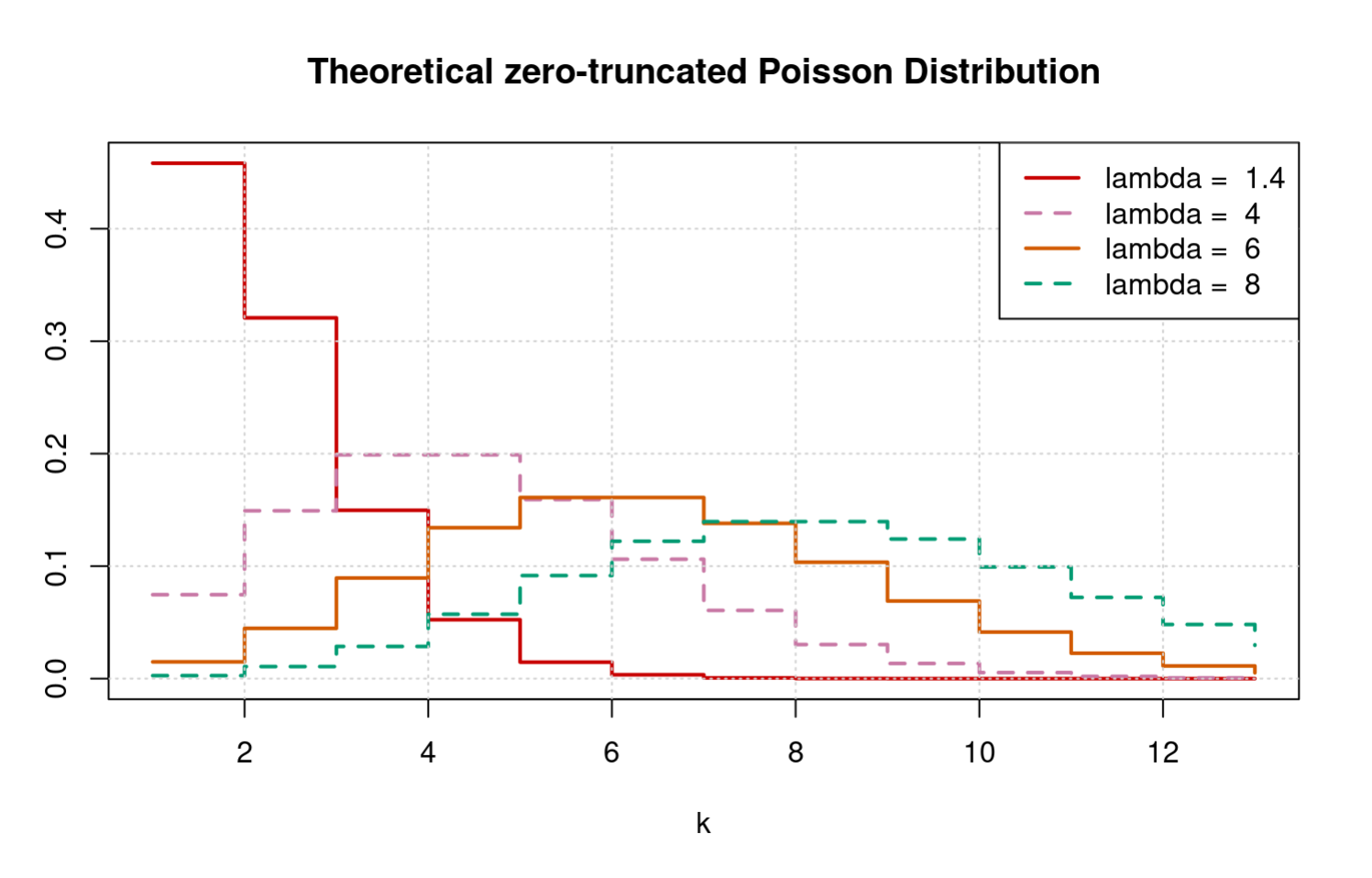
Assignment 2

Exercise 1: Discrete random variable

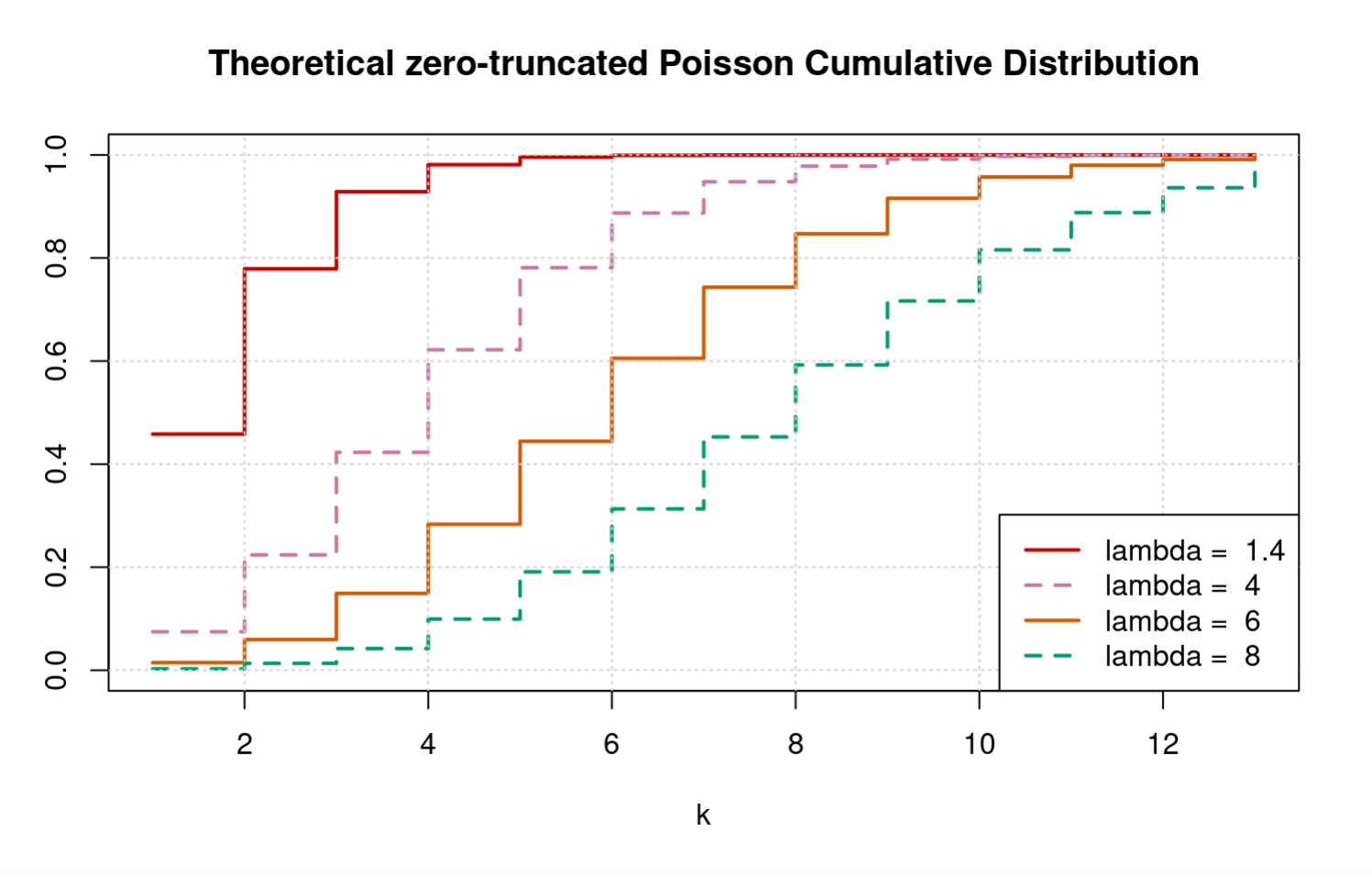
1. Write the R functions for the probability density and cumulative distribution functions, using the R naming convention.
2. Produce two plots showing the pdf and cdf, separately.
3. Compute the mean value and variance of the probability distribution using R. 4. Generate a sample of random numbers from this distribution and show them in an histogram. Evaluate the sample mean.

```
# Parameters for the poisson distribution
lambda_vec <- c(1.4,4,6, 8)
k <- 1:13


integrate_step <- function(f, lower, upper){
  integral = 0
  for(x in lower:(upper-1))
    integral = integral+f(x)
  return(integral)
}

dzeropois <- function(k, lambda){
  #to handle float numbers
  k = trunc(k)
  return(lambda^k*exp(-lambda)/factorial(k)/(1-exp(-lambda)))
}

#designed to handle vectors of q in input
pzeropois <- function(q, lambda) {
  #to handle float numbers
  q = as.integer(trunc(q))
  P_cum <- numeric(length(q))
  for (i in 1:length(q)) {
    k <- 1:q[i]
    P_cum[i] <- sum(dzeropois(k, lambda))
  }
  return(P_cum)
}

# Function used to invert the pzeropois
threshold_search <- function(f, guess, threshold, lambda) {
  x <- as.integer(trunc(guess))
  while(TRUE) {
    if(f(x, lambda) - threshold <= 0) {
      if(f(x + 1, lambda) - threshold >= 0)
        return(x+1)
      else
        x <- x + 1
    }
    else if(x>1)
      x <- x - 1
    else
      return(x)
  }
}

# Function to generate random samples from Poisson distribution
rzeropois <- function(n, lambda) {
  p <- runif(n, min = 1e-10, max = 1-1e-10)
  x <- sapply(p, function(probability) {
    threshold_search(pzeropois, lambda+1, probability, lambda)
  })
  return(x)
}


poisson_pdf <- sapply(lambda_vec, function(L) {dzeropois(k, L)})
poisson_cum <- sapply(lambda_vec, function(L) {pzeropois(k, L)})

linetype <- 2-(1:length(lambda_vec))%%2
```

```
# create the first straight line plot
plot(k,poisson_pdf[,1], type = "s", lwd = 2, col = color_vector[1],
     xlab = "k", ylab = "PDF", lty = linetype[1])

# create all the others
for (i in 2:length(lambda_vec)) {
  lines(k, poisson_pdf[, i], col=color_vector[i], type='s', lwd=2, lty = linetype[i])

}

grid()

# Create legend labels
legend_labels <- paste('lambda = ', lambda_vec)
# Add a legend
legend("topright", legend = legend_labels, col = color_vector[1:length(lambda_vec)], lty = linetype, l
# Add a title
title("Theoretical zero-truncated Poisson Distribution")
```



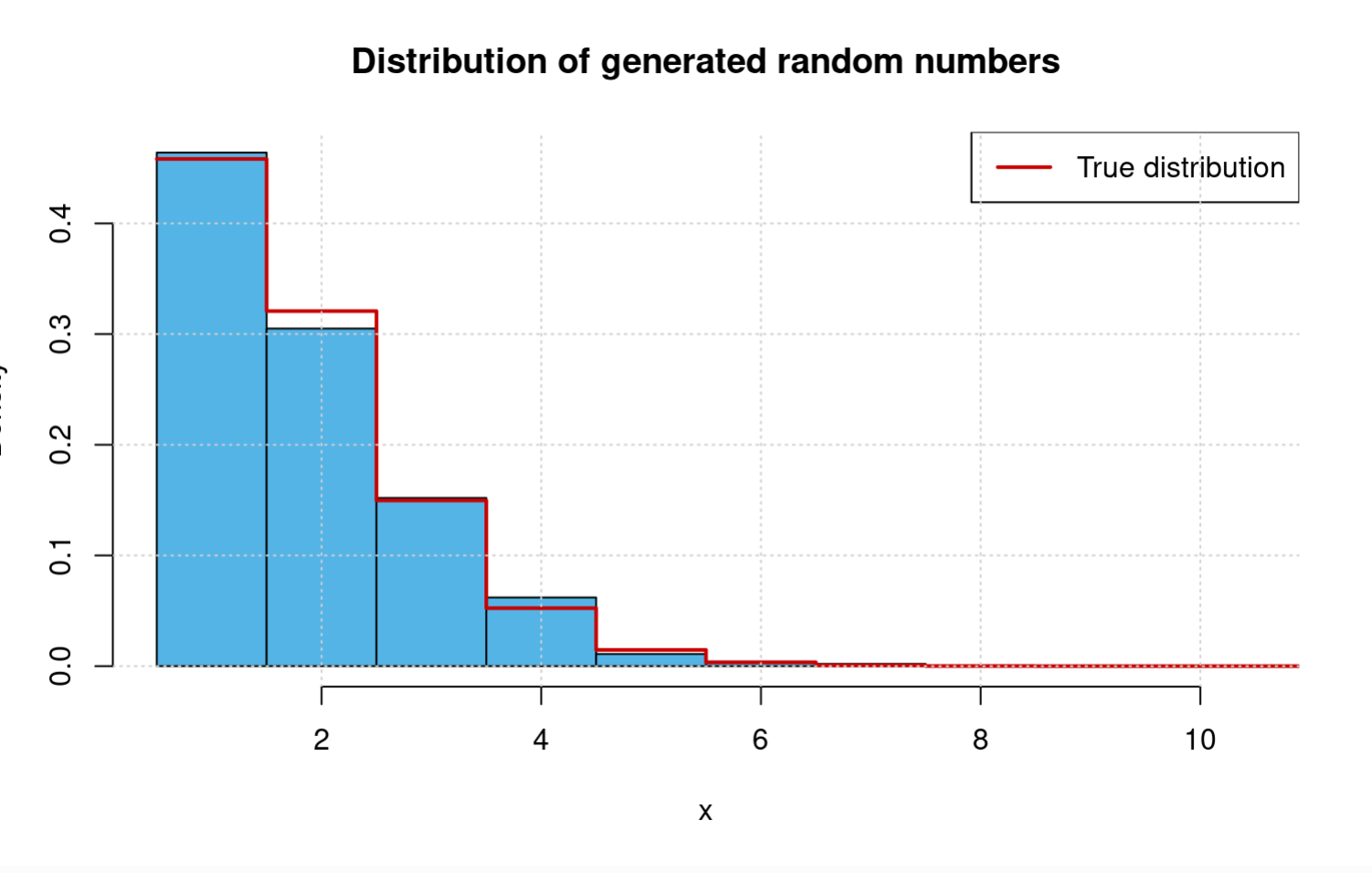

```
# create the first straight line plot
plot(k,poisson_cum[,1], type = "s", lwd = 2, col = color_vector[1],
     xlab = "k", ylab = "P", lty = linetype[1], ylim=c(0,1))

# create all the others
for (i in 2:length(lambda_vec)) {
  lines(k, poisson_cum[, i], col=color_vector[i], type='s', lwd=2, lty = linetype[i], ylim=c(0,1))

}

grid()

# Create legend labels
legend_labels <- paste('lambda = ', lambda_vec)
legend("bottomright", legend = legend_labels, col = color_vector[1:length(lambda_vec)], lty = linetype

title("Theoretical zero-truncated Poisson Cumulative Distribution")
```




```
lambda = 1.4
mean = integrate_step(function(x){x*dzeropois(x,lambda)}, lower = 1, upper = 15)
std  = integrate_step(function(x){x^2*dzeropois(x,lambda)}, lower = 1, upper = 15)

writeLines(sprintf(
"
Taking lambda = 1.4 and truncating the sum at k=15 we can obtain:
  - The mean value is %.2f
  - The standard deviation is %.2f
",
  mean[1], std[1])
)
```

```
##
## Taking lambda = 1.4 and truncating the sum at k=15 we can obtain:
##   - The mean value is 1.86
##   - The standard deviation is 4.46
```

```
n = 1000
random_numbers <- rzeropois(n, lambda)

breaks <- seq(1 - 0.5, 10 + 0.5, by = 1)
#plot
hist(random_numbers, breaks=breaks, col = color_vector[5], xlab = "x", ylab = "Density", freq = FALSE,
lines(k-0.5, poisson_pdf[,1], type='s', lwd = 2, col = color_vector[1], lty = linetype[1])

legend_labels <- paste('True distribution')
legend("topright", legend = legend_labels, col = color_vector[1], lty = linetype[1], lwd = 2)

grid()
title("Distribution of generated random numbers")
```



Exercise 2: Continuous random variable

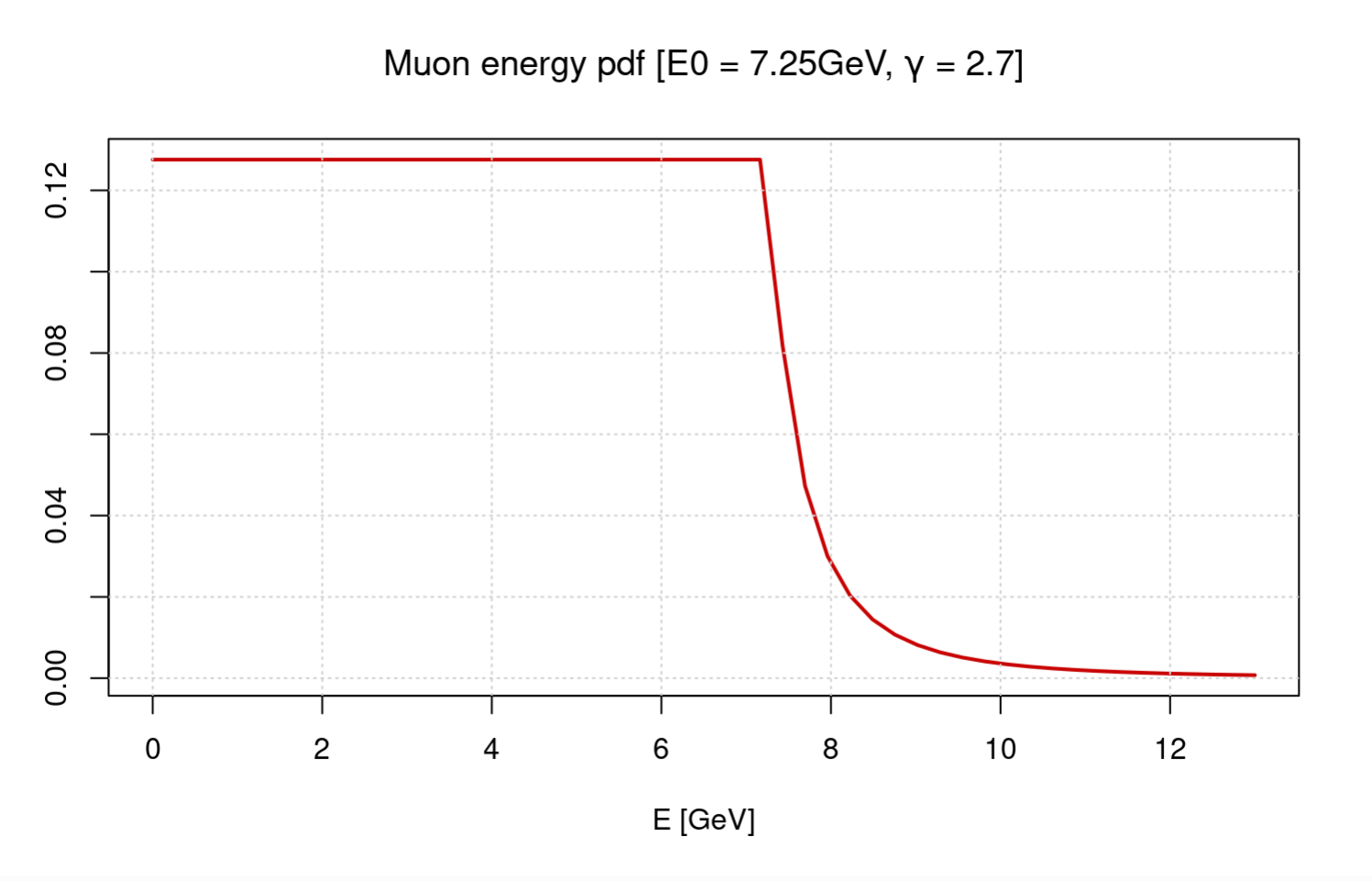
- Considering the energy distribution of CR muons at sea level:
  1. Compute the normalization factor;
  2. Plot the probability density function in R.
  3. Plot the cumulative density function in R.
  4. Compute the mean value using R
  5. [Optional] Generate 106 random numbers from this distribution, show them in an histogram and superimpose the pdf (with a line or with a sufficient number of points).

```
dmuonenergy <- function(E, N, E0, g){ ifelse(test = E<E0, yes = return(N), no = return(N*(E-E0+1)^(-g)
pmuonenergy <- function(q, N, E0, g){ return(integrate(function (x) {sapply(x, function(E) {dmuonenerg


qmuonenergy <- function(p, N, E0, g){ ifelse(p<N*E0, p/N, ((1-g)*(p/N-E0)+1)^(1/(1-g))+E0-1) }


rmuonenergy <- function(n, N, E0, g){
  uniform_sampling <- runif(n, min = 1.e-10, max = 1-1.e-10)
  return(sapply(uniform_sampling, function(p) qmuonenergy(p, N, E0, g)))
}

normalize <- function(E0, g){ pmuonenergy(Inf, 1, E0, g)^-1 }
```

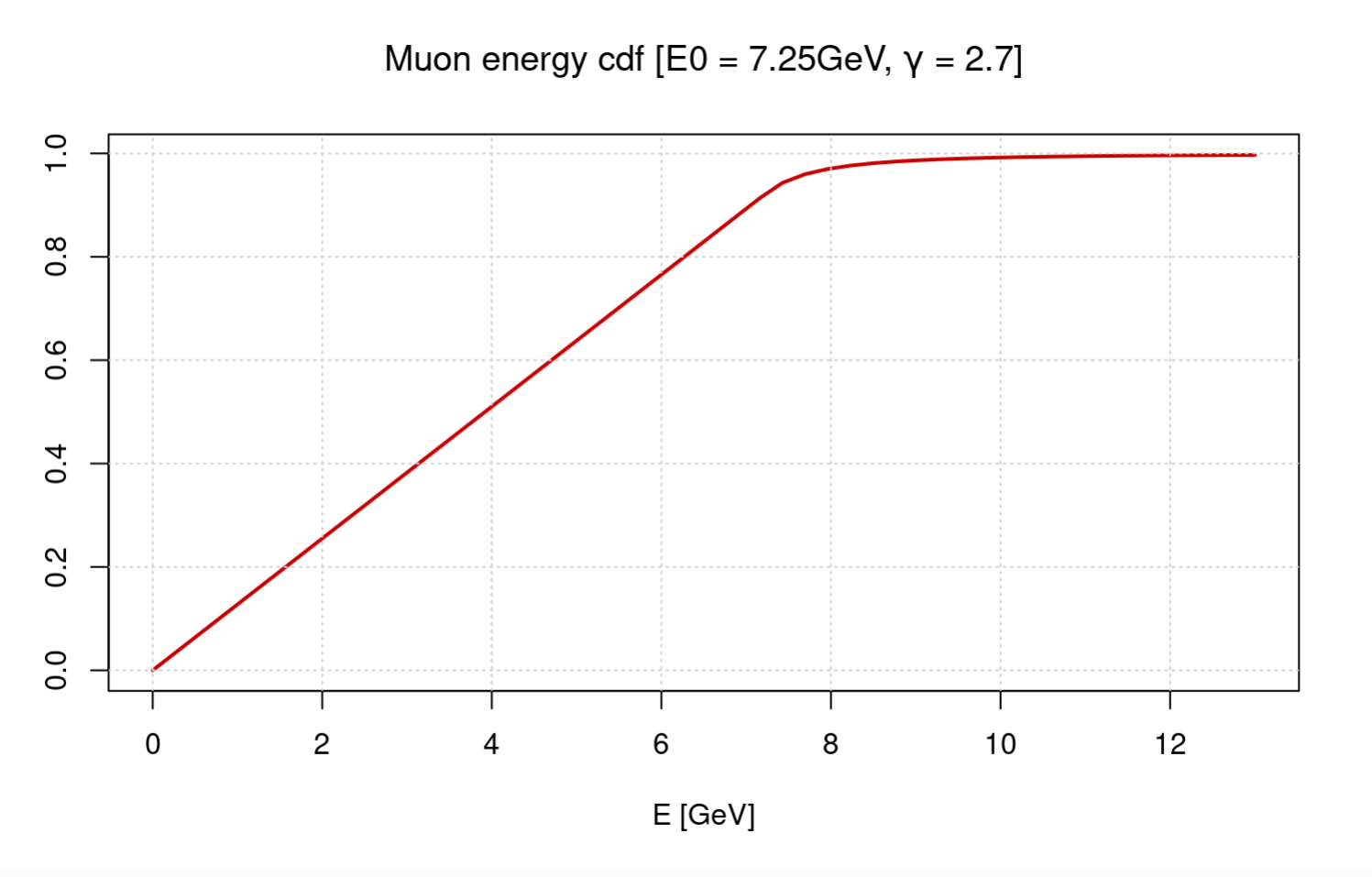
```
E0 <- 7.25 #GeV
g  <- 2.7  #gamma factor

#Compute the normalization factor
N = normalize(E0, g)
print(sprintf("The normalization factor is %.2f", N))
```

```
## [1] "The normalization factor is 0.13"
```

```
#Plotting the pdf
x_plot <- seq(from=0, to=13, length.out=50)
y_plot <- sapply(x_plot, function(E){dmuonenergy(E, N, E0, g)})

plot(x_plot, y_plot, type='l', lwd = 2, col = color_vector[1], lty = linetype[1], xlab="E [GeV]", ylab


grid()
title(expression(paste("Muon energy pdf [E0" , " = 7.25GeV, ", gamma, " = 2.7]")))
```



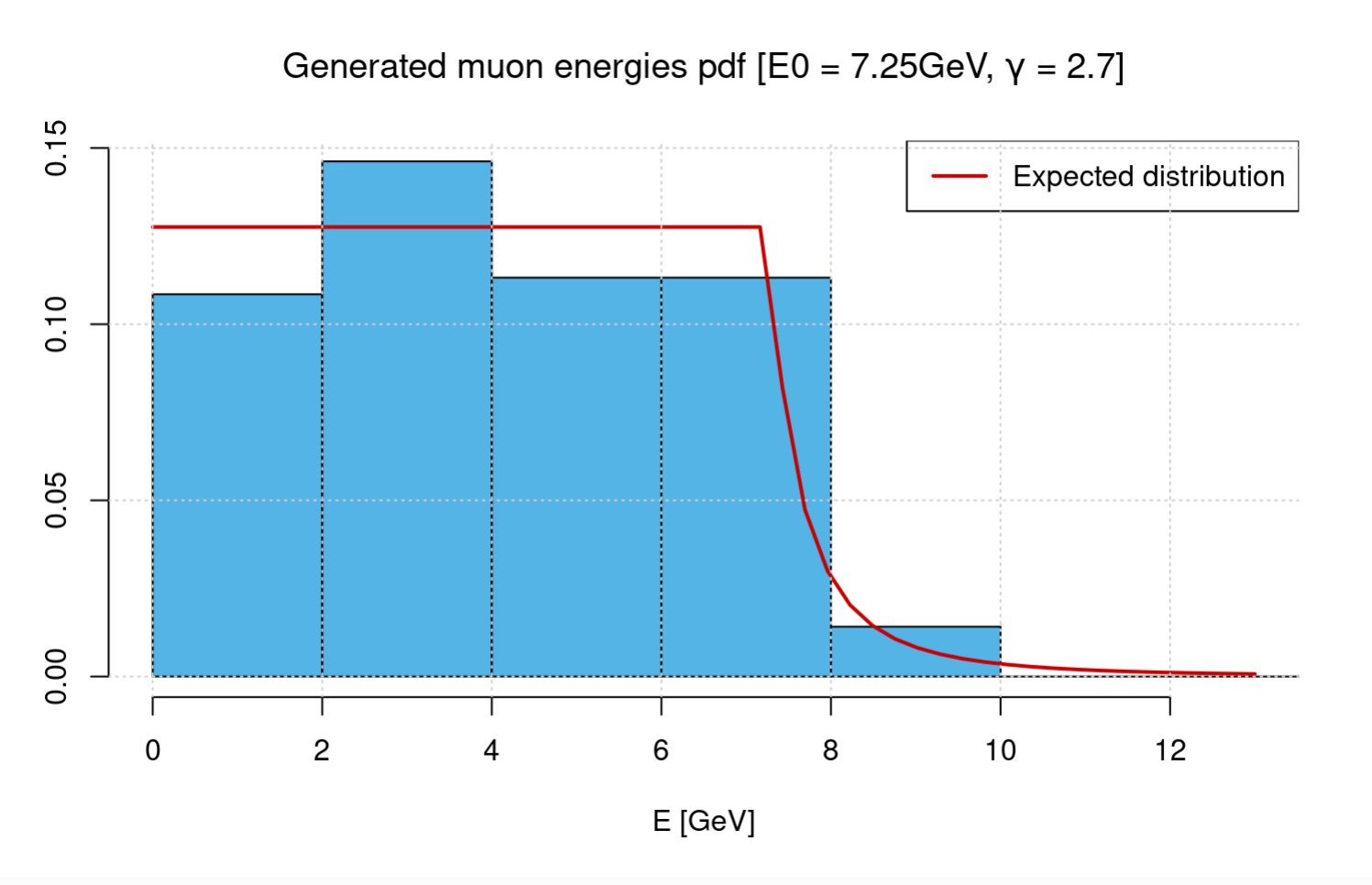

```
#Plotting the cumulative
x_plot <- seq(from=0, to=13, length.out=50)
y_plot <- sapply(x_plot, function(E){pmuonenergy(E, N, E0, g)})

#plot
plot(x_plot, y_plot, type='l', lwd = 2, col = color_vector[1], lty = linetype[1], xlab="E [GeV]", ylab

grid()
title(expression(paste("Muon energy cdf [E0" , " = 7.25GeV, ", gamma, " = 2.7]")))
```




```
#Comute the mean value
mean <- integrate(function(x){ sapply(x, function(E){ E * dmuonenergy(E, N, E0, g) }) }, lower = 0, up
sprintf("The mean value is %.2f", mean)
```

```
## [1] "The mean value is 4.00"
```

```
n=106
generated_energies <- rmuonenergy(n,  N, E0, g)
hist(generated_energies, xlab="E [GeV]", ylab="PDF", freq = FALSE, breaks = 10, xlim=c(0,13), main='',

#Plotting the pdf
x_plot <- seq(from=0, to=13, length.out=50)
y_plot <- sapply(x_plot, function(E){dmuonenergy(E, N, E0, g)})

lines(x_plot, y_plot, type='l', lwd = 2, col = color_vector[1], lty = linetype[1])
legend_labels <- paste('Expected distribution')
legend("topright", legend = legend_labels, col = color_vector[1], lty = linetype[1], lwd = 2)

grid()
title(expression(paste("Generated muon energies pdf [E0" , " = 7.25GeV, ", gamma, " = 2.7]")))
```



Exercise 3

- Suppose that the average number of accidents at an intersection is two per day.
  1. Using Markov's inequality, find a bound for the probability that at least five accidents will occur tomorrow.

  - Denoting with N the number of accidents per day, we are supposing that $E[N] = \mu = 2$. Since $N \geq 0$ by definition, from Markov's inequality the probability that at least five accidents will occur tomorrow is $P(N \geq 5) \leq \frac{\mu}{5} = \frac{2}{5}$.

  2. Using Poisson random variables, calculate the probability that at least five accidents will occur tomorrow. Compare this value with the bound obtained in the previous point a).

  - Recalling that the Poisson pdf is $f(N;\mu) = \frac{\mu^N e^{-\mu}}{N!}$, $P(N \geq 5) = 1 - P(N \leq 4) = 1 - \sum_{N=0}^{4} f(N;\mu) \approx 1 - 0.947 \approx 5.3\%$ which is way under the upper bound of 40%.

  3. Let the variance of the number of accidents be two per day. Using Chebyshev's inequality, find a bound on the probability that tomorrow at least five accidents will occur.

  - Considering $\sigma^2 = 2$, using Chebyshev's inequality $P(N \geq 5) = P(N - 2 \geq 3) = P(N - \mu \geq 3) \leq \frac{\sigma^2}{9} = \frac{2}{9}$

Exercise 4

The waiting period from the time a book is ordered until it is received is a random variable with mean seven days and standard deviation two days. If Helen wants to be 95% sure that she receives a book by certain date, how early should she order the book?

- Defining as T the random variable associated with the waiting period, let's $E[T] = \mu = 7, \sigma^2 = 4$ the mean and the variance, the upper bounds defined by Markov and Chebyshev's inequalities are respectively:
  - $P(T < k) = 1 - P(T \geq k) = 1 - \frac{\mu}{k} \geq \frac{k-7}{k}$;
  - $P(T < k) = 1 - P(T \geq k) = 1 - P(T - \mu \geq k - \mu) \geq 1 - P(|T - \mu| \geq k - \mu) \geq 1 - \frac{\sigma^2}{(k-\mu)^2}$

Plugging in the numbers we obtain the following inequalities:

$P(T < k) \geq 0.95 \rightarrow \frac{k-7}{k} \geq 0.95$. $P(T < k) \geq 0.95 \rightarrow 1 - \frac{4}{(k-7)^2} \geq 0.95 \rightarrow (k-7)^2 \geq 80$.

In particular, from the second condition, we can extrapolate that to reach the 95% of confidence level, the lower bound to the waiting period is about 16 days.

Exercise 5

An ordinary deck of 52 cards is divided randomly into 26 pairs. Using Chebyshev's inequality, find an upper bound for the probability that, at most, 10 pairs consist of a black and a red card.

```
#let's simulate the experiment
N <- 10000 #number of simulated events
deck <- c(replicate(26, 0), replicate(26, 1))
couples <- replicate(N, 0)

for (ie in 1:N) {
  deck <- sample(deck, 52)
    for(ic in seq(1,51, by=2)){
      if(deck[ic]!=deck[ic+1])
        couples[ie]=couples[ie]+1
    }
}

writeLines(sprintf(
'
Simulating %i events:
  - The mean is %.2f;
  - The standard deviation is %.2f.

Using the Chebyshev’s inequality, an upper bound of P(X<11) is: %.2f
',
  N, mean(couples), sd(couples), var(couples)/(10-mean(couples))^2
))
```

```
##
## Simulating 10000 events:
##   - The mean is 13.26;
##   - The standard deviation is 2.58.
##
## Using the Chebyshev’s inequality, an upper bound of P(X<11) is: 0.63
```

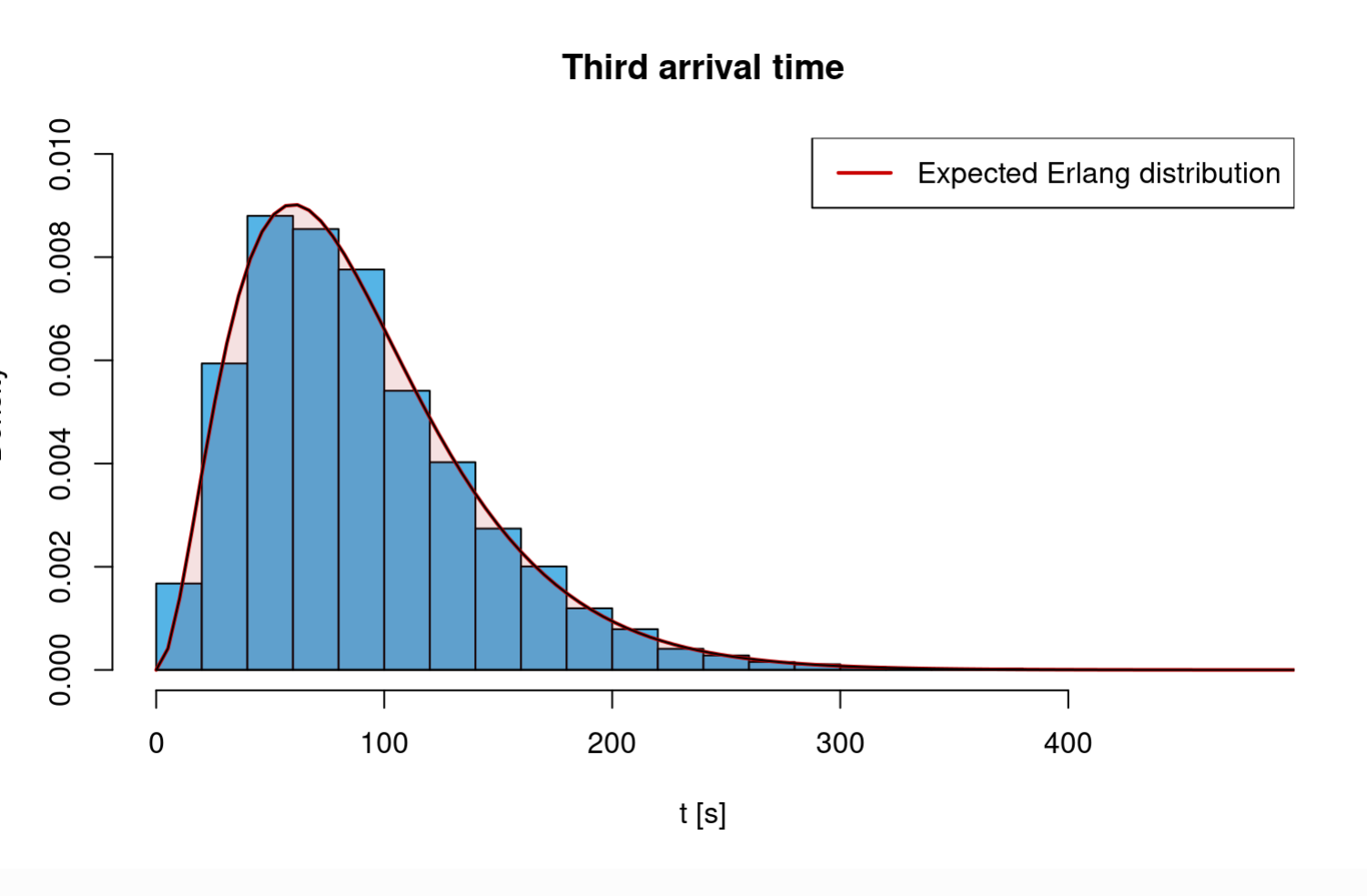
Exercise 6

- In a stationary bus at the departure station, a passenger gets on the bus, on average every 30 seconds.
  1. Compute the probability of getting more than 6 passenger after 2 minutes. Evaluate the probability of having less than 4 passenger after 3 minutes.
  2. Simulate the distribution of the arrival time of the third passenger and superimpose the corresponding pdf.
  3. Repeat the procedure of the point b) for the difference in arrival time between the fifth and the first passenger.

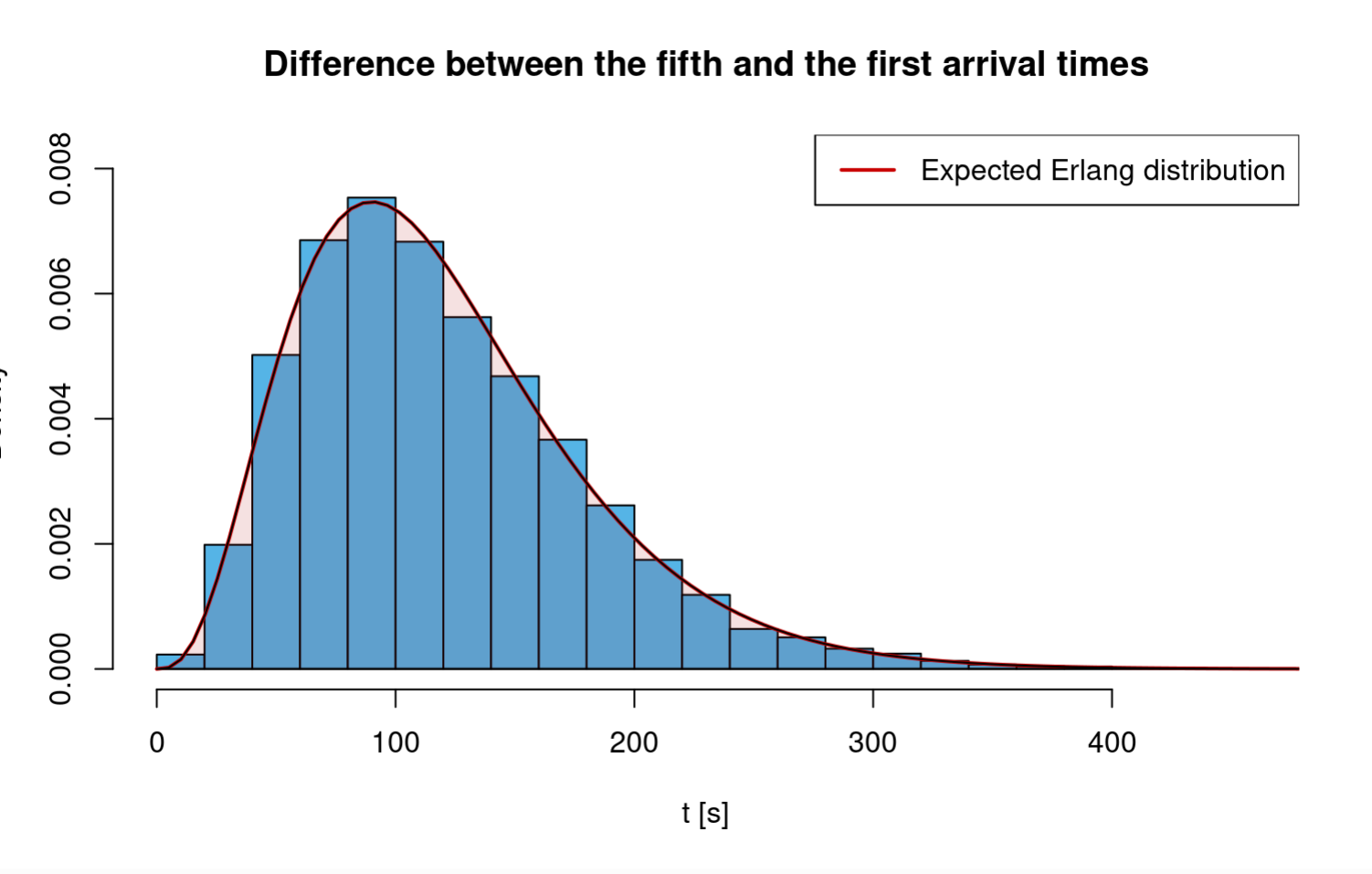
```
#Observations
rate <- 1/30. #passengers/second

#The number of passenger is a random value following a Poisson distribution.
P_more6 <- 1 - ppois(6, lambda = rate*120)
P_less4 <- ppois(3, lambda = rate*180)

writeLines(sprintf(
'
The probability of getting more than 6 passenger after 2 minutes is: %.2f;
The probability of having less than 4 passenger after 3 minutesis: %.2f.
',
  P_more6, P_less4
))
```

```
##
## The probability of getting more than 6 passenger after 2 minutes is: 0.11;
## The probability of having less than 4 passenger after 3 minutesis: 0.15.
```

```
N <- 10000 #number of simulated events

#let's simulate N times the waiting times between the arrival of three passengers
three_passengers_dt <- matrix(rexp(3 * N, rate), nrow = N, ncol = 3)
third_arrival_times <- rowSums(three_passengers_dt)

max_t <- max(third_arrival_times)

x_plot <- seq(from=0, to=max_t*1.1, length.out=100)
y_plot <- dgamma(x_plot, shape = 3, rate = rate)


hist(third_arrival_times, freq = FALSE, col = color_vector[5], xlab = "t [s]", ylab = "Density", main
lines(x_plot, y_plot, type='l', lwd = 2, col = color_vector[1], lty = linetype[1])
polygon(x_plot, y_plot, col = adjustcolor(color_vector[1], alpha.f = 0.1))

legend_labels <- paste('Expected Erlang distribution')
legend("topright", legend = legend_labels, col = color_vector[1], lty = linetype[1], lwd = 2)
```




```
N <- 10000 #number of simulated events

five_passengers_dt  <- matrix(rexp(5 * N, rate), nrow = N, ncol = 5)
first_arrival_times <- five_passengers_dt[,1]
fifth_arrival_times <- rowSums(five_passengers_dt)

first_fifth_time <- fifth_arrival_times - first_arrival_times

max_t <- max(first_fifth_time)

#from a theoretical POV, I can just consider the waiting time of the 4th passenger
x_plot <- seq(from=0, to=max_t*1.1, length.out=100)
y_plot <- dgamma(x_plot, shape = 4, rate = rate)


hist(first_fifth_time, freq = FALSE, col = color_vector[5], xlab = "t [s]", ylab = "Density", main = "
lines(x_plot, y_plot, type='l', lwd = 2, col = color_vector[1], lty = linetype[1])
polygon(x_plot, y_plot, col = adjustcolor(color_vector[1], alpha.f = 0.1))

legend_labels <- paste('Expected Erlang distribution')
legend("topright", legend = legend_labels, col = color_vector[1], lty = linetype[1], lwd = 2)
```



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2024-04-16