

INF102

Algorithms, Data Structures and Programming

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INF102, practical stuff

- ▶ Lecturer: Marc Bezem; Team: see homepage
- ▶ Homepage: [INF102](#) (hyperlinks in red)
- ▶ Also: [GitHub](#) (recommended); Dropbox: [slides](#), [schedule](#)
- ▶ Textbook: [Algorithms, 4th edition](#)
- ▶ Prerequisites: INF100 + 101 (\approx Ch. 1.1 + 1.2)
- ▶ Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- ▶ Exam: three compulsory exercises and a [written exam](#)
- ▶ Old exams: [2004–2013](#), [2014](#)
- ▶ [Table of Contents of these slides](#)

Resources

- ▶ Good textbook, USA-style: many pages, exercises etc.
- ▶ Average speed must be ca 50 pages p/w
- ▶ Lectures (ca 24) focus on the essentials
- ▶ Slides (ca 120, dense!) summarize the lectures
- ▶ Prepare yourself by reading in advance
- ▶ Workshops: selected exercises
- ▶ Test yourself by trying some exercises in advance
- ▶ If you can do the exercises (incl. compulsory), you are fine
- ▶ Review of exercises on Friday morning

Generic Bags, Queues and Stacks

- ▶ Generic programming in Java, example: **PolyPair.java**
- ▶ Bag, Queue and Stack are generic, iterable collections
- ▶ Queue and Stack: Ch. 9 in textbook INF100/1
- ▶ APIs include: `boolean isEmpty()` and `int size()`
- ▶ All three support adding an element
- ▶ Queue and Stack support removing an element (if any)
- ▶ FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- ▶ Dijkstra's Two-Stack Expression Evaluation **Movie**
- ▶ Example: $(1 + ((2 + 3) * (4 * 5)))$

Implementations

- ▶ `ResizingArray_Stack.java`
- ▶ Arrays give direct access, but have fixed size
- ▶ Resizing takes time and space proportional to size
- ▶ `LinkedList_Stack.java`
- ▶ No fixed size, but indirect access
- ▶ Pointers take space and dereferencing takes time
- ▶ Programming with pointers: make a picture
- ▶ `LinkedList_Queue.java`

Computation time and memory space

- ▶ Two central questions:
 - ▶ How long will my program take?
 - ▶ Will there be enough memory?
- ▶ Example: **ThreeSum.java**
- ▶ Inner loop (here $a[i] + a[j] + a[k] == 0$) is important
- ▶ Sorting helps: **ThreeSumOptimized.java**
- ▶ Run some experiments: `1Kints.txt`, `2Kints.txt`, ...

Methods of Analysis

- ▶ Empirical:
 - ▶ Run program with randomized inputs, measuring time & space
 - ▶ Run program repeatedly, doubling the input size
 - ▶ Measuring time: **StopWatch**
 - ▶ Plot, or log-log plot and **linear regression**
- ▶ Theoretical:
 - ▶ Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
 - ▶ Try to count/estimate/average this cost as function of the input (size)
 - ▶ Use $O(f(n))$ and $f(n) \sim g(n)$

ThreeSum, empirically

- ▶ Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4 ,51.1 sec
- ▶ The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- ▶ Basis of the logarithm should be the same for both
- ▶ Linear regression gives $y \approx 3x - 10$
- ▶ $\log(f(n)) = 3 \log(n) - 10$ iff

$$f(n) = 10^{\log(f(n))} = 10^{3 \log(n) - 10} = n^3 * 10^{-10}$$

- ▶ Conclusion: cubic in the input size, with constant $\approx 10^{-10}$
- ▶ Strong dependence on input can be a problem
- ▶ Constant 10^{-10} depends on computer, exponent 3 does not

ThreeSum, theoretically

- ▶ Number of different picks of triples: $g(n) = n(n-1)(n-2)/6$
- ▶ Inner loop $a[i] + a[j] + a[k] == 0$ executed $g(n)$ times
- ▶ $g(n) = n^3/6 - n^2/2 + n/3$
- ▶ Cubic term $n^3/6$ wins for large n
- ▶ Computational model # array accesses: $3 * n^3/6 = n^3/2$
- ▶ Cost array access t sec: time $t * n^3/2$ sec
- ▶ Cost models are abstractions! (NB cache)

Big Oh, and \sim

- ▶ Q: 'wins for large n ' uhh???
- ▶ A: Big Oh, and \sim will clear this up
- ▶ Costs are positive quantities, so $f, g, \dots : \mathbb{N} \rightarrow \mathbb{R}^+$
- ▶ MNF130: $f(n)$ is $O(g(n))$ if there exist $c \in \mathbb{R}^+$, $N \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (that is,, for n large enough)
- ▶ Example: n^2 and even $99n^3$ are $O(n^3)$, but n^3 is not $O(n^{2.9})$
- ▶ INF102: $f(n) \sim g(n)$ if $1 = \lim f(n)/g(n)$
- ▶ If $f(n) \sim g(n)$, then $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$
- ▶ Big Oh and \sim aim to capture 'order of growth'
- ▶ Big Oh abstracts from constant factors, \sim does not
- ▶ Large constant factors are important!

Important orders of growth

- ▶ constant: c , $f(n) = c$ for all n
- ▶ linear: n (compare all for $n = 20$ sec)
- ▶ linearithmetic: $n \log n$
- ▶ quadratic: n^2
- ▶ cubic: n^3
- ▶ exponential: 2^n
- ▶ general form: $an^b(\log n)^c$

Logarithms and Exponents

- ▶ Definition: $\log_x z = y$ iff $x^y = z$ for $x > 0$
- ▶ Inverses: $x^{\log_x y} = y$ and $\log_x x^y = y$
- ▶ Exponent: $x^{(y+z)} = x^y x^z$, $x^{(yz)} = (x^y)^z$
- ▶ Logarithm: $\log_x(yz) = \log_x y + \log_x z$, $\log_x z = \log_x y \log_y z$
- ▶ Base of logarithm: the x in \log_x
- ▶ Various bases: $\log_2 = \lg$, $\log_e = \ln$, $\log_{10} = \log$
- ▶ Double exponent: e.g. $2^{(2^n)}$ (not used in INF102)
- ▶ Double logarithm: $\log(\log n)$ (not used in INF102)

Worst case, average case, amortized cost

- ▶ Worst case: guaranteed, independent of input; Examples:
 - ▶ Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
 - ▶ Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- ▶ Average case: not guaranteed, dependent of input *distribution*
- ▶ Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost ≤ 21 , all others cost 1, amortized ≤ 3 p/o.
- ▶ Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- ▶ Special case of resizing array that is only growing:
 $1(2)2(4)3(4)4(8)5(6)6(7)7(8)8(16)9 \dots 16(32) \dots$, with (n) the new size.
 Resizing to (n) costs $2n$ array accesses, so in total
 $(1+4)+(1+8)+(2+16)+(4+32)+(8+64) \dots$, so 9 p/push.

Staying Connected

- ▶ We want efficient algorithms and datastructures for testing whether two objects are 'connected'
- ▶ MNF130: relation $E \subseteq V \times V$ is an *equivalence* if
 - ▶ E is *reflexive*: $\forall x \in V. E(x, x)$
 - ▶ E is *symmetric*: $\forall x, y \in V. E(x, y) \rightarrow E(y, x)$
 - ▶ E is *transitive*: $\forall x, y, z \in V. E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- ▶ We assume connectedness to be an equivalence
- ▶ Dynamic connectivity means (here) that E can grow
- ▶ Clear relationship with paths in graphs, (connected) components (MNF130)
- ▶ Input: N and pairs in $V = \{0, \dots, N-1\}$ defining E
- ▶ Challenge: efficient `boolean connected(int p, int q)`
- ▶ Example: $N = 10$, 4 3, 3 8, ... (`algs4-data/tinyUG.txt`)
- ▶ Picture on blackboard (don't print pairs that are already connected)

Union-Find

- ▶ Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- ▶ Union, idea: for any new pair $n\ m$ that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- ▶ API: **UF**; Cost model: number of array accesses
- ▶ Implementations:
 - ▶ **SlowUF.java**: `id[p]` identifier of p
`find()` ~ 1 , `union()` \sim between $n+3$ and $2n+1$
 - ▶ **FastUF.java**: `int[] id` pointers, `id[p]==p`: identifier
`find()` $\sim 1+2d$, `union()` $\sim 1 + \text{two find}()$'s
 - ▶ **WeightedUF.java**: `int[] id` pointers, `int[] sz` subtree sizes
`find()` and `union()` both $\sim \lg n$
- ▶ WeightedUF: height of subtree of size k is at most $\lg k$
- ▶ Path-compression: ultimate improvement of UF (almost $O(1)$, amortized)

Sorting

- ▶ Sorting: putting objects in a certain order
- ▶ MNF130: relation $R \subseteq V \times V$ is a *total order(ing)* if
 1. R is *reflexive*: $\forall x \in V. R(x, x)$
 2. R is *transitive*: $\forall x, y, z \in V. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
 3. R is *antisymmetric*: $\forall x, y \in V. R(x, y) \wedge R(y, x) \rightarrow x = y$
 4. R is *total*: $\forall x, y \in V. R(x, y) \vee R(y, x)$
- ▶ Natural orderings:
 - ▶ Numbers of any type: ordinary \leq and \geq
 - ▶ Strings: lexicographic
 - ▶ Objects of a Comparable type: `v.compareTo(w) <= 0`

Sorting (ctnd)

- ▶ Bubble sort: `ExampleSort.java`
- ▶ Certification: `assert isSorted(a)` in `main()`
- ▶ No guarantee against modifying the array (but `exch()` is safe)
- ▶ Costmodel 1: number of `exch()`'s and `less()`'s
- ▶ Costmodel 2: number of array accesses
- ▶ Pitfalls: cache misses, expensive `v.compareTo(w) < 0`
- ▶ Why studying sorting? (`java.util.Arrays.sort()`)
- ▶ Comparing sorting algorithms: `SortCompare.java`

Selection Sort

- ▶ Bubble sort: $\sim n^2/2$ compares, $0 \leq \text{exchanges} \leq \sim n^2/2$
- ▶ Selection sort:
 - ▶ Find index of a minimum in $a[0..n-1]$, exchange with $a[0]$
 - ▶ Find index of a minimum in $a[1..n-1]$, exchange with $a[1]$
 - ▶ ... until $n-2$
- ▶ Selection sort: $\sim n^2/2$ compares, $0 \leq \text{exchanges} \leq n-1$ (!)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=0; i<N-1; i++){  
        int min=i;  
        for (int j=i+1; j<N; j++) if (less(a[j],a[min])) min=j;  
        if (i != min) exch(a,i,min);  
    }  
}
```

Insertion sort

- ▶ Insertion sort:
 - ▶ Insert $a[1]$ on its correct place in (sorted) $a[0..0]$
 - ▶ Insert $a[2]$ on its correct place in (sorted) $a[0..1]$
 - ▶ ... until $a[n-1]$
- ▶ Very good for partially sorted arrays, costs:
 - ▶ Best case: $n-1$ compares and 0 exchanges
 - ▶ Worst case: $\sim n^2/2$ compares and exchanges
 - ▶ Average case: $\sim n^2/4$ compares and exchanges (distinct keys)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=1; i<N; i++){  
        for (int j=i; j>0 && less(a[j],a[j-1]); j--)  
            exch(a,j,j-1);  
    }  
}
```

Shell sort

- ▶ Insertion sort:
 - ▶ Very good for partially sorted arrays
 - ▶ Slow in transport: step by step `exch(a,j,j-1)`
- ▶ Idea: h-sort, `a[i], a[i+h], a[i+2h], ...` sorted (any `i`)

```
public static void hsort(int h, Comparable[] a) {  
    int N = a.length;  
    for (int i=h; i<N; i++)  
        for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)  
            exch(a,j,j-h);  
}
```

- ▶ Insertion sort: `hsort(1,a)`
- ▶ Shell sort: e.g., `hsort(10,a); hsort(1,a)`

Shell sort (ctnd)

- ▶ `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- ▶ Q: How is this possible?
- ▶ A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- ▶ What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- ▶ To be expected: depends on the length N of the array
- ▶ Best practice: $h = N/3, N/9, \dots, 364, 121, 40, 13, 4, 1$

Mergesort

- ▶ Top-down (recursive) algorithm:
 - ▶ Mergesort left half, mergesort right half
 - ▶ Merge the results
- ▶ Using an auxiliary array: [TopDownMergeSort.java](#), [Movie](#)
- ▶ Bottom-up algorithm (16 elements):
 - ▶ Merge $a[0], a[1]$, so $a[2], a[3]$, so $a[4], a[5]$, so ...
 - ▶ Merge $a[0..1], a[2..3]$, so $a[4..5], a[6..7]$, so ...
 - ▶ Merge $a[0..3], a[4..7]$, so $a[8..11], a[12..15]$
 - ▶ Merge $a[0..7], a[8..15]$, done!
- ▶ Also using an auxiliary array: [BottomUpMergeSort.java](#)

Run-time and memory use of mergesort

- ▶ Mergesort uses between $\sim (N/2) \lg N$ and $\sim N \lg N$ compares. Proof on bb. Important formula ($N = 2^n$):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- ▶ Mergesort uses at most $\sim 6N \lg N$ array accesses
- ▶ Mergesort uses $\sim 2N$ space (plus some var's)
- ▶ Q: How fast can compare-based sorting of N distinct keys be?
- ▶ A: $\lg N! \sim N \lg N$; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i], a[j])`, permutations in the leaves,
 $N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$

Quicksort

- ▶ Top-down (recursive) algorithm:
 - ▶ Choose a (pivot) value v in the array
 - ▶ Partition the array in non-empty parts $\leq v$ and $\geq v$
 - ▶ Quicksort the two parts
- ▶ Pros: in-place, average computation time $O(n \log n)$
- ▶ Cons: stack space for recursion, worst-case $O(n^2)$, not stable
- ▶ Implementation: **QuickSort.java**
- ▶ BTW: **Bug in java.util.Arrays.sort**

Quicksort, details

- ▶ Subtleties in `sort()`: shuffling protects against worst-case behaviour
- ▶ Termination of recursive `quicksort()`
- ▶ Subtleties in `partition()`:
 - ▶ Invariants $l \leq h$ in the two inner loops
 - ▶ Postcondition after the two inner loops
 - ▶ Invariant of the `for(;;)` loop
 - ▶ Termination of the `for(;;)` loop
 - ▶ There are some variations that are also correct

Run-time and memory use of quicksort

- ▶ Compare Quicksort to other sorts ($n = 10^2, 10^3, \dots$)
- ▶ Quicksort: time $O(n^2)$ if pivot is always smallest (or largest)
- ▶ Randomization: choose pivot randomly, or shuffle array
- ▶ If all keys are distinct and randomization is perfect, then quicksort uses on average $\sim 2n \ln n$ compares and $\sim (n/3) \ln n$ exchanges (proofs in book, complicated)
- ▶ Relevant improvements:
 - ▶ Cut-off to insertion sort for sizes ≤ 15 (ca.)
 - ▶ Median-of-three pivot
 - ▶ Taking advantage of duplicate keys (3-way partitioning)
- ▶ Quicksort is generally quite good
- ▶ In special situations other sorts are better (e.g., countsort)

Priority Queues

- ▶ Assume collecting and processing items having keys
- ▶ Examples of keys: time-stamp, price-tag, priority-tag
- ▶ Assume: keys can be ordered
- ▶ Reasonable: processing currently highest (or lowest)
- ▶ Special cases: items time-stamped when added
 - ▶ Queue: dequeue currently oldest (lowest time-stamp)
 - ▶ Stack: pop currently newest (highest time-stamp)
- ▶ Priority queue generalizes this
- ▶ Examples: highest priority, largest transaction, lowest price
- ▶ Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

Priority Queues

- ▶ Good info: [Wikipedia](#); API (the bare essentials):

```
public class  ArrayListPQ<Key extends Comparable<Key>>

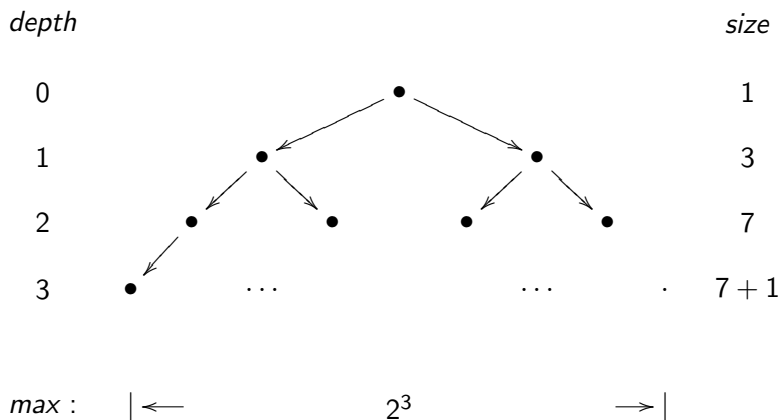
void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- ▶ Aim: operations in logarithmic time, no extra space
- ▶ In case of duplicate keys: 'a' largest, not 'the'
- ▶ Typical application: the 1K largest keys of 1G unsorted keys
- ▶ Client: [BottomM.java](#) (Q: why is the output slowing down?)

Binary Trees

- ▶ MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- ▶ MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height h has $2^{h+1}-1$ nodes.
- ▶ INF102: A binary tree is (left-) *complete* if all levels $< h$ are filled, level h may be partly empty from the right (picture bb). A (left-)complete binary tree of height h has between 2^h and $2^{h+1}-1$ nodes (from now on we leave out '(left-)').
- ▶ A complete binary tree of n nodes has height $\lfloor \lg n \rfloor$

Picture



Heap-ordered Binary Trees

- ▶ A binary tree is *heap-ordered* if the key in each node is \geq the keys in its children (if any). Thus the root has a maximal key.
- ▶ Array representation of heap-ordered complete binary tree (bb)
- ▶ Methods `swim()` and `sink()`: picture on bb, code below
- ▶ Implementation: `ArrayListPQ.java`

Run-time and memory use of heaps, applications

- ▶ In a heap of n elements (since height is $\leq \lfloor \lg n \rfloor$):
 - ▶ `swim()`, and hence `insert()`, takes $\leq 1 + \lfloor \lg n \rfloor$ compares and $\leq \lfloor \lg n \rfloor$ exchanges
 - ▶ `sink()`, and hence `delMax()`, takes $\leq 2\lfloor \lg n \rfloor$ compares
 - ▶ `sink()` takes $\leq \lfloor \lg n \rfloor$ exchanges, and `delMax()` $\leq 1 + \lfloor \lg n \rfloor$
- ▶ Heap construction by `insert()` can sometimes be improved
- ▶ Given an array of keys, right-to-left heap construction (bb) takes $< 2n$ compares and $< n$ exchanges
- ▶ Applications: **heapsort** and merging sorted streams (bb)
- ▶ Many variations with extended API (indexed priority queue)

Purpose of Sorting

- ▶ Sorting makes the following easier and more efficient:
 - ▶ Searching (binary search, example: `ThreeSumOptimized`)
 - ▶ Searching and looking up, e.g., the `pagenumber` in an index
 - ▶ Removing duplicates
 - ▶ Finding the median, quartiles etc.
- ▶ Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- ▶ We do *pointer sorting*, manipulating refs to objects.
 - ▶ Pro: not moving full objects
 - ▶ Cons: pointer dereferencing, no `sort(int[] a)`
- ▶ More flexibility: pass a `Comparator` object to `sort()`

Comparator object

- ▶ API: `void sort(Object[] a, Comparator c)`
- ▶ Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- ▶ Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- ▶ Obs: `import java.util.Comparator`
- ▶ Obs: `less(Object o1, Object o2, Comparator c)`
- ▶ Priority queues also with `Comparator`

```
public class Transaction {  
    ...  
    public static class MyOrder {  
        implements Comparator<Transaction>  
        public int compare(Transaction t, Transaction v){...}  
    } // End of Myorder  
    ...// similarly: WhenOrder, SizeOrder  
} // End of Transaction
```

Applications of Sorting

- ▶ Consider sorting first to make other problems easier
- ▶ Commercial computing (sort on price, departure time, ...)
- ▶ Search for information: web-indexing, search engines
- ▶ Job scheduling heuristic: longest processing time first
- ▶ To come: Prim's, Dijkstra's and Kruskal's algorithms
- ▶ Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur often.
Frequency counter: next chapter!
- ▶ Cryptology and genomics (e.g., longest repeated substring)

Symbol Tables

- ▶ Symbol table associates *keys* with *values*: *key-value pairs*
- ▶ Examples: keyword-page number, ID number-personal data
- ▶ Important operations:
 - ▶ Insert a key-value pair in the symbol table: `void put(k,v)`
 - ▶ Search the value for a given key (if any): `Value get(k)`
- ▶ Important conventions:
 - ▶ Inserting key-value for existing key: overwriting the value
 - ▶ No duplicate keys, no null keys
 - ▶ Value null: no value for this key
 - ▶ Lazy deletion: insert key-null; Eager: really delete key
- ▶ **API** of unordered symbol table
- ▶ Aim: all operations in time $\sim c \lg n$ with constant c small

ST Basics

- ▶ Archetypical ST-client: frequency counter (code: `main`)
- ▶ Cost model: number of compares
- ▶ Naive ST: unordered linked list, linear search (INF101, Ch.9)
 - ▶ Search miss: $\sim n$ compares
 - ▶ Search hit: between 1 and $\sim n$ compares
 - ▶ Random search hit: $(1 + \dots + n)/n \sim n/2$ compares
 - ▶ Inserting n distinct keys: $(1 + \dots + (n-1)) \sim n^2/2$ compares
- ▶ `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- ▶ Naive ST impracticable for genomics, internet
- ▶ Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- ▶ Better for unordered ST: hashing (in Ch. 3.4)

Ordered Symbol Table

- ▶ Ordered ST: keys are ordered
- ▶ **API** of ordered symbol table
- ▶ Binary search: `get(Key k)` takes $\sim \lg n$ comparisons
- ▶ What about `put(Key k, Value v)`? **ArrayListST**, good!
- ▶ TODO: test that `add(int i, E e)` is amortized $O(1)$
- ▶ Implementation with binary search in **ArrayListST.java**
- ▶ Experiments with `tinyTale.txt`, `tale.txt`, ...

Binary Search Trees

- ▶ Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- ▶ Search time: length of the path to the node where the key 'should' be
- ▶ Balanced binary tree with n keys has $\lg n$ height
- ▶ Unbalanced binary trees can have height n (max depth)
- ▶ Search hits in an unbalanced binary search tree of n random keys take on average $\sim 2 \ln n$ compares
- ▶ **UBST.java**: `put()`, `get()`, `size()`, `isEmpty()`
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E

Binary Search Trees (ctnd)

- ▶ Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- ▶ Node of minimum key: not `null`, and has left child `null`, and is root or left child of parent (picture on bb)

```
public Node min(Node x){ // subtree under x
    while (x!=null) x = x.left;
    return x;
} // cf. tail recursive min() in Alg. 3.3
```

- ▶ Delete minimum key, two cases:
(1) both children `null`; (2) left child `null`
- ▶ Delete is really difficult: bb + **BST.java**
- ▶ Don't forget: update `x.N` along the path to the root!

Balanced Search Trees: keep paths short!

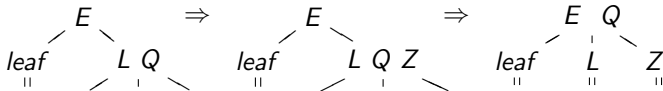
- ▶ NB tree balancing not as easy as in UF and Heap (4hrs!)
- ▶ A 2-3 search tree consists of 2-nodes and 3-nodes:
 - ▶ Each 2-node has two children and a key k such that all keys in the left subtree are $< k$, and all keys in the right subtree $> k$
 - ▶ Each 3-node has three children and two keys k_1, k_2 such that all keys in the left subtree are $< k_1$, all keys in the middle subtree $> k_1$ and $< k_2$, and all keys in the right subtree $> k_2$
- ▶ Examples and pictures on bb
- ▶ *Perfect* 2-3 search tree: paths from root to leaves equally long
- ▶ Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- ▶ Insert should keep tree perfect, rough idea:
 - ▶ into a 2-leaf: make it into a 3-leaf
 - ▶ into a 3-node: do something clever (explained below)

Insert in Balanced Search Trees

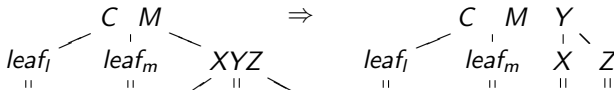
- ▶ Terminology: a *leaf* is a node all whose children are null
- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long

- ▶ Insert into a 2-leaf L either $\begin{array}{c} A \\ / \quad | \quad \backslash \\ L \end{array}$ or $\begin{array}{c} L \\ / \quad | \quad \backslash \\ Z \end{array}$

- ▶ into a 3-leaf whose parent is a 2-node: with new key Z



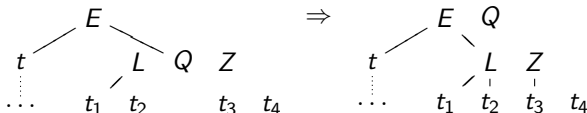
- ▶ into a 3-leaf whose parent is a 3-node: with new key Z



- ▶ into a 3-node whose parent is a 3-node: move up middle key!

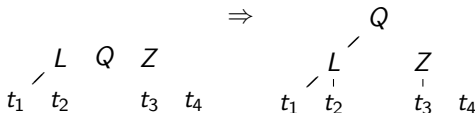
Insert (ctnd)

- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long
- ▶ Insert works up from the leaf where the key 'should' be
 - ▶ if 2-node on path to root: make it into a 3-node (two cases)



Equivalent

- ▶ otherwise: split the root

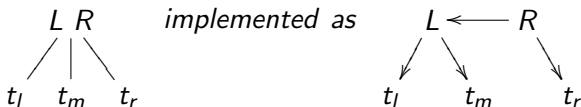


Insert, summary and examples

- ▶ Six operations for eliminating 4-nodes: Equivalent
 - ▶ if parent is root: root split
 - ▶ if parent is 2-node: move middle key up (left and right case)
 - ▶ if parent is 3-node: move middle key up (left, middle, right)
- ▶ Search and insert visit at most $\lfloor \lg n \rfloor$ nodes
- ▶ Proof: maximal path length is $\geq \lfloor \log_3 n \rfloor$ and $\leq \lfloor \log_2 n \rfloor$
- ▶ Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- ▶ Trace of inserts on bb: A C E H L M P R S X

Red-black trees

- ▶ Red-black trees implement 2-3 trees
- ▶ Idea: one 3-node = two 2-nodes + extra info
- ▶ Extra info coded in color, picture:



- ▶ A *red-black tree* is a binary search tree with red and black links such that:
 - ▶ Only left links can be red (but need not be)
 - ▶ Never $\leftarrow \leftarrow$
 - ▶ Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- ▶ Equivalent: red-black tree and perfect 2-3 search tree


Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

```
private class Node {  
    Key key;  
    Value value;  
    Node left, right;  
    boolean color; // true for red, false for black  
    int N;  
}  
private boolean isRed(Node n) {  
    if (n==null) {return false;} else {return x.color}
```

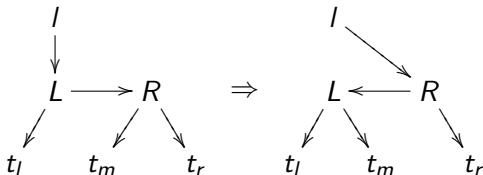
Rotating and Color Flipping

- ▶ Aim: restoring the data invariants of red-black search trees
 1. Only left red links, but never two
 2. Search tree invariant
 3. Perfect black balance
- ▶ Invariants get violated by temporary 4-nodes, e.g.,
 - ▶ inserting Z in $L \leftarrow R : L \leftarrow R \rightarrow Z$
 - ▶ inserting A in $L \leftarrow R : A \leftarrow L \leftarrow R$
 - ▶ inserting M in $L \longleftarrow R : L \longleftarrow R$


- ▶ Restoring:
 - ▶ Color flip $L \leftarrow R \rightarrow Z : L \leftarrow R \rightarrow Z$
 - ▶ Rotation right + color flip $A \leftarrow L \leftarrow R : A \leftarrow L \rightarrow R$
 - ▶ Rotation left into $M \leftarrow L \leftarrow R$, then as previous

Left Rotation

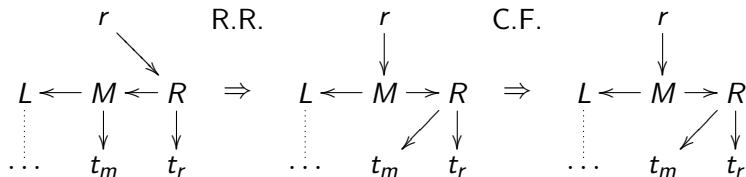
Call: `l = rotateLeft(l);`



```
private Node rotateLeft(Node l){
    Node r = l.right; l.right = r.left; r.left = l key;
    r.color = l.color; l.color = true // == RED
    r.N = l.N; l.N += size(l.right); // Why?
    return r;
}
```


Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- ▶ Code of `rotateRight()` like that of `rotateLeft()`
- ▶ NB1: operations are local (here only r , M , R)
- ▶ NB2: operations preserve data invariants
- ▶ NB3: root is a special case (always black)
- ▶ Deletions: complicated, but doable (Exc. 3.3.39–41)

Run-time and memory use of Red-Black BSTs

- ▶ The height of a red-black BST with n nodes is $\leq 2 \lg n$
Proof: the worst-case is one 3-node path and the rest 2-nodes
- ▶ The average length of path from the root to a *node* (?) in a red-black BST with n nodes is $\lg n$ ('empirical fact')
- ▶ In a red-black BST, search, insertion, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node.
- ▶ For red-black BSTs, logarithmic time is guaranteed!

Hashing

- ▶ A *hash function* maps keys to a (as unique as possible) array index
- ▶ A (hash) *collision* occurs when different keys are mapped to the same index
- ▶ In such a case we need *collision resolution*
- ▶ Hashing = hash function + collision resolution
- ▶ Symbol tables based on hashing: fast, but no support for `max, ...`

TODOs Chapter 1–3

- ▶ Find out if `add(int i, E e in ArrayList` is amortized $O(1)$
- ▶ Explain *indexed* priority queues
- ▶ Explain `delete(Key k` in red-black trees

ToC and topics of general interest

- ▶ Table of Contents on next slide (all items clickable)
- ▶ Practical stuff: slide 2

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