Completeness

Note Title — Yu Raz — 10/5/2015

SAT $\to$ Clique

3SAT $\to$ 3-COLORING

What about the other classes we defined?

$coNP, \Sigma_2, \Pi_2, \ldots \Sigma_i, \Pi_i, PH, PSPACE, \ldots$

Do they have complete problems?

YES! $\Sigma_i SAT$, $\Pi_i SAT$.

PSPACE

Recall

$$PSPACE = \bigcup_k SPACE(n^k)$$

L is PSPACE-complete if

(1) $L \in PSPACE$

(2) $\forall L' \in PSPACE, \; L' \hookrightarrow_p L$

polytime reduction

(Why polytime? If $L \in P \Rightarrow PSPACE = P$!)

What is a candidate PSPACE-complete L?

Life

Games

Totally Quantified Boolean Formulae

$$\left\{ F = \exists x_1 \forall x_2, x_3 \exists x_4 \ldots \forall x_n \left((x_1 \vee x_2) \wedge \ldots \right) \right\}$$
is TRUE

Every variable has a quantifier ($\exists$ or $\forall$) and the formula is TRUE

Thm TQBF is PSPACE-complete.

Pf. TQBF $\in$ PSPACE

Alternating TM that accepts true F.

i.e. A recursive algorithm to check if $F \in$ TQBF:

CHECK(F):
- If no variables, evaluate F
- Take first variable x_1
  - If $\exists x_1$, try CHECK($F(x_1=0, \ldots)$) and CHECK($F(x_1=1, \ldots)$)
    ACCEPT if one accepts
  - If $\forall x_1$, accept if BOTH accept.

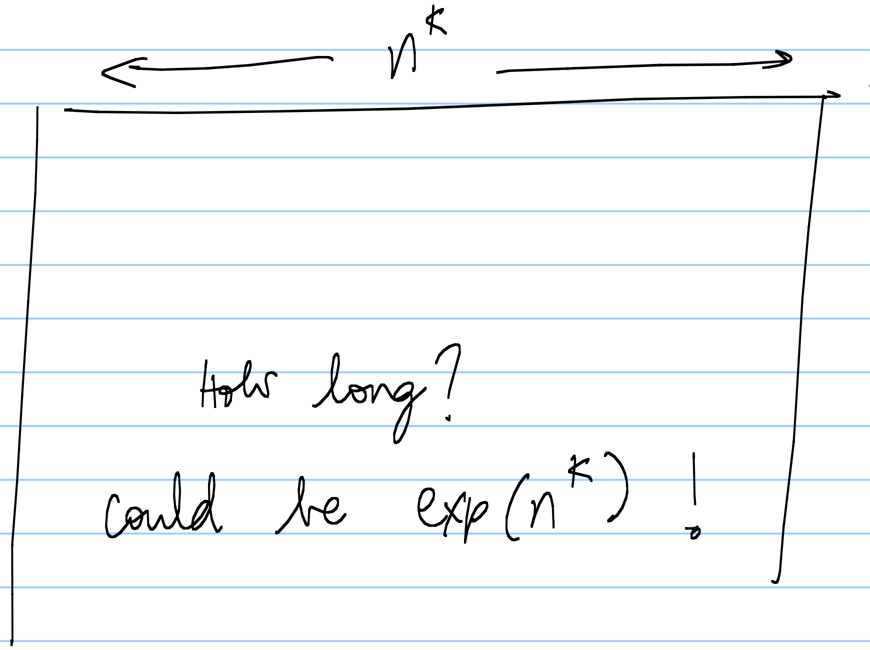
SPACE $\simeq$ DEPTH of RECURSION $\times$ poly

$\leq$ poly(|input|).

To show TQBF is PSPACE complete, consider some $L \in$ PSPACE.

$\exists$ TM M that decides L using polynomial space.

i.e. for $x \in L$, there exists a "transcript"



We get to check if such a transcript exists using a TQBF formula.

Initial configuration – as specified by x
Reaches q_{accept} somewhere
Every transition is valid

Too long! as there could be c^{n^k} positions to check.

We haven't used quantifiers..... hmmm....

Let's try the recursion we used for showing CONN $\in$ SPACE($\log^2 n$)

(NSPACE($s(n)$) $\subseteq$ SPACE($s(n)^2$))

PATH(u, v, T): $\exists$ valid sequence of transitions from u to v in $\leq T$ steps.

$$\begin{cases} T=0,\ T=1 \longrightarrow \text{easy checks,} \\ T>1,\ \exists w: \text{PATH}(u, w, \lfloor \frac{T}{2} \rfloor) \\ \qquad \wedge\ \text{PATH}(w, v, \lceil \frac{T}{2} \rceil) \end{cases}$$

Depth of this recursion is only $\log_2 T$

For $T = c^{n^k}$, this is $O(n^k)$.

But size of formula blows up! (doubles at each level).

Lets use $\forall$ as well:

$\exists w\ \forall (x, y) \in \{(u, w), (w, v)\}$ PATH$(x, y, \lceil \frac{T}{2} \rceil)$

How to write $(x, y) \in \{(u, w), (w, v)\}$ as a formula?

$$(X,Y)=(u,w) \vee (X,Y)=(w,v) \Rightarrow F$$

$$\left[\begin{array}{l} (X,Y) = (u,v) \iff (X=u) \wedge (Y=v) \\ \qquad \iff \left((X\wedge u)\vee(\bar{X}\wedge\bar{u})\right)\wedge\left((Y\wedge v)\vee(\bar{Y}\wedge\bar{v})\right) \\ A \Rightarrow B \iff (\bar{A} \vee B) \end{array}\right.$$

Applying this recursively, formula gets larger by $O(n^k)$ at every level of recursion. Depth $= O(n^k)$

SIZE OF final formula $= O(n^{2k})$

TIME to write FORMULA $= O(n^{2k})$.