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IE 554 Project – Mathematical Models

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1 Introduction

Graph domination problems play a central role in combinatorial optimization and graph theory. Two recently introduced variants – the **Dominator Partition Problem** and the **Dominator Coloring Problem** – focus on domination-based graph structures. In this project, we propose exact **integer programming formulations** for both problems, aiming to strengthen them with valid inequalities and analyze their computational performance.

2 Problem Definitions

2.1 Dominator Partition Problem

Given a graph $G = (V, E)$, a vertex $v \in V$ *dominates* a set $S \subseteq V$ if v is adjacent to every vertex in S. A **dominator partition** $\pi = \{V_1, V_2, \ldots, V_k\}$ of V satisfies that each vertex dominates at least one block V_i. The **dominator partition number** $\pi_d(G)$ is the minimum number k such that such a partition exists.

2.2 Dominator Coloring Problem

Given a graph $G = (V, E)$, a **dominator coloring** is a proper coloring such that each vertex dominates at least one color class. The objective is to minimize the number of colors used, known as the **dominator chromatic number** $\chi_d(G)$.

3 Literature Review

While dominator partitions and dominator colorings have been studied theoretically [1], focusing on bounds, properties, and complexity, there has been **no prior integer programming formulation** for either problem. It is known that deciding whether a graph admits a dominator partition or dominator coloring with bounded size is NP-complete. However, optimization-based approaches have not been explored in the literature.

Thus, this project addresses a novel research gap by developing IP models for both problems and improving their computational efficiency.

4 Mathematical Models

4.1 Model for Dominator Partition Problem

Let us define:

V: set of vertices.

a_{vu}: 1 if v and u are adjacent.

Decision Variables:

x_{vi}: 1 if vertex v belongs to block i.

y_i: 1 if block i is non-empty.

Objective and Constraints:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{vi} = 1 \quad \forall v \in V \\
& x_{ui} \le a_{vu} + (1 - x_{vi}) \quad \forall v \in V, u \in V, i = 1, \ldots, n \\
& x_{vi} \le y_i \quad \forall v \in V, i = 1, \ldots, n \\
& x_{vi}, y_i \in \{0, 1\} \quad \forall v \in V, i = 1, \ldots, n.
\end{aligned}
$$

4.2 Model for Dominator Coloring Problem

Sets and parameters are the same.

Decision Variables:

x_{vi}: 1 if vertex v is assigned color i.

y_i: 1 if color i is used.

d_{vi}: 1 if vertex v dominates color class i.

Objective and Constraints:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{vi} = 1 \quad \forall v \in V \\
& x_{ui} + x_{vi} \leq 2 - a_{uv} \quad \forall u \neq v \in V, i = 1, \ldots, n \\
& x_{vi} \leq y_i \quad \forall v \in V, i = 1, \ldots, n \\
& x_{ui} \leq a_{vu} + (1 - d_{vi}) \quad \forall v, u \in V, u \neq v, i = 1, \ldots, n \\
& d_{vi} \leq y_i \quad \forall v \in V, i = 1, \ldots, n \\
& \sum_{i=1}^{n} d_{vi} \geq 1 \quad \forall v \in V \\
& x_{vi}, y_i, d_{vi} \in \{0, 1\} \quad \forall v \in V, i = 1, \ldots, n.
\end{aligned}
$$

5 Planned Contributions

- **Exact Models**: This will be the first known IP models for dominator partition and dominator coloring.
- **Model Strengthening**: We aim to derive valid inequalities to strengthen the above formulation.
- **Computational Analysis**: We aim to compare models' performance on graphs of various sizes.
- **Alternative Approaches**: We aim to explore possible new formulations for both problems.

6 Conclusion

By formulating both of the dominator-based graph problems – the **Dominator Partition Problem** and the **Dominator Coloring Problem** – as integer programs and improving their solvebility through strengthening and computational testing, we aim to significantly contribute to the understanding and solution of these problems.

References

[1] S. M. Hedetniemi, S. T. Hedetniemi, R. Laskar, A. A. McRae, C. K. Wallis, *Dominator Partitions of Graphs*, 2008.