2024-2025 Spring

IE 554 Project – Mathematical Models

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1 Introduction

Graph coloring is a classical optimization problem with widespread applications in scheduling, network theory, and clustering. Recently, a new variant called the **Dominator Coloring Problem** was introduced by Hedetniemi and Haynes (2006) [1], building on the notion of domination in graphs. In dominator colorings, not only must the coloring be proper, but every vertex must dominate some color class. In this project, we propose an integer programming formulation to solve this problem exactly and try to improve it.

2 Problem Definition

Given a graph $G = (V, E)$, a vertex $v \in V$ *dominates* a set $S \subseteq V$ if it is adjacent to every vertex in S. A **dominator coloring** is a proper coloring where every vertex v dominates at least one color class. The objective is to find a dominator coloring with the **minimum number of colors**, called the dominator chromatic number $\chi_d(G)$.

For this project, we propose an **integer programming model** to solve the dominator partition problem exactly. Then, we try to improve the formulation such as making the model stronger such that it can be solved faster on bigger graphs.

The study process goes like this: first we tried to model the problem mathematically using integer programming. Then we plan to work on improving the model, mainly by adding valid inequalities. Maybe later we will find other ways to solve the problem and compare those methods with our original model.

3 Literature Review

The dominator coloring problem has been introduced and studied mainly from a combinatorial and theoretical standpoint [1]. It is known that deciding whether a graph admits a dominator coloring with at most k colors is NP-complete. However, to our knowledge, **no integer programming formulations have been proposed** for this problem. Our work fills this gap by providing a model and improvig it.

4 Dominator Coloring Model

4.1 Sets & Parameters

V: set of vertices in the graph, of size n

a_{vu}= 1 if vertices v and u are adjacent

4.2 Decision Variables

x_{vi}: 1 if vertex v is assigned color i

y_i: 1 if color i is used by at least one vertex

d_{vi}: 1 if vertex v dominates color-class i

4.3 Objective Function & Constraints

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i && \text{(minimize number of colors used)} \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{vi} = 1 && \forall v \in V \quad \text{(each vertex gets exactly one color)} \\
& x_{ui} + x_{vi} \leq 2 - a_{uv} && \forall u, v \in V,\ u \neq v \quad \text{(proper coloring)} \\
& x_{vi} \leq y_i && \forall v \in V,\ i = 1, \ldots, n \quad \text{(use color if assigned)} \\
& x_{ui} \leq a_{vu} \ + \ (1 - d_{vi}) && \forall v \in V,\ \forall u \in V,\ u \neq v,\ i = 1, \ldots, n \quad \text{(domination definition)} \\
& d_{vi} \leq y_i && \forall v \in V,\ i = 1, \ldots, n \quad \text{(cannot dominate an unused color)} \\
& \sum_{i=1}^{n} d_{vi} \geq 1 && \forall v \in V \quad \text{(each vertex dominates at least one class)} \\
& x_{vi},\ y_i,\ d_{vi} \in \{0, 1\} && \forall v \in V,\ i = 1, \ldots, n.
\end{aligned}
$$

5 Planned Contributions

- **Exact Modeling**: We provide the first known exact integer programming model for the dominator coloring problem.
- **Model Strengthening**: We plan to derive *valid inequalities* to strengthen the formulation and improve solver performance.
- **Computational Comparisons**: We will conduct experiments on graphs of varying sizes to:
  - Compare the performance of the **original** and **strengthened** formulations.
  - Analyze improvements in **runtime** and **solver robustness**.
- **Alternative Formulation Development**: Optionally, we may explore alternative models (e.g., flow-based or covering formulations) and compare them against our base model.

6 Conclusion

The Dominator Partition Problem is a recently introduced, yet still largely unexplored optimization problem with rich mathematical structure. Through developing mathematical formulations, strengthening them with valid inequalities, and performing extensive computational experiments, we aim to significantly contribute to the understanding and solution of this problem.

References

[1] S. M. Hedetniemi, S. T. Hedetniemi, R. Laskar, A. A. McRae, C. K. Wallis, *Dominator Partitions of Graphs*, 2008.