Delta method

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Intro

The delta method is a trick by which one can calculate mean and variance of a function of a random variable (see eg [1] for a rigorous definition). It works by writing the Taylor expansion of the function (say $g(X)$) around $E[X] = \mu$.

To compute the mean of $g(X)$ one does

$$E[g(X)] \approx E[g(\mu) + g'(\mu)(X-\mu)] = g(\mu)$$

. For the variance one has

$$\begin{aligned} Var[g(X)] = E[(g(X) - E[g(X)])^2] &\approx E[(g(X) - g(\mu))^2] \\ = E[(g(\mu) + g'(X-\mu) - g(\mu))^2] &= (g'(\mu))^2 E[(X-\mu)^2] \\ &= (g'(\mu))^2 \mathrm{Var}[X] \end{aligned}$$

Example: arcin transform of binomial random variable

Let's consider $Y \sim \text{Binomial}(N, \mu)$; hence $E[Y] = \mu$ and $\mathrm{Var}[Y] = N\mu(1-\mu)$. We also consider the transformed variable $Z = g(Y) = \arcsin(2Y/N - 1)$. Using the delta method as in the formulae above one has

$E[Z] \approx \arcsin(2\mu - 1)$. For the variance, $g'(\mu) = \frac{1}{\sqrt{1-(2\mu-1)^2}}$ hence

$$\mathrm{Var}[Z] \approx 4\frac{1}{N^2(1-(2\mu-1)^2)}\mu(1-\mu) = \frac{1}{N} \tag{1}$$

$\mathrm{Var}[Z]$ does not depend on μ hence arcsin is said to stabilize the variance of Y (this is a known result, [2]).

We can plot 1 against the variance of Z computed using the definition:

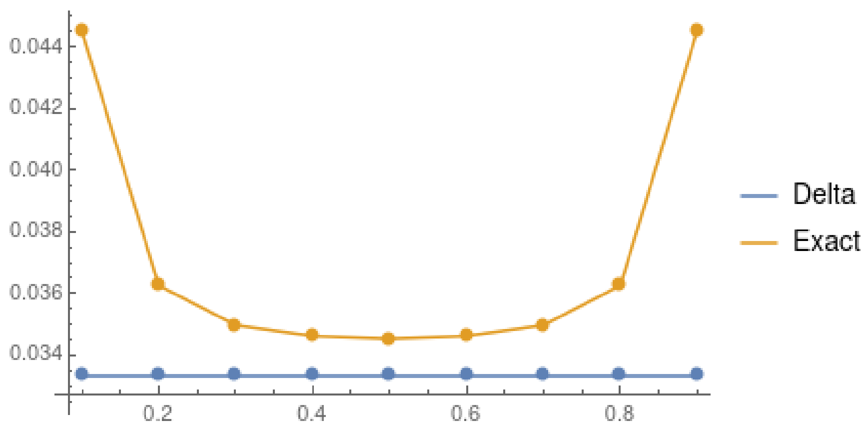
$$\text{Var}[Z] = \sum_{i=0}^{n} (Z_i - E[Z])^2 P(Z_i)$$

where $Z_i = g(i)$ and $P(Z_i) = P(Y = i) = \text{Binomial}(i|N, \mu)$, and

$$E[Z] = \sum_{i=0}^{n} Z_i P(Z_i)$$

.

The agreement between the variance computed with the delta method as in 1 and the exact variance can be seen in the figure below:



References

[1] L. Wasserman. *All of Statistics: A Concise Course in Statistical Inference*. Springer Texts in Statistics. Springer New York, 2013.

[2] Guan Yu. Variance stabilizing transformations of poisson, binomial and negative binomial distributions. *Statistics & Probability Letters*, 79(14):1621–1629, 2009.