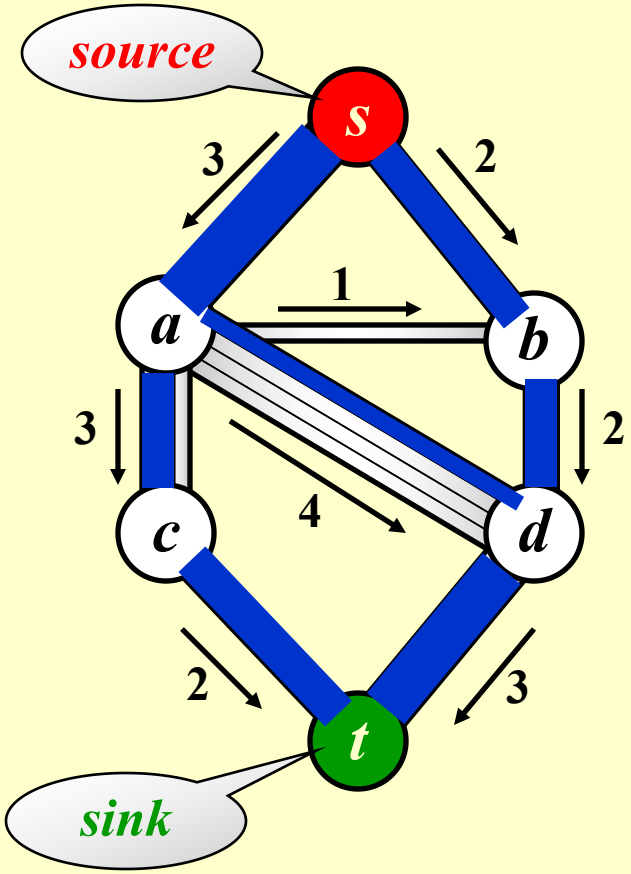
§ 4 Network Flow Problems

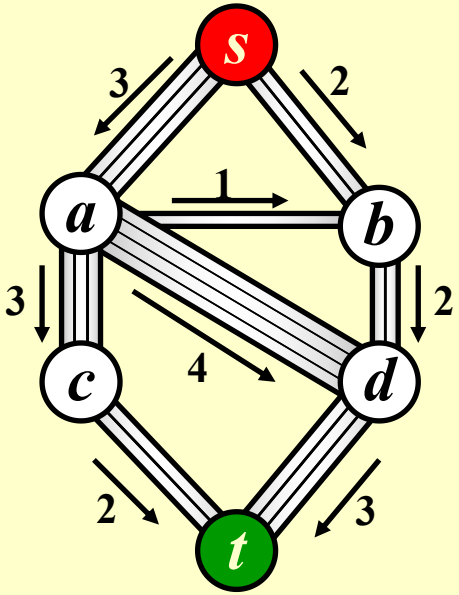
〖Example〗 Consider the following network of pipes:



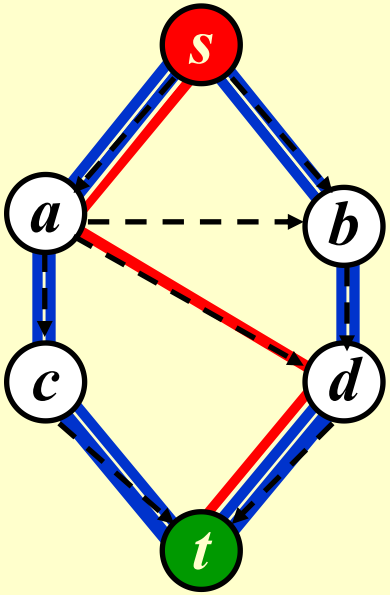
Note: Total coming in (v)
≡ Total going out (v)
where $v \notin \{ s, t \}$

Determine the maximum amount of flow that can pass from *s* to *t*.

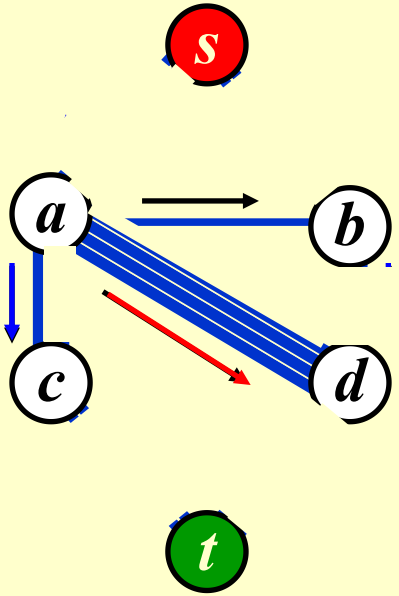
1. A Simple Algorithm



G



Flow G_f



Residual G_r

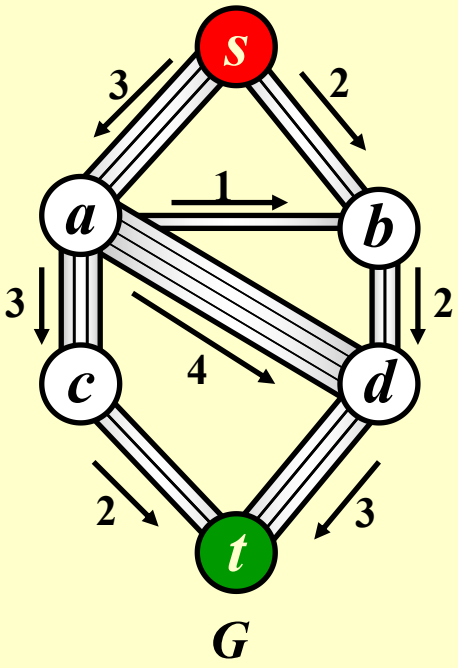
Step 1: Find any path $s \to t$ in G_r ;

Step 2: Take the minimum edge on this path as the amount of flow and add to G_f ;

augmenting path

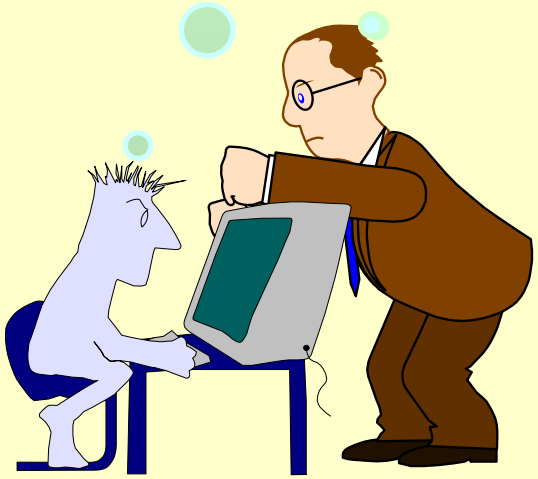
Step 3: Update G_r and remove the 0 flow edges;

Step 4: If (there is a path $s \to t$ in G_r)
Goto Step 1;
Else
End.



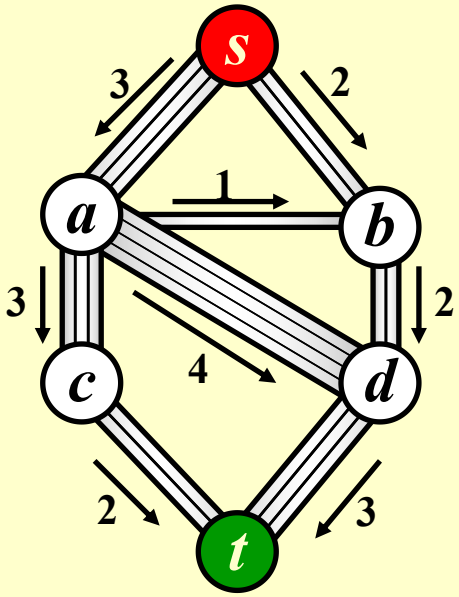
G

Uh-oh…
Seems we cannot be
greedy at this point.



2. A Solution – allow the algorithm to undo its decisions

For each edge (v, w) with flow $f_{v,w}$ in G_f , add an edge (w, v) with flow $f_{v,w}$ in G_r .



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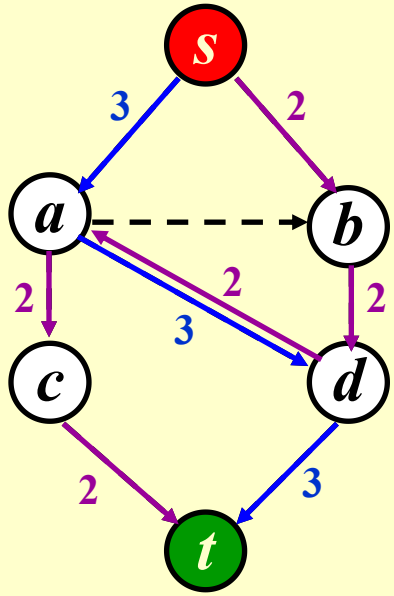
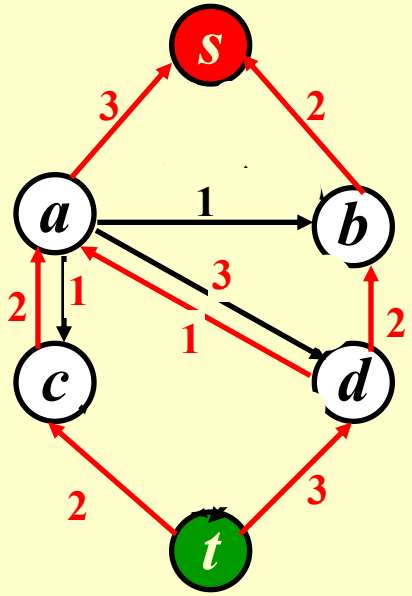


Flow G_f



Residual G_r

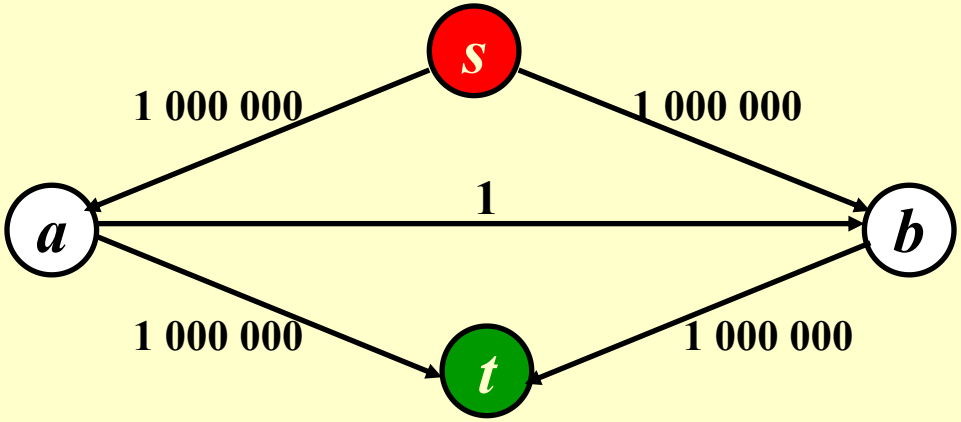
〖Proposition〗 If the edge capabilities are rational numbers, this algorithm always terminate with a maximum flow.

Note: The algorithm works for G with *cycles* as well.

3. Analysis (If the capacities are all integers)

☞ An augmenting path can be found by an unweighted shortest path algorithm.

$T = O(\ f \cdot |E|\ )$ where f is the maximum flow.



☞ Always choose the augmenting path that allows the largest increase in flow. /* modify Dijkstra's algorithm */

$$T = T_{augmentation} * T_{find\ a\ path}$$
$$= O(|E| \log cap_{max}) * O(|E| \log |V|)$$
$$= O(|E|^2 \log |V|) \text{ if } cap_{max} \text{ is a small integer.}$$

☞ **Always choose the augmenting path that has the least number of edges.**

$$T = T_{augmentation} * T_{find\ a\ path}$$
$$= O(|E|) * O(|E| \cdot |V|) \quad \text{/* unweighted shortest path algorithm */}$$
$$= O(|E|^2 |V|)$$

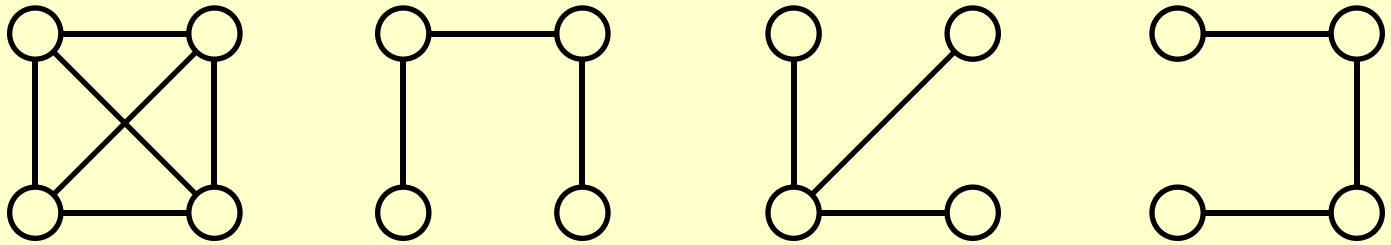
Note:

- If every $v \notin \{ s, t \}$ has either a single incoming edge of capacity 1 or a single outgoing edge of capacity 1, then time bound is reduced to $O(|E| |V|^{1/2})$.
- The *min-cost flow* problem is to find, among all maximum flows, the one flow of minimum cost provided that each edge has a cost per unit of flow.

§ 5 Minimum Spanning Tree

【Definition】 A *spanning tree* of a graph G is a tree which consists of V(G) and a subset of E(G)

〖Example〗 A complete graph and three of its spanning trees



Note:

- The minimum spanning tree is a *tree* since it is acyclic -- the number of edges is |V| – 1.
- It is *minimum* for the total cost of edges is minimized.
- It is *spanning* because it covers every vertex.
- A minimum spanning tree exists iff G is *connected*.
- Adding a non-tree edge to a spanning tree, we obtain a *cycle*.

Greedy Method

Make the best decision for each stage, under the following constrains :

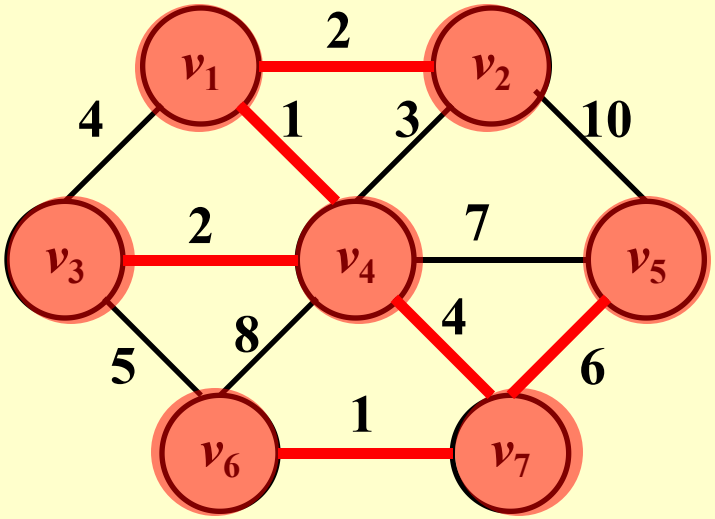
(1) we must use only edges within the graph;

(2) we must use exactly |V| –1 edges;

(3) we may not use edges that would produce a cycle.

1. *Prim*'s Algorithm – grow a tree

/* very similar to Dijkstra's algorithm */



2. *Kruskal*'s Algorithm – maintain a forest

$T = O(|E| \log |E|)$

```
void Kruskal ( Graph G )
{   T = { } ;
    while  ( T contains less than |V| –1 edges
                 && E is not empty ) {
        choose a least cost edge (v, w) from E ;   /* DeleteMin */
        delete (v, w) from E ;
        if  ( (v, w) does not create a cycle in T )
             add (v, w) to T ;      /* Union / Find */
        else
             discard (v, w) ;
    }
    if  ( T contains fewer than |V| –1 edges )
        Error ( "No spanning tree" ) ;
}
```

A more detailed pseudocode is given by Figure 9.58 on p.321