# Gauss' law

October 18, 2025

## 1 Background

Gauss' law is one of those really fundamental observations, that serves as one of those hints that humanity has managed to figure out something even more general than it was originallly awarded, namely Coulomb's law.

Gauss' law states the following: *The electric flux through a closed surface, is equal to the total charge enclosed by the surface, divided by the permittivity of vacuum, $\epsilon_0$.*

Stated mathematically:

$$\frac{Q_{enc}}{\epsilon_0} = \Phi$$

where $\Phi$ is the flux (we explain what the heck the flux is below). Now, what does this mean? Let's consider a concrete example, drawn in figure 1. We have a point charge $q$, and we would like to find the electric field that this causes. Of course, we could just used our trusted Coulomb's law, but this time, let's turn our attention to Gauss' law instead.

To apply Gauss' law, we need a hypothetical surface to consider. This is called a *Gaussian surface*. So, we make up one! Now, there is a lot of care that should go into picking the right kind of surface. In our case, because our problem is totally spherically symmetric, we pick a hypothetical sphere of radius $r$, centered on the point charge. To reiterate, this is just a hypothetical surface, that we're considering for the purpoise of solving this problem. Gauss' law doesn't care if the surface actually "exists".

**The charge:** The charge enclosed by this surface is simply $q$ (everything else inside the surface is just vacuum). Thus, we have the left-hand side of Gauss' law written out as $q/\epsilon_0$.

**The flux:** So what about the flux? What even is flux, anyway? Flux measures how much of a vector field flows through a surface. Imagine a surface sitting in a flowing river - flux tells you the total amount of water passing through that surface per unit time. So, the flux $\Phi$ is given be the component of your vector field perpendicular to the surface, $E_\perp$, multiplied by the area of the surface, $A$.

$$\Phi = E_\perp \times A$$

(A little more precisely, it's the integral of your vector field (e.g. the electric field, which is a vector field) dotted with the surface's normal vector over the entire area. The dot product captures the idea that flow perpendicular to the surface contributes fully to flux, while flow parallel to it contributes nothing. But this is overkill for our purposes.)

In the special case of a vector field of magnitude $E$ which is completely perpendicular to a surface of area $A$, the flux $\Phi$ just becomes $\Phi = E \times A$. And if the electric field is tangent to the surface, $\Phi = 0$. In general, $\Phi = E \times A \cos\theta$, where $\theta$ is the angle between the electric field and the normal to the surface (confirm that this agrees with our special cases!). However, in the vast majority of physics olympiad problems using Gauss' law, we are never going to consider this angle, because we are always going to pick Gaussian surfaces such that the electric field is either tangent or perpendicular to the surface (otherwise, thing would get frustratingly complicated to calculate, for no good reason).

So, let's calculate the flux through our Gaussian surface. Looking at figure 1, we know that we have an electric field (the magnitude of which we can call $E$) streaming out from the point charge, flowing through the sphere. And in fact, we observe (by spherical symmetry) that the electric field must be completely in

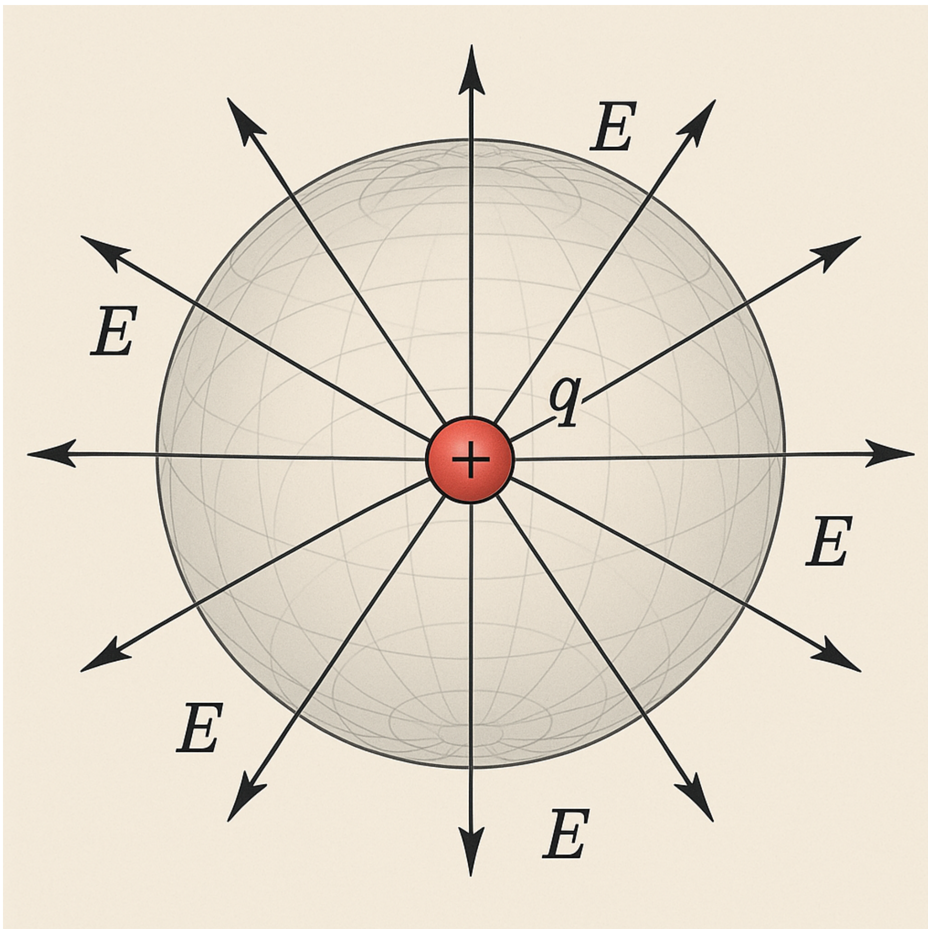

Figure 1: A Gaussian surface

the radial direction at all points. This, in turn, means that the electric field is always perpendicular to our Gaussian sphere (verify this for youreIf!). Thus, the flux through our Gaussian sphere just becomes:

$$\Phi = E \times 4\pi r^2$$

where $4\pi r^2$ is there area of the Gaussian sphere. Plugging our expression for the enclosed charge and the electric flux into Gauss' law, we find:

$$\frac{q}{\epsilon_0} = E \times 4\pi r^2$$

$$E = \frac{q}{4\pi r^2}$$

which is exactly what we get from Coulomb's law. Voilà!

There are many cases where using Gauss' law is much more convenient than Coulomb's law. These are typically cases where the problem has some particular symmetry, such as spherical symmetry in the problem above, or other kinds of symmetry in the problems below. The algorithm for finding electric fields using Gauss' law is usually the following:

1. Identify a Gaussian surface. This is the trickiest part - if you get this right, the rest is usually easy. A good Gaussian surface should:
   (a) Be simple. It should basically always be either a cylinder or a sphere (depending on whether the problem is spherically or cylindrically symmetric).
   (b) At every face of the Gaussian surface, the electric field should be either completely perpendicular to the face of the surface (making flux easy to calculate) or completely tangential (making the flux 0).
2. Calculate the charge enclosed.
3. Calculated the flux through the Gaussian surface in terms of the electric field $E$ (the magnitude of which is the only unknown) and other geometric variables (often there is some distance $d$ or $r$ involved, which parametrizes your surface).
4. Equate the charge enclosed with the flux thorugh the surface (Gauss' law), and solve for $E$ in terms of the enclosed charge and your geometric parameters.

# 2 Questions

1. Calculate the electric field $E$ at a perpendicular distance $d$ from an infinite line of linear charge density $\lambda$.
2. Calculate the electric field $E$ at a perpendicular distance $d$ from an infinite sheet of surface charge density $\sigma$.
3. Show, using Gauss' law, that the electric field outside of a uniformly charge spherical shell of charge $q$ is the same as the electric field of a unit charge $q$ placed in the center of the sphere.
4. Calculate the electric field $E$ inside of a uniformly charged spherical shell of charge $Q$.
5. As mentioned in last week's handout on *Electrodynamics*, the governing law of electricity, Coulomb's law, is functionally equivalent to the governing law of gravitational mechanics. That means there should exist a Gauss' law for gravity. Derive it.
6. In this question, you will derive the expression for the *capacitance* of a parallel plate capacitor of area $A$, with distance $d$ between the plates.
   (a) Calculate the electric field inside of the capacitor when it has a charge $Q$.

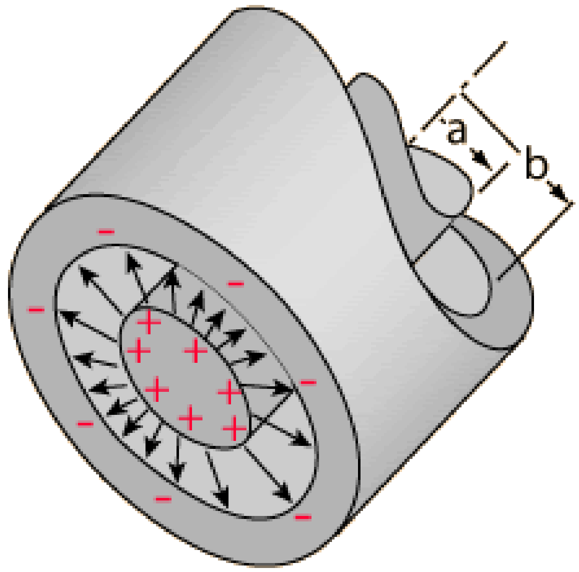


Figure 2: Cylindrical capacitor

(b) Hence, calculate the voltage difference between the two capacitor plates. *Hint: You may want to recall that in general the voltage difference between two point $x_1$ and $x_2$ $V = \int_{x_1}^{x_2} E\ dx$.*

(c) Hence, express the capacitance $C = Q/V$ in terms of $A$, $d$ and $\epsilon_0$.

7. Now, consider drawing a small Gaussian cylinder in the gap between the two parallel capacitor plates in question 4. There is no charge enclosed in this Gaussian cylinder, so the electric flux through it must be 0. Hence, the electric field must be 0. However, we found in question 5 that the electric field was not, in general, equal to 0. What's gone wrong?

8. Find the capacitance of a coaxial, cylindrical capacitor (see figure 2) of outer radius $b$, inner radius $a$. *Hint: Follow the same methodology as in question 4.*

9. Consider a solid sphere of uniform charge density. Find the ratio of the electrostatic potential at the surface to that at the center. (BAUPC).

## Answers

1. $E(d) = \dfrac{\lambda}{2\pi\varepsilon_0 d}$
2. $E = \dfrac{\sigma}{2\varepsilon_0}$
3. $E(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2} \quad (r > R)$
4. $E = 0 \quad (r < R)$
5. $\oint \mathbf{g} \cdot d\mathbf{S} = -4\pi G M_{\text{enc}}$
6. (a) $E = \dfrac{Q}{\varepsilon_0 A}$

   (b) $V = \dfrac{Qd}{\varepsilon_0 A}$

   (c) $C = \dfrac{\varepsilon_0 A}{d}$
7. $\Phi = EA - EA = 0 \quad \text{while} \quad E \neq 0$
8. $C' = \dfrac{2\pi\varepsilon_0}{\ln(b/a)} \quad \text{or} \quad C = \dfrac{2\pi\varepsilon_0 L}{\ln(b/a)}$
9. $\dfrac{V_{\text{surface}}}{V_{\text{center}}} = \dfrac{2}{3}$