

Quasi-thermodynamic Push-Pull Models

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I. BACKGROUND MODEL

In this model, we only have a substrate S^u and phosphorylation and dephosphorylation occur as background reactions.

Background Reactions:



Total Concentrations: We define total concentrations for each species and also separately for unphosphorylated and phosphorylated states.

- Total substrate: $[S_T] = [S^u] + [S^p]$

If we know the

One-way Reaction Rates: These are the reaction rates that cannot be described by binding energies.

- Background phosphorylation of substrate: k_{bg}^p
- Background dephosphorylation of substrate: k_{bg}^u

Reactions not in Detailed Balance: The kinetic equation for the phosphorylated substrate is

$$\frac{d[S_T^p]}{dt} = k_{bg}^p[S_T^u] - k_{bg}^u[S_T^p] \quad (2)$$

Setting this equation to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{bg}^p}{k_{bg}^p + k_{bg}^u} \quad (3)$$

$$= \frac{v_{bg}^p}{v_{bg}^p + 1} \quad (4)$$

where we have defined the reaction velocities relative to the background dephosphorylation rate,

$$v_{bg}^p = \frac{k_{bg}^p}{k_{bg}^u} \quad (5)$$

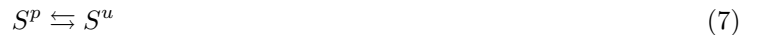
II. ONE-WAY PUSH MODEL

In this model, we have a substrate S^u and a writer (kinase) W . The writer is able to phosphorylate the substrate from S^u to S^p . In this model, we do not explicitly include an eraser (phosphatase), but there is a dephosphorylation background rate (and a phosphorylation background rate).

Reactions:



Background Reactions:



Total Concentrations: We define total concentrations for each species and also separately for unphosphorylated and phosphorylated states.

- Total writer: $[W_T] = [W] + [WS^u] + [WS^p]$
- Total substrate: $[S_T] = [S^u] + [S^p]$
- Total unphosphorylated substrate: $[S_T^u] = [S^u] + [WS^u]$
- Total phosphorylated substrate: $[S_T^p] = [S^p] + [WS^p]$

If we know the

Binding Energies: We assume the binding energies are the same for the unphosphorylated and phosphorylated states.

- Writer + substrate: $\Delta\epsilon_{WS}$

One-way Reaction Rates: These are the reaction rates that cannot be described by binding energies.

- Phosphorylation of substrate by writer: k_{WS}^p
- Background phosphorylation of substrate (independent of binding state): k_{bg}^p
- Background dephosphorylation of substrate (independent of binding state): k_{bg}^u

Partition Functions and Probabilities: We write down partition functions for each species based on the states listed in the total concentrations above. Partition function cannot mix unphosphorylated and phosphorylated states for the same species, so we do not write down a partition function for the total S . We assume the unbound state of each species is zero energy (although this choice doesn't matter).

- Writer: $Z_W = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])} + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])}$
- Unphosphorylated substrate: $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}$
- Phosphorylated substrate: $Z_{S^p} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}$

We define the reaction velocity

$$\alpha_{WS} = e^{\beta\Delta\epsilon_{WS}} \quad (9)$$

The associated conditional probabilities are then

$$p(W|S^u, S^p) = \frac{[W]}{[W_T]} = \frac{1}{Z_W} = \frac{1}{1 + \frac{[S^u] + [S^p]}{\alpha_{WS}}} \quad (10)$$

$$p(WS^u|S^u, S^p) = \frac{[WS^u]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{\frac{[S^u]}{\alpha_{WS}}}{1 + \frac{[S^u] + [S^p]}{\alpha_{WS}}} \quad (11)$$

$$p(WS^p|S^u, S^p) = \frac{[WS^p]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])}}{Z_W} = \frac{\frac{[S^p]}{\alpha_{WS}}}{1 + \frac{[S^u] + [S^p]}{\alpha_{WS}}} \quad (12)$$

$$p(S^u|W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + \frac{[W]}{\alpha_{WS}}} \quad (13)$$

$$p(WS^u|W) = \frac{[WS^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{\frac{[W]}{\alpha_{WS}}}{1 + \frac{[W]}{\alpha_{WS}}} \quad (14)$$

$$p(S^p|W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + \frac{[W]}{\alpha_{WS}}} \quad (15)$$

$$p(WS^p|W) = \frac{[WS^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^p}} = \frac{\frac{[W]}{\alpha_{WS}}}{1 + \frac{[W]}{\alpha_{WS}}} \quad (16)$$

Reactions in Detailed Balance: From the probabilities, we focus on the equations for the concentration of each species in its unbound state. These correspond to the kinetic equations for reactions not involving (de)phosphorylation.

$$\frac{[W]}{[W_T]} = \frac{1}{1 + \frac{[S^u] + [S^p]}{\alpha_{WS}}} \quad (17)$$

$$\frac{[S^u]}{[S_T^u]} = \frac{1}{1 + \frac{[W]}{\alpha_{WS}}} \quad (18)$$

$$\frac{[S^p]}{[S_T^p]} = \frac{1}{1 + \frac{[W]}{\alpha_{WS}}} \quad (19)$$

Reactions not in Detailed Balance: This is the kinetic equation for reactions involving (de)phosphorylation. It is possible for either S^u or WS^u to undergo this process, so we have two equations

$$\frac{d[S^p]}{dt} = k_{WS}^- [WS^p] - k_{WS}^+ [W][S^p] + k_{bg}^p [S^u] - k_{bg}^u [S^p] \quad (20)$$

$$\frac{d[WS^p]}{dt} = k_{WS}^+ [W][S^p] - k_{WS}^- [WS^p] + k_{WS}^p [WS^u] + k_{bg}^p [WS^u] - k_{bg}^u [WS^p] \quad (21)$$

$$\frac{d[S^u]}{dt} = k_{WS}^- [WS^u] - k_{WS}^+ [W][S^u] + k_{bg}^u [S^p] - k_{bg}^p [S^u] \quad (22)$$

$$\frac{d[WS^u]}{dt} = k_{WS}^+ [W][S^u] - k_{WS}^- [WS^u] - k_{WS}^p [WS^u] + k_{bg}^u [WS^p] - k_{bg}^p [WS^u] \quad (23)$$

where we have defined the forward and backward dissociation rates k_{WS}^+ and k_{WS}^- such that

$$\frac{1}{\alpha_{WS}} = \frac{k_{WS}^+}{k_{WS}^-} \quad (24)$$

Instead of writing these four equations, we can instead write the kinetic equations so that they only describe reactions that are not in detailed balance and do not depend on k_{WS}^+ and k_{WS}^- . To see what these equations are, we can simply write down the reaction rates for the total phosphorylated and unphosphorylated substrate. These will just be related by a negative sign since they sum to a constant, so we can write

$$\frac{d[S_T^p]}{dt} = k_{WS}^p [WS^u] + k_{bg}^p [S_T^u] - k_{bg}^u [S_T^p] \quad (25)$$

$$= k_{WS}^p [S_T^u] p(WS^u|W) + k_{bg}^p [S_T^u] - k_{bg}^u [S_T^p] \quad (26)$$

$$= k_{WS}^p ([S_T] - [S_T^p]) p(WS^u|W) + k_{bg}^p ([S_T] - [S_T^p]) - k_{bg}^u [S_T^p] \quad (27)$$

We note that this equation can also be obtained by summing together the first two kinetic equations. Similarly, the equation for $[S_T^u]$ can be obtained by summing the last two kinetic equations.

Setting this equation to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{WS}^p p(WS^u|W) + k_{bg}^p}{k_{WS}^p p(WS^u|W) + k_{bg}^p + k_{bg}^u} \quad (28)$$

$$= \frac{v_{WS}^p p(WS^u|W) + v_{bg}^p}{v_{WS}^p p(WS^u|W) + v_{bg}^p + 1} \quad (29)$$

where we have defined the reaction velocities relative to the background dephosphorylation rate,

$$v_{WS}^p = \frac{k_{WS}^p}{k_{bg}^u} \quad (30)$$

$$v_{bg}^p = \frac{k_{bg}^p}{k_{bg}^u} \quad (31)$$

Numerical Strategy:

We assume that $[W_T]$ and $[S_T]$ are known. Add together Eqs. (18) and (19) and defining $[S_f] = [S^p] + [S^u]$, we find a pair of equations that can be solved for $[W]$ and $[S_f]$,

$$\frac{[W]}{[W_T]} = \frac{1}{1 + \frac{[S_f]}{\alpha_{WS}}} \quad (32)$$

$$\frac{[S_f]}{[S_T]} = \frac{1}{1 + \frac{[W]}{\alpha_{WS}}} \quad (33)$$

Combining these equations, we find a quadratic equation in terms of S_f ,

$$0 = \left(\frac{[S_f]}{\alpha_{WS}} \right)^2 + \left(1 + \frac{[W_T] - [S_T]}{\alpha_{WS}} \right) \frac{[S_f]}{\alpha_{WS}} - \frac{[S_T]}{\alpha_{WS}} \quad (34)$$

Generally, we find that this has one positive solution for $[S_f]$. Using this solution, we solve for $[W]$ and then use the kinetic equation so solve for $[S_T^p]$. The rest of the concentrations are then straightforward to calculate without any further need to solve any equations.

A. Gradient

$$0 = \frac{\partial}{\partial \alpha_{WS}} \left[\left(\frac{[S_f]}{\alpha_{WS}} \right)^2 + \left(1 + \frac{[W_T] - [S_T]}{\alpha_{WS}} \right) \frac{[S_f]}{\alpha_{WS}} - \frac{[S_T]}{\alpha_{WS}} \right] \quad (35)$$

$$= -\frac{1}{\alpha_{WS}^2} \left[([W_T] - [S_T]) \frac{[S_f]}{\alpha_{WS}} - [S_T] \right] + \left[2 \frac{[S_f]}{\alpha_{WS}} + 1 + \frac{[W_T] - [S_T]}{\alpha_{WS}} \right] \left(\frac{1}{\alpha_{WS}} \frac{\partial [S_f]}{\partial \alpha_{WS}} - \frac{[S_f]}{\alpha_{WS}^2} \right) \quad (36)$$

$$= -\frac{1}{\alpha_{WS}} \left[2 \left(\frac{[S_f]}{\alpha_{WS}} \right)^2 + \left(1 + 2 \frac{[W_T] - [S_T]}{\alpha_{WS}} \right) \frac{[S_f]}{\alpha_{WS}} - \frac{[S_T]}{\alpha_{WS}} \right] + \left[2 \frac{[S_f]}{\alpha_{WS}} + 1 + \frac{[W_T] - [S_T]}{\alpha_{WS}} \right] \frac{1}{\alpha_{WS}} \frac{\partial [S_f]}{\partial \alpha_{WS}} \quad (37)$$

$$\frac{\partial [S_f]}{\partial \alpha_{WS}} = \frac{2 \left(\frac{[S_f]}{\alpha_{WS}} \right)^2 + \left(1 + 2 \frac{[W_T] - [S_T]}{\alpha_{WS}} \right) \frac{[S_f]}{\alpha_{WS}} - \frac{[S_T]}{\alpha_{WS}}}{2 \frac{[S_f]}{\alpha_{WS}} + 1 + \frac{[W_T] - [S_T]}{\alpha_{WS}}} \quad (38)$$

$$= \frac{\frac{[S_f]}{\alpha_{WS}} \left(\frac{[S_f]}{\alpha_{WS}} + \frac{[W_T] - [S_T]}{\alpha_{WS}} \right)}{2 \frac{[S_f]}{\alpha_{WS}} + 1 + \frac{[W_T] - [S_T]}{\alpha_{WS}}} \quad (39)$$

Notes need revision starting from here

$$\frac{\partial [W]}{\partial \alpha_{WS}} = -[W_T] \frac{[S_f]}{(1 + \alpha_{WS}[S_f])^2} - [W_T] \frac{\alpha_{WS}}{(1 + \alpha_{WS}[S_f])^2} \frac{\partial [S_f]}{\partial \alpha_{WS}} \quad (40)$$

$$= -[W_T] \frac{1}{(1 + \alpha_{WS}[S_f])^2} \left([S_f] + \alpha_{WS} \frac{\partial [S_f]}{\partial \alpha_{WS}} \right) \quad (41)$$

$$\frac{\partial p(W S^u | W)}{\partial \alpha_{WS}} = \frac{[W]}{1 + \alpha_{WS}[W]} - \frac{\alpha_{WS}[W]^2}{(1 + \alpha_{WS}[W])^2} + \left[\frac{\alpha_{WS}}{1 + \alpha_{WS}[W]} - \frac{\alpha_{WS}^2[W]}{(1 + \alpha_{WS}[W])^2} \right] \frac{\partial [W]}{\partial \alpha_{WS}} \quad (42)$$

$$= \frac{[W]}{(1 + \alpha_{WS}[W])^2} + \frac{\alpha_{WS}}{(1 + \alpha_{WS}[W])^2} \frac{\partial [W]}{\partial \alpha_{WS}} \quad (43)$$

$$= \frac{1}{(1 + \alpha_{WS}[W])^2} \left([W] + \alpha_{WS} \frac{\partial [W]}{\partial \alpha_{WS}} \right) \quad (44)$$

$$\frac{\partial[S_T^p]}{\partial\alpha_{WS}} = [S_T] \left[\frac{\alpha_{WS}^p}{\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1} - \frac{\alpha_{WS}^p (\alpha_{WS}^p p(W S^u|W) + v_{bg}^p)}{(\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1)^2} \right] \frac{\partial p(W S^u|W)}{\partial\alpha_{WS}} \quad (45)$$

$$= [S_T] \frac{\alpha_{WS}^p}{(\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1)^2} \frac{\partial p(W S^u|W)}{\partial\alpha_{WS}} \quad (46)$$

$$\frac{\partial[S_T^p]}{\partial\alpha_{WS}^p} = [S_T] \frac{p(W S^u|W)}{\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1} - [S_T] \frac{p(W S^u|W) (\alpha_{WS}^p p(W S^u|W) + v_{bg}^p)}{(\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1)^2} \quad (47)$$

$$= [S_T] \frac{p(W S^u|W)}{(\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1)^2} \quad (48)$$

$$\frac{\partial[S_T^p]}{\partial v_{bg}^p} = [S_T] \frac{1}{\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1} - [S_T] \frac{\alpha_{WS}^p p(W S^u|W) + v_{bg}^p}{(\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1)^2} \quad (49)$$

$$= [S_T] \frac{1}{(\alpha_{WS}^p p(W S^u|W) + v_{bg}^p + 1)^2} \quad (50)$$

III. GOLDBETER PUSH-PULL AMPLIFIER

In the Goldbeter model, we have a push-pull amplifier with a substrate S , a writer (kinase) W and an eraser (phosphatase) E with no background (de)phosphorylation [1]. In addition, we lack the intermediate states directly after phosphorylation $W S^p$ and dephosphorylation $E S^u$. These two states are assumed to be very short-lived.

Reactions:



Total Concentrations:

- Total writer: $[W_T] = [W] + [W S^u]$
- Total Eraser: $[E_T] = [E] + [E S^p]$
- Total substrate: $[S_T] = [S_T^u] + [S_T^p]$
- Total unphosphorylated substrate: $[S_T^u] = [S^u] + [W S^u]$
- Total phosphorylated substrate: $[S_T^p] = [S^p] + [E S^p]$

Binding Energies:

- Writer + substrate: $\Delta\epsilon_{WS}$
- Eraser + substrate: $\Delta\epsilon_{ES}$

One-way Reaction Rates:

- Phosphorylation of substrate by writer: k_{WS}^p
- Dephosphorylation of substrate by eraser: k_{ES}^u

Partition Functions and Probabilities:

- Writer: $Z_W = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}$
- Eraser: $Z_E = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])}$
- Unphosphorylated substrate: $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}$
- Phosphorylated substrate: $Z_{S^p} = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$

We define the reaction velocities

$$\alpha_{WS} = e^{-\beta\Delta\epsilon_{WS}} \quad (53)$$

$$\alpha_{ES} = e^{-\beta\Delta\epsilon_{ES}} \quad (54)$$

The associated conditional probabilities are then

$$p(W|S^u, S^p, E) = \frac{[W]}{[W_T]} = \frac{1}{Z_W} = \frac{1}{1 + \alpha_{WS}[S^u]} \quad (55)$$

$$p(W S^u|S^u, S^p, E) = \frac{[W S^u]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{\alpha_{WS}[S^u]}{1 + \alpha_{WS}[S^u]} \quad (56)$$

$$p(E|S^u, S^p, W) = \frac{[E]}{[E_T]} = \frac{1}{Z_E} = \frac{1}{1 + \alpha_{ES}[S^p]} \quad (57)$$

$$p(E S^p|S^u, S^p, W) = \frac{[E S^p]}{[E_T]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])}}{Z_E} = \frac{\alpha_{ES}[S^p]}{1 + \alpha_{ES}[S^p]} \quad (58)$$

$$p(S^u|E, W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + \alpha_{WS}[W]} \quad (59)$$

$$p(W S^u|E, W) = \frac{[W S^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{\alpha_{WS}[W]}{1 + \alpha_{WS}[W]} \quad (60)$$

$$p(S^p|E, W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + \alpha_{ES}[E]} \quad (61)$$

$$p(E S^p|E, W) = \frac{[E S^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}}{Z_{S^p}} = \frac{\alpha_{ES}[E]}{1 + \alpha_{ES}[E]} \quad (62)$$

Reactions in Detailed Balance:

$$\frac{[W]}{[W_T]} = \frac{1}{1 + \alpha_{WS}[S^u]} \quad (63)$$

$$\frac{[E]}{[E_T]} = \frac{1}{1 + \alpha_{ES}[S^p]} \quad (64)$$

$$\frac{[S^u]}{[S_T^u]} = \frac{1}{1 + \alpha_{WS}[W]} \quad (65)$$

$$\frac{[S^p]}{[S_T^p]} = \frac{1}{1 + \alpha_{ES}[E]} \quad (66)$$

Reactions not in Detailed Balance: We only focus on the equations for the total amounts of phosphorylated and unphosphorylated substrate, allowing us to ignore any of the reactions in detailed balance.

$$\frac{d[S_T^p]}{dt} = k_{WS}^p[W S^u] - k_{ES}^u[E S^p] \quad (67)$$

$$= k_{WS}^p[S_T^u]p(W S^u|E, W) - k_{ES}^u[S_T^p]p(E S^p|E, W) \quad (68)$$

$$= k_{WS}^p([S_T] - [S_T^p])p(W S^u|E, W) - k_{ES}^u[S_T^p]p(E S^p|E, W) \quad (69)$$

Setting this to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{WS}^p p(W S^u | E, W)}{k_{WS}^p p(W S^u | E, W) + k_{ES}^u p(ES^p | E, W)} \quad (70)$$

$$= \frac{\alpha_{WS}^p p(W S^u | E, W)}{\alpha_{WS}^p p(W S^u | E, W) + p(ES^p | E, W)} \quad (71)$$

$$(72)$$

where we have defined the reaction velocity

$$\alpha_{WS}^p = \frac{k_{WS}^p}{k_{ES}^u} \quad (73)$$

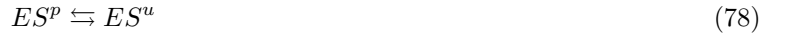
IV. COMPLETE PUSH-PULL AMPLIFIER

This model generalizes the Goldbeter model to include the intermediate states and background (de)phosphorylation.

Reactions:



Background:



Total Concentrations:

- Total writer: $[W_T] = [W] + [W S^p] + [W S^u]$
- Total Eraser: $[E_T] = [E] + [E S^p] + [E S^u]$
- Total substrate: $[S_T] = [S_T^u] + [S_T^p]$
- Total unphosphorylated substrate: $[S_T^u] = [S^u] + [W S^u] + [E S^u]$
- Total phosphorylated substrate: $[S_T^p] = [S^p] + [W S^p] + [E S^p]$

Binding Energies:

- Writer + substrate: $\Delta\epsilon_{WS}$
- Eraser + substrate: $\Delta\epsilon_{ES}$

One-way Reaction Rates:

- Phosphorylation of substrate by writer: k_{WS}^p
- Dephosphorylation of substrate by eraser: k_{ES}^u
- Background phosphorylation of substrate (independent of binding state): k_{bg}^p
- Background dephosphorylation of substrate (independent of binding state): k_{bg}^u

Partition Functions and Probabilities:

- Writer: $Z_W = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])} + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}$
- Eraser: $Z_E = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^u])}$
- Unphosphorylated substrate: $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$

- Phosphorylated substrate: $Z_{Sp} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$

We define the reaction velocities

$$\alpha_{WS} = e^{-\beta\Delta\epsilon_{WS}} \quad (79)$$

$$\alpha_{ES} = e^{-\beta\Delta\epsilon_{ES}} \quad (80)$$

The associated conditional probabilities are then

$$p(W|S^u, S^p, E) = \frac{[W]}{[W_T]} = \frac{1}{Z_W} = \frac{1}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (81)$$

$$p(WS^u|S^u, S^p, E) = \frac{[WS^u]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{\alpha_{WS}[S^u]}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (82)$$

$$p(WS^p|S^u, S^p, E) = \frac{[WS^p]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])}}{Z_W} = \frac{\alpha_{WS}[S^p]}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (83)$$

$$p(E|S^u, S^p, W) = \frac{[E]}{[E_T]} = \frac{1}{Z_E} = \frac{1}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (84)$$

$$p(ES^u|S^u, S^p, W) = \frac{[ES^u]}{[E_T]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^u])}}{Z_E} = \frac{\alpha_{ES}[S^u]}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (85)$$

$$p(ES^p|S^u, S^p, W) = \frac{[ES^p]}{[E_T]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])}}{Z_E} = \frac{\alpha_{ES}[S^p]}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (86)$$

$$p(S^u|E, W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (87)$$

$$p(WS^u|E, W) = \frac{[WS^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{\alpha_{WS}[W]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (88)$$

$$p(ES^u|E, W) = \frac{[ES^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}}{Z_{S^u}} = \frac{\alpha_{ES}[E]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (89)$$

$$p(S^p|E, W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (90)$$

$$p(WS^p|E, W) = \frac{[WS^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^p}} = \frac{\alpha_{WS}[W]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (91)$$

$$p(ES^p|E, W) = \frac{[ES^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}}{Z_{S^p}} = \frac{\alpha_{ES}[E]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (92)$$

Reactions in Detailed Balance:

$$\frac{[W]}{[W_T]} = \frac{1}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (93)$$

$$\frac{[E]}{[E_T]} = \frac{1}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (94)$$

$$\frac{[S^u]}{[S_T^u]} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (95)$$

$$\frac{[S^p]}{[S_T^p]} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (96)$$

Reactions not in Detailed Balance:

$$\frac{d[S_T^p]}{dt} = k_{WS}^p[WS^u] - k_{ES}^u[ES^p] + k_{bg}^p[S_T^u] - k_{bg}^u[S_T^p] \quad (97)$$

$$= k_{WS}^p[S_T^u]p(WS^u|E, W) - k_{ES}^u[S_T^p]p(ES^p|E, W) + k_{bg}^p[S_T^u] - k_{bg}^u[S_T^p] \quad (98)$$

$$= k_{WS}^p([S_T] - [S_T^p])p(WS^u|E, W) - k_{ES}^u[S_T^p]p(ES^p|E, W) + k_{bg}^p([S_T] - [S_T^p]) - k_{bg}^u[S_T^p] \quad (99)$$

Setting this to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{WS}^p p(W S^u | E, W) + k_{bg}^p}{k_{WS}^p p(W S^u | E, W) + k_{ES}^u p(ES^p | E, W) + k_{bg}^p + k_{bg}^u} \quad (100)$$

$$= \frac{\alpha_{WS}^p p(W S^u | E, W) + v_{bg}^p}{\alpha_{WS}^p p(W S^u | E, W) + \alpha_{ES}^u p(ES^p | E, W) + v_{bg}^p + 1} \quad (101)$$

$$(102)$$

where we have defined the reaction velocities

$$\alpha_{WS}^p = \frac{k_{WS}^p}{k_{bg}^u} \quad (103)$$

$$\alpha_{ES}^u = \frac{k_{ES}^u}{k_{bg}^u} \quad (104)$$

$$v_{bg}^p = \frac{k_{bg}^p}{k_{bg}^u} \quad (105)$$

Numerical Strategy:

We assume that $[W_T]$, $[E_T]$, and $[S_T]$ are known. Add together Eqs. (18) and (19) and defining $[S_f] = [S^p] + [S^u]$, we find three equations that can be solved for $[W]$, $[E]$, and $[S_f]$,

$$\frac{[W]}{[W_T]} = \frac{1}{1 + \alpha_{WS}[S_f]} \quad (106)$$

$$\frac{[E]}{[E_T]} = \frac{1}{1 + \alpha_{ES}[S_f]} \quad (107)$$

$$\frac{[S_f]}{[S_T]} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (108)$$

Combining these equations, we find a cubic equation in terms of S_f ,

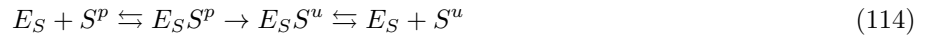
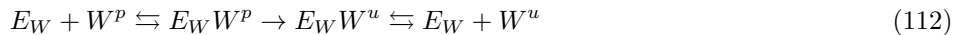
$$0 = \alpha_{WS}\alpha_{ES}[S_f]^3 + (\alpha_{WS} + \alpha_{ES} + \alpha_{WS}\alpha_{ES}([W_T] + [E_T] - [S_T]))[S_f]^2 \quad (109)$$

$$+ (1 + \alpha_{WS}([W_T] - [S_T]) + \alpha_{ES}([E_T] - [S_T]))[S_f] - [S_T] \quad (110)$$

Generally, we find that this has one positive solution for $[S_f]$. Using this solution, we solve for $[W]$ and $[E]$ and then use the kinetic equation to solve for $[S_T^p]$. The rest of the concentrations are then straightforward to calculate without any further need to solve any equations.

V. TWO-STEP AMPLIFIER

Reactions:



Are these viable?



Background:



Total Concentrations:

- Total writer: $[W_T] = [W_T^u] + [W_T^p]$
- Total unphosphorylated writer: $[W_T^u] = [W^u] + [RW^u] + [EW^u]$
- Total phosphorylated writer: $[W_T^p] = [W^p] + [W^p S^p] + [W^p S^u] + [RW^p] + [EW^p]$
- Total Writer-Eraser: $[E_{WT}] = [E_W] + [E_W W^p] + [E_W W^u]$
- Total Substrate-Eraser: $[E_{ST}] = [E_S] + [E_S S^p] + [E_S S^u]$
- Total substrate: $[S_T] = [S_T^u] + [S_T^p]$
- Total unphosphorylated substrate: $[S_T^u] = [S^u] + [W^p S^u] + [W^u S^u] + [ES^u]$
- Total phosphorylated substrate: $[S_T^p] = [S^p] + [W^p S^p] + [W^u S^p] + [ES^p]$

Binding Energies:

- Receptor + Writer: $\Delta\epsilon_{RW}$
- Writer-Eraser + Writer: $\Delta\epsilon_{E_W W}$
- Writer + substrate: $\Delta\epsilon_{WS}$
- Substrate-Eraser + substrate: $\Delta\epsilon_{ES S}$

One-way Reaction Rates:

- Phosphorylation of writer by receptor: k_{RW}^p
- Dephosphorylation of writer by eraser: $k_{E_W W}^u$
- Phosphorylation of substrate by writer: k_{WS}^p
- Dephosphorylation of substrate by eraser: $k_{ES S}^u$
- Background phosphorylation of writer (independent of binding state): k_W^p
- Background dephosphorylation of writer (independent of binding state): k_W^u
- Background phosphorylation of substrate (independent of binding state): k_{bg}^p
- Background dephosphorylation of substrate (independent of binding state): k_{bg}^u

Partition Functions and Probabilities:

- Unphosphorylated writer: $Z_{W^u} = 1 + e^{-\beta(\Delta\epsilon_{RW} - k_B T \log[R]) + e^{-\beta(\Delta\epsilon_{E_W W} - k_B T \log[E_W])}}$
- Phosphorylated writer: $Z_{W^p} = 1 + e^{-\beta(\Delta\epsilon_{RW} - k_B T \log[R]) + e^{-\beta(\Delta\epsilon_{E_W W} - k_B T \log[E_W])}} + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])} + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}$
- Eraser: $Z_E = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^u])}$
- Unphosphorylated substrate: $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$
- Phosphorylated substrate: $Z_{S^p} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$

We define the reaction velocities

$$\alpha_{WS} = e^{-\beta\Delta\epsilon_{WS}} \quad (120)$$

$$\alpha_{ES} = e^{-\beta\Delta\epsilon_{ES}} \quad (121)$$

The associated conditional probabilities are then

$$p(W^u|S^u, S^p, R, E_W, E_S) = \frac{[W^u]}{[W_T^u]} = \frac{1}{Z_{W^u}} = \frac{1}{1 + v_{RW}[R] + v_{E_W W}[E_W]} \quad (122)$$

$$p(RW^u|S^u, S^p, R, E_W, E_S) = \frac{[RW^u]}{[W_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{RW} - k_B T \log[R])}}{Z_{W^u}} = \frac{v_{RW}[R]}{1 + v_{RW}[R] + v_{E_W W}[E_W]} \quad (123)$$

$$p(E_W W^u|S^u, S^p, R, E_W, E_S) = \frac{[E_W W^u]}{[W_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{E_W W} - k_B T \log[E_W])}}{Z_{W^u}} = \frac{v_{E_W W}[E_W]}{1 + v_{RW}[R] + v_{E_W W}[E_W]} \quad (124)$$

Continue Here

$$p(W^p|S^u, S^p, R, E_W, E_S) = \frac{[W^p]}{[W_T^p]} = \frac{1}{Z_W} = \frac{1}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (125)$$

$$p(RW^p|S^u, S^p, R, E_W, E_S) = \frac{[RW^p]}{[W_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{RW} - k_B T \log[R])}}{Z_W} = \frac{v_{RW}[R]}{1 + v_{RW}[R] + v_{E_W W}[E_W]} \quad (126)$$

$$p(E_W W^p|S^u, S^p, R, E_W, E_S) = \frac{[E_W W^p]}{[W_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{E_W W} - k_B T \log[E_W])}}{Z_W} = \frac{v_{E_W W}[E_W]}{1 + v_{RW}[R] + v_{E_W W}[E_W]} \quad (127)$$

$$p(W^p S^u|S^u, S^p, R, E_W, E_S) = \frac{[W^p S^u]}{[W_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{\alpha_{WS}[S^u]}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (128)$$

$$p(W^p S^p|S^u, S^p, R, E_W, E_S) = \frac{[W^p S^p]}{[W_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])}}{Z_W} = \frac{\alpha_{WS}[S^p]}{1 + \alpha_{WS}([S^p] + [S^u])} \quad (129)$$

$$p(E|S^u, S^p, W) = \frac{[E]}{[E_T]} = \frac{1}{Z_E} = \frac{1}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (130)$$

$$p(ES^u|S^u, S^p, W) = \frac{[ES^u]}{[E_T]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^u])}}{Z_E} = \frac{\alpha_{ES}[S^u]}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (131)$$

$$p(ES^p|S^u, S^p, W) = \frac{[ES^p]}{[E_T]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])}}{Z_E} = \frac{\alpha_{ES}[S^p]}{1 + \alpha_{ES}([S^p] + [S^u])} \quad (132)$$

$$p(S^u|E, W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (133)$$

$$p(WS^u|E, W) = \frac{[WS^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{\alpha_{WS}[W]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (134)$$

$$p(ES^u|E, W) = \frac{[ES^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}}{Z_{S^u}} = \frac{\alpha_{ES}[E]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (135)$$

$$p(S^p|E, W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (136)$$

$$p(WS^p|E, W) = \frac{[WS^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^p}} = \frac{\alpha_{WS}[W]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (137)$$

$$p(ES^p|E, W) = \frac{[ES^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}}{Z_{S^p}} = \frac{\alpha_{ES}[E]}{1 + \alpha_{WS}[W] + \alpha_{ES}[E]} \quad (138)$$

Reactions not in Detailed Balance:

$$\frac{d[S_T^p]}{dt} = k_{WS}^p[WS^u] - k_{ES}^u[ES^p] + k_{bg}^p[S_T^u] - k_{bg}^u[S_T^p] \quad (139)$$

$$\frac{d[W_T^p]}{dt} = k_{RW}^p[RW^u] - k_{EW}^u[E_W W^p] + k_W^p[W_T^u] - k_W^u[w_T^p] \quad (140)$$