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Max flow – Min cut

Note Title 10/28/2015

s-t Cut :

set of edges whose removal disconnects s, t.

OR $S \subseteq V$ s.t. $s \in S$, $t \in \bar{S} = V \setminus S$

$(S, \bar{S}) = \{ (i,j) : i \in S, j \in \bar{S} \}$

– Every $(S, \bar{S})$ is a set of edges whose removal disconnects s from t.

– For every subset F of edges that disconnects s, t, let S = vertices reachable from s after removing F then $(S, \bar{S})$ is an s-t cut. $(S, \bar{S}) \subseteq F$.

– if set of edges is minimum (least weight) $(S, \bar{S}) = F$.

(Menger) : max # edge-disjoint s-t paths = min s-t cut

Flow. $G = (V, E)$ Capacities $c_{ij} \geq 0$ s, t.

$0 \leq f_{ij} \leq c_{ij}$, flow conservation at $u \neq s, t$.

Claim $f \leq c(S, \bar{S})$ $\forall S : s \in S, t \notin S$.

$f = f(s, V) - f(V, s)$, $c(S, \bar{S}) = \text{Cap}(S, \bar{S})$.

Pf. $f(s, V) - f(V, s) = f$

$\forall u \neq s, t \quad f(\cdot V) - f(V, \quad = 0$

$$\sum_{u \in S} f(u, V) - f(V, u) = f$$

$$= f(S, V) - f(V, S)$$

$$= f(S, \bar{S}) + f(S, S) - f(S, S) - f(\bar{S}, S)$$

$$= f(S, \bar{S}) - f(\bar{S}, S) \leq f(S, \bar{S}) \leq C(S, \bar{S}).$$

Q. IS f maximum?

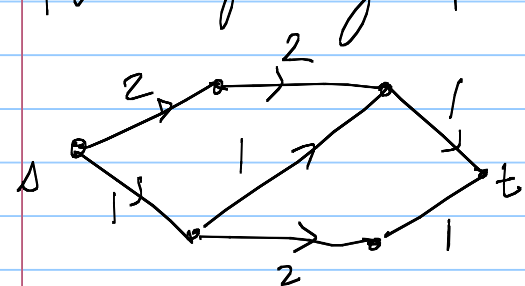
Thm max flow value f = min cut Capacity C

To prove this, we define augmenting paths

path $(u_0, u_1, u_2 \dots u_r)$ s.t.

- $u_0 = s$
- $(u_i, u_{i+1}) \in E \Rightarrow f_{i,i+1} < C_{i,i+1}$
- $(u_{i+1}, u_i) \in E \Rightarrow f_{i+1,i} > 0$

flow augmenting path if $u_r = t$.



Thm. A flow f is maximum iff $\nexists$ f-augmenting paths.

Pf. If $\exists$ f-augmenting path p,
then we can send some more flow from s to t.

Suppose no f-aug path.

Let $S = \{ u : \exists$ aug path from s to $u \}$

$\forall (i,j) \in (S, \bar{S}) \quad f_{ij} = C_{ij}$

$\forall (i,j) \in (\bar{S}, S) \quad f_{ij} = 0$

$\therefore f(S, \bar{S}) - f(\bar{S}, S) = C(S, \bar{S})$

$\Rightarrow f$ is maximum.

Algorithm

- start with $f = 0$

Repeat
- find f-aug path (how?)
- Augment f

Thm. Max flow = Min cut

Pf. maxflow $\leq$ min cut.

Let f be max flow. f has no f-aug paths.

$\therefore \exists$ cut $(S, \bar{S})$ s.t. $f = C(S, \bar{S})$.

Q.1 Does algorithm terminate?

Q.2 How to choose / find aug paths?

A1. with integer capacities and starting $f = 0$ (or integral)

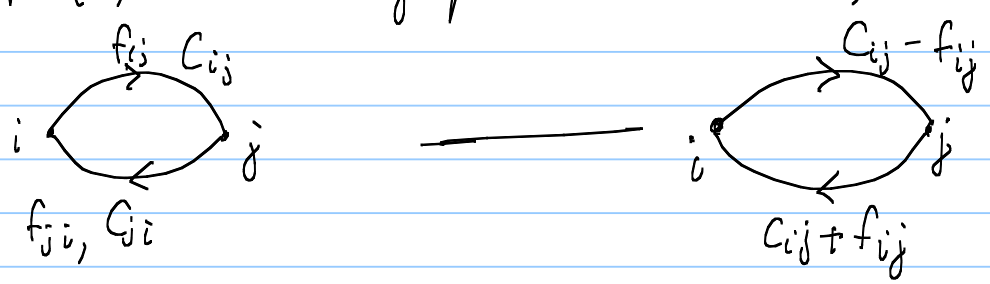
f remains integral after every augmentation

So process terminates. But could take *many* iterations.

With arbitrary starting flow, algorithm might go forever and never reach maxflow!

How to fing aug. paths.

Res(f) is a graph based on G, f



$f_{ij} \geq 0$, $f_{ji} = 0$

G Res(f)

Capacity(i,j) = flow that can now be sent.

Augmenting path for f in G $\equiv$ Directed path s to t in Res(G).

Finding one path is easy!
Which one to pick?

- max capacity
- shortest
- ??

<u>Algorithm</u>: Augment on max capacity f-augmenting path.
(Can we find this? YES).

<u>Lemma</u>: Any flow f can be decomposed into flow on at most m paths and cycles.

<u>Pf.</u> Find any s-t flow path, reduce f by max flow on this path; some edge capacity goes to 0, delete the edge.
This can be repeated at most m times, after which only cycles survive.

<u>Lemma</u>. For a flow f, there is a path of capacity $\geq \frac{f^* - f}{m}$ in Res(f), where f^* is the max flow.

<u>Pf.</u> $\exists$ flow of value $f^* - f$ in Res(f).
$\therefore \exists$ a path of capacity $\geq \frac{f^* - f}{m}$.

Thm. # augmentations of max capacity $= O(m \log nU)$

where $U = \max_{ij} C_{ij}$.

Pf. After reaching flow f, consider all augmentations of capacity $\geq \frac{f^* - f}{2m}$.

At most $2m$ such augmentations.

After these augmentations, max cap aug path $\leq \frac{f^* - f}{2m}$

$\Rightarrow$ remaining flow $\leq \frac{f^* - f}{2}$.

So remaining flow halves every $O(m)$ iterations.

$\therefore$ total # iterations $\leq 2m \log_2 (nU)$.

Since $f \leq nU$.

Time $= O(m^2 \log nU)$. polynomial!

$\boxed{\text{FRI:}}$ shortest aug path.