinct from each other. In fact, while redundancy is a concept associated with the task assigned to the manipulator, underactuation does not represent a limitation in the given task, but is related only to the impossibility of achieving desired accelerations through suitable inputs. Hence, **statement d is false**. The KUKA iiwa robot is actually fully actuated because it has exactly 7 actuators (one for each of its 7 joints). Each DOF is directly controlled by a dedicated motor, which provides complete control over all its degrees of freedom; this feature enables it to move smoothly and precisely, combining translational and rotational motions.

4 Involutive distributions and annihilator

In this section determine whether or not each of the given distributions is involutive and, if possible, find the annihilator for each distribution.

4.0.1 Exercise 4a.

Consider the distribution:

$$\Delta(x) = \text{span}\left\{\begin{bmatrix} -3x_2 \\ 1 \\ -1 \end{bmatrix}\right\}, \quad U \in \mathbb{R}^3 \tag{2}$$

with:

$$d = 1, \qquad \text{n} = 3, \qquad F = \begin{bmatrix} -3x_2 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{rank}(F) = 1, \qquad \dim(\Delta(x)) = 1$$

Distribution (2) is **involutive** since, by definition, any one dimensional distribution is involutive being $[f, f] = 0$ and the zero vector belongs to any distribution. Notice that, because of $\dim(\Delta(x)) = 1 = d$, (2) is **nonsingular** $\forall x \in U$. Since:

$$\dim(\Delta(x)) + \dim(\Delta^{\perp}(x)) = n \tag{3}$$

$$1 + \dim(\Delta^{\perp}(x)) = 3 \quad \Rightarrow \quad \dim(\Delta^{\perp}(x)) = 3 - 1 = 2 \tag{4}$$

the annihilator $\Delta^{\perp}(x)$ exists.

At this point, it is necessary to identify the set of co-vectors such that $w^*F(x)$=0:

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} -3x_2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

from which the following equation is derived:

$$-3x_2w_1 + w_2 - w_3 = 0 \quad \Rightarrow \quad w_2 = 3x_2w_1 + w_3 \tag{6}$$

Two linear independent co-vectors have to be determined. Avoiding the trivial solution $w_1 = w_2 = 0$, the following two choices may be made:
(A) If $w_1 = 0$, $w_3 = 1 \Rightarrow w_2 = 1$, and the first co-vector is:

$$w_A^* = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \tag{7}$$

(B) If $w_1 = 1$, $w_3 = 0 \Rightarrow w_2 = 3x_2$, and the second co-vector is:

$$w_B^* = \begin{bmatrix} 1 & 3x_2 & 0 \end{bmatrix} \tag{8}$$

Therefore, the annihilator is:

$$\Delta^{\perp}(x) = \text{span}\left\{ \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3x_2 & 0 \end{bmatrix} \right\} \tag{9}$$

4.0.2 Exercise 4b.

Consider the distribution:

$$\Delta(x) = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ -\alpha \\ x1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ -4 \\ x1 \end{bmatrix} \right\}, \quad U \in \mathbb{R}^3, \quad n = 3, \quad d = 2 \tag{10}$$

In order to determine the dimension of the distribution (10), it is necessary to construct matrix $F(x)$ and evaluate its rank:

$$F = \begin{bmatrix} -1 & x_2 \\ 0 & -\ 4 \\ x_3 & x_1 \end{bmatrix} \tag{11}$$

Since the matrix F has dimension 3×2, its rank can be at most 2. To determine whether the rank is indeed 2, it is necessary to verify the existence of at least one minor of order 2 whose determinant is different from zero. Let compute the determinant of the minor formed by the first two rows:

$$\det \begin{pmatrix} -1 & x_2 \\ 0 & -4 \end{pmatrix} = (-1) \cdot (-4) - x_2 \cdot 0 = 4$$

Since a minor of order 2 with nonzero determinant (equal to 4) was found, regardless of the values of the variables x_1, x_2 and x_3, it can be concluded that: $rank(F) = 2$ and, thus, that $dim(\Delta(x)) = rank(F) = 2 \; \forall x \in U$. Notice that, because of $\dim(\Delta(x)) = 2 = d$, (10) is **nonsingular** $\forall x \in U$. At this point one has to check whether the distribution (10) is involutive. For this purpose, the lie bracket is computed as follows:

$$[f_1, f_2] = \left(\frac{\partial f_2}{\partial x}\right) f_1 - \left(\frac{\partial f_1}{\partial x}\right) f_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -4 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -x_1 - 1 \end{bmatrix}$$

If $[f_1(x), f_2(x)] \in \Delta(x)$, then the distribution (10) is involutive. To verify the involutivity of (10), consider the following matrix whose columns are composed by the vectors $f_1, f_2, [f_1, f_2]$:

$$F_{\text{augmented}} = \begin{bmatrix} -1 & x_2 & 0 \\ 0 & -4 & 0 \\ x_3 & x_1 & -x_1 - 1 \end{bmatrix} \tag{12}$$

The determinant of matrix (12) is:

$$\det(F_{augmented}) = \det \begin{pmatrix} -1 & x_2 & 0 \\ 0 & -4 & 0 \\ x_3 & x_1 & -x_1 - 1 \end{pmatrix} = -4(x_1 + 1)$$

and it vanishes when $x_1 = -1$. Hence:

$$\begin{cases} \text{If } x_1 \neq -1, & \text{the rank of the matrix (12) is 3} \\ \text{If } x_1 = -1, & \text{the rank of matrix (12) is less than 3} \end{cases}$$

To verify if the rank is 2, check whether there exists a $(2{\times}2)$ minor with a nonzero determinant. Consider the minor:

$$\det \begin{pmatrix} -1 & x_2 \\ 0 & -4 \end{pmatrix} = (-1)(-4) - x_2(0) = 4 \neq 0$$

Since this determinant is nonzero, when $x_1 = -1$, the rank of the matrix is 2. In the end:

$$\begin{cases} \text{If } x_1 \neq -1, & \text{rank}(F_{\text{augmented}}) \neq \text{rank}(F) \text{ and } \textit{distribution}(10) \textbf{ is not involutive} \\ \text{If } x_1 = -1, & \text{rank}(F_{\text{augmented}}) = \text{rank}(F) \text{ and } \textit{distribution}(10) \textbf{ is involutive}. \end{cases}$$

To find the annihilator of (10), it is necessary before to compute its dimension:
Since:

$$\dim(\Delta(x)) + \dim(\Delta^{\perp}(x)) = n \tag{13}$$

$$2 + \dim(\Delta^{\perp}(x)) = 3 \quad \Rightarrow \quad \dim(\Delta^{\perp}(x)) = 3 - 2 = 1 \tag{14}$$

the annihilator $\Delta^{\perp}(x)$ exists and has a dimension equal to 1.
At this point, it is necessary to identify the set of co-vectors such that $w^* F(x){=}0$:

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} -1 & x_2 \\ 0 & -4 \\ x_3 & x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -w_1 + w_3 x_3 = 0 \\ w_1 x_2 - 4w_2 + w_3 x_1 = 0 \end{cases} \Rightarrow \begin{cases} w_1 = w_3 x_3 \\ w_2 = \frac{w_1 x_2}{4} + \frac{w_3 x_1}{4} \end{cases} \tag{15}$$

and then:

$$w^* = \begin{bmatrix} w_3 x_3 & \frac{w_3 x_2 x_3}{4} + \frac{w_3 x_1}{4} & w_3 \end{bmatrix} \tag{16}$$

Avoiding the trivial solution $w_3 = 0$, the choice $w_3 = 4$ leads to:

$$w^* = \begin{bmatrix} 4x_3 & x_2x_3 + x_1 & 4 \end{bmatrix} \tag{17}$$

Therefore, the annihilator is:

$$\Delta^\perp(x) = \text{span}\left\{\begin{bmatrix} 4x_3 & x_2x_3 + x_1 & 4 \end{bmatrix}\right\} \tag{18}$$

4.0.3 Exercise 4c.

Consider the distribution:

$$\Delta(x) = \text{span}\left\{\begin{bmatrix} 2x_3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2x_2 \\ x_1 \\ -1 \end{bmatrix}\right\}, \quad U \in \mathbb{R}^3, \quad n = 3, \quad d = 2 \tag{19}$$

In order to determine the dimension of the distribution (19), it is necessary to construct matrix $F(x)$ and evaluate its rank:

$$F = \begin{bmatrix} 2x_3 & -2x_2 \\ 1 & x_1 \\ 0 & -1 \end{bmatrix} \tag{20}$$

Since the matrix F has dimension 3×2, its rank can be at most 2. To determine whether the rank is indeed 2, it is necessary to verify the existence of at least one minor of order 2 that is different from zero. Let compute the determinant of the minor formed by the last two rows:

$$\det\begin{pmatrix} 1 & x_1 \\ 0 & -1 \end{pmatrix} = -1$$

Since a minor of order 2 with nonzero determinant (equal to 1) was found, regardless of the values of the variables x_1, x_2 and x_3, it can be concluded that: $rank(F) = 2$ and, thus, that $dim(\Delta(x)) = rank(F) = 2\ \forall x \in U$. Notice that, because of $\dim(\Delta(x)) = 2 = d$, (19) is **nonsingular** $\forall x \in U$. At this point one has to check whether the distribution (19) is involutive. For this purpose, the lie bracket is computed as follows:

$$[f_1, f_2] = \left(\frac{\partial f_2}{\partial x}\right) f_1 - \left(\frac{\partial f_1}{\partial x}\right) f_2 = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2x_3 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2x_2 \\ x_1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2x_3 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ 0 \end{bmatrix}$$

If $[f_1(x), f_2(x)] \in \Delta(x)$, then the distribution (19) is involutive. To verify the involutivity of (19), consider the following matrix whose columns are composed by the vectors $f_1, f_2, [f_1, f_2]$:

$$F_{\text{augmented}} = \begin{bmatrix} 2x_3 & -2x_2 & 0 \\ 1 & x_1 & 2x_3 \\ 0 & -1 & 0 \end{bmatrix} \tag{21}$$

The determinant of matrix (12) is:

$$\det(F_{augmented}) = \det\begin{pmatrix} 2x_3 & -2x_2 & 0 \\ 1 & x_1 & 2x_3 \\ 0 & -1 & 0 \end{pmatrix} = 4x_3^2$$

and it vanishes when $x_3 = 0$. Hence:

$$\begin{cases} \text{If } x_3 \neq 0, & \text{the rank of the matrix (21) is 3} \\ \text{If } x_3 = 0, & \text{the rank of matrix (21) is less than 3} \end{cases}$$

To verify if the rank is 2, check whether there exists a (2×2) minor with a nonzero determinant. Consider the minor:

$$\det \begin{pmatrix} 1 & x_1 \\ 0 & -1 \end{pmatrix} = -1 \neq 0$$

Since this determinant is nonzero, when $x_3 = 0$, the rank of the matrix is 2. In the end:

$$\begin{cases} \text{If } x_3 \neq 0, & \text{rank}(F_{\text{augmented}}) \neq \text{rank}(F) \text{ and } \textit{distribution}\text{(19)} \textbf{ is not involutive} \\ \text{If } x_3 = 0, & \text{rank}(F_{\text{augmented}}) = \text{rank}(F) \text{ and } \textit{distribution}\text{(19)} \textbf{ is involutive}. \end{cases}$$

To find the annihilator of (19), it is necessary before to compute its dimension:
Since:

$$\dim(\Delta(x)) + \dim(\Delta^{\perp}(x)) = n \tag{22}$$

$$2 + \dim(\Delta^{\perp}(x)) = 3 \quad \Rightarrow \quad \dim(\Delta^{\perp}(x)) = 3 - 2 = 1 \tag{23}$$

the annihilator $\Delta^{\perp}(x)$ exists and has a dimension equal to 1.
At this point, it is necessary to identify the set of co-vectors such that $w^* F(x)$=0:

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 2x_3 & -2x_2 \\ 1 & x_1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2w_1x_3 + w_2 = 0 \\ -2w_1x_2 + w_2x_1 - w_3 = 0 \end{cases} \Rightarrow \begin{cases} w_2 = -2w_1x_3 \\ w_3 = w_2x_1 - 2w_1x_2 \end{cases} \Rightarrow \tag{24}$$

$$\begin{cases} w_2 = -2w_1x_3 \\ w_3 = -2w_1x_1x_3 - 2w_1x_2 \end{cases} \tag{25}$$

and then:

$$w^* = \begin{bmatrix} w_1 & -2w_1x_3 & -2w_1x_1x_3 - 2w_1x_2 \end{bmatrix} \tag{26}$$

Avoiding the trivial solution $w_1 = 0$, the choices $w_1 = \frac{1}{2}$ leads to:

$$w^* = \begin{bmatrix} \frac{1}{2} & -x_3 & -x_1x_3 - x_2 \end{bmatrix} \tag{27}$$

Therefore, the annihilator is:

$$\Delta^{\perp}(x) = \text{span}\left\{ \begin{bmatrix} \frac{1}{2} & -x_3 & -x_1x_3 - x_2 \end{bmatrix} \right\} \tag{28}$$

5 A set of Pfaffian constraints that does not depend on the generalized coordinates is always integrable

Using the accessibility rank condition, in this section it will be demonstrated that a set of Pfaffian constraints that does not depend on the generalized coordinates, $A\dot{q} = 0$, with A constant, is always integrable (completely holonomic system).

Demonstration:

Under the assumption that A is constant and does not depend on q, the kinematic constraints in Pfaffian form for a mechanical system can be written as follows:

$$A^T \dot{q} = 0 \tag{29}$$

with $A^T \in \mathbb{R}^{k \times n}$ and $\dot{q} \in \mathbb{R}^n$.

From the Pfaffian matrix can be constructed the distribution:

$$\Delta = \text{span}\left\{g_j(q) \in \mathbb{R}^n : A^T g_j(q) = 0,\ j = 1, \ldots, m\right\} \tag{30}$$

where m denotes the number of control inputs.

On the assumption that A^T has full-rank, computing the derivative of $A^T g_j(q)$ with respect to q, yields:

$$\frac{\partial(A^T g_j(q))}{\partial q} = A^T \frac{\partial g_j(q)}{\partial q} = 0 \tag{31}$$

Since:

$$\frac{\partial g_j(q)}{\partial q} = 0 \tag{32}$$

then $g_j(q)$ is constant.

This result shows that the condition $A^T g_j(q) = 0$, with A^T constant, implies that $g_j(q)$ does not depend on q and thus, it can be written:

$$g_j(q) = g_j \tag{33}$$

Therefore, the kinematic model of the mechanical system subject to $A^T \dot{q} = 0$ is:

$$\dot{q} = G \cdot u \tag{34}$$

with $G = \sum_{j=1}^{m} g_j$ and $u = [u_1 \ \ldots \ u_m]^T \in \mathbb{R}^m$ control input vector.

Notice that the kinematic matrix G is constant and the kinematic model (34) constitutes a driftless affine control system ($f(x) = drif\ vector = 0$).

In order to prove the involutivity of the distribution (30), verify that the Lie bracket $[g_i, g_{i+1}]$ of any pair of vector fields $g_i, g_{i+1} \in \Delta$ is a vector field belonging to Δ.

Starting from the pair of vector fields $g_i, g_{i+1} \in \Delta$, the Lie bracket $[g_i, g_{i+1}]$ is given by:

$$[g_i, g_{i+1}] = \frac{\partial g_{i+1}}{\partial q} g_i - \frac{\partial g_i}{\partial q} g_{i+1} = 0 - 0 = 0 \tag{35}$$

in view of (33) and it trivially lies in Δ.

Hence, the accessibility distribution Δ_A, generated by the vector fields $f, g_1, g_2, \ldots, g_m$ and all Lie brackets that can be produced by these vector fields, does not grow beyond distribution Δ and coincides with Δ itself. Considering that, when Δ_A coincides with the distribution $\Delta = \text{span}\{g_1, \ldots, g_m\}$, that is, when the latter is involutive, holds:

$$m = \nu \tag{36}$$

with $\dim(\Delta) = m$ and $\dim(\Delta_A) = \nu < n$, then condition (36) implies that constraints (29) are completely integrable and, hence, holonomic.
To summarize, since the accessibility distribution Δ_A coincides with Δ and Δ is involutive (all Lie brackets vanish), the Pfaffian constraints

$$A^T \dot{q} = 0$$

are completely integrable. The system is therefore holonomic and the constraints reduce to the algebraic equations:

$$A^T q = c \qquad (c = constant)$$

which define a linear submanifold of admissible configurations. For constant A^T, constraints rigidly limit velocities to a fixed subspace and no new directions are generated through Lie bracket, confirming the absence of nonholonomic behavior.

6 Raibert's hooper robot

The goal of this section is to compute a kinematic model of Raibert's hooper robot and show whether this system is holonomic or not.
In order to satisfy the intended objective, through the use of MATLAB, the following steps were taken:

- **Variable Definition:**

  First, the symbolic variables were defined, namely: state variables (θ, ψ, I), their derivatives ($\dot{\theta}$, $\dot{\psi}$, $\dot{I}$), and physical parameters (m, l, d):

```
% Holonomic Analysis of Raibert's Hopper Robot
clear; clc;

% Symbolic variables definition
syms theta psi I real                    % State variables
syms theta_dot psi_dot I_dot real        % Derivatives of state variables
syms m l d real                          % Physical parameters of the system
```

Figure 3: Variable Definition

- **State Vector Definition:**

  Next, the configuration vector q and its derivative $\dot{q}$ were defined:

```
% State vector definition
q = [theta; psi; I];
q_dot = [theta_dot; psi_dot; I_dot];
```

Figure 4: State Vector Definition

- **Pfaffian Constraint Implementation:**

  At this point, the Pfaffian Constraint was established:

```
% 1. Pfaffian Constraint
A = [(I + m*(l + d)^2), m*(l + d)^2, 0];
constraint = A * q_dot;
```

Figure 5: Pfaffian Constraint Implementation

- **Null Space computation:**

  The null space of the constraint matrix A was then computed using MATLAB's *null*() function. The null space represents the set of all possible velocity vectors that satisfy the constraint. Each column of G (each basis vector of the null space) is printed next and its dimension is reported. For this system, a null space of dimension 2 (3 DOF minus 1 constraint) is expected.

```
% 2. Null Space Computation
G = null(A);
fprintf('\n  2. NULL SPACE (G):\n');
for i = 1:size(G,2)
    fprintf('   g%d = [%s; %s; %s]\n', i, ...
            string(G(1,i)), string(G(2,i)), string(G(3,i)));
end

G_rank = rank(G);
fprintf('\n   Rank of G: %d\n', G_rank);
fprintf('\n   Null space dimension: %d\n', size(G,2));
```

Figure 6: Null Space computation

- **Lie Bracket and Extended Distribution Matrix Computation:**

  After checking whether the null space has dimension 2, as expected, the two basis vectors g_1 and g_2 were extracted. From these, the Lie bracket $[g_1, g_2]$ was computed using the formula:

$$[g_1, g_2] = \left(\frac{\partial g_2}{\partial q}\right) \cdot g_1 - \left(\frac{\partial g_1}{\partial q}\right) \cdot g_2$$

  If the Lie bracket depends linearly on g_1 and g_2, the system is holonomic. On the other hand, if the Lie bracket introduces a new independent direction, the system is nonholonomic. To carry out the holonomy test, the extended distribution matrix, Δ, was constructed by adding the Lie bracket to G. Then, the rank of this extended matrix was computed.

```
% 3. Holonomy Test
if size(G,2) == 2
    g1 = G(:,1);
    g2 = G(:,2);

    % Lie Bracket Computation
    Lie = jacobian(g2,q)*g1 - jacobian(g1,q)*g2;

    fprintf('\n  3. LIE BRACKET:\n');
    fprintf('   [g1,g2] = [%s; %s; %s]\n\n', ...
            string(Lie(1)), string(Lie(2)), string(Lie(3)));

    % Extended Distribution Matrix
    Delta = [G Lie];
    delta_rank = rank(Delta);
    fprintf('   Distribution matrix rank: %d\n', delta_rank);
```

Figure 7: Lie Bracket and Extended Distribution Matrix

- **Holonomy Test Results:**

  To check whether the system is holonomic or nonholonomic, the rank of the extended distribution was compared with the rank of the G matrix (the null space of constraints).

  * **Case** rank($\boldsymbol{\Delta}$) > rank($\mathbf{G}$)**:** This means that the Lie bracket introduced a linearly independent direction not present in the original null space. In other words, the constraint is not integrable and the system is therefore classified as nonholonomic.

  * **Case** rank($\boldsymbol{\Delta}$) = rank($\mathbf{G}$)**:** This means that the Lie bracket has not introduced any new independent directions. The Lie bracket can be expressed as a linear combination of the vectors in G, the constraint is integrable and the system is then classified as holonomic.

```
% Test
if delta_rank > G_rank
    fprintf(' CONCLUSION: since delta_rank > rank(G), then SYSTEM is NONHOLONOMIC\n\n');
else
    fprintf(' CONCLUSION: since delta_rank = rank(G), then SYSTEM is HOLONOMIC \n\n');
end
```

Figure 8: Holonomy Test Results

Fig.9 shows the results obtained from the simulation:

```
Command Window
================================================
 HOLONOMY ANALYSIS - RAIBERT'S HOPPER ROBOT
================================================

1. PFAFFIAN CONSTRAINT:
   (I + m*(l+d)^2)*θ_dot + m*(l+d)^2*ψ_dot = 0


  2. NULL SPACE (G):
   g1 = [-(m*(d + l)^2)/(I + d^2*m + l^2*m + 2*d*l*m); 1; 0]
   g2 = [0; 0; 1]

   Rank of G: 2

   Null space dimension: 2

  3. LIE BRACKET:
   [g1,g2] = [-(m*(d + l)^2)/(I + d^2*m + l^2*m + 2*d*l*m)^2; 0; 0]

   Distribution matrix rank: 3

  4. HOLONOMY TEST:

   Explanation:
   - The system is nonholonomic if the Lie bracket introduces new independent directions (delta_rank > rank(G))
   - The system is holonomic if all Lie brackets remain within the original distribution (delta_rank = rank(G))

 CONCLUSION: since delta_rank > rank(G), then SYSTEM is NONHOLONOMIC

5. PHYSICAL INTERPRETATION:
   - The analysis shows that:
     * Non-integrable constraints exist
     * The system cannot instantaneously move in certain directions, but can eventually reach any configuration
```

Figure 9: Final Results

From the analysis of the results obtained, it was found that Raibert's hopper robot is a nonholonomic system, as demonstrated by the Lie bracket test. This result has significant implications for its dynamic behavior and control:

Non-integrable constraints: The relationship between the angular moment of the body and the leg is subject to a constraint that cannot be reduced to a purely positional relationship. This is evident from the fact that the rank of the extended distribution (3) is greater than the rank of the original null space (2).

Implications for control: Controlling this robot will require more sophisticated strategies than a holonomic system. It will not be possible to independently control all state variables instantaneously, but it will be necessary to plan trajectories that respect nonholonomic constraints.

Maneuverability: Despite constraints on instantaneous velocities, the robot can still reach any configuration in state space through appropriate sequences of movements.