Beta and binomial are conjugate distributions

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Consider a binomial distribution in the form

$$P(k|\theta) = \binom{N}{k}\theta^k(1-\theta)^{N-k}$$

We want to estimate θ; via Bayes theorem (and assuming $p(\theta) = 1$):

$$\begin{aligned} P(\theta|k) &= \frac{P(k|\theta)P(\theta)}{P(k)} \\ &= \frac{\binom{N}{k}\theta^k(1-\theta)^{N-k}}{\binom{N}{k}\int_0^1 \theta^k(1-\theta)^{N-k}\,d\theta} \end{aligned}$$

Now we introduce the beta function

$$\begin{aligned} \mathrm{B}(a,b) &= \int_0^1 \theta^{a-1}(1-\theta)^{b-1}\,d\theta \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \end{aligned}$$

(see [2])
hence, setting $k = a-1$ and $N-k = b-1$ one has

$$\begin{aligned} P(\theta|k) &= \frac{\theta^{a-1}(1-\theta)^{b-1}}{\mathrm{B}(a,b)} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1} \end{aligned}$$

which is called the Beta distribution (see [1]).

References

[1] Andrew Gelman, John B Carlin, Hal S Stern, David B Dunson, Aki Vehtari, and Donald B Rubin. *Bayesian data analysis*. Chapman and Hall/CRC, 2013.

[2] William H Press, Saul A Teukolsky, William T Vetterling, and Brian P Flannery. *Numerical recipes 3rd edition: The art of scientific computing*. Cambridge university press, 2007.