Computability → Complexity.

Note Title 9/21/2015

study of resources needed for computation

How much space?

——— time?

——— Randomness?

Q.1 Does more space increase the power of a TM? (i.e. can it recognize more languages?)

Q2. Does more time help?

Q3. Does Nondeterminism help?

Q4 ——— Randomness help?

Def. A function $f: \mathbb{N} \to \mathbb{N}$ is said to be space-constructible if $\exists$ TM M that on input 1^n outputs $f(n)$ and uses $O(f(n))$ space, (assuming $f(n) \geq \log n$).

$g(n) = O(f(n))$: $\exists n_0, C : \forall n \geq n_0,\ g(n) \leq C f(n)$

$g(n) = o(f(n))$: $\forall c > 0, \exists n_0 : \forall n \geq n_0$
$g(n) \leq c f(n)$.

Thm $\quad \text{Space}(s(n)) \subseteq \text{Time}(c^{s(n)})$

$$\text{TIME}(t(n)) = \left[\begin{array}{l} \text{Languages accepted by} \\ \text{TMs that run for} \\ \text{at most } t(n) \text{ steps} \end{array} \right.$$

$$\text{SPACE}(s(n)) = \left[\begin{array}{l} \text{Languages accepted by} \\ \text{TMs that use at most} \\ s(n) \text{ space.} \end{array} \right.$$

The "configuration" of a TM
is (q, head position, tape contents)
with space $s(n)$

$$\# \text{ possible configurations} \leq |Q| \cdot s(n) \cdot |\Gamma|^{s(n)}$$

$$= c_1 \cdot s(n) \cdot c_2^{s(n)}$$

$$\leq c_2^{s(n) + \log n}$$

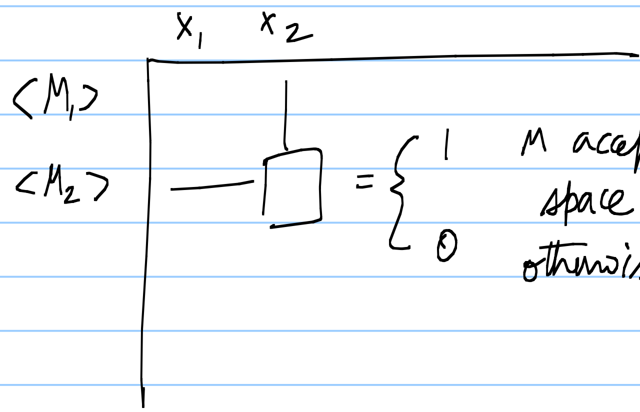
$$\leq c^{s(n)}.$$

If it goes longer, then it repeats configurations, i.e. it loops and will not accept.

Thm. (SPACE HIERARCHY THEOREM)

For any space constructible function f, $\exists$ language L that is decided by a TM using $O(f(n))$ space but not by any TM using $o(f(n))$ space.

Pf.



$$= \begin{cases} 1 & M \text{ accepts } x \text{ using } \le f(n) \text{ space and } \le 2^{f(n)} \text{ time.} \\ 0 & \text{otherwise} \end{cases}$$

Define $L_f = \{ \langle M \rangle, 1^n :$ M is a TM
M on $\langle M \rangle$ does not accept using space $\le f(n)$ and time $\le 2^{f(n)}$ $\}$

Claim 1. $\exists$ TM D that decides L_f using space $O(f(n))$.

D on input $\langle M \rangle, 1^n$

1. Mark $f(n)$ space on the tape
2. If M tries to use more, reject
3. If time used is $\ge 2^{f(n)}$ reject
4. Else if M accepts, reject; if M rejects, accept.

Claim 2. No TM using $o(f(n))$ space can decide L_f.

Suppose $\exists$ such a TM M.

What happens when D is run on $\langle M \rangle, 1^m$?

D accepts $\langle M \rangle$ if M rejects $\langle M \rangle$

— rejects $\langle M \rangle$ if M accepts $\langle M \rangle$.

So $L_D \neq L_M$. Contradiction!

FRI: TIME HIERARCHY THEOREM

Nondeterminism.

NTM. $Q, \Sigma, \Gamma, q_{start}, q_{accept}, q_{reject}$.
$\delta: Q \times \Gamma \longrightarrow (Q \times \Gamma)^* \times \{L, R\}$

$\delta(q, a) \longrightarrow \{ (q_1, a_1, L), (q_2, a_2, L), (q_3, a_3, R) \dots \}$
finite list of possibilities.

[NTM accepts input w if $\exists$ a sequence of transitions that lead to q_{accept}.

DTM computation is a path, ending in accept/reject or going forever.
NTM ——— is a tree: if any leaf is an accept, the input is accepted.

Thm $NTIME(t(n)) \subseteq TIME(c^{t(n)})$

Nondeterminism does not change the set of accepted languages.

Pf. Breadth-First Search

NTIME = Depth of tree

DTIME = Size of tree.

NTMs can guess an "accepting path" and check it.

<u>Examples</u> FACTORING

FINDING s-t paths

Ham cycles

What about space?

If an NTM takes space $s(n)$ what will a DTM take?

$c^{s(n)}$? this is as much time as it could take. If each step takes up some space, then...

<u>Specific Problem</u>: $\exists$ path in G from s to t?

$|V| = n$, then NTM needs $O(\log n)$ space.

(guess next vertex)

What about DTM? BFS, DFS?

$O(n \log n)$?

Thm [SAVITCH]

$$NSPACE(s(n)) \subseteq DSPACE(s(n)^2).$$

Thm Graph connectivity $\in DSPACE(O(\log^2 n))$.

Pf.

PATH(a, b, k):

If $k=0$: if $a=b$: accept else reject.

If $k=1$: if $(a,b) \in E$ or $a=b$: accept, else reject

else

For each $c \neq a, b$ in V:

If PATH$(a, c, \lfloor \frac{k}{2} \rfloor)$ accepts

and PATH$(c, b, \lceil \frac{k}{2} \rceil)$ accepts

then accept.

Reject.

PATH(a, b, k) accepts if $\exists$ path of length $\leq k$ between a and b in $G=(V,E)$

rejects otherwise.

Space used: $O(\log n)$ per level of recursion to store the name "c".

levels of recursion $= \log_2 k \leq \log_2 n$.

$\therefore$ total space $= O(\log^2 n)$.

Pf (of Savitch).

Consider graph of all possible configurations. $|V| = |Q| \times S(n) \times C^{S(n)}$

If $\exists$ path from start config to end config, it is of length $\leq |V| - 1$

space needed by a DTM $= O(\log^2 |V|)$

$$= O\left(\left(S(n) + \log(S(n))\right)^2\right)$$

$$= O(S(n)^2).$$

This bound has been improved for undirected graph connectivity, namely

UCONN can be solved using space $O(\log n)$ [REINGOLD]

CONN can be solved with Randomness in space $O(\log n)$.

Open to do this deterministically!