Computability

Note Title 9/14/2015

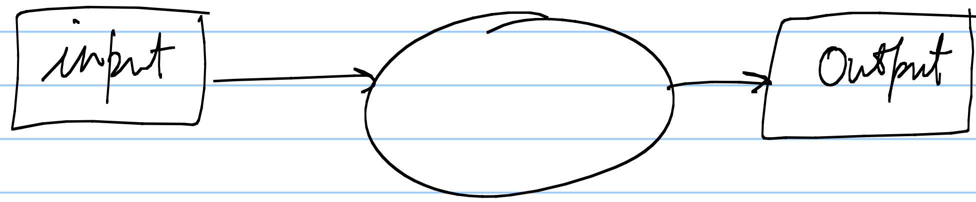
What can computers do?

- add
- Sort
- FFT
- ....

What can computers <u>not</u> do?

- examples

What is computation?



- a function mapping input to output
- can assume output is a single bit $\{0,1\}$

(if output has k bits, think of it as k functions: 1^{st} bit, 2^{nd} bit ....)

function : Domain (D) $\longrightarrow$ Range (R)

e.g. $\{0,1,2,3,\ldots\} \longrightarrow \{0,1\}$

<u>PRIME</u>: $f(x) = 1$ iff x is PRIME

<u>HAIR</u>: $f(x) = 1$ iff $\exists$ human with exactly x hairs on their head.

Computation is the evaluation of functions by a machine using a finite and well-defined set of operations.

Q. Can any function be computed?
(is every function computable?)

To formalize, fix a finite alphabet Σ.

e.g. $\Sigma = \{0, 1\}$

$\Sigma = \{0, 1, \ldots 9\}$

$\Sigma = \{A, B, C, \ldots Z\}$

$\Sigma = \text{ASCII}$

A function maps input strings to $\{0, 1\}$

For f, let $L_f = \{x \in \Sigma^* : f(x) = 1\}$

LANGUAGE.

Q. Are all languages computable?
i.e. given x, decide if $x \in L$.

Alphabet Σ e.g. $\{0,1\}$ $\{A,G,C,T\}$

strings Σ^* 01100010... AGGCAT
1001 ⋮

Number of strings? infinite

Thm 1. For any finite alphabet Σ, the set of strings is countable.

Countable : $\exists g : A \to \mathbb{N}$

g is 1-1 and into

$x \neq y \Rightarrow g(x) \neq g(y)$

e.g. the set of squares of all numbers is countable

Pf of Thm 1 Suppose $\Sigma = \{0,1\}$

then what is $x \in \Sigma^*$? 0110011

a binary number! Let $n(x)$ be the natural number it represents

Then $g = n$ i.e. $g(x) = n(x)$.
is a 1-1 and into map.

Σ is k-ary. then $x \in \Sigma^*$ is a k-ary number

$$x_1 x_2 \ldots x_\ell = x_\ell + k \cdot x_{\ell-1} + k^2 x_{\ell-2} + \ldots x_1 .$$

Programs, sets of instructions for machines, are strings over some finite alphabet. Not all strings are valid programs. But all programs are strings.

$\boxed{\text{The set of Programs is contable}}$

What about the set of functions?

Languages?

Set of all subsets of strings 2^{Σ^*}.
is this contable?

Thm. The set of languages over a finite alphabet is uncontable

Pf.

languages \ strings	0	00	01	11	......
L_1					
L_2					
$\vdots$					

Suppose not.

Then:

$T(x, L) = 1$ if $x \in L$
0 otherwise

Consider $L = \{ x_j : x_j \notin L_j \}$

Since languages are contable,
$\exists t \in \mathbb{N}$ s.t. $L = L_t$.

Q. Does $x_t \in L_t$?

$x_t \in L_t \iff x_t \notin L_t$

Contradiction!

L_t cannot be defined.

Languages cannot be enumerated
(are not contable).

$\exists$ languages that do not have
programs to decide them

PROGRAMS $\subseteq$ STRINGS
contable

Languages $\subseteq$ functions

uncontable
