yuntan

Linear Programming

Note Title 11/4/2015

$\max\ c^T x$	$\max\ c^T x$	$\max\ c^T x$
$Ax \le b$	$Ax \le b$	$Ax = b$
	$x \ge 0$	$x \ge 0$

$$a_i^T x = b_i \iff \begin{array}{l} a_i^T x \le b_i \\ a_i^T x \ge b_i \end{array}$$

$$a_i^T x \le b_i \iff \begin{array}{l} a_i^T x + s_i = b_i \\ s_i \ge 0 \end{array}$$

Dec. Problem : $\exists x :\ \begin{array}{c} Ax \le b \\ x \ge 0 \end{array},\quad c^T x \ge z$

$\in NP$ ✓ give x (caution: # bits should be poly in input size)

$\in$ co-NP ?? How to prove $\nexists x$?

$$\max\quad x_1 + 5x_2 - x_3$$

$$\begin{array}{llcl} (1) & 2x_1 - x_2 - 2x_3 & \le & 10 \\ (2) & 4x_1 + x_2 + x_3 & \le & 20 \\ (3) & -x_1 + 2x_2 + x_3 & \le & 4 \end{array}$$

$$x_1, x_2, x_3 \ge 0$$

$(1, 2, 1)$ $z = 10$

$5\times(2)$ $\quad 20x_1 + 5x_2 + 5x_3 \leq 100$

$\qquad\qquad\qquad \text{VI}$

$\qquad\qquad\qquad x_1 + 5x_2 - x_3$

$\qquad\qquad\qquad \wedge$

$(2) + 2\times(3)$ $\quad 2x_1 + 5x_2 + 3x_3 \leq 28$

$z \leq 28$

More generally

$y_1 \times (1) + y_2 \times (2) + y_3 \times (3)$ is a valid inequality

it is useful as an upper bound if the coefficients of x_1, x_2, x_3 are $\geq$ the coefficients in the objective function.

$$(2y_1 + 4y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-2y_1 + y_2 + y_3)x_3 \leq 10y_1 + 20y_2 + 4y_3$$

So we would like

$$\min \quad 10y_1 + 20y_2 + 4y_3$$

s.t.

$$2y_1 + 4y_2 - y_3 \geq 1$$

$$-y_1 + y_2 + 2y_3 \geq 5$$

$$-2y_1 + y_2 + y_3 \geq -1$$

$$y_1, y_2, y_3 \geq 0$$

Dual Linear Program!

Weak Duality $\max c^T x \leq \min b^T y$

(P) $Ax \leq b$, $x \geq 0$ $\quad$ $A^T y \geq c$, $y \geq 0$ (D)

Cor. if (P) is unbounded, (D) is infeasible
— (D) — (P) —.

(P) $\max c^T x$, $Ax = b$, $x \geq 0$ $\quad$ (D) $\min c^T y$, $A^T y \geq c$

Thm (LP duality). If both (P) and (D) are feasible then their values are equal!

This is an Co-NP certificate.

To prove it, we first prove a more basic fact.

Lemma (FARKAS) Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ one of the following is true:

(1) $\exists x \geq 0$ s.t. $Ax = b$

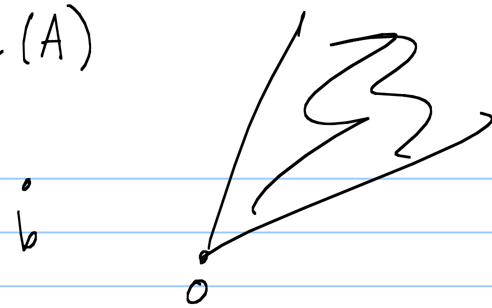
(2) $\exists y$: $A^T y \geq 0$ $\quad b^T y < 0$

Proof

$$C(A) = \left\{ x_1 \begin{pmatrix} a_1 \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_n \end{pmatrix}, \ x \geq 0 \right\}$$

Cone generated by A.

Either $b \in$ Cone(A)
or it does not.



If $b \notin$ Cone A, $\exists$ a halfspace that contains the cone but not b. (HW exercise).
(each a_i)

Pf (of LP duality).

Let (P), (D) be feasible
and $z =$ OPT (D).

Consider
$$Ax = b$$
$$c^T x = z$$

By Farkas' lemma, either $\exists x \geq 0$ s.t. $\begin{pmatrix} A \\ c^T \end{pmatrix} x = \begin{pmatrix} b \\ z \end{pmatrix}$

(OR) $\exists (y, \alpha)$ $\begin{pmatrix} A^T & c \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} \geq 0$ $\begin{pmatrix} b^T & z \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} < 0$ ✓

i.e. $A^T y + \alpha c \geq 0$ $\qquad$ $b^T y + \alpha z < 0$

Take feasible x, i.e. $x \geq 0$, $Ax = b$

$$x^T A^T y + \alpha x^T c = b^T y + \alpha (x^T c) \geq 0$$

If $\alpha > 0$ $\implies$ $x^T c > z$ contradicts weak duality.

If $\alpha < 0$
$$A^T y \geq -\alpha c \qquad b^T y < -\alpha z$$
$$A^T \left(-\frac{y}{\alpha}\right) \geq c \qquad b^T \left(-\frac{y}{\alpha}\right) < z$$
contradicts optimality of z!