PHYSICAL HUMAN-ROBOT INTERACTION

Passivity-based control

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Outline

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Passivity

Interconnection of passive systems

Passivity for LTI systems

Positive real function,
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Scattering Operator

Signals & Systems

Definition ($\mathcal{L}_2$ Spaces)

The function space $\mathcal{L}_2(\mathbb{R}^n)$ is the set of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}^n$ such that

f is a function, so f(t) is a vector (nothing new)

$$\int_{-\infty}^{+\infty} \|f(t)\|^2 dt < \infty$$

where $\|\cdot\|$ is the Euclidean norm ($\|x\| = \sqrt{\langle x, x\rangle_{\mathbb{R}^n}} = \sqrt{x^T x}$).

The space $\mathcal{L}_2(\mathbb{R}^n)$ with the *scalar (or inner) product* $\langle\cdot,\cdot\rangle : \mathcal{L}_2(\mathbb{R}^n) \times \mathcal{L}_2(\mathbb{R}^n) \to \mathbb{R}$

scalar product of functions
(f is a function and g is a function)

$$\langle f, g\rangle := \int_{-\infty}^{+\infty} \langle f(t), g(t)\rangle_{\mathbb{R}^n} dt = \int_{-\infty}^{+\infty} f^T(t) g(t) dt$$

scalar product of vectors

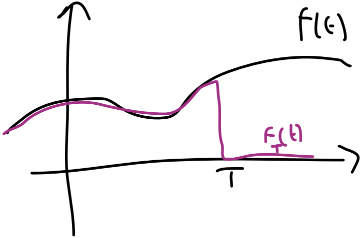
is a *Hilbert space* (i.e. *a real or complex inner product space that is a complete metric space with respect to the distance function induced by the inner product*).

given 2 functions the distance between them can be defined as a function of the inner product
Ci sono spazi in cui questa prop non vale. Negli spazi di hilbert vale

The norm $\|\cdot\|_2$ and the distance $d(\cdot,\cdot)$ can be defined as

$$\|f\|_2 = \sqrt{\langle f, f\rangle}, \qquad d(f,g) = \|f - g\|_2$$

Truncated function



Definition (Truncated function)

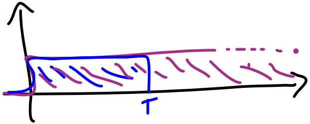
Given the function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ and a time $T \in \mathbb{R}$, the function

$$\begin{aligned} f_T : \mathbb{R} & \rightarrow \mathbb{R}^n \\ t & \mapsto \begin{cases} f(t), & t < T \\ 0, & t \geq T \end{cases} \end{aligned}$$

is called truncation of f on the interval $[-\infty, T]$.

Definition ($\mathcal{L}_{2e}$ Spaces)

The function space $\mathcal{L}_{2e}(\mathbb{R}^n)$ is the set of Lebesgue measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$ such that $f_T \in \mathcal{L}_2(\mathbb{R}^n)$ for all $T \in [-\infty, +\infty)$. It is called the extension of $\mathcal{L}_2$ or the extended $\mathcal{L}_2$ space.



$\delta_{-1}(t) \notin \mathcal{L}_2$

$\delta_{-1}(t)\big|_T \in \mathcal{L}_2$

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Definition (Causal map)

An input-output map $G : \mathcal{L}_{2e}^m(\mathcal{U}) \to \mathcal{L}_{2e}^p(\mathcal{Y})$ is causal if (I dont need th future to compute the past => the truncation does not change the G output)

$$(G(u))_T = (G(u_T))_T, \qquad \forall T \geq 0, u \in \mathcal{L}_{2e}^m(\mathcal{U}).$$

T = truncation

Lemma

The input-output map $G : \mathcal{L}_{2e}^m(\mathcal{U}) \to \mathcal{L}_{2e}^p(\mathcal{Y})$ *is causal if and only if*

$$u, v \in \mathcal{L}_{2e}^m(\mathcal{U}), u_T = v_T \Rightarrow (G(u))_T = (G(v))_T, \forall T \geq 0.$$

Definition (Time-invariant map)

An input-output map $G : \mathcal{L}_{2e}^m(\mathcal{U}) \to \mathcal{L}_{2e}^p(\mathcal{Y})$ is time-invariant if

$$\sigma_\tau G = G\sigma_\tau, \qquad \forall \tau \geq 0,$$

where $\sigma_\tau : \mathcal{L}_{2e}^n(\mathcal{W}) \to \mathcal{L}_{2e}^n(\mathcal{W})$ is the temporal shift operator (i.e. $(\sigma_\tau w)(t) = w(t+\tau)$).

Passivity

Definition (Passivity)

The input-output map G where inputs and outputs have the same dimensions $m = p$

$$G : \mathcal{L}_{2e}(\mathbb{R}^m) \quad \rightarrow \quad \mathcal{L}_{2e}(\mathbb{R}^m) \qquad\qquad u \mapsto y = G(u)$$

is passive if there exists a constant β such that

$$\int_0^T y^T(s)u(s)ds \geq -\beta, \qquad\qquad \forall T \geq 0, \forall u(\cdot), \tag{1}$$

What is the sign of β?

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What is the sign of β?

Since the inequality (1) must hold for any $u \in \mathcal{L}_{2e}$, and so also for $u \equiv 0$, we have $\beta \geq 0$.

Is the integral well defined for all finite T?

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passivity is a property of G, it does not depend on the input

What is the sign of β?

Since the inequality (1) must hold for any $u \in \mathcal{L}_{2e}$, and so also for $u \equiv 0$, we have $\beta \geq 0$.

Is the integral well defined for all finite T? $\infty \overset{\checkmark}{>} \int_0^T \cdots\cdots$

Yes, because since $u, y \in \mathcal{L}_{2e}$, the following inequality (Schwarz) holds

$$\int_0^T y^T(s)u(s)ds \leq \int_0^T [y^T(s)y(s) + u^T(s)u(s)]ds.$$

Passivity - Meaning -

If u and y are conjugate variables (e.g., current-voltage, linear velocity-force, angular velocity-torque)

e.g. their product is a "power"

$\downarrow$

$y^T(t)u(t)$ is the power at time t

$\downarrow$

$\int_0^T y^T(s)u(s)ds$ is the energy produced by the system over the time interval $[0, T]$

$\downarrow$

$\int_0^T y^T(s)u(s)ds \geq -\beta$, Passivity means that the amount of energy dissipated by the system has a lower bound given by $-\beta$

$\downarrow$

The energy that can be extracted from a passive system is equal or smaller than the one provided to it plus its initial energy

Passivity - Meaning -

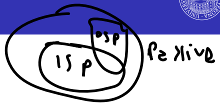
An electric circuit which consists of passive components (resistors, capacitors, inductors) is *passive*

A mechanical system build up on passive components (masses, springs, dampers) is *passive*

$$\downarrow$$

Interconnection will play a key role

Strict Passivity



Definition (Input Strictly Passive (ISP))

The input-output map G where inputs and outputs have the same dimensions $m = p$

$$G : \mathcal{L}_{2e}(\mathbb{R}^m) \quad \rightarrow \quad \mathcal{L}_{2e}(\mathbb{R}^m) \qquad\qquad u \mapsto y = G(u)$$

is Input Strictly Passive (ISP) if there exist two constants β and $\delta > 0$ such that

$$\underbrace{\int_0^T y^T(s)u(s)ds \geq -\beta}_{\text{prev. condition}} + \underbrace{\delta \int_0^T u^T(s)u(s)ds,}_{\text{additional condition}} \qquad \forall T \geq 0, \forall u(\cdot), \tag{2}$$

Definition (Output Strictly Passive (OSP))

[...] is Output Strictly Passive (OSP) if there exist two constants β and $\varepsilon > 0$ such that

$$\int_0^T y^T(s)u(s)ds \geq -\beta + \varepsilon \int_0^T y^T(s)y(s)ds, \qquad \forall T \geq 0, \forall u(\cdot). \tag{3}$$

Strict Passivity

According to the definition of truncated functions and inner product, we can write

$$
\begin{aligned}
\int_0^T y^T(s)u(s)ds &= \int_0^\infty y_T^T(s)u_T(s)ds = \langle y_T, u_T\rangle =: \langle y, u\rangle_T \\
\int_0^T u^T(s)u(s)ds &= \|u_T\|^2 =: \|u\|_T^2 \\
\int_0^T y^T(s)y(s)ds &= \|y_T\|^2 =: \|y\|_T^2
\end{aligned}
$$

and so

$$
\begin{aligned}
G \text{ is passive} \quad &\Leftrightarrow \quad \exists \beta \text{ such that } \langle y, u\rangle_T \geq -\beta, \forall T \geq 0, \forall u \in \mathcal{L}_{2e}(\mathcal{U}) \\
G \text{ is ISP} \quad &\Leftrightarrow \quad \exists \beta, \delta > 0 \text{ such that } \langle y, u\rangle_T \geq -\beta + \delta\|u\|_T^2, \forall T \geq 0, \forall u \in \mathcal{L}_{2e}(\mathcal{U}) \\
G \text{ is OSP} \quad &\Leftrightarrow \quad \exists \beta, \varepsilon > 0 \text{ such that } \langle y, u\rangle_T \geq -\beta + \varepsilon\|y\|_T^2, \forall T \geq 0, \forall u \in \mathcal{L}_{2e}(\mathcal{U})
\end{aligned}
$$

Differential form of (Strict) Passivity

If there exists a function $V(\cdot) \in \mathcal{C}$, $V(\cdot) \geq 0$ such that

$$V(t) - V(0) \leq \int_0^t y^T(s)u(s)ds, \qquad \forall t \geq 0, \forall u(\cdot), \forall V(0)$$

then the input-output system $G : \mathcal{L}_{2e}(\mathbb{R}^m) \rightarrow \mathcal{L}_{2e}(\mathbb{R}^m)$ is passive.

The initial condition $V(0)$ plays the role of the constant β in the previous definition.

Theorem

Let $V(\cdot) \geq 0$ be a continuously differentiable function and there exists a measurable function $D(t)$ such that $\int_0^t D(s)ds \geq 0$, $\forall t \geq 0$ (i.e. dissipative term). Then

$$G \text{ is passive} \quad \Leftarrow \quad \dot{V}(t) \leq y^T(t)u(t) - D(t), \qquad \forall t \geq 0, \forall u(\cdot)$$

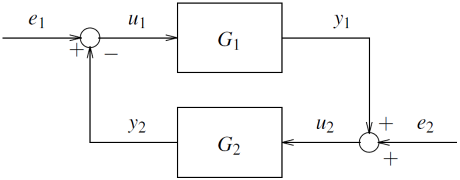
$$G \text{ is ISP} \quad \Leftarrow \quad \exists \delta > 0,\ \dot{V}(t) \leq y^T(t)u(t) - \delta u^T(t)u(t) - D(t), \forall T \geq 0, \forall u(\cdot)$$

$$G \text{ is OSP} \quad \Leftarrow \quad \exists \varepsilon > 0,\ \dot{V}(t) \leq y^T(t)u(t) - \varepsilon y^T(t)y(t) - D(t), \forall T \geq 0, \forall u(\cdot)$$

Interconnection of passive systems

Theorem

Let $G_1 : \mathcal{L}_{2e}(\mathbb{R}^m) \to \mathcal{L}_{2e}(\mathbb{R}^m)$, $G_2 : \mathcal{L}_{2e}(\mathbb{R}^m) \to \mathcal{L}_{2e}(\mathbb{R}^m)$ be two passive systems interconnected via feedback



with exogenous inputs $e_1, e_2 \in \mathcal{L}_{2e}(\mathbb{R}^m)$ such that $u_1, u_2 \in \mathcal{L}_{2e}(\mathbb{R}^m)$. Then the closed-loop system with input $e := \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and output $y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is passive.

Proof.

Given $T \geq 0$, the following relations hold

$$\begin{aligned}
\langle y, e \rangle_T &= \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T \\
&= \langle y_1, u_1 + y_2 \rangle_T + \langle y_2, u_2 - y_1 \rangle_T \\
&= \langle y_1, u_1 \rangle_T + \langle y_2, u_2 \rangle_T + \langle y_1, y_2 \rangle_T + \langle y_2, -y_1 \rangle_T,
\end{aligned}$$

where the last two terms cancel out. Since G_1 and G_2 are passive, we have

$$\langle y_1, u_1 \rangle_T \geq -\beta_1 \qquad \langle y_2, u_2 \rangle_T \geq -\beta_2$$

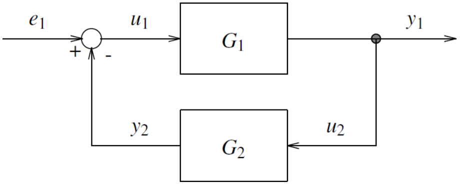
and so

$$\langle y, e \rangle_T \geq -\beta := -\beta_1 - \beta_2.$$

which proves the passivity of the feedback connection. $\square$

Remark

The previous block diagram becomes the standard feedback connection

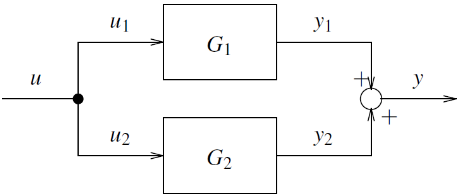


when e_2 is equal to zero.

Following the same line of reasoning, it is possibile to prove the passivity of such system with input $e = e_1$ and output $y = y_1$.

Theorem

Let $G_1 : \mathcal{L}_{2e}(\mathbb{R}^m) \to \mathcal{L}_{2e}(\mathbb{R}^m)$, $G_2 : \mathcal{L}_{2e}(\mathbb{R}^m) \to \mathcal{L}_{2e}(\mathbb{R}^m)$ be two passive system interconnected in parallel.



The parallel system with input u and output $y = y_1 + y_2$ is passive.

Proof.

Given $T \geq 0$, the following relations hold

$$\begin{aligned}\langle y, u\rangle_T &= \langle y_1 + y_2, u\rangle_T \\ &= \langle y_1, u_1\rangle_T + \langle y_2, u_2\rangle_T \\ &\geq -\beta_1 - \beta_2 =: -\beta\end{aligned}$$

beta = b1+b2 makes sense if we think of beta as the initial energy of the system
beta must be the sum of the initial energy of the singular systems

which prove the passivity . $\square$

Remark

The series connection of passive system can be NOT passive.

Passivity for LTI systems

LTI systems

If the input-output map $G : \mathcal{L}_{2e}(\mathbb{R}^m) \to \mathcal{L}_{2e}(\mathbb{R}^m)$ is a LTI system, the conditions on passivity can be given in term of the corresponding transfer function $G(s)$.

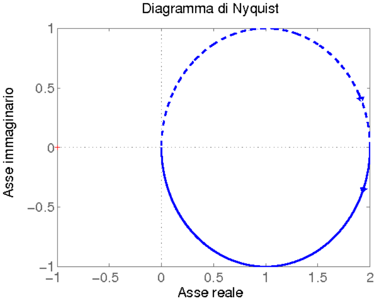
If $G(s)$ is LTI and SISO we can resort to the Nyquist diagram where

$\mathrm{Re}\{G(j\omega)\}$ are on the x axis

$\mathrm{Im}\{G(j\omega)\}$ are on the y axis

Passivity implies stability, but not the contrary

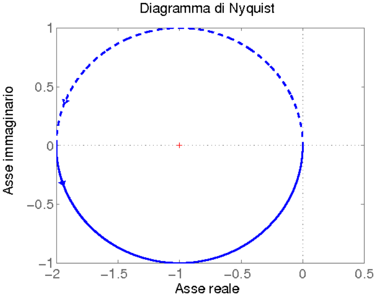
PASSIVE => STABLE



$$F(s) = \frac{2}{s+1}$$

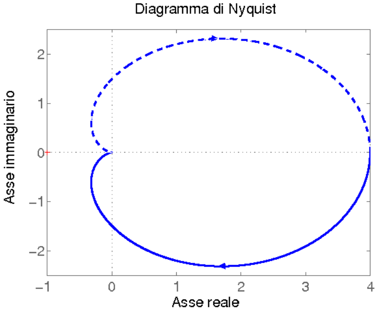
F(s) is bibo stable, so I can use the theorem (next slides)

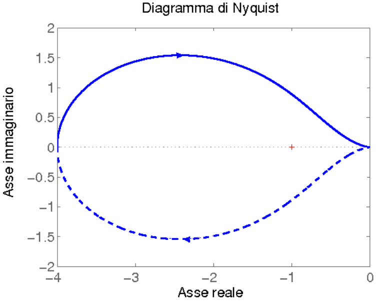
UNSTABLE => NOT PASSIVE



$$F(s) = \frac{2}{s-1}$$

F(s) not BIBO stable, so I cannot use the theorem stable but not passive





$$F(s) = \frac{20}{(s+1)(s+5)}$$

BIBO stable

$$F(s) = \frac{20}{(s+1)(s-5)}$$

not BIBO stable

Exercise

Using the matlab function `nyquist.m`, plot the Nyquist diagram of the different kinds of PID controllers and of the DC motor.

Theorem

Let $G(s)$ be the transfer function of a SISO LTI BIBO stable system. Then:

- *the system is passive if and only if* the nyqiuist diagram is in the Right Half Plane

$$\mathrm{Re}\{G(j\omega)\} \geq 0, \qquad \forall \omega \in [-\infty, +\infty]; \tag{4}$$

- *the system is input strictly passive (ISP) if and only if*

$$\exists \delta > 0, \text{ tale che } \mathrm{Re}\{G(j\omega)\} \geq \delta > 0, \qquad \forall \omega \in [-\infty, +\infty]; \tag{5}$$

- *the system is output strictly passive (OSP) if and only if*

$$\exists \varepsilon > 0, \text{ tale che } \mathrm{Re}\{G(j\omega)\} \geq \varepsilon |G(j\omega)|^2, \qquad \forall \omega \in [-\infty, +\infty]. \tag{6}$$

Proof.

To prove the theorem we need to resort to a well know result of Fourier analysis called Parseval's theorem.
This theorem relates the integral over time (e.g. passivity) with the integral of the harmonic functions over frequency

Theorem (Teorema di Parseval)

Let $f, g \in \mathcal{L}_2(\mathbb{R})$ be two functions with Fourier transforms $F(j\omega), G(j\omega) \in \mathcal{L}_2(j\mathbb{R})$. The following equation holds

$$\int_{-\infty}^{+\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)G^*(j\omega)d\omega.$$

In the present case we have

$\square$

Proof.

$$\langle y, u\rangle_T = \int_0^T y_T(s)u_T(s)ds = \int_{-\infty}^{+\infty} y_T(s)u_T(s)ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y_T(j\omega)U_T^*(j\omega)d\omega.$$

Since the system is causal we have $Y_T(j\omega) = G(j\omega)U_T(j\omega)$ and so

$$\begin{aligned}
\langle y, u\rangle_T &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)U_T(j\omega)U_T^*(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \{\mathrm{Re}\{G(j\omega)\} + j\mathrm{Im}\{G(j\omega)\}\}\,|U_T(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}\{G(j\omega)\}|U_T(j\omega)|^2 d\omega
\end{aligned}$$

where the last equation is due to the fact that the left term of the equation is a real number. □

Theorem

Let $G(s)$ be the transfer function of a MIMO LTI BIBO stable system. Then:

- *the system is passive if and only if*

$$\lambda_{min}\{G(j\omega)+G^*(j\omega)\} \geq 0, \qquad \forall \omega \in [-\infty, +\infty]; \tag{8}$$

- *the system is input strictly passive (ISP) if and only if*

$$\exists \delta > 0, \text{ such that } \lambda_{min}\{G(j\omega)+G^*(j\omega)\} \geq \delta, \qquad \forall \omega \in [-\infty, +\infty]. \tag{9}$$

Theorem

Let $G(s)$ be the transfer function of a SISO LTI system. The system is passive if and only if

1. *$G(s)$ has no poles on the right half plane*
2. $\mathrm{Re}\{G(j\omega)\} \geq 0, \forall \omega \in [-\infty, +\infty]$ *such that $j\omega$ is not a pole of $G(s)$,*
3. *If $j\omega_0$ is a finite pole of $G(s)$, then it must be simple and the residual must be strictly positive.*

$$\lim_{s \to j\omega_0} (s - j\omega_0)G(s) > 0;$$

if $j\omega_0$ is a pole of $G(s)$ at infinity (i.e. degree of the polynomial at the numerator strictly larger than degree of the polynomial at the denominator), then it must be simple and the residual must be strictly positive

$$\lim_{\omega \to \infty} \frac{G(j\omega)}{j\omega} > 0.$$

Positive real function, KYP Lemma

Positive real function

Definition (Positive real)

A transfer function $G(s)$ is positive real (PR) if

1. $G(s)$ is analytic in $\mathrm{Re}\{s\} > 0$,
2. $G(s)$ is real for $s \in \mathbb{R}_+$ (i.e. positive real s),
3. $\mathrm{Re}\{G(s)\} \geq 0$, $\forall s$ such that $\mathrm{Re}\{s\} > 0$.
   [$G(s) + G^*(s) \geq 0$, $\forall s$ such that $\mathrm{Re}\{s\} > 0$.]

Definition (Strictly positive real)

A transfer function $G(s)$ is strictly positive real (SPR) if there exists $\varepsilon > 0$ such that $G(s - \varepsilon)$ is positive real.

Theorem

Let $G(s)$ be a SISO LTI transfer function with $G(s) \neq 1, \forall s$ such that $\mathrm{Re}\{s\} > 0$. The following items are equivalent

- *$G(s)$ is passive,*
- *$G(s)$ is positive real,*
- *the Popov function $\Pi(s) := G^*(s) + G(s) = G^T(-s) + G(s)$ satisfies*

$$y^* \Pi(j\omega) y = y^* \left(G^*(j\omega) + G(j\omega) \right) y \geq 0, \qquad \forall \omega \in \mathbb{R}, \forall y \in \mathbb{C},$$

if $G(s)$ is BIBO stable (no poles on $j\mathbb{R}$)

Definition (Positive/negative (semi)definite function)

Let $W \subseteq \mathcal{X} = \mathbb{R}^n$ be an open set containing the origin. A continuous function $V : W \to \mathbb{R}$ is

- positive semi-definite if $V(0) = 0$ and there exists an open set containing the origin where V is non-negative (i.e. positive or zero),
- positive definite if it is positive semi-definite and is equal to zero only in the origin,
- negative semi-definite if $-V$ is positive semi-definite,
- negative definite if $-V$ is positive definite,
- indefinite otherwise.

For quadratic functions

$$V(x) = x^T P x.$$

the analysis of the type can be done by studying the symmetric matrix $P \in \mathbb{R}^{n\times n}$.

Definition (Positive/negative (semi)definite matrix)

A symmetric square matrix $P = P^T \in \mathbb{R}^{n\times n}$ is

- positive definite if all its eigenvalues are strictly positive,
- positive semi-definite if all its eigenvalues are non-negative,
- negative definite if all its eigenvalues are strictly negative,
- negative semi-definite if all its eigenvalues are non-positive,
- indefinite otherwise.

The Lyapunov stability theory (*Lyapunov's second method for stability*) is based on the study of a *candidate Lyapunov function*

$$\begin{aligned} V : W \subseteq \mathbb{R}^n &\rightarrow \mathbb{R} \\ x &\mapsto V(x) \end{aligned}$$

and of its time derivative along the trajectory of the system.

The gradient of $V(x)$ with respect to the state x is

$$\nabla V(x) = \left[\frac{\partial V}{\partial x}\right]^T = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}^T \tag{10}$$

whereas its time derivative will be

$$\begin{aligned} \dot{V} : W \subseteq \mathbb{R}^n &\rightarrow \mathbb{R} \\ x &\mapsto \frac{dV}{dt}. \end{aligned}$$

Since $\dot{x}(t) = f(x(t))$, it is possible to write

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \nabla V(x(t))^T f(x(t))$$

where we assumed

- V continuous with its derivative in W ($V \in \mathcal{C}^1(W)$)
- f continuous in W ($f \in \mathcal{C}^0(W)$)

Theorem (Lyapunov stability criterion or the Direct Method)

Let $x = 0$ be an equilibrium point for the autonomous system $\dot{x}(t) = f(x(t))$ and $V : W \to \mathbb{R}$ be a positive definite continuously differentiable function in the open set W containing the origin. Then

- *if $\dot{V} : W \to \mathbb{R}$ is negative semi-definite, then the origin $x = 0$ is stable;*
- *if $\dot{V} : W \to \mathbb{R}$ is negative definite, then the origin $x = 0$ is asymptotically stable.*

Example

Given the autonomous system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2^5 \end{aligned}$$

with the origin as equilibrium point ($f(0) = 0$). Is the origin a stable equilibrium point or not?

Let $V(x_1, x_2) = x_1^2 + x_2^2$ be the candidate Lyapunov function ($V = x^t P x$ with $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ positive definite). Computing the derivative along the trajectory of the system we get

$$\dot{V}(x_1, x_2) = 2x_1 x_2 + 2x_2(-x_1 - x_2^5) = -2x_2^6.$$

$\dot{V}(x_1, x_2)$ is negative semi-definite and so we can conclude that the origin is at least stable.

Theorem (Lyapunov equation)

The system $\dot{x}(t) = Ax(t)$, $A \in \mathbb{R}^{n\times n}$ is asymptotically stable (all the eigenvalues of A have negative real part) if and only if for all matrices $Q = Q^T \in \mathbb{R}^{n\times n}$ positive definite, there exists a unique solution $P = P^T \in \mathbb{R}^{n\times n}$ positive definite to the matrix equation

$$A^T P + PA = -Q. \tag{11}$$

Example

Given the autonomous system

$$\dot{x} = \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix} x.$$

Let choose the positive definite matrix $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Solving the Lyapunov equation

$$\begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix}^T P + P \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in the unknown matrix P with the `lyap.m` Matlab function, we have

$$P = \begin{bmatrix} 0.1833 & 0.1167 \\ 0.1167 & 0.3833 \end{bmatrix}$$

which has strictly positive eigenvalues (i.e. P is positive definite). Then system is asymptotically stable (in fact the eigenvalues of A are -2 e -3).

Lemma (Kalman–Yakubovich–Popov Lemma)

Given the MIMO LTI transfer function $G(s) \in \mathcal{R}^{m\times m}$ *with minimal (reachable and observable) state-space representation*

$$\begin{cases} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx + Du \end{cases}$$

and let $n = \dim\{x\}$ *be the size of the vector space. The following statements are equivalent:*

- $G(s)$ *is positive real (PR),*
- *there exist matrices* $P \in \mathbb{R}^{n\times n}$, $P = P^T \succ 0$ *and* $L \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$ *such that*

$$PA + A^T P = -LL^T \tag{12}$$

$$PB - C^T = -LW \tag{13}$$

$$D + D^T = W^T W. \tag{14}$$

Exercise

Compute the state-space representation for the PID controller and DC motor (`ss.m`) and check if they are or not PR.

Lemma (Kalman–Yakubovich–Popov Lemma for SPR systems)

Given the MIMO LTI transfer function $G(s) \in \mathcal{R}^{m\times m}$ with minimal (reachable and observable) state-space representation

$$\begin{cases} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx + Du \end{cases}$$

and let $n = \dim\{x\}$ be the size of the vector space. The following statements are equivalent:

- *$G(s)$ is strictly positive real (SPR),*
- *there exist matrices $P \in \mathbb{R}^{n\times n}$, $P = P^T \succ 0$ and $L \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$, and a positive constant ε such that*

$$PA + A^T P = -LL^T - 2\varepsilon P \tag{15}$$

$$PB - C^T = -LW \tag{16}$$

$$D + D^T = W^T W. \tag{17}$$

Scattering Operator

Definition

A bilinear form on the finite-dimensional space $V \subseteq \mathbb{R}^n$ on the field $\mathbb{R}$ is a bilinear function

$$\ll \cdot, \cdot \gg : V \times V \mapsto \mathbb{R}$$

i.e. a function which is linear with respect to each variable considered separately:

$$\begin{aligned} \ll v_1 + v_2, w \gg &= \ll v_1, w \gg + \ll v_2, w \gg \\ \ll v, w_1 + w_2 \gg &= \ll v, w_1 \gg + \ll v, w_2 \gg \\ \ll \lambda v, w \gg &= \ll v, \lambda w \gg = \lambda \ll v, w \gg . \end{aligned}$$

The bilinear form is symmetric if

$$\ll v, w \gg = \ll w, v \gg, \qquad \forall v, w \in V.$$

Let $(U, \mathbb{R})$, $U \subseteq \mathbb{R}^m$ be a *vector space*. Its *dual space* $U^\star$ is the vector space of all linear functionals on U, together with the vector space structure of pointwise addition and scalar multiplication by constants.

The bilinear form we are interested in is defined on the vector space $V = U \times U^\star$

$$\ll \cdot, \cdot \gg : (U \times U^\star) \times (U \times U^\star) \mapsto \mathbb{R}$$

and is defined using the *dual product* $\langle \cdot | \cdot \rangle$ on U and $U^\star$

bilinear form: $$\ll (u_1, u_1^\star), (u_2, u_2^\star) \gg := \langle u_1^\star | u_2 \rangle + \langle u_2^\star | u_1 \rangle. \tag{18}$$

If $U = \mathbb{R}^m$, then $U = U^\star = \mathbb{R}^m$ and so the dual product is the standard inner product $\langle u_1^\star | u_2 \rangle = \langle u_1^\star, u_2 \rangle = (u_1^\star)^T u_2$. The bilinear form can be written as

$$\langle u_1^\star | u_2 \rangle_{\mathbb{R}^m \times \mathbb{R}^m} + \langle u_2^\star | u_1 \rangle_{\mathbb{R}^m \times \mathbb{R}^m} \quad = \quad \langle u_1^\star, u_2 \rangle_{\mathbb{R}^m} + \langle u_2^\star, u_1 \rangle_{\mathbb{R}^m}$$

Scattering

Given the input-output map

$$G : U \to Y, \qquad U^* = Y$$

with $U = \mathbb{R}^m$, $Y = \mathbb{R}^m$.

Let $\{e_1, \ldots, e_m\}$ be a basis of the input vector space U and $\{e'_1, \ldots, e'_m\}$ be the corresponging dual basis[1] of $Y = U^\star$, the matrix representation of the bilinear form (18) is

$$\ll (u_1, y_1), (u_2, y_2) \gg = \begin{bmatrix} u_1^T & y_1^T \end{bmatrix} \underbrace{\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}.$$

with $u_1, u_2 \in U$, $y_1, y_2 \in Y$.

The matrix B is simmetric and m eigenvalues equal to $+1$ m eigenvalues equal to -1.

Scattering is based on decomposing the combined vector $(u, y) \in U \times Y$ with respect to the positive and negative singular values of the bilinear form.

[1] $e'_i(e_j) = \delta_{ij}$, nel caso $U = U^\star = \mathbb{R}^m$ è il prodotto scalare $(e'_i)^T e_j = \delta_{ij}$

Definition (Scattering subspaces)

Any pair (W, Z) of subspaces $W, Z \subset U \times Y$ is called a pair of scattering subspaces if

(i) $W \oplus Z = U \times Y$

(ii) $\ll w_1, w_2 \gg > 0$, for all $w_1 \neq 0$, $w_2 \neq 0$, $w_1, w_2 \in W$

(iii) $\ll z_1, z_2 \gg < 0$, for all $z_1 \neq 0$, $z_2 \neq 0$, $z_1, z_2 \in Z$

(iv) $\ll w, z \gg = 0$, for all $w \in W$, $z \in Z$.

By the definition, it follow that both scattering subspaces have dimension m and are orthogonal

$$W \perp Z$$

Then the $\oplus$ in

$$U \times Y = W \oplus Z$$

is the *orthogonal direct sum*.

m evalues = +1 -> dim W = m

m evalues = -1 -> dim Z = m

dim W + dim Z = dim UxY = 2m

The subspace $W \subset U \times Y$ refers to the positive eigenvalues of the bilinear form, whereas $Z \subset U \times U^{\star}$ refers to the negative eigenvalues.

$\rightarrow W \oplus Z$ is a sort of change of coordinate of $U \times Y$

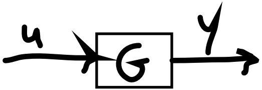
Each pair of vectors $(u, y) \in U \times Y$ can be written *in a unique way* as a pair of vectors $(w, z) \in W \oplus Z$.

- w is the projection along Z of the combined vector $(u, y) \in U \times Y$ on W
- z is the projection along W of the combined vector $(u, y) \in U \times Y$ on Z

The representation $(u, y) = w \oplus z$ is the *scattering representation* of (u, y), and w, z are called the *wave vectors* of the combined vector (u, y).

power variables

u(t) * y(t) is the power at time t



Scattering

If U (and so also Y) is endowed with the inner producr, then

$$\begin{aligned} \ll \cdot,\cdot \gg_W &= \langle \cdot,\cdot \rangle_W \\ -\ll \cdot,\cdot \gg_Z &= \langle \cdot,\cdot \rangle_Z \end{aligned}$$

i.e. the bilinear form computed on one of the scattering subspaces W o Z is equal to the inner product on that subspaces.

Using orthogonality between W and Z, we have the relationship

$$\begin{aligned} \ll (u_1, y_1), (u_2, y_2) \gg &= \langle y_1 | u_2 \rangle + \langle y_2 | u_1 \rangle \\ &= \langle w_1, w_2 \rangle_W - \langle z_1, z_2 \rangle_Z . \end{aligned}$$

dove $(u_1, y_1) = (w_1 \oplus z_1)$, $(u_2, y_2) = w_2 \oplus z_2$.

For $u = u_1 = u_2$ and $y = y_1 = y_2$) the following equalities hold

$$\begin{aligned} \ll (u,y),(u,y) \gg \quad &= \quad 2\langle y|u\rangle \\ &= \quad \langle w,w\rangle_W - \langle z,z\rangle_Z \\ &= \quad \|w\|_W^2 - \|z\|_Z^2. \end{aligned} \tag{19}$$

Energetic interpretation
Since $\langle y|u\rangle$ is the power, the vector w thus can be regarded as the *incoming wave vector*, with half times its norm being the *incoming power* (positive sign), and the vector z is the *outgoing wave vector*, with half times its norm being the *outgoing power* (negative sign).

Let $y = G(u)$ be a causal system, the equation (19) can be written as

$$\langle G(u)|u\rangle_T \quad = \quad \frac{1}{2}\|w_T\|_W^2 - \frac{1}{2}\|z_T\|_Z^2.$$

for truncated signals, $T \geq 0$.

Orthonormal bases W and Z

In order to compute $(w \oplus z)$ from (u, y) and the other way around, it is necessary to find *orthonormal bases* for W and Z such that $W \oplus Z = U \times U^\star$.

Let $\{e_1, e_2, \dots, e_m\}$ be a base for U and $\{e_1^\star, e_2^\star, \dots, e_m^\star\}$ be a base for $U^\star$, then a base for $U \times U^\star$ as a subspace of $\mathbb{R}^{2m}$ is

$$\left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \begin{bmatrix} e_2 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} e_m \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_1^\star \end{bmatrix}, \begin{bmatrix} 0 \\ e_2^\star \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ e_m^\star \end{bmatrix} \right\}.$$

The new base for W and Z

$$\{e_1^W, e_2^W, \dots, e_m^W, e_1^Z, e_2^Z, \dots, e_m^Z\}, \qquad e_i^W, e_i^Z \in \mathbb{R}^{2m}$$

should be such that

$$\begin{bmatrix} w \\ z \end{bmatrix} \in \operatorname{span}\{e_i^W, e_i^Z\} \qquad \Longleftrightarrow \qquad w \perp z.$$

Orthonormal bases W and Z

The new base for W and Z can be constructed from the base of $U \times U^\star$ as

$$\left\{ \begin{bmatrix} e_1/\sqrt{2} \\ e_1^\star/\sqrt{2} \end{bmatrix}, \begin{bmatrix} e_2/\sqrt{2} \\ e_2^\star/\sqrt{2} \end{bmatrix}, \dots, \begin{bmatrix} e_m/\sqrt{2} \\ e_m^\star/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -e_1/\sqrt{2} \\ e_1^\star/\sqrt{2} \end{bmatrix}, \dots, \begin{bmatrix} -e_m/\sqrt{2} \\ e_m^\star/\sqrt{2} \end{bmatrix} \right\}$$

or, using a different notation,

$$\begin{aligned} W &= \operatorname{span}\left\{ \left(\frac{e_i}{\sqrt{2}}, \frac{e_i^\star}{\sqrt{2}} \right), \ i = 1, \dots, m \right\} \\ Z &= \operatorname{span}\left\{ \left(-\frac{e_i}{\sqrt{2}}, \frac{e_i^\star}{\sqrt{2}} \right), \ i = 1, \dots, m \right\}. \end{aligned}$$

The square root of 2, $\sqrt{2}$, is needed to have an orthonormal base.

Example

Let $U = U^\star = \mathbb{R}^2$. The canonical base is

$$e_1 = \begin{bmatrix}1\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\end{bmatrix}, e_1^\star = \begin{bmatrix}1\\0\end{bmatrix}, e_2^\star = \begin{bmatrix}0\\1\end{bmatrix}$$

and so we have for W and Z the versors

$$e_1^W = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}, e_2^W = \begin{bmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}, \qquad e_1^Z = \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}, e_2^Z = \begin{bmatrix}0\\-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

or, using a different notation,

$$\begin{aligned} W &= \mathrm{span}\left\{\left(\begin{bmatrix}1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\0\end{bmatrix}\right), \left(\begin{bmatrix}0\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{2}\end{bmatrix}\right)\right\} \\ Z &= \mathrm{span}\left\{\left(\begin{bmatrix}-1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\0\end{bmatrix}\right), \left(\begin{bmatrix}0\\-1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{2}\end{bmatrix}\right)\right\}. \end{aligned}$$

It is easy to see that any vector $w \in W$ is orthogonal to any vector $z \in Z$.

G is the original system that is mapping the power variable U into the power variable Y

Let $G : \mathcal{L}_{2e}(U) \to \mathcal{L}_{2e}(Y)$ be an input-output system with $Y = U^\star \subseteq \mathbb{R}^n$. The relation (19)

$$\ll (u, y), (u, y) \gg = 2\langle y|u\rangle = \langle w, w\rangle_W - \langle z, z\rangle_Z = \|w\|_W^2 - \|z\|_Z^2.$$

considering truncated signals is equal to

$$2\langle y_T, u_T\rangle = \|w_T\|_2^2 - \|z_T\|_2^2, \qquad \forall T \geq 0. \tag{20}$$

Then we can relate the definition of passivity involving the term on the left side of the equation with the $\mathcal{L}_2$ norm of the transfer function mapping w_T into z_T involving the terms on the right side

$$\begin{aligned} S : \mathcal{L}_{2e}(W) &\to \mathcal{L}_{2e}(Z) \\ w_T &\mapsto z_T \end{aligned}$$

The map S is called *Scattering operator*.

S is mapping the wave variable W into the wave variable Z

Scattering operator

Using the change of coordinates introduced above, the relationship between (u, y) and (w, z) is

$$
\begin{aligned}
w &= \frac{1}{\sqrt{2}}(u + y) \\
z &= \frac{1}{\sqrt{2}}(-u + y)
\end{aligned}
$$

Since $y = G(u)$ it is possible to explicitly take into account the input-output map G as

$$
\begin{aligned}
w &= \frac{1}{\sqrt{2}}(G + I)(u) \\
z &= \frac{1}{\sqrt{2}}(G - I)(u)
\end{aligned}
$$

where now the relationship is just a function of the input u.

Scattering operator

If the map $(G + I)$ is invertible, z can be obtained as a function of w using the mappings

$$w \xrightarrow{(G+I)^{-1}} u \xrightarrow{(G-I)} z$$

which allows to derive the expression for the scattering operator

$$S = (G - I)(G + I)^{-1}.$$

If the input-output map G is LTI and SISO we get the transfer function

$$S(s) = \frac{G(s) - 1}{G(s) + 1}.$$

It is common to resort to a more general relationship between the *wave variables* $\{w, z\}$ and the *power variables* $\{u, y\}$

$$
\begin{aligned}
w &= \frac{1}{\sqrt{b}\sqrt{2}}(B(u)+y) \\
z &= \frac{1}{\sqrt{b}\sqrt{2}}(-B(u)+y)
\end{aligned}
$$

where the map $B(\cdot)0$ is an impedance.

In the SISO LTI case, the impedance can be just a positive constant $b > 0$

$$
\begin{aligned}
w &= \frac{1}{\sqrt{b}\sqrt{2}}(bu+y) && (21)\\
z &= \frac{1}{\sqrt{b}\sqrt{2}}(-bu+y) && (22)
\end{aligned}
$$

and the scattering operator will have the transfer function

$$
S(s) = \frac{G(s)-b}{G(s)+b} = \frac{\frac{G(s)}{b}-1}{\frac{G(s)}{b}+1} \qquad (23)
$$

Scattering operator

Using the equations (21)-(22) it is possible to obtain the power variables as a function of the wave variables since the tranformation is invertible (i.e. 1-to-1)

$$\begin{aligned} y &= \frac{\sqrt{2b}}{2}(w+z) \\ u &= \frac{\sqrt{2b}}{2b}(w-z) \end{aligned}$$

Such expressions allow to write the integral defining the passivity as a function of the wave variables

$$\begin{aligned} \langle u, y\rangle_T = \int_0^T y(s)u(s)ds &= \int_0^T \frac{\sqrt{2b}}{2}(w+z)\frac{\sqrt{2b}}{2b}(w-z)ds \\ &= \frac{1}{2}\int_0^T (w(s)+z(s))(w(s)-z(s))ds \\ &= \frac{1}{2}\int_0^T w^2(s)ds - \frac{1}{2}\int_0^T z^2(s)ds. \end{aligned}$$

Scattering operator

If the initial energy is zero, i.e. $\beta = 0$ or $V(0) = 0$, the passivity condition

$$\int_0^T y(s)u(s)ds \geq 0$$

is equivalent to

$$\int_0^T w^2(s)ds \geq \int_0^T z^2(s)ds.$$

This expression is due to (20)

$$\langle y_T, u_T \rangle = \frac{1}{2}\|w_T\|_2^2 - \frac{1}{2}\|z_T\|_2^2, \qquad \forall T \geq 0.$$

Theorem

Let $G(s)$ be the transfer function of SISO LTI and BIBO stable system which scattering operator $S(s)$. Then

- *G is passive if and only if $|S(s)| \leq 1, \forall\omega \in [-\infty, +\infty]$;*
- *G is input strictly passive (ISP) and there exists a constant $\gamma > 0$ such that $|G(j\omega)| \leq \gamma, \forall\omega \in [-\infty, +\infty]$, if and only if there exists a constant $\zeta \in (0, 1)$ such that $|S(j\omega)|^2 \leq 1 - \zeta, \forall\omega \in [-\infty, +\infty]$.*

when the norm of the scattering operator is smaller or equal to 1 then the norm of the input wave variable is always larger than the norm of the output wave variable so that the system does not generate more energy than the one that enters the system