Governing equations for the Galerkin discretization for the steady state incompressible Navier-Stokes equations

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In fluid mechanics the Navier-Stokes equations are the partial differential equations governing the fluid motion.

$$\rho \frac{\partial \vec{U}}{\partial t} + \rho(\vec{U} \cdot \nabla)\vec{U} = -\nabla p + \eta \nabla^2 \vec{U} + (\lambda + \eta)\nabla(\nabla \cdot \vec{U}) + \vec{f} \tag{1}$$

To simplify we make a few assumptions:

- Incompressible flow
- Steady state flow
- no source terms

$$(\vec{U} \cdot \nabla)\vec{U} - \nu \nabla^2 \vec{U} = -\frac{1}{\rho}\nabla p \tag{2}$$

1 SIMPLE solver

The SIMPLE (Semi Implicit method for pressure linked equations) solver works in three stages:

1. Momentum predictor
2. Pressure corrector
3. explicit calculation of momentum

1.1 Momentum predictor

write equation 2 as a $MU = -\nabla p$ and solve for U given an initial pressure field. Note: to make the matrix M we can use different methods: Finite Volumes, finite methods. In this report we use finite methods. Note2: M is function the velocity profile.

1.2 pressure corrector

to correct the pressure equations we first calculate the residual with is the M matrix without the main diagonal multiplied with the momentum.

$$H = AU - MU$$

substitute $MU = -\nabla p$ we can derive an explicit calculation for U:

$$A^{-1}H - A^{-1}\nabla p = U$$

Plugging the above equation in the continuity equation gives a Poisson equation for pressure.

$$\nabla(A^{-1}H) = \nabla(A^{-1}\nabla p)$$

1.3 explicit calculation of momentum

basically evaluate the second expression in the section above

$$A^{-1}H - A^{-1}\nabla p = U$$

2 Galerkins method - Momentum

Galerkins method is a method to transform equation (2) into a matrix vector product.

Step 1 is to transform (2) into a weak form. we start with multiplying the equation with a testfunction φ and integrating the strong form (equation(2)) over the entire domain.

$$\int (U \cdot \nabla)U\varphi dV - \int \nu\nabla^2 U\varphi dV = -\int \nabla p\varphi dV$$

The convection and pressure terms can be integrated by parts to lower the order of the momentum and pressure terms

$$-\int \nu\nabla^2 U\varphi dV = -\int \nabla(\nu\nabla U\varphi)dV + \int \nu\nabla U\nabla\varphi dV$$

applying Gauss theorem gives a expression with a surface integral

$$-\int \nu\nabla^2 U\varphi dV = -\int \nu\nabla U \cdot \hat{n}\varphi dS + \int \nu\nabla U\nabla\varphi dV$$

putting it all together gives:

$$\int (U \cdot \nabla)U\varphi dV - \int \nu\nabla U \cdot \hat{n}\varphi dS + \int \nu\nabla U\nabla\varphi dV = -\int \nabla p\varphi dV$$

Now we split the weakform in two equations, one for U_x and one for U_y

$$\int (U \cdot \nabla)U_x\varphi dV - \int \nu\nabla U_x \cdot \hat{n}\varphi dS + \int \nu\nabla U_x\nabla\varphi dV = -\int \nabla_x p\varphi dV$$

$$\int (U\cdot\nabla)U_y\varphi dV - \int \nu\nabla U_y\cdot\hat{n}\varphi dS + \int \nu\nabla U_y\nabla\varphi dV = -\int \nabla_x p\varphi dV$$

We discretized U_x as follows: $U_x = \sum_{j=1}^{n} c_j\phi_j$

$$\sum_{j=1}^{n} c_j\left[\int (U\cdot\nabla)\phi_j\varphi_i dV - \int \nu\nabla\phi_j\cdot\hat{n}\varphi_i dS + \int \nu\nabla\phi_j\nabla\varphi_i dV\right] = -\int \nabla_x p\varphi_i dV$$

This now can be written as $S\vec{c} = \vec{f}$

$$S_{ij} = \int \varphi_i\nabla\phi_j\cdot U dV + \int \nu\nabla\phi_i\nabla\varphi_j dV - \int \nu\varphi_i\nabla\phi_j\cdot\hat{n}dS$$

$$f_i = -\int \nabla_x p\varphi_i dV$$

The stiffness matrix S is made in a routine summing over all elements. The functions φ and ϕ can be written as $\alpha + \beta x + \gamma y$

Element matrix

$$S_{ij}^{e_k} = \int_{e_k} \varphi_i\nabla\phi_j\cdot U dV + \int_{e_k} \nu\nabla\phi_i\nabla\varphi_j dV$$

Lets start with the first integral. we first expand U with the solution approximation

$$\int_{e_k} \varphi_i(\nabla\phi_j\cdot U)dV = \int_{e_k} \varphi_i(\beta_i U_x + \gamma_i U_y)dV = \int_{e_k} \varphi_i\left(\beta_i\sum_{n=1}^{3} c_{x,n}\phi_n + \gamma_i\sum_{n=1}^{3} c_{y,n}\phi_n\right)dV$$

Now we can move the summation out of the integration and extract constant from the integral

$$\int_{e_k} \varphi_i\left(\beta_i\sum_{n=1}^{3} c_{x,n}\phi_n + \gamma_i\sum_{n=1}^{3} c_{y,n}\phi_n\right)dV = \sum_{n=1}^{3}(\beta_i c_{x,n} + \gamma_i c_{y,n})\int_{e_k}\varphi_i\phi_n dV$$

now we can apply Holand & Bell to the integral and get our final answer

$$\int_{e_k} \varphi_i(\nabla\phi_j\cdot U)dV = \sum_{n=1}^{3}(\beta_i c_{x,n} + \gamma_i c_{y,n})\frac{|\Delta|}{24}(1+\delta_{jn})$$

Now the second integral, this one is less involved as the previous one. First we evaluate the dot product for the gradients for the test ans solution approximation functions.

$$\int_{e_k} \nu\nabla\phi_i\nabla\varphi_j dV = \int_{e_k} \nu(\beta_i\beta_j + \gamma_i\gamma_j)dV$$

In the integral are only constants, therefore we can remove all from the integral and use the definition for the area of the element $A(e_k) = |\Delta|/2$

$$\int_{e_k} \nu\nabla\phi_i\nabla\varphi_j dV = \nu(\beta_i\beta_j + \gamma_i\gamma_j)\frac{|\Delta|}{2}$$

lastly putting it all together

$$S_{ij}^{e_k} = \sum_{n=1}^{3}(\beta_i c_{x,n} + \gamma_i c_{y,n})\frac{|\Delta|}{24}(1+\delta_{jn}) + \nu(\beta_i\beta_j + \gamma_i\gamma_j)\frac{|\Delta|}{2}$$

Boundary element matrix

$$S_{ij}^{b_{el}} = -\int_{b_{el}} \nu\varphi_i \nabla\phi_j \cdot \hat{n} dS$$

first calculate $\nabla\phi_j \cdot \hat{n}$, by definition of the solution approximation

$$\nabla\phi_j \cdot \hat{n} = (\beta_j n_x + \gamma_j n_y)$$

the gradient is constant over the boundary therefore it can be pulled out of the integral. Holand & Bell theorem states

$$\int_{b_{el}} \varphi_i dS = \frac{|\Delta|}{2}$$

$$S_{ij}^{b_{el}} = -\nu(\beta_j n_x + \gamma_j n_y)\frac{|\Delta|}{2}$$

Last thing that needs to be done is apply the boundary conditions

no-slip, inflow boundary conditions

In no-slip and inflow boundary conditions the solution is prescribed therefore the testfunction in the weakform has to be 0, i.e. $\int_{b_{el}} \varphi_i dS = 0$

$$S_{ij}^{b_{el}} = 0$$

outflow boundary conditions

in the outflow boundary there is a zero gradient boundary condtion, i.e. $\beta n_x + \gamma n_y = 0$

$$S_{ij}^{b_{el}} = 0$$

slip boundary condition

in the slip boundary condition the

Element vector

$$f_i^{e_k} = -\int_{e_k} \nabla_x p \varphi_i dV$$

first calculate $\nabla_x p$ on the element, defined by the solution approximation to be

$$(\nabla_x p)_{el} = \sum_{n_1=1}^{3} c_{p,n_1}\beta_{n_1}$$

the pressure gradient is constant over the element therefore it can be pulled out of the integral. Holand & bell theorem states

$$\int_{e_k} \varphi_i dV = \frac{|\Delta|}{6}$$

putting it all together gives

$$f_i^{e_k} = -\sum_{n_1=1}^{3} c_{p,n_1}\beta_{n_1}\frac{|\Delta|}{6}$$

Boundary element vector

$$f_i^{b_{el}} = 0$$

3 Galerkins method - pressure

As described above the pressure differential equation is a Poisson equation

$$\nabla \cdot (A^{-1}H) = \nabla \cdot (A^{-1}\nabla p)$$

calculating the weak form is fairly easy, same procedure as in the section above

$$\int A^{-1}\nabla p \cdot \hat{n}\varphi dS - \int A^{-1}\nabla p \nabla\varphi dV = \int A^{-1}H \cdot \hat{n}\varphi dS - \int A^{-1}H\nabla\varphi dV$$

defining Galerkins discretization $p = \sum_{j=1}^{n} c_{p,n}\varphi_j; \nabla p = \sum_{j=1}^{n} c_{p,n}\nabla\varphi_j; \varphi = \varphi_i$ applying this discretization and rearranging the terms gives

$$\sum_{j=1}^{n} c_{p,j}\left[\int A^{-1}\nabla\phi_j \cdot \hat{n}\varphi_i dS - \int A^{-1}\nabla\phi_j\nabla\varphi_i dV\right] = \int A^{-1}H \cdot \hat{n}\varphi_i dS - \int A^{-1}H \cdot \nabla\varphi_i dV$$

Element matrix

$$-\int_{e_k} A^{-1}\nabla\phi_j\nabla\varphi_i dV = -(\beta_i\beta_j + \gamma_i\gamma_j)\int \frac{1}{a_j}dV$$

note: $A^{-1}\nabla p = \sum_{j=1}^{n} c_{p,j}\frac{1}{a_j}\nabla\phi_j$, where $\frac{1}{a_j}$ are the diagonal coefficients of A^{-1}

$$S_{ij}^{e_k} = -(\beta_i\beta_j + \gamma_i\gamma_j)\sum_{n=1}^{3}\frac{1}{a_n}\frac{|\Delta|}{6}$$

Element vector

$$-\int_{e_k} A^{-1}H \cdot \nabla\varphi_i dV = -\int_{e_k}\sum_{n=1}^{3}\frac{1}{a_n}H_n \cdot \nabla\varphi_i\varphi_n dV = -\sum_{n=1}^{3}\frac{1}{a_n}H_n \cdot \nabla\varphi_i\int_{e_k}\varphi_n dV$$

$$f_i^{e_k} = -\sum_{n=1}^{3}\frac{1}{a_n}H_n \cdot \nabla\varphi_i\frac{|\Delta|}{6}$$

Boundary element matrix

$$\int_{b_{el}} A^{-1}\nabla\phi_j \cdot \hat{n}\varphi_i dS = (\beta_j n_x + \gamma_j ny)\int_{b_{el}}\frac{1}{a}\varphi_i dS$$

another newton cotes integration session

$$S_{ij}^{b_{el}} = (\beta_j n_x + \gamma_j ny)\sum_{n=1}^{3}\frac{1}{a_n}\frac{|\Delta|}{6}(1 + \delta_{in})$$

Boundary element vector

$$\int A^{-1}H \cdot \hat{n}\varphi_i dS = \int \sum_{n=1}^{3} \frac{1}{a_n} H_n \cdot \hat{n}\varphi_n\varphi_i dS = \sum_{n=1}^{3} \frac{1}{a_n} H_n \cdot \hat{n} \int \varphi_n\varphi_i dS$$

last newton cotes integration of the day

$$f_i^{b_{el}} = \sum_{n=1}^{3} \frac{1}{a_n} H_n \cdot \hat{n} \frac{|\Delta|}{6}(1 + \delta_{in})$$