L6Z2

a) $a_0 = a_1 = 1$

$a_{n+1} = |\sqrt{a_n^2 + a_{n-1}^2}|$

$b_{n+1} = a_{n+1}^2$

$$b_{n+1} = b_n + b_{n-1}$$

$E\langle b_n\rangle = \langle b_{n+1}\rangle = \langle b_n\rangle + \langle b_{n-1}\rangle$

$E^2\langle b_n\rangle = \langle b_{n+2}\rangle = \langle b_{n+1}\rangle + \langle b_n\rangle = E\langle b_n\rangle + \langle b_n\rangle$

$(E^2 - E - 1)\langle b_n\rangle = \langle 0\rangle$

$\Delta = 5$

$x_1 = \frac{1-\sqrt{5}}{2}$, $x_2 = \frac{1+\sqrt{5}}{2}$

$(E - \frac{1-\sqrt{5}}{2})(E - \frac{1+\sqrt{5}}{2})\langle b_n\rangle = \langle 0\rangle$

$b_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \alpha + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \beta$

$\begin{cases} 1 = \alpha + \beta \\ 1 = \left(\frac{1-\sqrt{5}}{2}\right)\alpha + \left(\frac{1+\sqrt{5}}{2}\right)\cdot\beta \end{cases}$

/* $\beta = 1 - \alpha$

$1 = \left(\frac{1-\sqrt{5}}{2}\right)\alpha + \left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1+\sqrt{5}}{2}\right)\alpha$

$\frac{1-\sqrt{5}}{2} = -\sqrt{5}\alpha$

$\alpha = \frac{\sqrt{5}-1}{2\sqrt{5}} = \frac{5-\sqrt{5}}{10}$

$\beta = \frac{5+\sqrt{5}}{10}$ */

$b_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \cdot \left(\frac{5-\sqrt{5}}{10}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \left(\frac{5+\sqrt{5}}{10}\right)$

$a_n^2 = b_n$

$a_n = \sqrt{\left(\frac{1-\sqrt{5}}{2}\right)^n \cdot \left(\frac{5-\sqrt{5}}{10}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \left(\frac{5+\sqrt{5}}{10}\right)}$

b) $\begin{cases} b_0 = 8 \\ b_{n+1} = \left| \sqrt{b_n^2 + 3} \right| \end{cases}$

$$a_{n+1} = b_{n+1}^2$$

$$a_{n+1} = a_n + 3$$

$$E\langle a_n \rangle = \langle a_{n+1} \rangle = \langle a_n \rangle + \langle 3 \rangle$$

$$(E-1)(E-1)\langle a_n \rangle = \langle 0 \rangle$$

$$a_n = \alpha \cdot n + \beta$$

$$a_1 = 67$$

$$\begin{cases} 67 = \alpha + \beta \\ 64 = \beta \end{cases}$$

$$\alpha = 3$$

$$a_n = 3n + 64$$

$$b_n = \sqrt{3n + 64}$$

c) $\begin{cases} c_0 = 0 \\ c_1 = 1 \\ c_{n+1} = (n+1)c_n + (n^2+n)c_{n-1} \end{cases}$

$$c_{n+1} = (n+1)c_n + n(n+1)c_{n-1} \quad / : (n+1)!$$

$$\frac{c_{n+1}}{(n+1)!} = \frac{c_n}{n!} + \frac{c_{n-1}}{(n-1)!}$$

$$a_n = \frac{c_n}{n!}$$

$$a_{n+1} = a_n + a_{n-1}$$

z podpunktu a)

$$a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \cdot \alpha + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \beta$$

$$\begin{cases} 0 = \alpha + \beta \\ 1 = \left(\frac{1-\sqrt{5}}{2}\right)\alpha + \left(\frac{1+\sqrt{5}}{2}\right)\beta \end{cases}$$

$$\alpha = -\beta$$

$$1 = \frac{\sqrt{5}-1}{2}\beta + \frac{1+\sqrt{5}}{2}\beta$$

$$1 = \sqrt{5}\beta$$

$$\beta = \frac{\sqrt{5}}{5}, \quad \alpha = -\frac{\sqrt{5}}{5}$$

$$a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \cdot \left(-\frac{\sqrt{5}}{5}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \frac{\sqrt{5}}{5}$$

$$c_n = a_n \cdot n! = \left(\left(\frac{1-\sqrt{5}}{2}\right)^n \cdot \left(-\frac{\sqrt{5}}{5}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \frac{\sqrt{5}}{5}\right) \cdot n!$$

L6Z4

k kolejnych liczb naturalnych

$n+1, n+2, \dots, n+k$

$$k! \mid (n+1)(n+2)\cdot\ldots\cdot(n+k)$$

$$k! \mid \frac{(n+k)!}{n!}$$

$$\frac{(n+k)!}{n!\cdot k!} = \frac{(n+k)!}{k!(n+k-k)!} = \binom{n+k}{k} \in \mathbb{N}$$

czyli

$$k! \mid (n+1)(n+2)\cdot\ldots\cdot(n+k)$$

L6 Z6

$$\begin{cases} a_0 = 2 \\ a_n^2 = 2a_{n-1}^2 + 1 \end{cases}$$

zał $a_n > 0$ dla $n \in \mathbb{N}$

$a_n^2 = b_n$

$b_n = 2b_{n-1} + 1$

$(E-2)\langle b_n\rangle = \langle b_{n+1}\rangle - 2\langle b_n\rangle = 2\langle b_n\rangle + \langle 1\rangle - 2\langle b_n\rangle = \langle 1\rangle$

$(E-1)(E-2)\langle b_n\rangle = \langle 0\rangle$

$b_n = 2^n\alpha + \beta$

$2^2 = \alpha + \beta \Rightarrow 4 = \alpha + \beta$

$b_1 = 2 \cdot b_0 + 1 = 9$

$$\begin{cases} 4 = \alpha + \beta \\ 9 = 2\alpha + \beta \end{cases}$$

$\beta = 4 - \alpha$

$9 = 2\alpha + 4 - \alpha$

$$\begin{cases} \alpha = 5 \\ \beta = -1 \end{cases}$$

$b_n = 2^n \cdot 5 - 1$

$a_n = \sqrt{2^n \cdot 5 - 1}$

L6 Z7

w_n - liczba wyrazów z n liter

(odd) o_n - liczba wyrazów z n liter zawierająca nieparzystą ilość "a"

(even) e_n - — " — parzystą ilość "a"

$$w_n = o_n + e_n = 25^n$$

$$\begin{cases} e_{n+1} = o_n + 24 \cdot e_n & / \cdot 24 \\ o_{n+1} = 24 o_n + e_n \end{cases}$$

$$\begin{cases} 24 e_{n+1} = 24 o_n + 576 e_n \\ o_{n+1} = 24 o_n + e_n \end{cases} \quad -$$

$$24 e_{n+1} - o_{n+1} = 575 e_n$$

$$o_{n+1} = 24 e_{n+1} - 575 e_n$$

$$e_{n+1} = 24 e_n - 575 e_{n-1} + 24 e_n = 48 e_n - 575 e_{n-1}$$

$$e_{n+1} = 24e_{n+1} - 575e_n$$

$$e_{n+1} = 24e_n - 575e_{n-1} + 24e_n = 48e_n - 575e_{n-1}$$

$$\begin{cases} e_0 = 1 \\ e_1 = 24 \\ e_{n+1} = 48e_n - 575e_{n-1} \quad n = 2, 3, \dots \end{cases}$$

$$(E^2 - 48E + 575)\langle e_n \rangle = \langle e_{n+2} \rangle - 48\langle e_{n+1} \rangle + 575\langle e_n \rangle =$$

$$= 48\langle e_{n+1} \rangle - 575\langle e_n \rangle - 48\langle e_{n+1} \rangle + 575\langle e_n \rangle =$$

$$= \langle 0 \rangle$$

$$\Delta = 2304 - 2300 = 4$$

$$x_1 = \frac{48-2}{2} = 23 \quad , \quad x_2 = \frac{48+2}{2} = 25$$

$$(E-23)(E-25)\langle e_n \rangle = \langle 0 \rangle$$

$$e_n = 23^n \alpha + 25^n \beta$$

$$e_0 = 1 \quad , \quad e_1 = 24$$

$$\begin{cases} 1 = \alpha + \beta \\ 24 = 23\alpha + 25\beta \end{cases}$$

$$\alpha = 1 - \beta$$

$$24 = 23 - 23\beta + 25\beta$$

$$1 = 2\beta$$

$$\beta = \frac{1}{2} \quad , \quad \alpha = \frac{1}{2}$$

$$e_n = 23^n \cdot \frac{1}{2} + 25^n \cdot \frac{1}{2}$$

L6 Z8

a) $\begin{cases} a_0 = a_1 = 0 \\ a_{n+2} = 2a_{n+1} - a_n + 3^n - 1 \end{cases}$

$$a_n = 2a_{n-1} - a_{n-2} + 3^{n-2} - 1$$

$$(E^2 - 2E + 1)(E-3)(E-1)\langle a_n \rangle = (E-1)^3 (E-3) \langle a_n \rangle$$

$$a_n = \alpha \cdot n^2 + \beta n + \gamma + \delta \cdot 3^n$$

b)

$$a_0 = a_1 = 1$$

$$a_{n+2} = 4a_{n+1} - 4a_n + n \cdot 2^{n+1}$$

$$(E^2-2E+1)(E-3)(E-1)\langle a_n\rangle = (E-1)^3(E-3)\langle a_n\rangle$$

$$a_n = \alpha \cdot n^2 + \beta n + \gamma + \delta \cdot 3^n$$

b)

$$a_0 = a_1 = 1$$

$$a_{n+2} = 4a_{n+1} - 4a_n + n \cdot 2^{n+1}$$

$$(E^2-4E+4)(E-2)^2\langle a_n\rangle = \langle 0\rangle \qquad // \ (E-2)\langle n\cdot 2^{n+1}\rangle = \langle 2^{n+2}\rangle$$

$$(E-2)^4\langle a_n\rangle = \langle 0\rangle$$

$$a_n = 2^n(\alpha \cdot n^3 + \beta \cdot n^2 + \gamma n \cdot \delta)$$

c) $a_0 = a_1 = 1$

$$a_{n+2} = \frac{1}{2^{n+1}} - 2a_{n+1} - a_n$$

c) $a_0 = a_1 = 1$

$$a_{n+2} = \frac{1}{2^{n+1}} - 2a_{n+1} - a_n$$

$$(E^2 + 2E + 1)\langle a_n \rangle = \langle \frac{1}{2^{n+1}} \rangle$$

$$(E - \tfrac{1}{2})\langle \frac{1}{2^{n+1}} \rangle = \langle 0 \rangle$$

$$(E^2 + 2E + 1)(E - \tfrac{1}{2})\langle a_n \rangle = \langle 0 \rangle$$

$$(E + 1)^2 (E - \tfrac{1}{2})\langle a_n \rangle = \langle 0 \rangle$$

$$a_n = (-1)^n(\alpha n + \beta) + (\tfrac{1}{2})^n \gamma$$

$$\begin{cases} 1 = -\alpha - \beta + \frac{1}{2}\gamma \\ 1 = \beta + \gamma \\ -\frac{5}{2} = 2\alpha + \beta + \frac{1}{4}\gamma \end{cases}$$

$$a_2 = \frac{1}{2} - 2 - 1 = -2\tfrac{1}{2}$$

.....

$\alpha = -\frac{5}{3}$

$\beta = \frac{7}{9}$

$\gamma = \frac{2}{9}$

$$a_n = (-1)^n \left(-\frac{5}{3} \cdot n + \frac{7}{9}\right) + \left(\frac{1}{2}\right)^n \cdot \frac{2}{9}$$