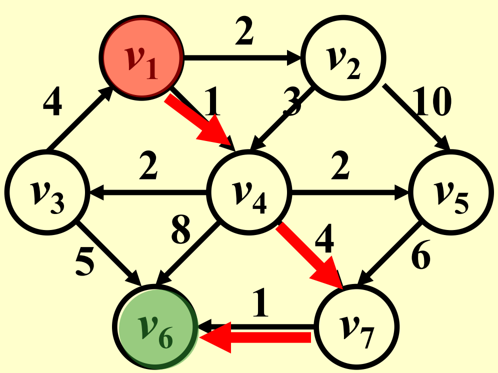
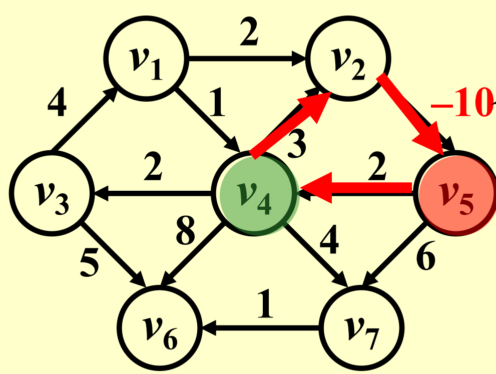
§ 3 Shortest Path Algorithms

Given a digraph G = (V, E), and a cost function *c*(*e*) for *e* ∈ E(G). The length of a path *P* from source to destination is $\sum_{e_i \subset P} c(e_i)$ **(also called weighted path length).**

1. Single-Source Shortest-Path Problem

Given as input a weighted graph, G = (V, E), and a distinguished vertex, *s*, find the shortest weighted path from *s* to every other vertex in G.

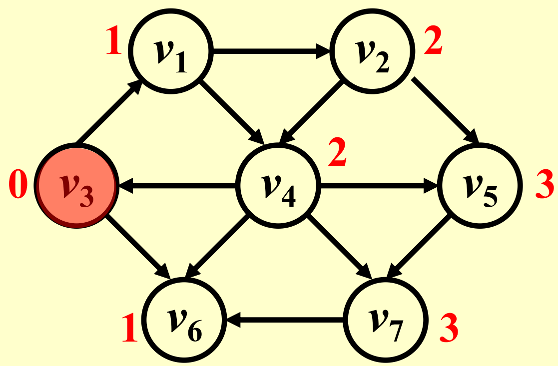




Note: If there is no negative-cost cycle, the shortest path from *s* to *s* is defined to be zero.

➢ Unweighted Shortest Paths

❖ Sketch of the idea



0: ☞ v_3

1: ☞ v_1 **and** v_6

2: ☞ v_2 **and** v_4

3: ☞ v_5 **and** v_7

Breadth-first search

❖ Implementation

Table[i].Dist ::= distance from *s* to v_i **/* initialized to be ∞ except for *s* */**

Table[i].Known ::= 1 if v_i is checked; or 0 if not

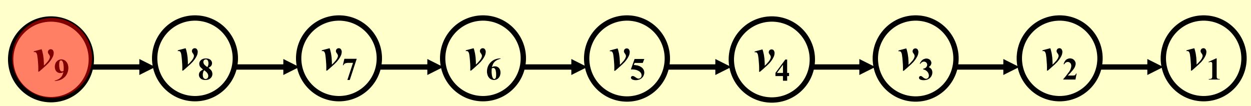
Table[i].Path ::= for tracking the path **/* initialized to be 0 */**

```
void Unweighted( Table T )
{   int  CurrDist;
    Vertex  V, W;
    for ( CurrDist = 0; CurrDist < NumVertex; CurrDist ++ ) {
        for ( each vertex V )
            if ( !T[ V ].Known && T[ V ].Dist == CurrDist ) {
                T[ V ].Known = true;
                for ( each W adjacent to V )
                    if ( T[ W ].Dist == Infinity ) {
                        T[ W ].Dist = CurrDist + 1;
                        T[ W ].Path = V;
                    } /* end-if Dist == Infinity */
            } /* end-if !Known && Dist == CurrDist */
    }  /* end-for CurrDist */
}
```

If V is unknown yet has Dist < Infinity, then Dist is either CurrDist or CurrDist+1.

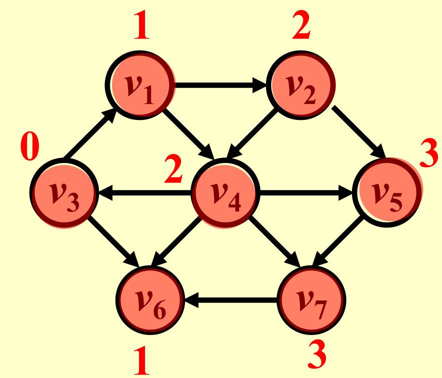
$T = O(\,|V|^2\,)$

The worst case:



❖ Improvement

```
void Unweighted( Table T )
{   /* T is initialized with the source vertex S given */
    Queue  Q;
    Vertex  V, W;
    Q = CreateQueue (NumVertex );  MakeEmpty( Q );
    Enqueue( S, Q ); /* Enqueue the source vertex */
    while ( !IsEmpty( Q ) ) {
        V = Dequeue( Q );
        T[ V ].Known = true; /* not really necessary */
        for ( each W adjacent to V )
            if ( T[ W ].Dist == Infinity ) {
                T[ W ].Dist = T[ V ].Dist + 1;
                T[ W ].Path = V;
                Enqueue( W, Q );
            } /* end-if Dist == Infinity */
    } /* end-while */
    DisposeQueue( Q ); /* free memory */
}
```



	Dist	Path
v_1	1	v_3
v_2	2	v_1
v_3	0	0
v_4	2	v_1
v_5	3	v_2
v_6	1	v_3
v_7	3	v_4

$T = O(|V| + |E|)$

➢ *Dijkstra*'s Algorithm (for weighted shortest paths)

Let S = { s and v_i's whose shortest paths have been found }

For any $u \notin S$, define distance [u] = minimal length of path { $s \rightarrow (v_i \in S) \rightarrow u$ }. If the paths are generated in non-decreasing order, then

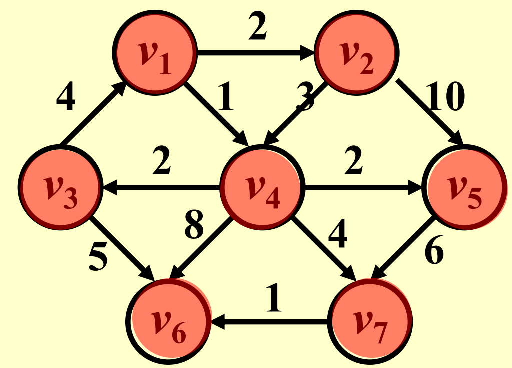
① the shortest path must go through ONLY $v_i \in S$;

Why? If it is not true, then there must be a vertex w on this path that is not in S. Then ...

② u is chosen so that distance[u] = min{ $w \notin S$ | distance[w] } (If u is not unique, then we may select any of them) ; /* Greedy Method */

③ if distance [u_1] < distance [u_2] and we add u_1 into S, then distance [u_2] may change. If so, a shorter path from s to u_2 must go through u_1 and distance' [u_2] = distance [u_1] + length($< u_1, u_2 >$).

```
void Dijkstra( Table T )
{   /* T is initialized by Figure 9.30 on p.303 */
    Vertex  V, W;
    for ( ; ; ) {    /* O( |V| ) */
        V = smallest unknown distance vertex;
        if ( V == NotAVertex )
            break;
        T[ V ].Known = true;
        for ( each W adjacent to V )
            if ( !T[ W ].Known )
                if ( T[ V ].Dist + Cvw < T[ W ].Dist ) {
                    Decrease( T[ W ].Dist  to
                              T[ V ].Dist + Cvw );
                    T[ W ].Path = V;
                } /* end-if update W */
    } /* end-for( ; ; ) */
}   /* not work for edge with negative cost */
```



	Dist	Path
v_1	0	0
v_2	2	v_1
v_3	3	v_4
v_4	1	v_1
v_5	3	v_4
v_6	6	v_7
v_7	5	v_4

Please read Figure 9.31 on p.304 for printing the path.

❖ Implementation 1

V = smallest unknown distance vertex;
/* simply scan the table – O(|V|) */

$T = O(\ |V|^2 + |E|\ )$

Good if the graph is dense

❖ Implementation 2

V = smallest unknown distance vertex;
/* keep distances in a priority queue and call DeleteMin – O(log|V|) */

Decrease(T[W].Dist to T[V].Dist + Cvw);

Good if the graph is sparse

/* Method 1: DecreaseKey – O(log|V|) */

$T = O(\ |V| \log|V| + |E| \log|V|\ ) = O(\ |E| \log|V|\ )$

/* Method 2: insert W with updated Dist into the priority queue */

/* Must keep doing DeleteMin until an unknown vertex emerges */

$T = O(\ |E| \log|V|\ )$ but requires |E| DeleteMin with |E| space

❖ **Other improvements: Pairing heap (Ch.12) and Fibonacci heap (Ch. 11)**

➢ Graphs with Negative Edge Costs

$T = O(\,|V| \times |E|\,)$

```
void  WeightedNegative( Table T )
{   /* T is initialized by Figure 9.30 on p.303 */
    Queue  Q;
    Vertex  V, W;
    Q = CreateQueue (NumVertex );  MakeEmpty( Q );
    Enqueue( S, Q ); /* Enqueue the source vertex */
    while ( !IsEmpty( Q ) ) {  /* each vertex can dequeue at most |V|
        V = Dequeue( Q );         times */
        for ( each W adjacent to V )
            if ( T[ V ].Dist + Cvw < T[ W ].Dist ) {   /* no longer once
                T[ W ].Dist = T[ V ].Dist + Cvw;          per edge */
                T[ W ].Path = V;
                if ( W is not already in Q )
                    Enqueue( W, Q );
            } /* end-if update */
    } /* end-while */
    DisposeQueue( Q ); /* free memory */
}   /* negative-cost cycle will cause indefinite loop */
```

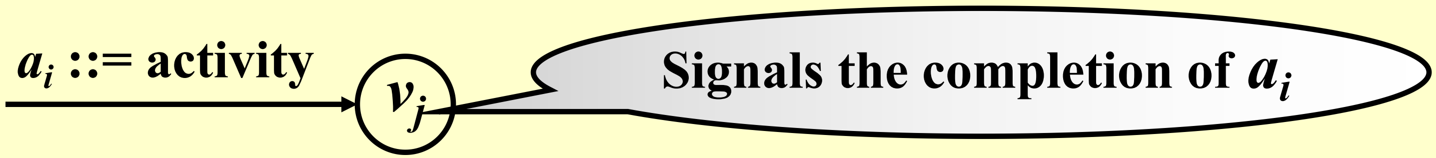
➤ Acyclic Graphs

If the graph is acyclic, vertices may be selected in topological order since when a vertex is selected, its distance can no longer be lowered without any incoming edges from unknown nodes.

T = O(|E| + |V|) and no priority queue is needed.

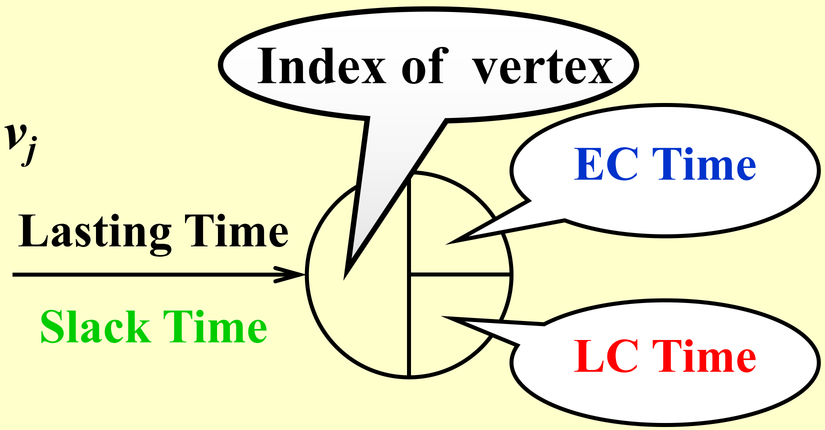
❖ **Application: AOE (Activity On Edge) Networks**

—— scheduling a project

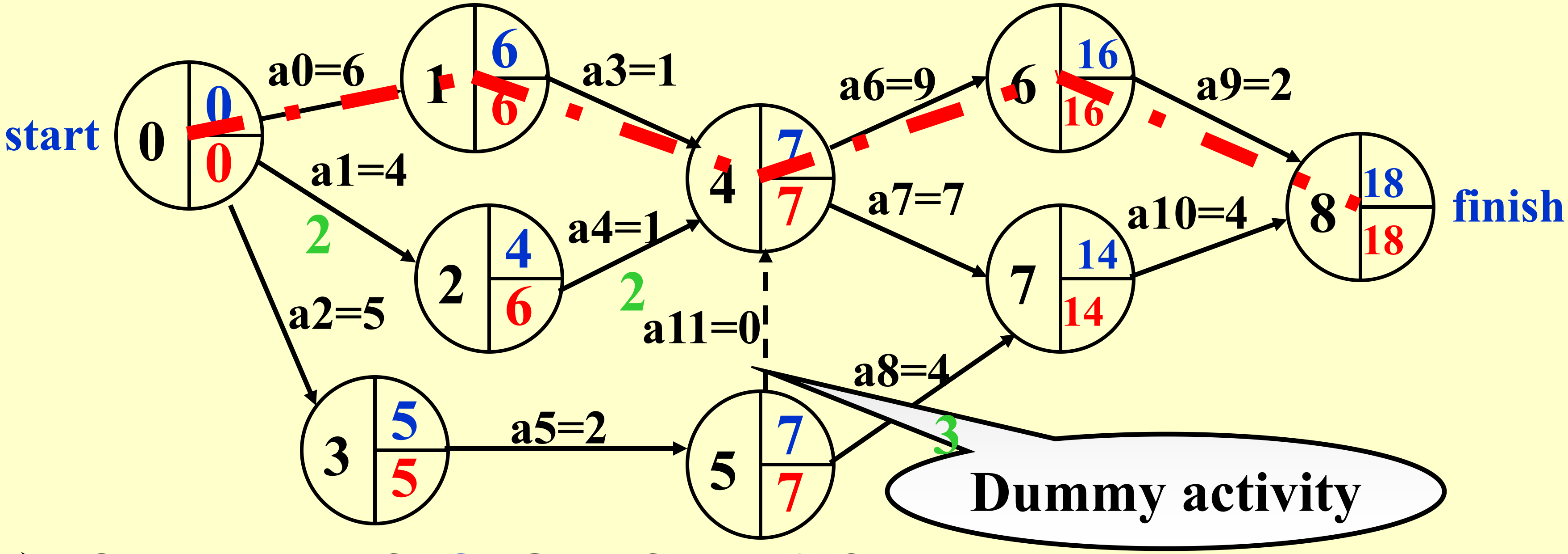


✎ **EC[j] \ LC[j] ::= the earliest \ latest completion time for node v_j**

🛠 **CPM (Critical Path Method)**



〖Example〗 AOE network of a hypothetical project



- Calculation of EC: Start from v0, for any a_i = <v, w>, we have

$$EC[w] = \max_{(v,w)\in E}\{EC[v] + C_{v,w}\}$$

- Calculation of LC: Start from the last vertex v8, for any a_i = <v, w>, we have $$LC[v] = \min_{(v,w)\in E}\{LC[w] - C_{v,w}\}$$

- Slack Time of <v,w> = $LC[w] - EC[v] - C_{v,w}$

- Critical Path ::= path consisting entirely of zero-slack edges.

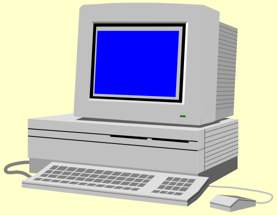
2. All-Pairs Shortest Path Problem

For all pairs of v_i and v_j ($i \neq j$), find the shortest path between.

Method 1 **Use single-source algorithm for |V| times.**

T = O($|V|^3$) – works fast on sparse graph.

Method 2 **O($|V|^3$) algorithm given in Ch.10, works faster on dense graphs.**



Laboratory Project 3

Normal: Dijkstra Sequence

Hard: Transportation Hub

Due: Monday, May 6th, 2024 at 10:00pm