Monty Hall et al.

Alberto Garfagnini

Università di Padova

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The Monty Hall Problem

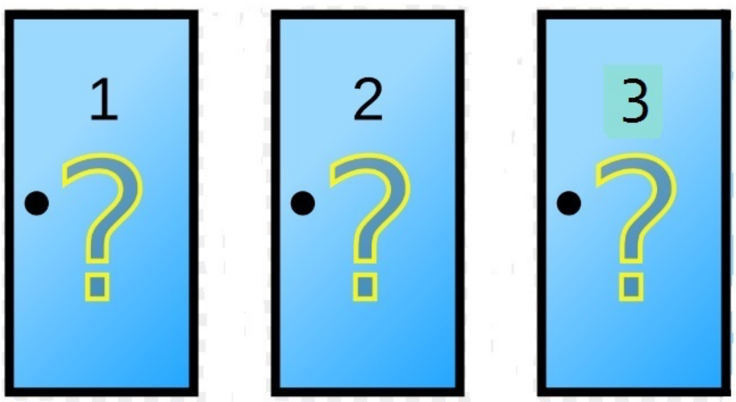
The Game Show

- there are 3 door, closed
- behind one door there is a prize, an expensive car
- but behind the other doors there is a goat

The Rules of the Game

- you select one door, but you cannot open it, yet
- the game show host, that knows where the car is, open one of the other two doors, revealing a goat behind it
- you are given the opportunity to change your choice of door, before opening it

What is your choice ?



The Monty Hall Problem Solution

The Game Propositions

- we select door number 1
- the host opens door number 2
- we are asked to choose between door 1 and 3

W : the CAR is behind door 1

C : we select the car by changing door

$$
\begin{aligned}
P(C|I) &= P(CW|I) + P(C\overline{W}|I) \\
&= P(C|WI) \cdot P(W|I) + P(C|\overline{W}I) \cdot P(\overline{W}|I)
\end{aligned}
$$

Our Knowledge

$P(W|I) = 1/3 \;\; \rightarrow \;\; P(\overline{W}|I) = 1 - P(W|I) = 2/3$

$P(C|WI) = 0 \;\; \rightarrow \;\; P(C|\overline{W}I) = 1$

- therefore

$$P(C|I) = 2/3$$

The Monty Hall Problem - Variation I

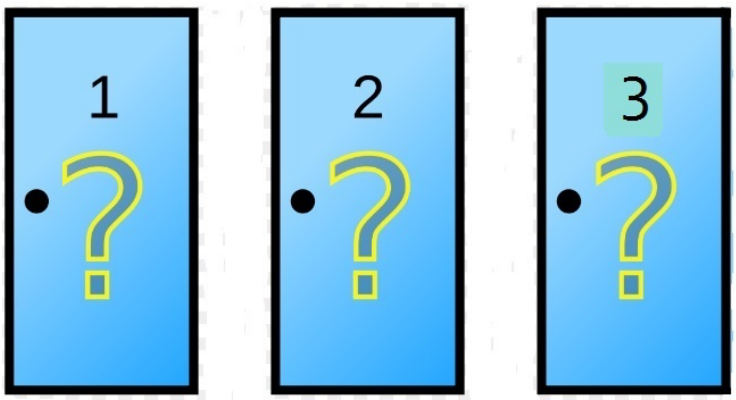
The Game Show

- there are 3 door, closed
- behind one door there is a prize, an expensive car
- but behind the other doors there is a goat

The Rules of the Game

- you select one door, but you cannot open it, yet
- the game show host, that does NOT know which door hides the prize, opens one of the other two doors. The door happens to have a goat behind it
- you are given the opportunity to change your original choice, switching to the other unopened door, before opening it

What is your choice ?



The Monty Hall Problem Variation - Solution

The Game Propositions

- we have selected door number 1
- the host opens door number 2, revealing a goat
- we are asked to choose between door 1 and 3

G_k : a goat is behind door k
C_k : a car is behind door k

- we need to evaluate the probability that door 3 hides a car, if door 2 hides a goat

$$P(C_3 \mid G_2 I) = \frac{P(G_2 \mid C_3 I)\, P(C_3 \mid I)}{\sum_{j=1}^{3} P(G_2 \mid C_j I)\, P(C_j \mid I)}$$

Our Knowledge

$$P(G_2 \mid C_1) = 1 \quad P(G_2 \mid C_2) = 0 \quad P(G_2 \mid C_3) = 1$$
$$P(C_1 \mid I) = 1/3 \quad P(C_2 \mid I) = 1/3 \quad P(C_3 \mid I) = 1/3$$

➜ therefore: $P(C_3 \mid G_2 I) = \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{2}$

The Monty Hall Problem generalization

- it is easy to generalize the problem to the case of n doors
- the game show host opens k doors, revealing as many goats ($0 \leq k \leq n-2$)
- there is still ONE car

➜ what is the probability of winning if we switch to another closed door, randomly chosen ?

C : we select the CAR by changing door
W : the CAR is behind door 1
we have:

$$P(W \mid I) = 1/n \quad P(\overline{W} \mid I) = 1 - 1/n = (n-1)/n$$

and

$$P(C \mid W\, I) = 0 \quad P(C \mid \overline{W}\, I) = 1/(n-k-1)$$

- therefore

$$P(C \mid I) = \frac{1}{n-k-1} \frac{n-1}{n}$$

- the probability of winning is increased from $1/n$ whenever one or more doors are opened. ➜ we should always switch doors

Three prisoner's dilemma

The problem

- Three prisoners are in jail and condamned to death
- Since on the next day it will be the KIng's birthday, his majesty as decided to release one prisoner
- Only the chief guard knows who is going to be freed, but is not allowed to tell anybody who he is

☐ Prisoner **A** begs the chief of the guards to tell him the name of one of the two prisoners that will be executed.

☐ He hopes that, by interrogating the guard, his chance of surviving increases

- The chief guard agrees to tell him the name of one of the two prisoners condamned to death. So they have the following deal:
  - if **B** is going to be pardoned, give me **C** name
  - if **C** is going to be pardoned, give me **B** name
  - if I am (**A**) going to be pardoned, flip a coin and tell me either **B** or **C** name
- The guard says: **B** is going to be executed.
  Has **A** probability of surviving increased or not ?

Three prisoner's dilemma (2)

The solution

- we define the propositions:
  - **A** will be freed
  - **B** will be freed
  - **C** will be freed
  - β the guard tells A that **B** will be executed

We want to compute

$$P(A \mid \beta) = \frac{P(\beta \mid A)P(A)}{P(\beta)}$$

$$= \frac{P(\beta \mid A)P(A)}{P(\beta \mid A)P(A) + P(\beta \mid B)P(B) + P(\beta \mid C)P(C)}$$

☐ we know that

- $P(A) = P(B) = P(C) = 1/3$
- $P(\beta \mid A) = 1/2$, $P(\beta \mid B) = 0$ and $P(\beta \mid C) = 1$

☐ therefore:

$$P(A \mid \beta) = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}$$

Three prisoner's dilemma (3)

The solution

- and what about prisoner **C** ? Does his survival probability changes ?

  We want to compute

$$
\begin{aligned}
P(C \,|\, \beta) &= \frac{P(\beta \,|\, C)P(C)}{P(\beta)} \\
&= \frac{P(\beta \,|\, C)P(C)}{P(\beta \,|\, A)P(A) + P(\beta \,|\, B)P(B) + P(\beta \,|\, C)P(C)} \\
&= \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{2}{3}
\end{aligned}
$$

☐ we know that

- $P(A) = P(B) = P(C) = 1/3$
- $P(\beta \,|\, A) = 1/2$, $P(\beta \,|\, B) = 0$ and $P(\beta \,|\, C) = 1$

key points:

➜ **A** is freed: the guard can say that **B** or **C** will be executed

➜ **C** is freed: the guard can only say that **B** will be executed