July 7, 2016

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Problem 1. Show that there is exactly one positive k such that no graph contains exactly k spanning trees.

Solution. Let G be a connected graph.

Case. $k = 1$: G is a Spanning Tree
If G itself is a spanning tree then $k = 1$.

Case. $2 < k$
Suppose that G is not itself a spanning tree and k is strictly greater than 2.
Let $C_k \subset G$, then C_k is a fundamental cycle.
Since removing one edge from the cycle gives a spanning tree, and since there is a *fundamental cycle* for every edge in the spanning tree, there is a one-to-one correspondence between the edges in the spanning tree and the edges which are not in the spanning tree.
So there are k spanning trees in G.

Case. $k = 2$
The only case left to show is one where $k = 2$.
Suppose that T_1 and T_2 exist and are the only spanning trees of G.
If we take either one of these trees, say T_1 and add an edge to it then it will be a cycle.

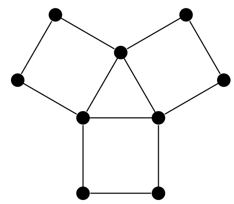
Rewrite: *Then there must be an edge which is not in $T_1 + e$, however since T_1 is now a cycle it must have a length of at least three and so will have at least three spanning trees.*

REWRITE: Then there must be an edge which is not in $T_1 + e$, however since T_1 is now a cycle it must have a length of at least three and so will have at least three spanning trees.

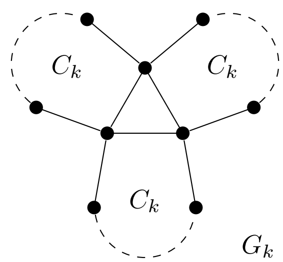
Problem 2.

(a) Find the number of spanning trees in the graph G depicted below.

(b) Find the number of spanning trees in the graph G_k for $k \geq 5$ depicted below.



G



G_k

Note. Note that 2 is the case where $k = 4$.
Notation. C_k stands for a cycle on k vertices.

Solution.

- We skip 2 since proving part b will cover its case.
- There are three cycles $C_{k_1}, C_{k_2}, C_{k_3}$, removing an edge from any of these cycles will not produce a spanning tree since all share edges with X (are not when combined with X fundamental cycles).
- Let X be the set of edges in the center triangle.

Case. The edge deleted is not in X
There are three ways to remove an edge from X.
If when breaking each of the cycles the first edge deleted is not in X, then we will have to also remove an edge from X.
The order of each cycle is k so the number of ways we can remove an edge from a cycle is $k - 1$.
Since there are three cycles $C_{k_1}, C_{k_2}, C_{k_3}$ we get $3(k-1)^3$ different spanning trees from this case.

Case. Only one C_k removes an edge from X
There are only three ways to remove an edge from X.
Then the remaining two cycles have $(k-1)^2$ ways, ($(k-1)$ ways each) to choose an edge not in X.
As in the first case, the remaining two cycles will not produce trees until an additional edge

from X has been removed.
So we have the total number of spanning trees produced in this case:

$$3 \times 2 \times (k-1)^2 = 6(k-1)^2$$

Case. Only one C_k does not remove an edge from X
The cycle which does not remove an edge from X (first) must remove one of its own edges (chosen in $(k-1)$ ways), then one edge from X (3 possible choices).
So we get $3(k-1)$ spanning trees from this cycle.
The remaining two cycles choose an edge from X first which produces a tree right away and has been accounted for in .

The above cases give the total number of spanning trees as:

$$3(k-1)^2 + 6(k-1)^2 + 3(k-1) \tag{1}$$

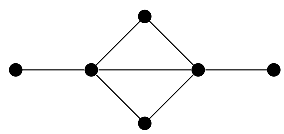
Applying the general formula to 2 gives the total number of spanning trees:

$3 \times (4-1)^3 + 2 \times 3 \times (4-1)^2 + 3 \times (4-1)$
$= 3(3)^3 + 2 \times 3 \times (3)^2 + 3(3)$
$= 3^4 + 2(3)^3 + 3^2$
$= 144$

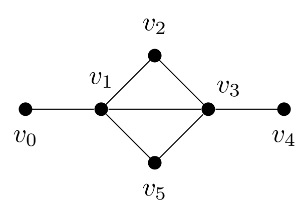
Problem 3. Let T and T' be two spanning trees of a connected graph G of order n. Show that there exists a sequence $T = T_0, T_1, \ldots, T_k = T'$ of spanning trees of G such that T_i and T_{i+1} have $n-2$ edges in common for each i with $1 \leq i \leq k-1$.

Solution. Let T and T' be finite spanning trees of G.
Start with $T = T_0$ we add edges to T and try to make it look like T'.
Suppose we have added edges up to the $i-th$ tree T_i.
If $T_i = T'$ then we are done, so assume $T_i \neq T'$.
If $T_i \neq T'$ then adding an edge to T_i will create a cycle C, meaning that there is an edge e' in C which is not an edge in T'.
Let $T_{i+1} = T_i + e - e'$.
Then T_{i+1} has one more edge in common with T' than it does with T_i.
Since these spanning trees were finite by supposition this process will eventually terminate; and since there is a one-to-one correspondence between edges in a spanning tree and edges not in a spanning tree there will be some (mid)-point where $T' = T_k$ must be true for some k.

Problem 4. Count the number of spanning trees of the depicted graph using Matrix Tree Theorem.



Solution.



$$L = 10000 - 1020 - 10 - 1001 - 1000 - 1 - 14 - 1 - 1000 - 12 - 1 - 10 - 1 - 1 - 14$$

The characteristic polynomial of the Laplacian is:

$$x^6 - 14x^5 + 70x^4 - 152x^3 + 144x^2 - 48x$$

The eigenvalues are:

$$\lambda_1 = 3 + \sqrt{5}, \lambda_2 = 3 + \sqrt{3}, \lambda_3 = 2, \lambda_4 = 3 - \sqrt{5}, \lambda_5 = 3 - \sqrt{3}, \lambda_6 = 0$$

The number of spanning trees of the graph is equal to $\frac{1}{n}\lambda_1 \dots \lambda_{n-1}$ where the eigenvalues are the largest.

$$\frac{1}{6} \times \lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4 \times \lambda_5 = \frac{48}{6} = 8$$

Problem 5. Count the number of spanning trees of K_n using Matrix Tree Theorem.

Solution. Note: Cayley's theorem is a special case of the Matrix Tree (Kirchhoff's) theorem.

There are n vertices in the graph and each is connected to every other vertex in the graph. The vectors span a space with dimension $n - 1$. There is only one eigenvalue equal to zero every other eigenvalue corresponds to n. Reducing the matrix by the ith-row and jth-column gives us a span with dimension $n-2$. So we have that for n vertices each has $n-1$ neighbors, and in the reduction each has $n-2$.