Sets and Data.

- $G = (V, E)$: undirected graph with vertex set V and edge set E.
- $K \in \mathbb{Z}_{\geq 1}$: number of blocks; $\Pi \coloneqq \{1, 2, \ldots, K\}$.
- For $v \in V$, $\mathrm{CN}(v) \subseteq V$: (closed) neighborhood used in the code (given).
- Optional weights $\alpha_{v,i}$ and $\beta_{v,i}$ (aligned with `ALPHA` and `BETA`).

Decision Variables.

$$x_{vi} \in \{0,1\} \quad (v \in V,\ i \in \Pi), \qquad d_{vi} \in \{0,1\} \quad (v \in V,\ i \in \Pi).$$

Here, $x_{vi} = 1$ if vertex v is assigned to block i, and $d_{vi} = 1$ if (by the model's logic) vertex v "dominates" block i.

Objective.

$$\min \sum_{v \in V} \sum_{i \in \Pi} \alpha_{vi}\, x_{vi} + \sum_{v \in V} \sum_{i \in \Pi} \beta_{vi}\, d_{vi}. \tag{1}$$

Core Constraints. (1) Assignment: each vertex to exactly one block

$$\sum_{i \in \Pi} x_{vi} = 1 \qquad \forall v \in V. \tag{2}$$

(2) Nonempty blocks

$$\sum_{v \in V} x_{vi} \geq 1 \qquad \forall i \in \Pi. \tag{3}$$

(3) Domination condition for non-edges

$$x_{ui} + d_{vi} \leq 1 \qquad \forall i \in \Pi,\ \forall v, u \in V :\ v \neq u,\ \{v, u\} \notin E. \tag{4}$$

(4) Each vertex dominates exactly one block

$$\sum_{i \in \Pi} d_{vi} = 1 \qquad \forall v \in V. \tag{5}$$

(5) Block usage order

$$\sum_{v \in V} x_{v,i} \ \geq \ \sum_{v \in V} x_{v,i+1} \qquad \forall i \in \Pi \setminus \{K\}. \tag{6}$$

Valid Inequalities

VI-1: Minimum fill of the first block.

$$\sum_{v \in V} x_{v,1} \;\geq\; \left\lceil \frac{|V|}{K} \right\rceil. \tag{7}$$

VI-2: Per-block upper bound.

$$\sum_{v \in V} x_{v,i} \;\leq\; \left\lfloor \frac{|V| - K + i}{i} \right\rfloor \qquad \forall i \in \Pi. \tag{8}$$

VI-3: Neighborhood-based lower link on d.

$$d_{vi} \;\geq\; \sum_{u \in \mathrm{CN}(v)} x_{ui} \;-\; \big(|\mathrm{CN}(v)| - 1\big) \qquad \forall v \in V,\; \forall i \in \Pi. \tag{9}$$

VI-4: Neighborhood-based upper link on d.

$$d_{vi} \;\leq\; \sum_{u \in \mathrm{CN}(v)} x_{ui} \qquad \forall v \in V,\; \forall i \in \Pi. \tag{10}$$

VI-5: Maximal independent set (MIS) domination cap. Let $\mathcal{I}$ be the family of given maximal independent sets. Then

$$\sum_{v \in S} d_{vi} \;\leq\; 1 \qquad \forall S \in \mathcal{I},\; \forall i \in \Pi. \tag{11}$$

VI-6: MIS-based cover inequality. For every $i \in \Pi$, $v \in V$, and every $S \in \mathcal{I}$ with $v \in S$,

$$\sum_{u \in \mathrm{CN}(v)} x_{ui} \;\leq\; |\mathrm{CN}(v)| - 1 \;+\; d_{vi} \;-\; \sum_{\substack{y \in S \\ y \neq v}} d_{yi}. \tag{12}$$

Variable Domains.

$$x_{vi} \in \{0,1\}, \quad d_{vi} \in \{0,1\} \qquad \forall v \in V,\; \forall i \in \Pi. \tag{13}$$