PI AND THE GREAT PYRAMID

It was John Taylor who first proposed the idea that the number ***&pi*** might have been intentionally incorporated into the design of the Great Pyramid of Khufu at Giza. He discovered that if one divides the perimeter of the Pyramid by its height, one obtains a close approximation to 2***&pi***. He compared this to the fact that if one divides the circumference of a circle by its radius, one obtains 2***&pi***. He suggested that perhaps the Great Pyramid was intended to be a representation of the spherical Earth, the height corresponding to the radius joining the center of the Earth to the North Pole and the perimeter corresponding to the Earth's circumference at the Equator. Taylor's ideas were presented in his book The Great Pyramid: Why Was It Built? And Who Built It?, published in 1859. These ideas were further promulgated and elaborated by Charles Piazzi Smyth - Professor of Astronomy at Edinburgh University and Astronomer Royal of Scotland - in his book Inheritance in the Great Pyramid, published in 1864. In more recent times, books such as Secrets of the Great Pyramid by Peter Tompkins discuss the relationship between ***&pi*** and the Great Pyramid and also another relationship involving the number ϕ (the famous Golden Mean) at considerable length, almost seeming to take it for granted that those relationships are really intentional.

It is true that if one divides the Great Pyramid's perimeter by its height, one indeed obtains a very good approximation to 2***&pi***. An equivalent statement is that the slope of each face of the Great Pyramid is very close to 4/***&pi***=1.273239.... This relationship is accurate to within .04% or better (depending on the data that one uses). That level of accuracy seems very impressive and is certainly the reason that Taylor's idea has been widely promoted and believed. Could such an accurate and elegant relationship be just a mere coincidence?

The main point that I will make in this essay is that when one takes into account what we know about ancient Egyptian mathematics (based primarily on the Rhind Papyrus), especially their ways of representing lengths and slopes, then the relationship between ***&pi*** and the Great Pyramid no longer seems very remarkable. The essential point is that the measurement system which the ancient Egyptians used would lead the architects to use certain slopes in the design of pyramids. One of those slopes just happens to be an excellent approximation to the number 4/***&pi***, and if the architect chooses that slope, then the pyramid would exhibit the famous ***&pi*** relationship. From this point of view, the probability that the architect might choose that particular slope for at least one of the pyramids is actually rather high. It then becomes quite reasonable to believe that the relationship between ***&pi*** and the Great Pyramid is just an accidental consequence of their Mathematics.

How can one calculate the probability that an architect building a pyramid would choose a slope which is so close to 4/***&pi***? This is actually a very tricky question. It is impossible to do such a calculation without making some assumptions. The answer will depend crucially on those assumptions. I will discuss a number of hypothetical sets of assumptions to illustrate this important point.

Suppose that an architect is designing a pyramidal structure and has decided that the base should be a perfect square and that the apex should be directly above the center of the square base. This would mean that the pyramid would be very symmetrical. All of the faces would be isosceles triangles and slanted at the same angle. The architect would then have to choose that angle. For this discussion, I will assume that the builders will be able to carry out the architect's design with perfect precision. The architect would have to somehow specify his choice of angle to those builders. One could imagine many different ways in which this could be done, but I will consider five illustrative examples.

1. For example, let's assume that the architect decides to chose an angle between 43^{o} and 55^{o}. (The Egyptian pyramids built during the 4th dynasty and still standing have the angles of their faces in that range.) Let's also assume that the architect will specify the angle with the familiar notation $A^{o}B'C''$, where B and C are integers in the range 0 to 59, and that it is equally likely for the architect to pick any of the angles so represented. The architect would then have more than 43,000 possible choices for the angle. To achieve a slope of 4/***&pi*** with an accuracy of at least .04%, the angle would have to be chosen in the range $51^{o}50'34''$ to $51^{o}51'54''$- a total of only 81 possibilities. Under all of the above assumptions, the probability that the architect will choose one of those angles is about 1/530. That is, the odds that the face of the pyramid would turn out to have a slope of 4/***&pi*** with at least an accuracy of .04% are 530 to 1.

2. Let's assume instead that the architect decides to make it easier for the builders by choosing an angle in the above range, but with B=0 and C=0. That means that the architect is restricting the choice to just the thirteen angles 43^{o}, 44^{o}, ...,54^{o}, 55^{o}. Assume again that any one of these choices is equally likely. It would then be impossible for the architect to achieve a slope of 4/***&pi*** to within .04%. The probability is 0!! The closest that the slope could come to 4/***&pi*** would occur if the architect chooses an angle of 52^{o}. Then the slope would be TAN(52^{o})=1.279941...., which is approximately 4/***&pi***, but the error is more than .5%.

3. Now let's assume instead that the architect decides to specify the angle by its slope. Since TAN(43^{o})=.932515... and Tan(55^{o})=1.428148..., I will assume that the architect decides to choose the slope between .932515 and 1.428148 and that any number in that range is equally likely to be chosen. The probability that the architect would choose a slope which is close to 4/***&pi*** with an accuracy of at least .04% now turns out to be about 1/490.

4. Let's again vary the assumption by imagining that the architect would (perhaps more realistically) represent the slope by a number with just two decimal places. (That is, the slope will be a fraction with a denominator dividing 100.) Thus the slope could be any of the following numbers .94, .95, .96,..., 1.41, or 1.42, allowing 49 possibilities for the

architect to choose from. The closest that the architect could come to a slope of 4/***&pi***=1.273239... is by choosing the slope to be 1.27. That would be an approximation to 4/***&pi*** which is only accurate to about .25%. It would again be impossible for the architect to achieve a slope of 4/***&pi*** to within .04%. That is, the probability for that to happen would be 0 under the above assumption.

5. The final example is a variation on the preceding example. Let's assume that the architect decides to specify the angle by the *inverse* of its slope and also decides to represent that inverse-slope by a fraction with a denominator which divides 28. Let's continue to assume that the architect wants the angle to be between 43^o and 55^o. The smallest possible value for that inverse-slope would then be 20/28=5/7. That corresponds to the angle ARCTAN(7/5)=54^{o}27'44". The largest possible value for the inverse-slope is 30/28=15/14, corresponding to the angle ARCTAN(14/15)=43^{o}01'30". Thus, under the above assumptions, the inverse-slope would be chosen as one of the fractions 20/28, 21/28, ..., 30/28. There are 11 possible choices. Let's assume that the architect is equally likely to choose any of these possibilities. Thus, with a probability of 1/11, the architect might choose 22/28=11/14 as the inverse-slope. The faces of the resulting pyramid built with that specification would have a slope of 14/11. This turns out to be extremely close to 4/***&pi***. In fact, 4/***&pi***=1.273239... and 14/11=1.272727... and the approximation of 4/***&pi*** by 14/11 is accurate to within .04%. Under this scenario, which might seem rather strange at first, the probability that the architect's pyramid would have a slope of 4/***&pi*** with the same accuracy as that exhibited by the Great Pyramid of Khufu is 1/11. The probability is even higher if one assumes that the architect wants to make the angle rather steep. That would correspond to choosing a *smaller* value for the inverse-slope and therefore make 22/28 a more likely choice.

The scenario in the last example may actually be fairly close to the truth. It is strikingly supported by the measurements of a number of the pyramids built during the 4th dynasty. Also, it is strongly suggested by what we know about Egyptian mathematics from the Rhind Papyrus - specifically, the fact that they used a system of measuring lengths in terms of "cubits," "palms," and "fingers." A cubit is equal to seven palms and a palm is equal to four fingers. Thus one cubit is equal to 28 fingers, and that is where denominators dividing 28 would come from. I will discuss this in more detail later, after reviewing the measurements of the pyramids from the 4th dynasty.

The numbers 20/28, 21/28, and 22/28 do indeed occur as the inverse-slopes of the faces for some of the 4-th dynasty pyramids which are still standing. The first of these pyramids is the BENT PYRAMID built by the Pharaoh Sneferu. Its name comes from the fact that the builders changed the angle of slant of the faces at some stage in the construction of that pyramid. The lower portion is the steepest of all of these pyramids. It is possible that it was found to be too steep. In any case, the inverse-slope of the faces for the lower portion is extremely close to 20/28=.714285.... In his book, The Complete Pyramids, Mark Lehner gives the angle of slant as 54^{o}27'44". Based on this, the inverse-slope of each face is .714288.

The second of the three pyramids at Giza is the PYRAMID OF KHAFRE. Lehner gives the angle of slant for each face as 53°10'00" - less steep than the Bent Pyramid, but steeper than the Great Pyramid, which was built by Khafre's father Khufu. The inverse-slope of each face is then .749003 which is approximately 21/28=.750000....

For the GREAT PYRAMID OF KHUFU at Giza, the angle of slant for each face is 51°50'40" and the inverse-slope of each face then turns out to be .785667. This is quite close to 22/28=.785714.... The third of the pyramids built at Giza is the PYRAMID OF MENKAURE. Its base is rectangular, but not a perfect square. The lengths of the longest and shortest sides differ by more than 2 meters and so the slant-angles of the faces are not equal. There are two angles of slant, the steepest of which is 51°49'38", quite close to the angle of slant for each face of the Great Pyramid. The inverse-slope of the two faces having that slant-angle is again quite close to 22/28. (It is now .786154.)

The above calculations are based, as I mentioned, on the data given in Lehner's book. I am not sure how reliable that data is. Additional measurements obtained by modern technology would be interesting. Did the Egyptian builders really achieve 20/28=5/7 as the inverse-slope of the faces of the Bent Pyramid (the lower portion) with the astounding accuracy indicated above?

Although the measurements of the above-mentioned pyramids seem to fit in very well with the pattern in example 5 (i.e., inverse-slopes having denominators which divide 28), the issue of how the architects chose slopes for the pyramids becomes somewhat more complicated on closer examination. For example, the so-called RED PYRAMID and the upper portion of the Bent Pyramid both have the same slant-angle, namely 43°22'00". The inverse-slope corresponding to that slant-angle is equal to 1.058703, which does not fit into the pattern discussed above. (If one multiplies that number by 28, the result is not close to an integer!) That slant-angle may have been chosen in a different way, and one theory about this is discussed in my essay Slopes of the Egyptian Pyramids. The slant-angle of the other two (less steep) faces of the Pyramid of Menkaure also does not fit into the above pattern. But that pyramid turns out to be quite an interesting example for the theory discussed in my essay. Concerning the Great Pyramid of Khufu, the theory proposed in my essay turns out to make 22/28 a very logical choice as the inverse-slope for the slant-angle of the faces.

Now I will discuss the pyramid exercises from the Rhind Papyrus. This papyrus as well as the other extant mathematical papyri were written hundreds of years after the 4-th dynasty. One can ask how accurately they represent the mathematical knowledge of the architect who built the Great Pyramid. It is obvious that the architects undertaking the building of a pyramid would need a good mathematical knowledge of the geometry associated with such a structure, and one can indeed find this in these later papyri. As an example, it would seem necessary to know in advance the quantity of stone that would need to be quarried, and this requires computing the volume of a pyramid. One of the extant mathematical papyri contains an example of such a computation, indicating that the

Egyptians knew how to do this. The volume is equal to one third of the height multiplied by the area of the base. They could compute the area of the base just knowing the length of each side. The height is then related to the slant-angle of each face which can be specified by the "seked" of the pyramid. In the Rhind papyrus, the exercises about pyramids concern how to compute the height knowing the seked, or the seked knowing the height (assuming that the length of the sides is also known). It would then seem reasonable to believe that the methods used by the 4-th dynasty architects are similar to those illustrated in these papyri. It has been said that the mathematical knowledge represented by the papyri actually goes back to Imhotep - the architect of the the Step Pyramid of Djoser (3-rd dynasty).

The seked corresponds to the inverse-slope of each face of the pyramid. It is the cotangent of the slope-angle. One can find a more complete explanation here. Two of the five pyramid exercises from the Rhind papyrus can be found here. As these exercises show, the seked is represented as a certain number of palms and fingers. This is really the horizontal change in the distance for each change of one cubit in the vertical distance. The exercises make it clear that one cubit is equal to 7 palms and that one palm is equal to 4 fingers. Therefore, one cubit is equal to 28 fingers. (Not so different from the English measurement system in which one yard is 3 feet and one foot is 12 inches.) Thus, for example, a seked of five palms, two fingers would correspond to an inverse-slope of (22 fingers)/(28 fingers), or 22/28. As I mentioned above, the Great Pyramid indeed has this seked, and with a high degree of accuracy.

These considerations suggest that the famous relationship between ***&pi*** and the Great Pyramid of Khufu is a coincidence which has its roots in two facts - one purely mathematical and the other historical, but both involving the number 7:

1. *The rational number 22/7 happens to be an excellent approximation to the number pi.*

2. *The Egyptian measurement system involves dividing one unit of measurement(the cubit) into 7 equal units (palms)*

Concerning the first fact, it is actually somewhat remarkable that an irrational number such as ***&pi*** can be approximated so well by a rational number with a small denominator. (The denominator is 7 in this case.) Concerning the second fact, the reason why Egyptians chose to divide a cubit into 7 parts might be quite hard to trace. It would seem to go back long before the construction of the Great Pyramid, at least to the time of Imhotep.

There is also a frequently mentioned relationship between the Great Pyramid and the number ϕ, which I will now discuss. The legend that the architect who designed the Great Pyramid of Khufu intentionally incorporated the Golden Mean (which is this number ϕ) into the proportions of that structure seems to have its roots in a misunderstanding or distortion of the writings of Herodotus. The story goes as follows: Herodotus learned from Egyptian priests that the Great Pyramid was built so that the square whose side is the height of the Great Pyramid will have an area equal to that of each of the Pyramid's faces.

Based on this statement, it is not difficult to derive the famous relationship between the Great Pyramid and the number ϕ. The derivation makes use of the Pythagorean theorem and is described here.

But what Herodotus actually wrote is quite different. Based on a translation from the Greek by A.D. Godley, the relevant statement is actually as follows:

Its base is square, each side eight hundred feet long, and its height is the same;

The full statement can be found here: The Histories. In the above notation, it states rather clearly that h=s. In fact, this is quite far from true.

Nevertheless, the relationship f/(s/2)= ϕ is approximately true. Just as with the relationship involving ***&pi***, it corresponds to choosing a seked of five palms, two fingers for the pyramid. And so the probability of making that particular choice is rather high according to the earlier discussion. As with the relationship involving ***&pi***, this relationship is therefore much less remarkable than might seem at first glance.