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Double Descent in the case of ridge regression

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Abstract

We consider $y^\star : x \in \mathcal{X} = \mathbb{R}^D \to \mathbb{E}_{(x,y)\sim\mathcal{P}}(y|x) \in \mathcal{Y} = \mathbb{R}$, with x and y random variable with joint distribution $(x, y) \hookrightarrow \mathcal{P}$. We have a **dataset** $\mathcal{D} := \{(x_n, y_n) \in \mathcal{X} \times \mathcal{Y}\}_{n=1}^N$ where $y_n := y(x_n)$, and $y = y^\star + \epsilon$ where ϵ represents the noise as $\mathbb{E}(\epsilon|x) = 0$, et $\mathbb{V}(\epsilon) = \sigma^2$. We want to create an estimator $\hat{y}_\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ which minimises the empirical risk in order to minimise the true risk $\mathcal{R}(\hat{y}) = \mathbb{E}_{(x,y)\sim\mathcal{P}}((y - \hat{y}(x))^2)$. Working in the context of [1]

Source Code

The reviewed source code and documentation for this algorithm are available from the web page of this article[1]. Usage instruction are included in the README.txt file of the archive.

1 Model

- $\mathcal{F} := \{f_p, 1 \leq p \leq P\}$ a feature set.
- $X = [x_1, \cdots, x_N]^T \in \mathcal{M}_{N,D}(\mathbb{R})$, $Y = [y_1, \cdots y_N]^T$
- $\Phi_P(x) = [f_1(x), \cdots, f_P(x)]^T$ and $Z_P = [\Phi_P(x_1), \cdots, \Phi_P(x_N)]^T = [f_j(x_i)] \in \mathcal{M}_{N,P}(\mathbb{R})$
- $\hat{y}_\mathcal{D}(x) = \Phi_P^T(x)\hat{\beta}_P$, with $\hat{\beta}_P = \begin{bmatrix} Z_P \\ \sqrt{N\lambda} I_P \end{bmatrix}^\dagger \begin{bmatrix} Y \\ O_P \end{bmatrix}$
- Loss function $\mathcal{L}(y) = \frac{1}{N}||Y - Z\beta||_2^2 + \lambda||\beta||_2^2$ with $\lambda \geq 0$

[1] https://ipolcore.ipol.im/demo/clientApp/demo.html?id=77777000515

2 Algorithms

Algorithm 1: Generation of the orthonormal polynomial basis (Gram- Schmidt)

function generate_orthonormal_basis$(\mathrm{D}, \mathrm{Deg}, \mathrm{C})$
  Input D, Deg, C**:** $D \in \mathbb{N}^\star$,$Deg \in \mathbb{N}^\star$, $C \subset \mathbb{R}^D$
  $\mathrm{Basis_{D'}} = [\Pi_{d=1}^{D} X_d^{\alpha_d}, \sum \alpha_d \leq D']$
  $\mathrm{P} = \mathrm{LENGHT}(\mathrm{Basis_{Deg}})$
  for $1 \leq p \leq P$ **do**
    $f'_p = \mathrm{Basis_{Deg}[p]} - \sum_{i=1}^{p-1} [\int_C \mathrm{Basis\_ortho_{Deg}[i]} \cdot \mathrm{Basis_{Deg}[i]}] \times \mathrm{Basis\_ortho_{Deg}[i]}$
    $\mathrm{Basis\_ortho_{Deg}[p]} = \frac{f'_p}{||f'_p||_2}$
  return $\mathrm{Basis\_ortho_{Deg}}$

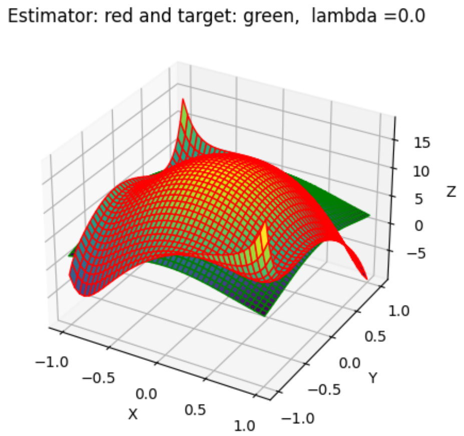
Algorithm 2: Dataset initialisation

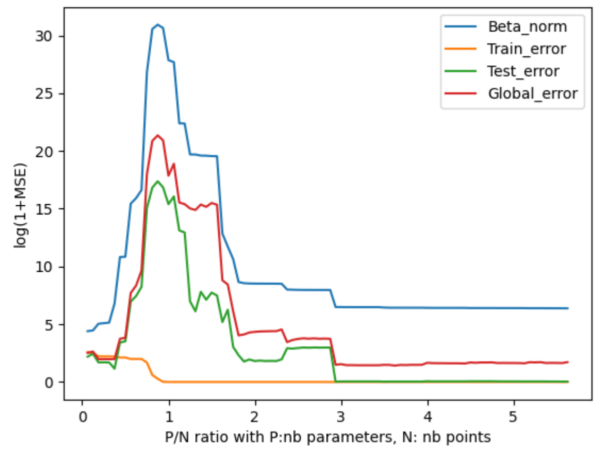
function dataset_initialisation$(f, C, M, ratio_data)$
  Input $\mathrm{y, C, M, ratio\_data}$**:** $y : \mathbb{R}^D \to \mathbb{R}$, $C = [[a_d, b_d], 1 \leq d \leq D]$, $\mathrm{ratio\_data} \in [0, 1]$
  $U = \mathcal{U}([0, 1]^{(M,D)})$
  $X = Diag(a_d) + U Diag(b_d - a_d)$ # $X \hookrightarrow \mathcal{U}(C^M)$
  $Y = y(X)$
  $X_{train}, X_{test}, Y_{train}, Y_{test} = \mathrm{test\_split}(X, Y, \mathrm{ratio\_data})$
  return $\mathrm{X_{train}, X_{test}, Y_{train}, Y_{test}}$

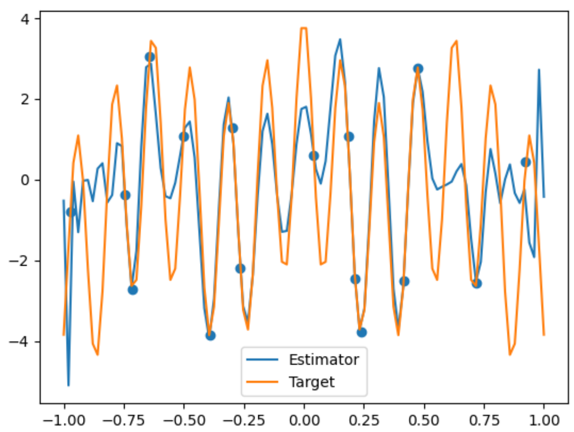
Algorithm 3: Ridge Regression

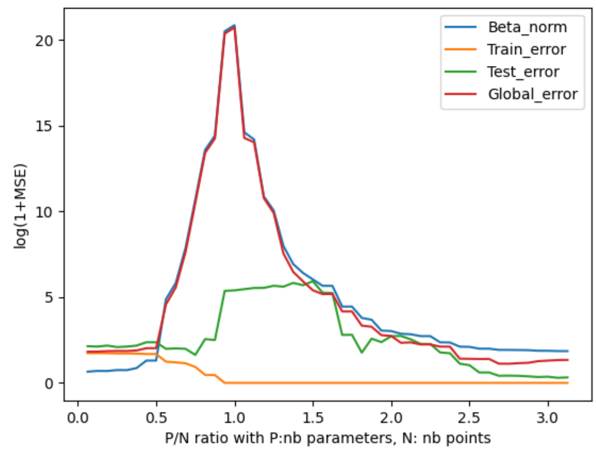
function ridge_regression$(\mathrm{P_{min}, P_{max}, Features}, \lambda, \mathrm{Data, Data_global})$
  $X_{train}, X_{test}, Y_{train}, Y_{test} = Data$
  $X_{global}, Y_{global} = Data\_global$
  for $P_{min} \leq p \leq P_{max}$ **do**
    $\Phi_p = [f_p, 1 \leq p \leq P]$ # (f_p) an orthonormal basis for p.s. on C
    $Z_p = \Phi_p(X_{train})^T$
    $\hat{\beta}_p = \begin{bmatrix} Z_p \\ \sqrt{N\lambda I_p} \end{bmatrix}^\dagger \begin{bmatrix} Y_{train} \\ O_p \end{bmatrix}$
    $\mathrm{Train\_error[p]} = \log(1 + \mathrm{MSE}(\mathrm{Y_{train}}, \mathrm{Z_p}\hat{\beta}_\mathrm{p}))$
    $\mathrm{Test\_error[p]} = \log(1 + \mathrm{MSE}(\mathrm{Y_{test}}, \Phi_\mathrm{p}(\mathrm{X_{test}})^\mathrm{T}\hat{\beta}_\mathrm{p}))$
    $\mathrm{Global\_error[p]} = \log(1 + \mathrm{MSE}(\mathrm{Y_{global}}, \Phi_\mathrm{p}(\mathrm{X_{global}})^\mathrm{T}\hat{\beta}_\mathrm{p}))$
    $\mathrm{Beta\_norm[p]} = \log(1 + ||\hat{\beta}_\mathrm{p}||_2^2)$
  $\hat{y} = \Phi_p^T \hat{\beta}_{P_{max}}$
  return Train_error, Test_error, Beta_norm, Global_error, $\hat{\mathrm{y}}$

3 Examples (random seed : 23334)

<table>
<tr><th>Function plot</th><th>Graph error</th></tr>
<tr><td>


$y(x) = 2e^{2-(x_1^2+x_2^2)}, Basis : cano$
</td><td>


$M = 20, r = 0.2, n = 40, a = 1, Deg = 12, D = 2, \lambda = 0$
</td></tr>
<tr><td>


$y(x) = 3cos(40x) + cos(10x) - x^2, Basis : ortho$
</td><td>


$M = 20, r = 0.2, n = 100, a = 1, Deg = 50, D = 1, \lambda = 0$
</td></tr>
</table>

References

[1] Emett Haddad, *Github Double Descent*. https://github.com/EmettGabrielH/Double-descent---Emett-Haddad, 2024. [Online].