

# Delta method

Emanuele

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## Intro

The delta method is a trick by which one can calculate mean and variance of a function of a random variable (see eg [1] for a rigorous definition). It works by writing the Taylor expansion of the function (say  $g(X)$ ) around  $E[X] = \mu$ .

To compute the mean of  $g(X)$  one does

$$E[g(X)] \approx E[g(\mu) + g'(\mu)(X - \mu)] = g(\mu)$$

. For the variance one has

$$\begin{aligned} \text{Var}[g(X)] &= E[(g(X) - E[g(X)])^2] \approx E[(g(X) - g(\mu))^2] \\ &= E[(g(\mu) + g'(\mu)(X - \mu) - g(\mu))^2] = (g'(\mu))^2 E[(X - \mu)^2] \\ &= (g'(\mu))^2 \text{Var}[X] \end{aligned}$$

## Example: arcin transform of binomial random variable

Let's consider  $Y \sim \text{Binomial}(N, \mu)$ ; hence  $E[Y] = \mu$  and  $\text{Var}[Y] = N\mu(1-\mu)$ . We also consider the transformed variable  $Z = g(Y) = \arcsin(2Y/N - 1)$ . Using the delta method as in the formulae above one has

$E[Z] \approx \arcsin(2\mu - 1)$ . For the variance,  $g'(\mu) = \frac{1}{\sqrt{1-(2\mu-1)^2}}$  hence

$$\text{Var}[Z] \approx 4 \frac{1}{N^2(1 - (2\mu - 1)^2)} \mu(1 - \mu) = \frac{1}{N} \quad (1)$$

$\text{Var}[Z]$  does not depend on  $\mu$  hence arcsin is said to stabilize the variance of  $Y$  (this is a known result, [2]).

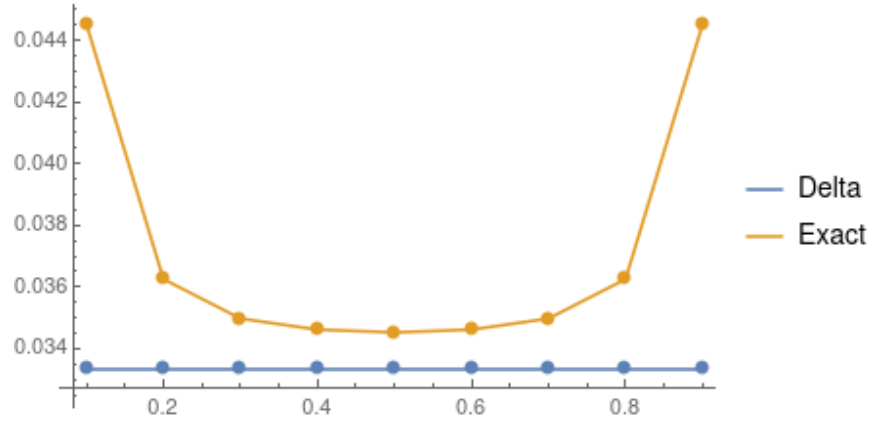
We can plot 1 against the variance of  $Z$  computed using the definition:

$$\text{Var}[Z] = \sum_{i=0}^n (Z_i - E[Z])^2 P(Z_i)$$

where  $Z_i = g(i)$  and  $P(Z_i) = P(Y = i) = \text{Binomial}(i|N, \mu)$ , and

$$E[Z] = \sum_{i=0}^n Z_i P(Z_i)$$

The agreement between the variance computed with the delta method as in 1 and the exact variance can be seen in the figure below:



## References

- [1] L. Wasserman. *All of Statistics: A Concise Course in Statistical Inference*. Springer Texts in Statistics. Springer New York, 2013.
- [2] Guan Yu. Variance stabilizing transformations of poisson, binomial and negative binomial distributions. *Statistics & Probability Letters*, 79(14):1621–1629, 2009.