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IE 554 Project – Proposal

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1 Introduction

Graph partitioning problems are fundamental in combinatorial optimization and graph theory, offering a wide range of applications from network design to clustering. In this project, we focus on a novel variant called the **Dominator Partition Problem**, recently introduced in the literature by Hedetniemi and Haynes (2006) [1]. We propose integer programming formulation for the problem and try to improve it.

2 Problem Definition

Given a graph $G = (V, E)$, a vertex $v \in V$ *dominates* a set $S \subseteq V$ if v is adjacent to every vertex in S. A **dominator partition** $\pi = \{V_1, V_2, \ldots, V_k\}$ of V is a partition such that each vertex $v \in V$ dominates at least one block V_i in the partition. The *dominator partition number* $\pi_d(G)$ is the minimum number k such that G has a dominator partition of order k, and our goal is to find it.

For this project, we propose an **integer programming model** to solve the dominator partition problem exactly. Then, we try to improve the formulation such as making the model stronger such that it can be solved faster on bigger graphs.

The study process goes like this: first we tried to model the problem mathematically using integer programming. Then we plan to work on improving the model, mainly by adding valid inequalities. Maybe later we will find other ways to solve the problem and compare those methods with our original model. Also, after fixing the value of k, we want to try different formulations and compare their runtime performances.

3 Literature Review

Since dominator partitions were introduced, most studies have been about their properties, complexity and special graph classes like trees and cycles [1]. It was shown that checking if a graph has a dominator partition with at most k blocks is NP-complete, even when the graph is bipartite, planar, or chordal.

However, **no one has formulated the dominator partition problem using an integer programming model before**, to our knowledge. The existing work is mostly theoretical, without optimization models. So, our study brings a new angle by using integer programming to find dominator partitions.

4 Dominator Partition Model (variable k)

4.1 Sets & Parameters

V: set of vertices in the graph, of size n

a_{vu}: 1 if vertices v and u are adjacent

4.2 Decision Variables

x_{vi}: 1 if vertex v is assigned to the i^{th} block

y_i: 1 if the i^{th} block is used (non-empty)

d_{vi}: 1 if vertex v dominates block i

4.3 Objective Function & Constraints

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i && \text{(minimize the number of blocks used)} \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{vi} = 1, \quad \forall v \in V && \text{(each vertex is assigned to exactly one block)} \\
& x_{vi} \le y_i \quad \forall v \in V, i \in \{1, 2, \ldots, n\} && \text{(vertex assigned only if block used)} \\
& \sum_{v \in V} x_{vi} \ge y_i \quad \forall i \in \{1, 2, \ldots, n\} && \text{(block used only if it is not empty)} \\
& x_{ui} \le a_{vu} + (1 - d_{vi}) \quad \forall u, v \in V, i \in \{1, 2, \ldots, n\} && \text{(domination condition)} \\
& \sum_{i=1}^{n} d_{vi} \ge 1 \quad \forall v \in V && \text{(each vertex dominates at least one block)} \\
& d_{vi} \le y_i \quad \forall v \in V, i \in \{1, 2, \ldots, n\} && \text{(cannot dominate an empty block)} \\
& x_{vi}, y_i, d_{vi} \quad \forall v \in V, i \in \{1, 2, \ldots, n\} && \text{(binary variables)}
\end{aligned}
$$

5 Planned Contributions

- **Exact Modeling**: We provide the first known exact integer programming model for the dominator coloring problem.
- **Model Strengthening**: We aim to derive *valid inequalities* to strengthen the above formulation.

- **Computational Comparisons**: We will perform computational experiments on different graph sizes (small, medium, large) to:
  - Compare the **original model** and **strengthened model**.
  - Analyze improvements in **computational time** and **solver performance**.
- **New Formulation Development**: Optionally, we may propose a **new mathematical model** based on alternative representations (e.g., alternative graph properties) and compare it against the original one in terms of both **formulation strength** and **computational efficiency**.

6 Conclusion

The Dominator Partition Problem is a recently introduced, yet still largely unexplored optimization problem with rich mathematical structure. Through developing mathematical formulations, strengthening them with valid inequalities, and performing extensive computational experiments, we aim to significantly contribute to the understanding and solution of this problem.

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References

[1] S. M. Hedetniemi, S. T. Hedetniemi, R. Laskar, A. A. McRae, C. K. Wallis, *Dominator Partitions of Graphs*, 2008.