

# Quasi-thermodynamic Push-Pull Models

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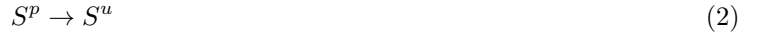
## I. ONE-WAY PUSH MODEL

In this model, we have a substrate  $S^u$  and a writer (kinase)  $W$ . The writer is able to phosphorylate the substrate from  $S^u$  to  $S^p$ . In this model, we do not explicitly include an eraser (phosphatase), but there is a dephosphorylation background rate (and a phosphorylation background rate).

**Reactions:**



**Background Reactions:**



**Total Concentrations:** We define total concentrations for each species and also separately for unphosphorylated and phosphorylated states.

- Total writer:  $[W_T] = [W] + [WS^u] + [WS^p]$
- Total substrate:  $[S_T] = [S^u] + [S^p]$
- Total unphosphorylated substrate:  $[S_T^u] = [S^u] + [WS^u]$
- Total phosphorylated substrate:  $[S_T^p] = [S^p] + [WS^p]$

If we know the

**Binding Energies:** We assume the binding energies are the same for the unphosphorylated and phosphorylated states.

- Writer + substrate:  $\Delta\epsilon_{WS}$

**One-way Reaction Rates:** These are the reaction rates that cannot be described by binding energies.

- Phosphorylation of substrate by writer:  $k_{WS}^p$
- Background phosphorylation of substrate (independent of binding state):  $k_S^p$
- Background dephosphorylation of substrate (independent of binding state):  $k_S^u$

**Partition Functions and Probabilities:** We write down partition functions for each species based on the states listed in the total concentrations above. Partition function cannot mix unphosphorylated and phosphorylated states for the same species, so we do not write down a partition function for the total  $S$ . We assume the unbound state of each species is zero energy (although this choice doesn't matter).

- Writer:  $Z_W = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])} + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])}$
- Unphosphorylated substrate:  $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}$
- Phosphorylated substrate:  $Z_{S^p} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}$

We define the reaction velocity

$$v_{WS} = e^{-\beta\Delta\epsilon_{WS}} \quad (6)$$

The associated conditional probabilities are then

$$p(W|S^u, S^p) = \frac{[W]}{[W_T]} = \frac{1}{Z_W} = \frac{1}{1 + v_{WS}([S^u] + [S^p])} \quad (7)$$

$$p(WS^u|S^u, S^p) = \frac{[WS^u]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{v_{WS}[S^u]}{1 + v_{WS}([S^u] + [S^p])} \quad (8)$$

$$p(WS^p|S^u, S^p) = \frac{[WS^p]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])}}{Z_W} = \frac{v_{WS}[S^p]}{1 + v_{WS}([S^u] + [S^p])} \quad (9)$$

$$p(S^u|W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + v_{WS}[W]} \quad (10)$$

$$p(WS^u|W) = \frac{[WS^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{v_{WS}[W]}{1 + v_{WS}[W]} \quad (11)$$

$$p(S^p|W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + v_{WS}[W]} \quad (12)$$

$$p(WS^p|W) = \frac{[WS^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^p}} = \frac{v_{WS}[W]}{1 + v_{WS}[W]} \quad (13)$$

**Reactions in Detailed Balance:** From the probabilities, we focus on the equations for the concentration of each species in its unbound state. These correspond to the kinetic equations for reactions not involving (de)phosphorylation.

$$\frac{[W]}{[W_T]} = \frac{1}{1 + v_{WS}([S^u] + [S^p])} \quad (14)$$

$$\frac{[S^u]}{[S_T^u]} = \frac{1}{1 + v_{WS}[W]} \quad (15)$$

$$\frac{[S^p]}{[S_T^p]} = \frac{1}{1 + v_{WS}[W]} \quad (16)$$

**Reactions not in Detailed Balance:** This is the kinetic equation for reactions involving (de)phosphorylation. It is possible for either  $S^u$  or  $WS^u$  to undergo this process, so we have two equations

$$\frac{d[S^p]}{dt} = k_{WS}^- [WS^p] - k_{WS}^+ [W][S^p] + k_S^p [S^u] - k_S^u [S^p] \quad (17)$$

$$\frac{d[WS^p]}{dt} = k_{WS}^+ [W][S^p] - k_{WS}^- [WS^p] + k_{WS}^p [WS^u] + k_S^p [WS^u] - k_S^u [WS^p] \quad (18)$$

$$\frac{d[S^u]}{dt} = k_{WS}^- [WS^u] - k_{WS}^+ [W][S^u] + k_S^u [S^p] - k_S^p [S^u] \quad (19)$$

$$\frac{d[WS^u]}{dt} = k_{WS}^+ [W][S^u] - k_{WS}^- [WS^u] - k_{WS}^p [WS^u] + k_S^u [WS^p] - k_S^p [WS^u] \quad (20)$$

where we have defined the forward and backward dissociation rates  $k_{WS}^+$  and  $k_{WS}^-$  such that

$$v_{WS} = \frac{k_{WS}^+}{k_{WS}^-} \quad (21)$$

Instead of writing these four equations, we can instead write the kinetic equations so that they only describe reactions that are not in detailed balance and do not depend on  $k_{WS}^+$  and  $k_{WS}^-$ . To see what these equations are, we can simply write down the reaction rates for the total phosphorylated and unphosphorylated substrate. These will just be related by a negative sign since they sum to a constant, so we can write

$$\frac{d[S_T^p]}{dt} = k_{WS}^p [WS^u] + k_S^p [S_T^u] - k_S^u [S_T^p] \quad (22)$$

$$= k_{WS}^p [S_T^u] p(WS^u|W) + k_S^p [S_T^u] - k_S^u [S_T^p] \quad (23)$$

$$= k_{WS}^p ([S_T] - [S_T^p]) p(WS^u|W) + k_S^p ([S_T] - [S_T^p]) - k_S^u [S_T^p] \quad (24)$$

We note that this equation can also be obtained by summing together the first two kinetic equations. Similarly, the equation for  $[S_T^u]$  can be obtained by summing the last two kinetic equations.

Setting this equation to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{WS}^p p(W S^u | W) + k_S^p}{k_{WS}^p p(W S^u | W) + k_S^p + k_S^u} \quad (25)$$

$$= \frac{v_{WS}^p p(W S^u | W) + v_S^p}{v_{WS}^p p(W S^u | W) + v_S^p + 1} \quad (26)$$

where we have defined the reaction velocities relative to the background dephosphorylation rate,

$$v_{WS}^p = \frac{k_{WS}^p}{k_S^u} \quad (27)$$

$$v_S^p = \frac{k_S^p}{k_S^u} \quad (28)$$

If the total concentrations  $[W_T]$  and  $[S_T]$  are known, this equation plus Eqs. (14)-(15) and the four equations for concentration comprise a system of 8 equations for the remaining 7 unknown concentrations. Since this system is over-determined, either one of the concentrations or one of Eqs. (14)-(15) maybe neglected.

## II. GOLDBETER PUSH-PULL AMPLIFIER

In the Goldbeter model, we have a push-pull amplifier with a substrate  $S$ , a writer (kinase)  $W$  and an eraser (phosphatase)  $E$  with no background (de)phosphorylation [1]. In addition, we lack the intermediate states directly after phosphorylation  $W S^p$  and dephosphorylation  $E S^u$ . These two states are assumed to be very short-lived.

**Reactions:**



**Total Concentrations:**

- Total writer:  $[W_T] = [W] + [W S^u]$
- Total Eraser:  $[E_T] = [E] + [E S^p]$
- Total substrate:  $[S_T] = [S^u] + [S_T^p]$
- Total unphosphorylated substrate:  $[S_T^u] = [S^u] + [W S^u]$
- Total phosphorylated substrate:  $[S_T^p] = [S^p] + [E S^p]$

**Binding Energies:**

- Writer + substrate:  $\Delta\epsilon_{WS}$
- Eraser + substrate:  $\Delta\epsilon_{ES}$

**One-way Reaction Rates:**

- Phosphorylation of substrate by writer:  $k_{WS}^p$
- Dephosphorylation of substrate by eraser:  $k_{ES}^u$

**Partition Functions and Probabilities:**

- Writer:  $Z_W = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}$
- Eraser:  $Z_E = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])}$
- Unphosphorylated substrate:  $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}$

- Phosphorylated substrate:  $Z_{Sp} = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$

We define the reaction velocities

$$v_{WS} = e^{-\beta\Delta\epsilon_{WS}} \quad (31)$$

$$v_{ES} = e^{-\beta\Delta\epsilon_{ES}} \quad (32)$$

The associated conditional probabilities are then

$$p(W|S^u, S^p, E) = \frac{[W]}{[W_T]} = \frac{1}{Z_W} = \frac{1}{1 + v_{WS}[S^u]} \quad (33)$$

$$p(WS^u|S^u, S^p, E) = \frac{[WS^u]}{[W_T]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{v_{WS}[S^u]}{1 + v_{WS}[S^u]} \quad (34)$$

$$p(E|S^u, S^p, W) = \frac{[E]}{[E_T]} = \frac{1}{Z_E} = \frac{1}{1 + v_{ES}[S^p]} \quad (35)$$

$$p(ES^p|S^u, S^p, W) = \frac{[ES^p]}{[E_T]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])}}{Z_E} = \frac{v_{ES}[S^p]}{1 + v_{ES}[S^p]} \quad (36)$$

$$p(S^u|E, W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + v_{WS}[W]} \quad (37)$$

$$p(WS^u|E, W) = \frac{[WS^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{v_{WS}[W]}{1 + v_{WS}[W]} \quad (38)$$

$$p(S^p|E, W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + v_{ES}[E]} \quad (39)$$

$$p(ES^p|E, W) = \frac{[ES^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}}{Z_{S^p}} = \frac{v_{ES}[E]}{1 + v_{ES}[E]} \quad (40)$$

**Reactions in Detailed Balance:**

$$\frac{[W]}{[W_T]} = \frac{1}{1 + v_{WS}[S^u]} \quad (41)$$

$$\frac{[E]}{[E_T]} = \frac{1}{1 + v_{ES}[S^p]} \quad (42)$$

$$\frac{[S^u]}{[S_T^u]} = \frac{1}{1 + v_{WS}[W]} \quad (43)$$

$$\frac{[S^p]}{[S_T^p]} = \frac{1}{1 + v_{ES}[E]} \quad (44)$$

**Reactions not in Detailed Balance:** We only focus on the equations for the total amounts of phosphorylated and unphosphorylated substrate, allowing us to ignore any of the reactions in detailed balance.

$$\frac{d[S_T^p]}{dt} = k_{WS}^p[WS^u] - k_{ES}^u[ES^p] \quad (45)$$

$$= k_{WS}^p[S_T^u]p(WS^u|E, W) - k_{ES}^u[S_T^p]p(ES^p|E, W) \quad (46)$$

$$= k_{WS}^p([S_T] - [S_T^p])p(WS^u|E, W) - k_{ES}^u[S_T^p]p(ES^p|E, W) \quad (47)$$

Setting this to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{WS}^p p(WS^u|E, W)}{k_{WS}^p p(WS^u|E, W) + k_{ES}^u p(ES^p|E, W)} \quad (48)$$

$$= \frac{v_{WS}^p p(WS^u|E, W)}{v_{WS}^p p(WS^u|E, W) + p(ES^p|E, W)} \quad (49)$$

$$(50)$$

where we have defined the reaction velocity

$$v_{WS}^p = \frac{k_{WS}^p}{k_{ES}^u} \quad (51)$$

### III. COMPLETE PUSH-PULL AMPLIFIER

This model generalizes the Goldbeter model to include the intermediate states and background (de)phosphorylation.

**Reactions Reactions:**



**Background:**



**Total Concentrations:**

- Total writer:  $[W_T] = [W] + [WS^p] + [WS^u]$
- Total Eraser:  $[E_T] = [E] + [ES^p] + [ES^u]$
- Total substrate:  $[S_T] = [S^u] + [S^p]$
- Total unphosphorylated substrate:  $[S_T^u] = [S^u] + [WS^u] + [ES^u]$
- Total phosphorylated substrate:  $[S_T^p] = [S^p] + [WS^p] + [ES^p]$

**Binding Energies:**

- Writer + substrate:  $\Delta\epsilon_{WS}$
- Eraser + substrate:  $\Delta\epsilon_{ES}$

**One-way Reaction Rates:**

- Phosphorylation of substrate by writer:  $k_{WS}^p$
- Dephosphorylation of substrate by eraser:  $k_{ES}^u$
- Background phosphorylation of substrate (independent of binding state):  $k_S^p$
- Background dephosphorylation of substrate (independent of binding state):  $k_S^u$

**Partition Functions and Probabilities:**

- Writer:  $Z_W = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^p])} + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[S^u])}$
- Eraser:  $Z_E = 1 + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^p])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[S^u])}$
- Unphosphorylated substrate:  $Z_{S^u} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$
- Phosphorylated substrate:  $Z_{S^p} = 1 + e^{-\beta(\Delta\epsilon_{WS} - k_B T \log[W])} + e^{-\beta(\Delta\epsilon_{ES} - k_B T \log[E])}$

We define the reaction velocities

$$v_{WS} = e^{-\beta \Delta \epsilon_{WS}} \quad (60)$$

$$v_{ES} = e^{-\beta \Delta \epsilon_{ES}} \quad (61)$$

The associated conditional probabilities are then

$$p(W|S^u, S^p, E) = \frac{[W]}{[W_T]} = \frac{1}{Z_W} = \frac{1}{1 + v_{WS}([S^p] + [S^u])} \quad (62)$$

$$p(W S^u|S^u, S^p, E) = \frac{[W S^u]}{[W_T]} = \frac{e^{-\beta(\Delta \epsilon_{WS} - k_B T \log[S^u])}}{Z_W} = \frac{v_{WS}[S^u]}{1 + v_{WS}([S^p] + [S^u])} \quad (63)$$

$$p(W S^p|S^u, S^p, E) = \frac{[W S^p]}{[W_T]} = \frac{e^{-\beta(\Delta \epsilon_{WS} - k_B T \log[S^p])}}{Z_W} = \frac{v_{WS}[S^p]}{1 + v_{WS}([S^p] + [S^u])} \quad (64)$$

$$p(E|S^u, S^p, W) = \frac{[E]}{[E_T]} = \frac{1}{Z_E} = \frac{1}{1 + v_{ES}([S^p] + [S^u])} \quad (65)$$

$$p(E S^u|S^u, S^p, W) = \frac{[E S^u]}{[E_T]} = \frac{e^{-\beta(\Delta \epsilon_{ES} - k_B T \log[S^u])}}{Z_E} = \frac{v_{ES}[S^u]}{1 + v_{ES}([S^p] + [S^u])} \quad (66)$$

$$p(E S^p|S^u, S^p, W) = \frac{[E S^p]}{[E_T]} = \frac{e^{-\beta(\Delta \epsilon_{ES} - k_B T \log[S^p])}}{Z_E} = \frac{v_{ES}[S^p]}{1 + v_{ES}([S^p] + [S^u])} \quad (67)$$

$$p(S^u|E, W) = \frac{[S^u]}{[S_T^u]} = \frac{1}{Z_{S^u}} = \frac{1}{1 + v_{WS}[W] + v_{ES}[E]} \quad (68)$$

$$p(W S^u|E, W) = \frac{[W S^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta \epsilon_{WS} - k_B T \log[W])}}{Z_{S^u}} = \frac{v_{WS}[W]}{1 + v_{WS}[W] + v_{ES}[E]} \quad (69)$$

$$p(E S^u|E, W) = \frac{[E S^u]}{[S_T^u]} = \frac{e^{-\beta(\Delta \epsilon_{ES} - k_B T \log[E])}}{Z_{S^u}} = \frac{v_{ES}[E]}{1 + v_{WS}[W] + v_{ES}[E]} \quad (70)$$

$$p(S^p|E, W) = \frac{[S^p]}{[S_T^p]} = \frac{1}{Z_{S^p}} = \frac{1}{1 + v_{WS}[W] + v_{ES}[E]} \quad (71)$$

$$p(W S^p|E, W) = \frac{[W S^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta \epsilon_{WS} - k_B T \log[W])}}{Z_{S^p}} = \frac{v_{WS}[W]}{1 + v_{WS}[W] + v_{ES}[E]} \quad (72)$$

$$p(E S^p|E, W) = \frac{[E S^p]}{[S_T^p]} = \frac{e^{-\beta(\Delta \epsilon_{ES} - k_B T \log[E])}}{Z_{S^p}} = \frac{v_{ES}[E]}{1 + v_{WS}[W] + v_{ES}[E]} \quad (73)$$

**Reactions in Detailed Balance:**

$$\frac{[W]}{[W_T]} = \frac{1}{1 + v_{WS}([S^p] + [S^u])} \quad (74)$$

$$\frac{[E]}{[E_T]} = \frac{1}{1 + v_{ES}([S^p] + [S^u])} \quad (75)$$

$$\frac{[S^u]}{[S_T^u]} = \frac{1}{1 + v_{WS}[W] + v_{ES}[E]} \quad (76)$$

$$\frac{[S^p]}{[S_T^p]} = \frac{1}{1 + v_{WS}[W] + v_{ES}[E]} \quad (77)$$

**Reactions not in Detailed Balance:**

$$\frac{d[S_T^p]}{dt} = k_{WS}^p[WS^u] - k_{ES}^u[ES^p] + k_S^p[S_T^u] - k_S^u[S_T^p] \quad (78)$$

$$= k_{WS}^p[S_T^u]p(WS^u|E, W) - k_{ES}^u[S_T^p]p(ES^p|E, W) + k_S^p[S_T^u] - k_S^u[S_T^p] \quad (79)$$

$$= k_{WS}^p([S_T] - [S_T^p])p(WS^u|E, W) - k_{ES}^u[S_T^p]p(ES^p|E, W) + k_S^p([S_T] - [S_T^p]) - k_S^u[S_T^p] \quad (80)$$

Setting this to zero, we get

$$\frac{[S_T^p]}{[S_T]} = \frac{k_{WS}^p p(WS^u|E, W) + k_S^p}{k_{WS}^p p(WS^u|E, W) + k_{ES}^u p(ES^p|E, W) + k_S^p + k_S^u} \quad (81)$$

$$= \frac{v_{WS}^p p(WS^u|E, W) + v_S^p}{v_{WS}^p p(WS^u|E, W) + p(ES^p|E, W) + v_S^p + v_S^u} \quad (82)$$

$$(83)$$

where we have defined the reaction velocities

$$v_{WS}^p = \frac{k_{WS}^p}{k_{ES}^u} \quad (84)$$

$$v_S^p = \frac{k_S^p}{k_{ES}^u} \quad (85)$$

$$v_S^u = \frac{k_S^u}{k_{ES}^u} \quad (86)$$

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[1] A. Goldbeter and D. E. Koshland, [Proc. Natl. Acad. Sci. \*\*78\*\*, 6840 \(1981\)](#).