Heimadæmi 10

Töluleg Greining

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Dæmi 1

(a) Apply Romberg Integration to find R_{33} for the integral

$$\int_0^1 xe^x dx$$

.
(b) Show that the extrapolation of the composite Trapezoid Rules in R_{11} and R_{21} yields the composite Simpson's Rule(with step size h_2) in R_{22}

Svar

(a) Byrjum á að athuga að $f(a) = f(0) = 0$ og $f(b) = f(1) = e$ og að $h_i = \frac{1}{2^{i-1}}(b-a)$. Við vitum síðan að

$$R_{11} = \frac{h_1}{2}(f(a) + f(b)) = \frac{e}{2}$$

nú þurfum við að finna eftirfarandi gildi á R: $R_{21}, R_{22}, R_{31}, R_{32}$ og loks getum við fundið R_{33}. Hefjumst handa:

$$R_{j1} = \frac{1}{2}R_{j-1,1} + h_j \sum_{i=1}^{2^{j-2}} f(a + (2i-1)h_j).$$

$$R_{22} = \frac{2^2 R_{21} - R_{11}}{3}$$

$$R_{32} = \frac{2^2 R_{31} - R_{21}}{3}$$

$$R_{21} = \frac{h_2}{2}(f(a) + f(b) + 2f\left(\frac{a+b}{2}\right))$$

$$R_{31} = \frac{1}{2}R_{21} + h_3 \sum_{i=1}^{2} f(a + (2i-1)h_3)$$

Fyllum nú í jöfnurnar:

$$
\begin{aligned}
R_{21} &= \frac{1}{4}(0 + e + 2(\frac{1}{2}e^{1/2})) \\
&= \frac{1}{4}(e + \sqrt{e}) \\
R_{31} &= \frac{1}{8}(e + \sqrt{e}) + \frac{1}{4}\left[(\frac{1}{4}e^{1/4}) + (\frac{3}{4}e^{3/4}\right] \\
&= \frac{1}{8}(e + \sqrt{e}) + \frac{1}{16}(e^{1/4} + 3e^{3/4}) \\
R_{22} &= \frac{4(\frac{1}{4}(e + \sqrt{e})) - \frac{1}{2}e}{3} \\
&= \frac{1}{6}(e + 2\sqrt{e}) \\
R_{32} &= \frac{4(\frac{1}{8}(e + \sqrt{e}) + \frac{1}{16}(e^{1/4} + 3e^{3/4}) - \frac{1}{4}(e + \sqrt{e})}{3} \\
&= \frac{1}{12}(e + \sqrt{e} + e^{1/4} + 3e^{3/4}) \\
R_{33} &= \frac{16(\frac{1}{12}(e + \sqrt{e} + e^{1/4} + 3e^{3/4})) - \frac{1}{6}(e + 2\sqrt{e})}{15} \\
&= \frac{7e + 6\sqrt{e} + 8e^{1/4} + 24e^{3/4}}{90} \\
&\approx 1.0000056
\end{aligned}
$$

(b) Athugum nú trapizuregluna:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Þar sem $h = (b-a)/n$, látum nú n vera slétta tölu og skrifum upp trapizuregluna fyrir $h = 2h$:

$$\int_a^b f(x)dx \approx \frac{h}{4}[f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)]$$

Summan inniheldur aðeins x_{2k}. Því verður villan fjórföld í seinni jöfnunni. Drögum nú fjórðung af seinni jöfnunni frá fyrri:

$$
\begin{aligned}
&\frac{h}{8}[2f(x_0) + 4f(x_1) + 4f(x_2) + \cdots + 4f(x_{n-1}) + 2f(x_n)] - [f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)] \\
&= \frac{h}{8}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \\
&= \frac{h}{8}\left[f(x_0) + f(x_n) + 4\sum_{i=0}^{n/2-1} f(x_{2i+1}) + 2\sum_{i=0}^{n/2-1} f(x_{2i})\right] \\
&= \frac{3}{4}\int_a^b f(x)dx
\end{aligned}
$$

og ef við margföldum nú með $4/3$ fáum við Simpsons regluna:

$$\int_a^b f(x)dx = \frac{h}{6}\left[f(x_0) + f(x_n)4\sum_{i=0}^{n/2-1} f(x_{2i+1}) + 2\sum_{i=0}^{n/2-1} f(x_{2i})\right]$$

Dæmi 2

Approximate the integrals, using n = 4 Gaussian Quadrature.

$$\int_1^4 \ln x dx$$

Svar

Notum jöfnu 5.46 úr bók og fáum:

$$\begin{aligned}\int_1^4 \ln x dx &= \frac{4-1}{2}\int_{-1}^{1} \ln\left(\frac{(4-1)t+1+4}{2}\right)dt \\ &= \frac{3}{2}\int_{-1}^{1} \ln\left(\frac{3t+5}{2}\right)dt\end{aligned}$$

Nú er x_i lausnir $p_4(x)$ úr bók:

$$\begin{aligned}x_1 &= -\sqrt{\frac{15+2\sqrt{30}}{35}} \\ x_2 &= -\sqrt{\frac{15-2\sqrt{30}}{35}} \\ x_3 &= \sqrt{\frac{15-2\sqrt{30}}{35}} \\ x_4 &= \sqrt{\frac{15+2\sqrt{30}}{35}}\end{aligned}$$

Við fáum einnig c_i úr bók:

$$\begin{aligned}c_1 &= \frac{90-5\sqrt{30}}{180} \\ c_2 &= \frac{90+5\sqrt{30}}{180} \\ c_3 &= \frac{90+5\sqrt{30}}{180} \\ c_4 &= \frac{90-5\sqrt{30}}{180}\end{aligned}$$

Svo beitum við Gaussian Quadrature aðferðinni $\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} c_i f(x_i)$

$$\begin{aligned}\int_1^4 \ln x dx &= \frac{3}{2}\int_{-1}^{1} \ln\left(\frac{3t+5}{2}\right)dt \\ &= 0.09872679636 + 0.67315948323 + 1.07792808398 + 0.69543956996 \\ &= 2.54525393353\end{aligned}$$

sem er nokkuð nálægt rétta svarinu sem er 2.5451774445