INF102
Algorithms, Data Structures and Programming

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Fall 2015

INF102, practical stuff

- Lecturer: Marc Bezem; Team: see homepage
- Homepage: INF102 (hyperlinks in red)
- Also: GitHub (recommended); Dropbox: slides, schedule
- Textbook: Algorithms, 4th edition
- Prerequisites: INF100 + 101 ($\approx$ Ch. 1.1 + 1.2)
- Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- Exam: three compulsory exercises and a written exam
- Old exams: 2004–2013, 2014
- Table of Contents of these slides

Resources

- Good textbook, USA-style: many pages, exercises etc.
- Average speed must be ca 50 pages p/w
- Lectures (ca 24) focus on the essentials
- Slides (ca 120, dense!) summarize the lectures
- Prepare yourself by reading in advance
- Workshops: selected exercises
- Test yourself by trying some exercises in advance
- If you can do the exercises (incl. compulsory), you are fine
- Review of exercises on Friday morning

Generic Bags, Queues and Stacks

- Generic programming in Java, example: PolyPair.java
- Bag, Queue and Stack are generic, iterable collections
- Queue and Stack: Ch. 9 in textbook INF100/1
- APIs include: `boolean isEmpty()` and `int size()`
- All three support adding an element
- Queue and Stack support removing an element (if any)
- FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- Dijkstra's Two-Stack Expression Evaluation Movie
- Example: (1 + ((2 + 3) * (4 * 5)))

Implementations

- ResizingArray_Stack.java
- Arrays give direct access, but have fixed size
- Resizing takes time and space proportional to size
- LinkedList_Stack.java
- No fixed size, but indirect access
- Pointers take space and dereferencing takes time
- Programming with pointers: make a picture
- LinkedList_Queue.java

Computation time and memory space

- Two central questions:
  - How long will my program take?
  - Will there be enough memory?
- Example: ThreeSum.java
- Inner loop (here `a[i]+a[j]+a[k]==0`) is important
- Sorting helps: ThreeSumOptimized.java
- Run some experiments: `1Kints.txt, 2Kints.txt, ...`

Methods of Analysis

- Empirical:
  - Run program with randomized inputs, measuring time & space
  - Run program repeatedly, doubling the input size
  - Measuring time: StopWatch
  - Plot, or log-log plot and linear regression
- Theoretical:
  - Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
  - Try to count/estimate/average this cost as function of the input (size)
  - Use $O(f(n))$ and $f(n) \sim g(n)$

ThreeSum, empirically

- Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4 ,51.1 sec
- The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- Basis of the logarithm should be the same for both
- Linear regression gives $y \approx 3x - 10$
- $\log(f(n)) = 3\log(n) - 10$ iff

$$f(n) = 10^{\log(f(n))} = 10^{3\log(n)-10} = n^3 * 10^{-10}$$

- Conclusion: cubic in the input size, with constant $\approx 10^{-10}$
- Strong dependence on input can be a problem
- Constant 10^{-10} depends on computer, exponent 3 does not

ThreeSum, theoretically

- Number of different picks of triples: $g(n) = n(n-1)(n-2)/6$
- Inner loop `a[i]+a[j]+a[k]==0` executed $g(n)$ times
- $g(n) = n^3/6 - n^2/2 + n/3$
- Cubic term $n^3/6$ wins for large n
- Computational model # array accesses: $3 * n^3/6 = n^3/2$
- Cost array access t sec: time $t * n^3/2$ sec
- Cost models are abstractions! (NB cache)

Big Oh, and $\sim$

- Q: 'wins for large n' uhh???
- A: Big Oh, and $\sim$ will clear this up
- Costs are positive quantities, so $f, g, \ldots : \mathbb{N} \to \mathbb{R}^+$
- MNF130: $f(n)$ is $O(g(n))$ if there exist $c \in \mathbb{R}^+,\ N \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (that is,, for n large enough)
- Example: n^2 and even $99n^3$ are $O(n^3)$, but n^3 is not $O(n^{2.9})$
- INF102: $f(n) \sim g(n))$ if $1 = \lim f(n)/g(n)$
- If $f(n) \sim g(n))$, then $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$
- Big Oh and $\sim$ aim to capture 'order of growth'
- Big Oh abstracts from constant factors, $\sim$ does not
- Large constant factors are important!

Important orders of growth

- constant: c, $f(n) = c$ for all n
- linear: n (compare all for $n = 20$ sec)
- linearithmetic: $n \log n$
- quadratic: n^2
- cubic: n^3
- exponential: 2^n
- general form: $an^b(\log n)^c$

Logarithms and Exponents

- Definition: $\log_x z = y$ iff $x^y = z$ for $x > 0$
- Inverses: $x^{\log_x y} = y$ and $\log_x x^y = y$
- Exponent: $x^{(y+z)} = x^y x^z$, $x^{(yz)} = (x^y)^z$
- Logarithm: $\log_x(yz) = \log_x y + \log_x z$, $\log_x z = \log_x y \log_y z$
- Base of logarithm: the x in $\log_x$
- Various bases: $\log_2 = \lg$, $\log_e = \ln$, $\log_{10} = \log$
- Double exponent: e.g. $2^{(2^n)}$ (not used in INF102)
- Double logarithm: $\log(\log n)$ (not used in INF102)

Worst case, average case, amortized cost

- Worst case: guaranteed, independent of input; Examples:
  - Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
  - Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- Average case: not guaranteed, dependent of input *distribution*
- Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost ≤ 21, all others cost 1, amortized ≤ 3 p/o.
- Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- Special case of resizing array that is only growing: 1(2)2(4)34(8)5678(16)9 ... 16(32) ..., with (n) the new size. Risizing to (n) costs $2n$ array accesses, so in total (1+4)+(1+8)+(2+16)+(4+32)+(8+64) ..., so 9 p/push.

Staying Connected

- We want efficient algorithms and datastructures for testing whether two objects are 'connected'
- MNF130: relation $E \subseteq V \times V$ is an *equivalence* if
  - E is *reflexive*: $\forall x \in V.\ E(x, x)$
  - E is *symmetic*: $\forall x, y \in V.\ E(x, y) \rightarrow E(y, x)$
  - E is *transitive*: $\forall x, y, z \in V.\ E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- We assume connectedness to be an equivalence
- Dynamic connectivity means (here) that E can grow
- Clear relationship with paths in graphs, (connected) components (MNF130)
- Input: N and pairs in $V = \{0, \ldots, N-1\}$ defining E
- Challenge: efficient `boolean connected(int p, int q)`
- Example: $N = 10$, 4 3, 3 8, ... (`algs4-data/tinyUG.txt`)
- Picture on blackboard (don't print pairs that are already connected)

Union-Find

- Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- Union, idea: for any new pair n m that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- API: UF; Cost model: number of array accesses
- Implementations:
  - SlowUF.java: `id[p]` identifier of `p`
    `find()` ~ 1, `union()` $\sim$ between $n+3$ and $2n+1$
  - FastUF.java: `int[] id` pointers, `id[p]==p`: identifier
    `find()` $\sim 1+2d$, `union()` $\sim 1+$ two `find()`'s
  - WeightedUF.java: `int[] id` pointers, `int[] sz` subtree sizes
    `find()` and `union()` both $\sim \lg n$
- WeightedUF: height of subtree of size k is at most $\lg k$
- Path-compression: ultimate improvement of UF (almost $O(1)$, amortized)

Sorting

- Sorting: putting objects in a certain order
- MNF130: relation $R \subseteq V \times V$ is a *total order(ing)* if
  1. R is *reflexive*: $\forall x \in V.\ R(x,x)$
  2. R is *transitive*: $\forall x,y,z \in V.\ R(x,y) \wedge R(y,z) \rightarrow R(x,z)$
  3. R is *antisymmetic*: $\forall x,y \in V.\ R(x,y) \wedge R(y,x) \rightarrow x = y$
  4. R is *total*: $\forall x,y \in V.\ R(x,y) \vee R(y,x)$
- Natural orderings:
  - Numbers of any type: ordinary $\leq$ and $\geq$
  - Strings: lexicographic
  - Objects of a Comparable type: `v.compareTo(w) <= 0`

Sorting (ctnd)

- Bubble sort: ExampleSort.java
- Certification: `assert isSorted(a)` in `main()`
- No guarantee against modifying the array (but `exch()` is safe)
- Costmodel 1: number of `exch()`'s and `less()`'s
- Costmodel 2: number of array accesses
- Pitfalls: cache misses, expensive `v.compareTo(w) < 0`
- Why studying sorting? (`java.util.Arrays.sort()`)
- Comparing sorting algorithms: SortCompare.java

Selection Sort

- Bubble sort: $\sim n^2/2$ compares, $0 \leq$ exchanges $\leq\sim n^2/2$
- Selection sort:
  - Find index of a minimum in `a[0..n-1]`, exchange with `a[0]`
  - Find index of a minimum in `a[1..n-1]`, exchange with `a[1]`
  - ... until `n-2`
- Selection sort: $\sim n^2/2$ compares, $0 \leq$ exchanges $\leq n-1$ (!)

```
public  static void sort(Comparable[] a) {
 int N = a.length;
 for (int i=0; i<N-1; i++){
   int min=i;
   for (int j=i+1; j<N; j++) if less(a[j],a[min])) min=j;
   if (i != min) exch(a,i,min);
 }
}
```

Insertion sort

- Insertion sort:
  - Insert `a[1]` on its correct place in (sorted) `a[0..0]`
  - Insert `a[2]` on its correct place in (sorted) `a[0..1]`
  - ... until `a[n-1]`
- Very good for partially sorted arrays, costs:
  - Best case: $n-1$ compares and 0 exchanges
  - Worst case: $\sim n^2/2$ compares and exchanges
  - Average case: $\sim n^2/4$ compares and exchanges (distinct keys)

```
public  static void sort(Comparable[] a) {
 int N = a.length;
 for (int i=1; i<N; i++){
  for (int j=i; j>0 && less(a[j],a[j-1]); j--)
   exch(a,j,j-1);
 }
}
```

Shell sort

- Insertion sort:
  - Very good for partially sorted arrays
  - Slow in transport: step by step `exch(a,j,j-1)`
- Idea: h-sort, `a[i],a[i+h],a[i+2h],...` sorted (any `i`)

```
public static void hsort(int h, Comparable[] a) {
 int N = a.length;
 for (int i=h; i<N; i++)
  for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)
    exch(a,j,j-h);
}
```

- Insertion sort: `hsort(1,a)`
- Shell sort: e.g., `hsort(10,a); hsort(1,a)`

Shell sort (ctnd)

- `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- Q: How is this possible?
- A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- To be expected: depends on the length N of the array
- Best practice: $h = N/3, N/9, \ldots, 364, 121, 40, 13, 4, 1$

Mergesort

- Top-down (recursive) algorithm:
  - Mergesort left half, mergesort right half
  - Merge the results
- Using an auxiliary array: TopDownMergeSort.java, Movie
- Bottom-up algorithm (16 elements):
  - Merge `a[0]`,`a[1]`, so `a[2]`,`a[3]`, so `a[4]`,`a[5]`, so ...
  - Merge `a[0..1]`,`a[2..3]`, so `a[4..5]`,`a[6..7]`, so ...
  - Merge `a[0..3]`,`a[4..7]`, so `a[8..11]`,`a[12..15]`
  - Merge `a[0..7]`,`a[8..15]`, done!
- Also using an auxiliary array: BottomUpMergeSort.java

Run-time and memory use of mergesort

- Mergesort uses between $\sim (N/2) \lg N$ and $\sim N \lg N$ compares. Proof on bb. Important formula ($N = 2^n$):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- Mergesort uses at most $\sim 6N \lg N$ array accesses
- Mergesort uses $\sim 2N$ space (plus some var's)
- Q: How fast can compare-based sorting of N distinct keys be?
- A: $\lg N! \sim N \lg N$; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i],a[j])`, permutations in the leaves,

$$N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$$

Quicksort

- Top-down (recursive) algorithm:
  - Choose a (pivot) value v in the array
  - Partition the array in non-empty parts $\leq v$ and $\geq v$
  - Quicksort the two parts
- Pros: in-place, average computation time $O(n \log n)$
- Cons: stack space for recursion, worst-case $O(n^2)$, not stable
- Implementation: QuickSort.java
- BTW: Bug in java.util.Arrays.sort

Quicksort, details

- Subtleties in `sort()`: shuffling protects against worst-case behaviour
- Termination of recursive `quicksort()`
- Subtleties in `partition()`:
  - Invariants `l<=h` in the two inner loops
  - Postcondition after the two inner loops
  - Invariant of the `for(;;)` loop
  - Termination of the `for(;;)` loop
  - There are some variations that are also correct

Run-time and memory use of quicksort

- Compare Quicksort to other sorts ($n = 10^2, 10^3, \ldots$)
- Quicksort: time $O(n^2)$ if pivot is always smallest (or largest)
- Randomization: choose pivot randomly, or shuffle array
- If all keys are distinct and randomization is perfect, then quicksort uses on average $\sim 2n \ln n$ compares and $\sim (n/3) \ln n$ exchanges (proofs in book, complicated)
- Relevant improvements:
  - Cut-off to insertion sort for sizes ≤ 15 (ca.)
  - Median-of-three pivot
  - Taking advantage of duplicate keys (3-way partitioning)
- Quicksort is generally quite good
- In special situations other sorts are better (e.g., countsort)

Priority Queues

- Assume collecting and processing items having keys
- Examples of keys: time-stamp, price-tag, priority-tag
- Assume: keys can be ordered
- Reasonable: processing currently highest (or lowest)
- Special cases: items time-stamped when added
  - Queue: dequeue currently oldest (lowest time-stamp)
  - Stack: pop currently newest (highest time-stamp)
- Priority queue generalizes this
- Examples: highest priority, largest transaction, lowest price
- Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

Priority Queues

- Good info: Wikipedia; API (the bare essentials):

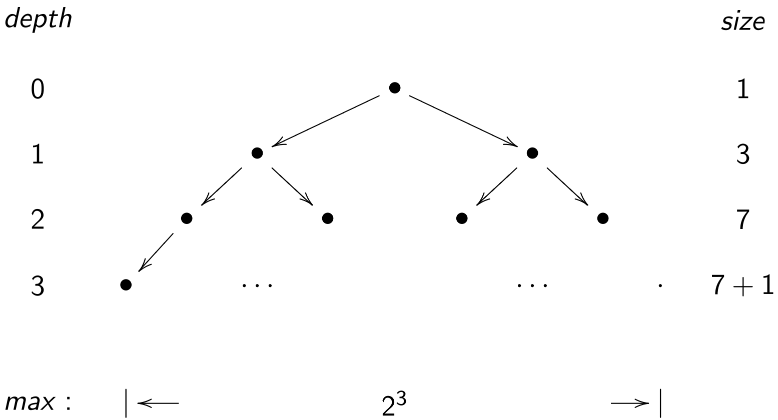
```
public class  ArrayListPQ<Key extends Comparable<Key>>

void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- Aim: operations in logarithmic time, no extra space
- In case of duplicate keys: ‘a’ largest, not ‘the’
- Typical application: the 1K largest keys of 1G unsorted keys
- Client: BottomM.java (Q: why is the output slowing down?)

Binary Trees

- MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height h has $2^{h+1}-1$ nodes.
- INF102: A binary tree is (left-)*complete* if all levels $< h$ are filled, level h may be partly empty from the right (picture bb). A (left-)complete binary tree of height h has between 2^h and $2^{h+1}-1$ nodes (from now on we leave out '(left-)').
- A complete binary tree of n nodes has height $\lfloor \lg n \rfloor$

Picture



Heap-ordered Binary Trees

- A binary tree is *heap-ordered* if the key in each node is $\geq$ the keys in its children (if any). Thus the root has a maximal key.
- Array representation of heap-ordered complete binary tree (bb)
- Methods `swim()` and `sink()`: picture on bb, code below
- Implementation: ArrayListPQ.java

Run-time and memory use of heaps, applications

- In a heap of n elements (since height is $\leq \lfloor \lg n \rfloor$):
  - `swim()`, and hence `insert()`, takes $\leq 1 + \lfloor \lg n \rfloor$ compares and $\leq \lfloor \lg n \rfloor$ exchanges
  - `sink()`, and hence `delMax()`, takes $\leq 2\lfloor \lg n \rfloor$ compares
  - `sink()` takes $\leq \lfloor \lg n \rfloor$ exchanges, and `delMax()` $\leq 1+\lfloor \lg n \rfloor$
- Heap construction by `insert()` can sometimes be improved
- Given an array of keys, right-to-left heap construction (bb) takes $< 2n$ compares and $< n$ exchanges
- Applications: heapsort and merging sorted streams (bb)
- Many variations with extended API (indexed priority queue)

Purpose of Sorting

- Sorting makes the following easier and more efficient:
  - Searching (binary search, example: `ThreeSumOptimized`
  - Searching and looking up, e.g., the pagenumber in an index
  - Removing duplicates
  - Finding the median, quartiles etc.
- Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- We do *pointer sorting*, manipulating refs to objects.
  - Pro: not moving full objects
  - Cons: pointer dereferencing, no `sort(int[] a)`
- More flexibility: pass a `Comparator` object to `sort()`

Comparator object

- API: `void sort(Object[] a, Comparator c)`
- Call: `sort(a, new Transaction.WhenOrder())`
- Call: `sort(a, new Transaction.SizeOrder())`
- Obs: `import java.util.Comparator`
- Obs: `less(Object o1, Object o2, Comparator c)`
- Priority queues also with `Comparator`

```
public class Transaction {
 ...
 public static class WhenOrder {
 implements Comparator<Transaction>
  public int compare(Transaction t, Transaction v){...}
 } // End of Myorder
 ...
} // End of Transaction
```

Applications of Sorting

- Consider sorting first to make other problems easier
- Commercial computing (sort on price, departure time, ...)
- Search for information
- Job scheduling heuristic: longest processing time first
- Combinatorial search in AI
- To come: Prim's and Dijkstra's algorithms
- Data compressions
- Cryptology and genomics (e.g., longest repeating substring)

Symbol Tables

- Symbol table associates *keys* with *values: key-value pairs*
- Examples: keyword-page number, ID number-personal data
- Important operations:
  - Insert a key-value pair in the symbol table: `void put(k,v)`
  - Search the value for a given key (if any): `Value get(k)`
- Important conventions:
  - Inserting key-value for existing key: overwriting the value
  - No duplicate keys, no null keys
  - Value null: no value for this key
  - Lazy deletion: insert key-null; Eager: really delete key
- API of unordered symbol table
- Aim: all operations in time $\sim c(\lg n)$ with small constant c

ST Basics

- Archetypical ST-client: frequency counter (code: later)
- Cost model: number of compares
- Naive ST: unordered linked list, linear search
  - Search miss: $\sim n$ compares
  - Search hit: between 1 and $\sim n$ compares
  - Random search hit: $(1 + \cdots + n)/n \sim n/2$ compares
  - Inserting n distinct keys: $(1 + \cdots + n) \sim n^2/2$ compares
- `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- Naive ST impracticable for genomics, internet
- Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- Better: hashing (in Ch. 3.4)
- Better: ordered ArrayList, binary search, ArrayListST.java
- Binary search: $O(\lg n)$; ArrayList: insert amortized $O(1)$

Binary Search Trees

- Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- Search time: lenght of the path to the node where the key 'should' be
- Balanced binary tree with n keys has $\lg n$ height
- Unbalanced binary trees can have height n (so long paths)
- API of ordered symbol table
- UBST.java: `put()`, `get()`, `size()`, `isEmpty()`

Binary Search Trees (ctnd)

- Interrelated, increasing difficulty: `min()`, `deleteMin()`, `delete(Key k)`
- Node of minimum key: not `null`; has left child `null`; is root or left child of parent (picture on bb)

```
public Value min(Node x){//min in subtree with root x
 if (x==null) return null; // secures x!=null below
 Value v;
 do {v = x.value; x = x.left;} while (x!=null);
 return v;
} // cf. tail recursive min() in Alg. 3.3
```

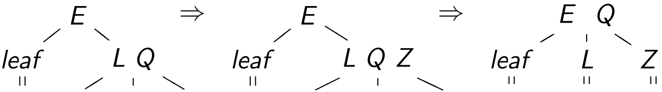
- Delete minimum key, two cases: (1) both children `null`; (2) left child `null`
- Delete is really difficult: bb + BST.java
- Don't forget: update `x.N` along the path to the root!

Balanced Search Trees: keep paths short!

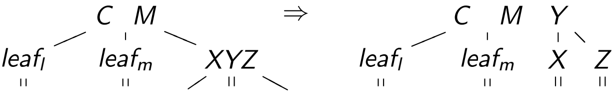
- NB tree balancing not as easy as in UF and Heap (4hrs!)
- A *2-3 search* tree consists of 2-nodes and 3-nodes:
  - Each 2-node has two children and a key k such that all keys in the left subtree are $< k$, and all keys in the right subtree $> k$
  - Each 3-node has three children and two keys k_1, k_2 such that all keys in the left subtree are $< k_1$, all keys in the middle subtree $> k_1$ and $< k_2$, and all keys in the right subtree $> k_2$
- Examples and pictures on bb
- *Perfect* 2-3 search tree: paths from root to leaves equally long
- Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- Insert should keep tree perfect, rough idea:
  - into a 2-leaf: make it into a 3-leaf
  - into a 3-node: do something clever (explained below)

Insert in Balanced Search Trees

- Terminology: a *leaf* is a node all whose children are `null`
- Data invariant 1: 2-3 search tree
- Data invariant 2: paths from root to leaves equally long
  - Insert into a 2-leaf L either $A\ L$ or $L\ Z$
  - into a 3-leaf whose parent is a 2-node: with new key Z



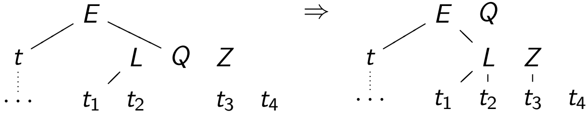
  - into a 3-leaf whose parent is a 3-node: with new key Z



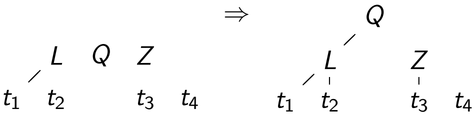
  - into a 3-node whose parent is a 3-node: move up middle key!

Insert (ctnd)

- Data invariant 1: 2-3 search tree
- Data invariant 2: paths from root to leaves equally long
- Insert works up from the leaf where the key ‘should’ be
  - if 2-node on path to root: make it into a 3-node (two cases)



Equivalent
  - otherwise: split the root

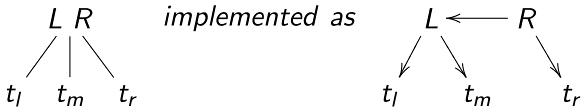


Insert, summary and examples

- Six operations for eliminating 4-nodes:Equivalent
  - if parent is root: root split
  - if parent is 2-node: move middle key up (left and right case)
  - if parent is 3-node: move middle key up (left, middle,right)
- Search and insert visit at most $\lfloor \lg n \rfloor$ nodes
- Proof: maximal path length is $\geq \lfloor \log_3 n \rfloor$ and $\leq \lfloor \log_2 n \rfloor$
- Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- Trace of inserts on bb: A C E H L M P R S X

Red-black trees

- Red-black trees implement 2-3 trees
- Idea: one 3-node = two 2-nodes + extra info
- Extra info coded in color, picture:



- A *red-black tree* is a binary search tree with red and black links such that:
  - Only left links can be red (but need not be)
  - Never $\leftarrow \leftarrow$
  - Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- Equivalent: red-black tree and perfect 2-3 search tree

Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

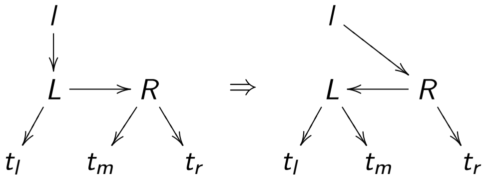
```
private class Node {
  Key key;
  Value value;
  Node left, right;
  boolean color; // true for red, false for black
  int N;
}
private boolean isRed(Node n) {
 if (n==null) {return false;} else {return x.color}
```

Rotating and Color Flipping

- Aim: restoring the data invariants of red-black search trees
  1. Only left red links, but never two
  2. Search tree invariant
  3. Perfect black balance
- Invariants get violated by temporary 4-nodes, e.g.,
  - inserting Z in $L \longleftarrow R$: $L \longleftarrow R \longrightarrow Z$
  - inserting A in $L \longleftarrow R$: $A \longleftarrow L \longleftarrow R$
  - inserting M in $L \longleftarrow\!\!\!\longleftarrow R$:
$$\begin{tikzcd} L \arrow[dr] & & R \arrow[ll] \\ & M & \end{tikzcd}$$
- Restoring:
  - Color flip $L \longleftarrow R \longrightarrow Z$: $L \longleftarrow R \longrightarrow Z$
  - Rotation right + color flip $A \longleftarrow L \longleftarrow R$: $A \longleftarrow L \longrightarrow R$
  - Rotation left into $M \longleftarrow L \longleftarrow R$, then as previous

Left Rotation

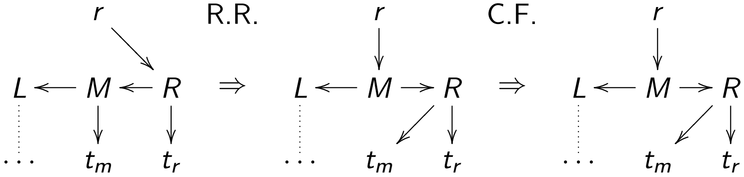
Call: `l = rotateLeft(l);`




```
private Node rotateLeft(Node l){
  Node r = l.right; l.right = r.left; r.left = l key;
  r.color = l.color; l.color = true // == RED
  r.N = l.N; l.N += size(l.right); // Why?
  return r;
}
```

Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- Code of `rotateRight()` like that of `rotateLeft()`
- NB1: operations are local (here only `r`, `M` , `R`)
- NB2: operations preserve data invariants
- NB3: root is a special case (always black)
- Deletions: complicated, but doable (Exc. 3.3.39–41)

Run-time and memory use of Red-Black BSTs

- The height of a red-black BST with n nodes is $\leq 2 \lg n$
  Proof: the worst-case is one 3-node path and the rest 2-nodes
- The average length of path from the root to a *node (?)* in a red-black BST with n nodes is $\lg n$ ('empirical fact')
- In a red-black BST, search, insertion, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node.
- For red-black BSTs, logarithmic time is guaranteed!

Hashing

- A *hash function* maps keys to a (as unique as possible) array index
- A (hash) *collision* occurs when different keys are mapped to the same index
- In such a case we need *collision resolution*
- Hashing = hash function + collision resolution
- Symbol tables based on hashing: fast, but no support for `max`, . . .

ToC and topics of general interest

- Table of Contents on next slide (all items clickable)
- Practical stuff: slide 2