yvRave Matchings

Note Title 10/19/2015

$G = (V, E)$ undirected, simple

$M \subseteq E$ is a matching if no two edges share a vertex

M is perfect if every vertex has an edge in M

$|M| = \frac{|V|}{2} = \frac{n}{2}$.

Problem: Determine whether G has a p.m.

NP? Co-NP?

Bipartite graphs $G = (U, V, E)$

all edges between U and V.

Thm. (König-Hall)

Bipartite G has a p.m. iff $|U| = |V|$

and $\forall X \subseteq U$, $|N(X)| \geq |X|$.

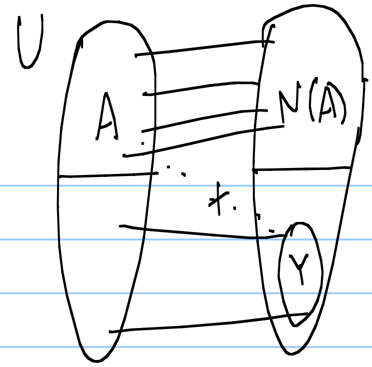
Pf. $\Rightarrow$: follows from the p.m.

$\Leftarrow$: If $|N(X)| > |X|$ $\forall X \subset U$,

then take any $e \in E$, add it to matching

$e = (u, v)$, consider $G' = G - \{u, v\}$. Still satisfies $|N(X)| \geq |X|$, $\exists$ p.m. M' in G'. $M' \cup \{e\}$ is a p.m. of G.

Else $\exists A \subset U$: $|N(A)| = |A|$



$|A| = |N(A)|$

$|B| = |U \setminus A| = |V \setminus N(A)|$

$\forall X \subseteq A, |N(X)| \geq |X|$

$\exists$ p.m. in G_A. M_A.

<u>claim:</u> $\forall Y \subseteq V \setminus N(A)$, $|N(Y)| \geq |Y|$

Suppose not, then take $X = U \setminus N(Y)$

$|Y| > |N(Y)|$

$N(X) \subseteq V \setminus Y$

$|X| > |N(X)|$. Contradicts assumption.

$\exists$ p.m. M_B in G_B.

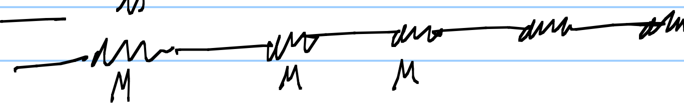
$M_A \cup M_B$ is a p.m. of G.

So p.m. $\in$ NP $\cap$ Co-NP

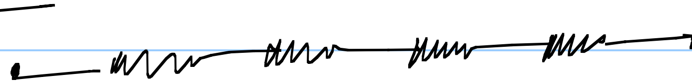
What about an algorithm?

Given G and a matching M (not necessarily perfect),

an <u>alternating path</u> is

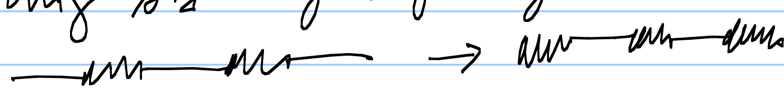


an <u>augmenting path</u>



starts with unmatched and ends with unmatched.

Given an <u>aug path</u>, we can increase the matching size by augmenting (matched $\rightleftarrows$ unmatched)



Algorithm.

Start with $M = \{\}$

Repeat
- Find aug path for M
- augment M

Lemma. M is maximum iff G has no aug paths wrt M.

Pf. $\Rightarrow$ ✓ since $\exists$ aug path $\Rightarrow$ M is not maximum.

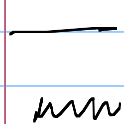
$\Leftarrow$: M has no aug path.

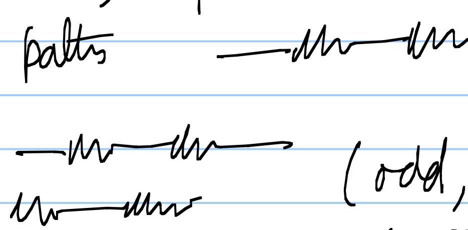
Suppose N is maximum matching.

Consider $M \oplus N = \{\text{edges in } M \text{ and not in } N\}$
$\cup \{\text{edges in } N \text{ and not in } M\}$

each vertex has degree 0, 1 or 2.

ie. isolated vertices, paths, cycles.

M

N

paths

(even) $|M| = |N|$

(odd, cannot exist since no aug paths for M or N!)

cycles

(even)


(odd: cannot exist!).

$\therefore \quad |M| = |N|$. ∎

Algorithm.

How to find an augmenting path?

"Hungarian Method"

- start with unmatched vertices on one side.
- alternately add ~~add~~ edges (unmatched) and matching edges
- If unmatched vertex of B is reached $\Rightarrow$ augmenting path.

$G = (A, B, E)$.

Maintain a forest F.

1. $F = A^U$ (unmatched.
2. Add edges to B, maintaining a forest (one edge to each new vertex)
3. If unmatched vertex of B is reached, output aug. path.
4. Add matching edges from B
5. Repeat 2, 3, 4 till no more edges can be added.

Claim: If no aug path found, i.e. no further edges can be added, then M is maximum.

Pf. $S = (A \setminus V(F)) \cup (B \cap V(F))$ is a vertex cover of G. $|S| = |M|$, but $|S| \geq |M| \; \forall S, M$

$\therefore$ M is maximum.