Matching Algorithms

10/21/2015

Find Aug path
Augment
] Repeat.

How to find Aug. Path?

Bipartite Graphs $G = (A, B, E)$, Matching M

Maintain an alternating forest.

- Start with all unmatched vertices of A.
- Add edges to B, maintaining a forest (one edge to each new vertex)
- If edge to unmatched vertex in B, then aug path!
- Add Matching edges from vertices in B

↑ Repeat

Claim: If Algorithm terminates with no Aug. path, then G has no aug path w.r.t. M and is maximum.

Pf. We will show that G has a vertex cover S s.t. $|S| = |M|$. Since $|VC| \geq |M|$ for any v.c. and any matching, the current M must be maximum.

$S = A \setminus V(F) \cup B \cap V(F)$

Suppose $e = (a,b)$ is not covered.

then $a \in V(F)$ and $b \notin V(F)$

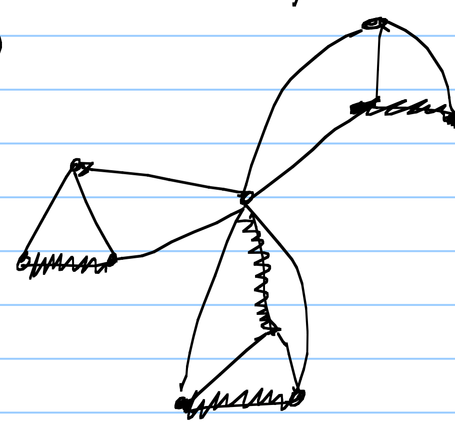
so (a,b) can be added to F and the algorithm continues.

Example.

Tue: $O(|E||V|)$ Friday: $O(|E|\sqrt{|V|})$

What about general graphs?

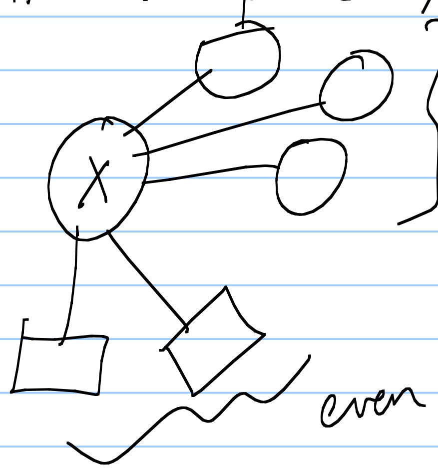
Where does this fail?



$|M| = 4$

$|V.C.| \geq 6$

How to prove this graph has no p.m.?!



odd components

$\forall X \subseteq V,$

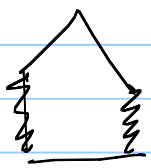
$|odd(G \setminus X)| \leq |X|$

is necessary.

even components.

Edmonds algorithm for Matchings in general graphs.

Blossom: odd cycle with $2K+1$ edges
and K of them matched.



Idea: Cycle shrinking.

<u>Lemma</u>. B is a blossom disjoint from the rest of M in G. G' is obtained by shrinking B to a single vertex. M' is the induced matching. Then G has an aug path for M iff G' has an aug path for M'.

<u>Pf</u>.

Algorithm to find an aug. path.

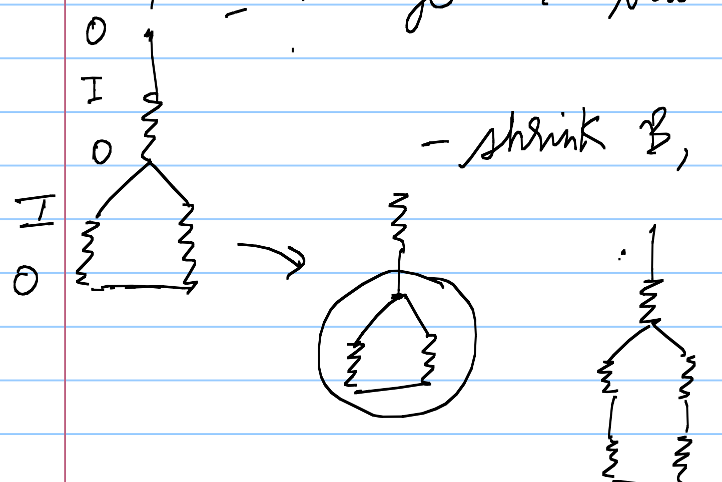
- Start with all unmatched vertices U as separate components of F.

- For $X \in F$, $Y \notin F$, $(X, Y) \in F$ then add (X, Y) and (Y, Z), the matching edge at Y.

INNER: odd level vertices (starts at 0)
OUTER: even level vertices (includes U)

- inner vertices have degree 2
  : one unmatched, one matched.

- If outer vertices X, Y from different components share an edge, then aug path!
- If outer vertices X, Y from same component share an edge → blossom B



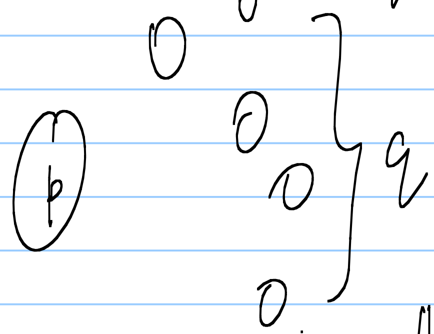
- shrink B, swap edges on path to root.

- If every outer vertex only has inner vertices as neighbors, then M is maximum!

$p = \#$ inner vertices $\quad q = \#$ outer vertices

$q - p = |U|$, since every outer vertex is matched to an inner vertex below it.

Consider removing the inner vertices.
Then we get q isolated vertices



So $q-p$ of them can't be matched.

Hence M is maximum.

<u>Cor.</u> Tutte's thm.

G has a p.m. iff $\forall X \subseteq V$
$|\mathrm{odd}(G \setminus X)| \leq |X|$.