Journal of Computational and Applied Mathematics 426 (2023) 115102

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A fractional-order model to study the dynamics of the spread of crime

Sugandha Arora *, Trilok Mathur, Kamlesh Tiwari

Birla Institute of Technology and Science Pilani, India

ARTICLE INFO

Article history:
Received 14 August 2022
Received in revised form 7 December 2022

Keywords:
Crime transmission modeling
Recidivism
Caputo's fractional-order derivative
Reproduction number
Stability analysis
Lyapunov function

ABSTRACT

Numerous crucial factors and parameters influence the dynamic process of the spread of crime. Various integer-order differential models have been proposed to capture crime spread. Most of these introduced dynamic systems have not considered the history of the criminal and the impact of crime on society. To address these shortcomings, a fractional-order crime transmission model is proposed in this manuscript considering five different classes *viz* law-abiding citizens, non-incarcerated criminals, incarcerated criminals, prison-released and recidivists. The primary focus of the proposed model is to study the effect of recidivism in society and decide the adequate imprisonment for repeat offenders. The existence, uniqueness, non-negativity and boundedness of the solution of the proposed model are examined. The local stability of the equilibrium points is also analysed using Routh–Hurwitz Criteria with Matignon conditions. Further, the threshold condition for the uniform asymptotic stability of the system is evaluated by the Lyapunov stability method. Moreover, the long-term impact of the imprisonment of criminals on society is also examined in the current study. The numerical simulations of the model for a range of fractional orders are obtained using power series expansion method to strengthen the theoretical results.

1. Introduction

There is no such simple or universal definition of crime. For various purposes, legitimate definitions of crime have been stipulated. The most widely held belief is that crime is a legal category, that is, something is a crime if it is proclaimed as such by the relevant and applicable legislation. Human trafficking, theft, fraud, kidnapping, rape, conspiracy, first-degree murder, domestic abuse, child abuse are major crimes. It has been analyzed that inequality in society and the lack of employment are also leading to an increase in materialistic crimes like vehicle theft, robbery etc. [1]. Modeling the prevalence of crime is a multidisciplinary task as crime is a social issue. Thus it necessitates a comprehension of the social structure and economic factors promoting the dissemination of crime.

Several models for analyzing the spread of crime have been discussed in [2]. 'Broken window theory' is the most widely adopted criminological theory, introduced by Wilson and Kelling [3]. It states that multiple broken windows are very likely to occur due to a single damaged window. They also linked the broken window to the insufficient monitoring of the target population. As a result of this negligence, the rate of crime rises. They have used a variety of examples and anecdotes to demonstrate the notion. They have explained how minor features of urban chaos can lead to anti-social

* Corresponding author.
E-mail addresses: p20180024@pilani.bits-pilani.ac.in (S. Arora), tmathur@pilani.bits-pilani.ac.in (T. Mathur), kamlesh.tiwari@pilani.bits-pilani.ac.in (K. Tiwari).

https://doi.org/10.1016/j.cam.2023.115102

behavior and perilous crimes. Blumstein's work is focused on modeling the process of recidivism with the help of criminal data [4]. They have proposed the modification of deployment of police forces and accordingly advised the incarceration policymakers to reduce crime. In general, it is believed that the increase in prison length leads to a decrease in the rate of crime spread. But it has been reviewed and concluded that length of imprisonment deters crime but weakly [5]. This means that a specific optimal value of the length of the prison exists, which can reduce the crime spread. Thus models are now developed to obtain an optimal prison length [6]. Over the past few years, several mathematical models have been introduced to control and comprehend criminal activities [7–10]. For example, the evolution of regional heterogeneity in criminal behavior was studied using innovative reaction–diffusion systems [11,12]; approaches based on game theory have been applied to better understand criminal behavior [13–15]; and differential systems have been developed to model the dynamics of the spread of crime in a society [16–18]. Extensive use of statistical tools has also been seen in the analysis of criminal data [19,20].

Although numerous factors influence crime transmission in society, this is evident that it is propagating like an infectious disease. There is a spike in the number of criminal activities, thus motivating the jurisdiction's design of several new policies. To control the spread of crime, it is required to analyze the factors leading to its propagation including the several past stages of criminal behavior. Very few models [21,22] have considered the history of criminals and criminal justice, which is one of the significant factors that decide the future state of crime in society. In these works, fractional differential equation-based systems have been developed to model crime propagation. It is known that fractional derivatives can effectively model the dynamic systems because of their memory property and non-local behavior [23,24]. Non-locality means that the future state of the function is determined by the current and past behavior of the function. In addition, the models based on fractional differentiation provide an additional degree of freedom due to different fractional order. Further, it is noteworthy that fractional-order derivatives model the physical engineering processes superior to that of ordinary derivatives [25].

Pritam et al. [21] proposed an essential 3-D fractional-order crime transmission model. Based on crime involvement and imprisonment, along with the consideration of the criminal history of any member of the population, they divided the population into three compartments, *viz*. non-criminal population, criminal population, and imprisoned population. They used phase-plane analysis to evaluate the equilibria of the model and employed the Lyapunov function to determine the threshold conditions. Bansal et al. [22] proposed a 4-D fractional-order model for analyzing the spread of crime with an additional factor of time delay between the occurrence of criminal activity by an individual and their imprisonment. In addition to criminal, non-criminal, and imprisoned, they considered the population released from prison separately. They have analyzed the stability of equilibria of the delayed model. The existing crime transmission fractional-order differential models have not considered the possibility of recidivism in society. It has been observed that past offenders are relapsing into crime either due to ineffective or severe punishments [26]. Hence, it is important to evaluate the adequate subjection amount of 'stick' and 'carrot' on criminals [27]. In other words, in addition to imprisoning criminals with appropriate prison length, there is a need to rehabilitate and integrate past offenders into society. To address the aforementioned issues, a fractional-order crime transmission model has been developed to analyze crime spread in the presence of recidivists. The contributions of the manuscript are highlighted below:

1. In this work, a 5-D fractional-order crime dissemination model has been developed to comprehend the complexity of criminal activity and behaviors in society. The history dependencies of crime transmission in the society are modeled by exploiting the memory property of fractional derivatives. The total population has been divided into criminal, non-criminal, imprisoned, prison-released, and recidivists.
2. Matignon criteria and Routh–Hurwitz criteria are employed for the analysis of equilibrium points of the model and for evaluating the threshold conditions. The extended Lyapunov function approach is also used to determine the global stability of the fractional-order system, and the corresponding threshold condition for a crime-free society. Results show that the fractional-order crime dissemination model is uniformly asymptotic stable.
3. The primary purpose of crime modeling here is to help understand the facts and analyse how crime rates are expected to vary when specific parameters are modified in the presence of recidivists. The ablation study has been performed to obtain adequate imprisonment, especially for repeat offenders.
4. Numerical simulations are also carried out for the stability analysis of the model with different fractional-orders α. The memory effect due to different orders is clearly observed in the spread of crime and thus on the population of criminals. Numerical results show that the derivative order α can play the role of precautionary measure against crime transmission and ineffective punishments.

The rest of this paper is structured as follows: Section 2 consists of the basic definitions and background of fractional derivatives along with theorems that are being used in the rest of the sections; Section 3 describes the proposed fractional-order crime dissemination model; The existence, uniqueness, non-negativity and boundedness of the solution are proved in Section 4; Section 5 explains the theoretical results related to the stability of crime-free and endemic equilibrium of the proposed model. The stability analysis of endemic equilibrium is done with the help of numerical simulations in Section 6; The Ablation study is discussed in Section 7 accompanied by the conclusion and future scope.

2. Preliminaries

Fractional calculus is a branch of mathematics that deals with the integration and differentiation of non-integer order. The memory property of fractional-order derivatives and differential equations [25] motivates the application of fractional calculus to model real-life problems. Several works and monographs clearly explain the theoretical and application details of fractional operators [28,29]. Fractional derivatives has been used to model several communicable diseases like Ebola [30], Dengue [31], Tuberculosis [32], Hepatitis [33], Covid-19 [34,35], HIV/AIDS [36], Cancer [37] and other contagious phenomenon [23,24,38]. In this section, the versions of fractional derivatives and integrals which are frequently used in the fractional-order modeling, and the theorems and lemmas which are used in further sections are presented.

2.1. Riemann–Liouville (R–L) fractional integral

For a piece-wise continuous function g on (a, ∞) which is integrable on any finite sub-interval of $[a, \infty)$, the fractional-order integral of order α (Re $(\alpha) > 0$) is defined as,

$$ {}^{RL}_{a}D_t^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} g(\xi)\, d\xi \tag{1} $$

where $t > a$ and $t, a \in \mathbb{R}$. The above expression becomes improper integral for $0 <$ Re $(\alpha) < 1$, it is required for the function g to be piece-wise continuous only on (a, t) for the accommodation of functions like $(t-a)^{\mu}$ for $-1 < \mu < 0$. The above expression depicts Riemann version of fractional integral if $a = 0$ and Liouville version if $a = -\infty$ [39]. This expression is not defined for α being a negative integer or zero. Thus this does not work for regular differentiation and separate expressions for fractional differentiation have been introduced.

2.2. Riemann–Liouville (R–L) fractional derivative

This variant of fractional derivative is the natural generalization of the regular differentiation. The R–L fractional derivative of order α is evaluated as

$$ {}^{RL}_{a}D_t^{\alpha} g(t) = D^n \left[{}^{RL}_{a}D_t^{-(n-\alpha)} g(t) \right], $$

i.e.

$$ {}^{RL}_{a}D_t^{\alpha} g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t \dfrac{g(\xi)}{(t-\xi)^{\alpha+1-n}}\, d\xi, & n-1 < \alpha < n \\ \dfrac{d^n}{dt^n} g(t), & \alpha = n \end{cases} \tag{2} $$

where $\alpha > 0$, $t > a$ and $n \in \mathbb{N}$ [39]. For integer n such that $n-1 < \alpha < n$, the $(n-\alpha)$ order integral is differentiated n times in order to evaluate the α-order derivative of the function. The main disadvantage of aforementioned definition is that the R–L derivative of a constant is non-zero. To overcome the limitations of this definition, Caputo [40] proposed another way of evaluating fractional derivative.

2.3. Caputo's fractional derivative

The Caputo's fractional derivative for function g of order α is evaluated as

$$ {}^{C}_{a}D_t^{\alpha} g(t) = {}_{a}D_t^{-(m-\alpha)} \left[D^m g(t) \right], $$

i.e.

$$ {}^{C}_{a}D_t^{\alpha} g(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t \dfrac{g^{(m)}(\xi)}{(t-\xi)^{\alpha+1-m}}\, d\xi, & m-1 < \alpha < m \\ \dfrac{d^m}{dt^m} g(t), & \alpha = m \end{cases} \tag{3} $$

where $\alpha > 0$, $t > a$ and $m \in \mathbb{N}$ [40]. The Caputo's fractional derivative of a constant is equal to zero, thus this property increases the applicability of fractional derivative in solving real-world problems. Due to this practical applicability, this version of fractional operator is used in this study.

2.4. Grunwald letnikov derivative

Anton Karl Grünwald and Aleksey Vasilievich Letnikov has given the limit definition of fractional derivative in 1867 and 1868 respectively [41]. Without any assumptions on differentiability of the function for $\alpha \ : \ n-1 < \alpha < n > 0$, the GL derivative of order α for any function $g(t)$ is expressed as follows

$$ {}_0D_t^{\alpha} g(t) = \lim_{\substack{h\to 0 \\ nh=t}} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} g(t-rh) \tag{4}$$

where $\binom{\alpha}{r} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1).\Gamma(r+1)}$, h is the step size and $\Gamma(.)$ is the Gamma function (extension of factorial function for non-integers).

2.5. Memory property

The first-order derivative can be computed using only two points as shown below

$$ \frac{dg(t)}{dt} = \lim_{\Delta t\to 0} \frac{g(t)-g(t-\Delta t)}{\Delta t}, \quad t > 0. \tag{5}$$

Thus, it is clear from the definition that it can measure instantaneous change and the system will exhibit short-term memory. On the contrary, fractional derivative considers the function's behavior on all previous states t to a. So, fractional derivative is non-local [42–44].

Theorem 2.1. *Consider the following initial value problem*

$$ {}_0^CD_t^{\alpha} = g(t, x(t)), \qquad x^{(k)}(a) = x_k \tag{6}$$

where $k = 0, 1, \ldots, \lceil\alpha\rceil - 1$ for some $\nu > 0$ on the interval $[a, T]$ and ${}_0^CD_t^{\alpha}$ denotes the Caputo fractional differential operator of order α, and if $g : [a, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function which satisfies the Lipschitz condition (8) with respect to the second variable, i.e.,

$$ |g(y, x_1) - g(y, x_2)| \le l|x_1 - x_2| \tag{7}$$

for some constant $l > 0$. Then, the fractional order initial value problem has a unique continuous solution on $[a, T]$ [45].

Theorem 2.2. *Let x_0 be an equilibrium point of the non-autonomous fractional-order dynamical system where x_0 belongs to domain $\Phi \subset \mathbb{R}$. Then if for any continuously differentiable function $F(x, t) : \Phi \times [\,0, \infty) \to \mathbb{R}$, there exists continuous positive definite functions $F_1(x), F_2(x), F_3(x)$ defined on domain Φ such that $F_1(x) \le F(x, t) \le F_2(x)$ and ${}_0^CD_t^{\alpha}\ F(x, t) \le -F_3(x)$ for every $\alpha \in (0, 1)$. Then the equilibrium point x_0 is uniformly asymptotically stable [45,46].*

3. Five dimensional fractional-order crime dissemination model

Crime spreads in society like a communicable disease. A person with a criminal background may affect or influence a person with no criminal background. Thus it is required to imprison the criminals to control the spread of crime for some period of time. But after imprisonment, they get released and then can assimilate back into the society or can again indulge in criminal activities due to ineffective or severe punishments. Hence, it is required to keep a check on the criminals released from the prison and then further on the recidivists. For modeling the dynamics of the crime spread in the society, these five sub-populations are needed to be focused and the policies are to designed for each set of population separately such as different rehabilitation programs, different prison length *etc*. Unlike existing crime transmission differential models, the developed fractional dynamic system (8) helps in the investigation of historic and simultaneous effects of imprisonment, length of prison, recidivism, contagion, history of criminals, and jurisdiction on crime transmission. The conventional integer-order crime transmission model is taken from [6], and the fractionalizing procedure given by Dokoumetzidis et al. [47] is adopted for developing the proposed crime transmission fractional differential model and for analyzing the dynamical aspects of the final fractionalized differential system. The proposed fractional-order differential system demonstrates crime propagation in the society based on an individual's criminal record and imprisonment by classifying the total population into the following classes:

- A : Law abiding people
- O_1 : Non-Incarcerated Criminals
- P : Incarcerated criminals
- O_2 : Repeat offenders
- R : Released people

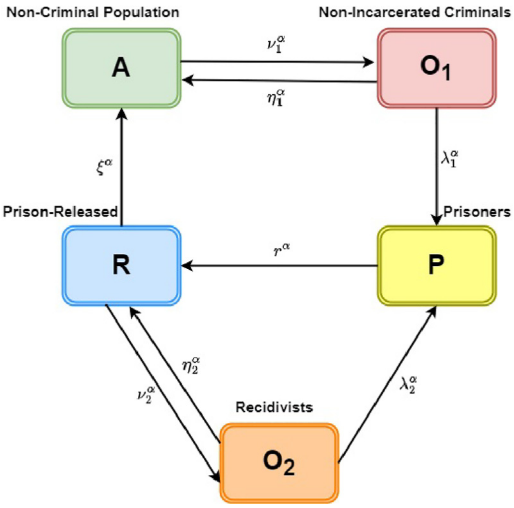

Fig. 1. Schematic Diagram of the Fractional-order Crime Dissemination Model: Five sub-populations *viz.* Non-Criminals, Non-Incarcerated Criminals, Prisoners, Recidivists and Prison-released are shown in green, red, yellow, orange, and blue colored boxes respectively. The flow of population to other compartments are shown by arrows with rate of flow over it.

The system (8) represents Caputo's version-based fractional-order crime transmission model with five classes where α is the fractional-order which lies in $(0, 1]$. The dimension of all the population-based parameters A, O_1, P, O_2, R and N used in the proposed fractional crime transmission model are aligned with the conventional integer-order crime transmission model [6]. Therefore, the incoming and outgoing flux for a class are not violating the population balance. It can be also seen from the L.H.S. of the system that the fractionalized sytem satisfies the power-law kind of system dynamics and the population of the classes are in fractional time with dimension $t^{-\alpha}$. Further, every parameter in the R.H.S. of the system has power α to keep the system dimensionally balanced because the rate of change of any population concerning time has dimension t^{-1}. Fig. 1 displays the flow of population between the bifurcated classes. See Table 1 for the description of population flow rates in between classes.

$$\begin{aligned}
\frac{d^\alpha A}{dt^\alpha} &= \eta_1^\alpha O_1 - \nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)} + \xi^\alpha R \\
\frac{d^\alpha O_1}{dt^\alpha} &= -\eta_1^\alpha O_1 + \nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)} - \lambda_1^\alpha O_1 \\
\frac{d^\alpha P}{dt^\alpha} &= \lambda_1^\alpha O_1 + \lambda_2^\alpha O_2 - r^\alpha P \\
\frac{d^\alpha R}{dt^\alpha} &= r^\alpha P + \eta_2^\alpha O_2 - \nu_2^\alpha R - \xi^\alpha R \\
\frac{d^\alpha O_2}{dt^\alpha} &= \nu_2^\alpha R - \lambda_2^\alpha O_2 - \eta_2^\alpha O_2 \\
N &= A + O_1 + O_2 + P + R
\end{aligned} \tag{8}$$

The equation first of the system (8) depicts the flux of non-criminal class A. The law-abiding citizens can indulge in criminal activities due to social interaction with other criminals at rate ν_1. The term $\nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)}$ captures the change in population due to contact of innocent citizens with other criminals. People indulging in criminal activity for the first time can move back to the non-criminal class without punishment, at rate η_1, and the term $\eta_1^\alpha O_1$ captures this flux. The first-time offenders are caught at the incarceration rate λ_1. Eq. (2) is the sum of terms relating to a contagion effect, non-contagion effect, resistance, and incarceration, which depicts the flux of criminal population/ offenders. Eq. (3) of the system captures the change in the prison population occurring due to imprisonment of first-time and repeat offenders and the release of prisoners after punishment at a rate r. Eq. (4) captures the change in the current prison-released population. The prison-released people can indulge in criminal activities again at rate ν_2 due to their natural tendency, independent of any interactions with other criminals. The term $\nu_2^\alpha A$ captures the change in population due to the non-contagion effects. The prison-released people move back to the society at rate η_2 after punishment and rehabilitation programs, the term $\eta_2^\alpha O_2$ captures this flux. The repeat offenders are caught at the incarceration rate λ_2. The last equation of the system shows the assumption that the total population remains constant. It can be observed that all the equations of the system are dimensionally balanced and thus, the model is well-posed.

Table 1
Parameter elucidation: parameterization of the population flows.

Parameter	Elucidation
ν_1	crime indulgence rate due to social interactions
η_1	rate of assimilating back in society from criminal class O_1
λ_1	law-enforcement rate
r	prison-release rate
ν_{20}	rate of crime indulgence of prison-released people
η_2	rate at which recidivists become criminally-inactive
λ_2	law-enforcement rate on recidivists
ξ	rate of moving back to society after releasing from prison

4. Analysis of solution

This section shows that the solutions of the system are not only existent but also unique, non-negative and bounded. Let $X(t)=[A(t),O_1(t),P(t),R(t),O_2(t)]^T$ and $\Omega_+=\{X(t)\in\Omega\subseteq\mathbb{R}^5 : X(t)\geq 0\}$.

4.1. Existence and uniqueness of solution

Theorem 4.1. *If $X(t)=(A(t),O_1(t),P(t),R(t),O_2(t))$ there exists a unique solution of system (8) with initial condition $X(t_0=0)$.*

Proof. Let

$$\begin{aligned}
\frac{d^\alpha A}{dt^\alpha} &= \eta_1^\alpha O_1-\nu_1^\alpha\frac{A(O_1+O_2)}{(N-P)}+\xi^\alpha R = f_1(t,A(t),O_1(t),P(t),R(t),O_2(t))\\
\frac{d^\alpha O_1}{dt^\alpha} &= -\eta_1^\alpha O_1+\nu_1^\alpha\frac{A(O_1+O_2)}{(N-P)}-\lambda_1^\alpha O_1 = f_2(t,A(t),O_1(t),P(t),R(t),O_2(t))\\
\frac{d^\alpha P}{dt^\alpha} &= \lambda_1^\alpha O_1+\lambda_2^\alpha O_2-r^\alpha P = f_3(t,A(t),O_1(t),P(t),R(t),O_2(t))\\
\frac{d^\alpha R}{dt^\alpha} &= r^\alpha P+\eta_2^\alpha O_2-\nu_2^\alpha R-\xi^\alpha R = f_4(t,A(t),O_1(t),P(t),R(t),O_2(t))\\
\frac{d^\alpha O_2}{dt^\alpha} &= \nu_2^\alpha R-\lambda_2^\alpha O_2-\eta_2^\alpha O_2 = f_5(t,A(t),O_1(t),P(t),R(t),O_2(t))
\end{aligned}\tag{9}$$

Being the total population to be constant, the sub-population $A(t),O_1(t),P(t),R(t),O_2(t)$ are bounded and clearly, $f_i(t,A(t),O_1(t),P(t),R(t),O_2(t))$ are continuous on $(0,\infty)$. Thus $f_i(t,A(t),O_1(t),P(t),R(t),O_2(t))$ are bounded on $(0,\infty)$. Consider

$$\begin{aligned}
&|f_1(t,A,O_1,P,R,O_2)-f_1(t,A',O_1',P',R',O_2')|\\
&=\left|\left(\eta_1^\alpha O_1-\nu_1^\alpha\frac{A(O_1+O_2)}{(N-P)}+\xi^\alpha R\right)-\left(\eta_1^\alpha O_1'-\nu_1^\alpha\frac{A(O_1'+O_2')}{(N-P')}+\xi^\alpha R'\right)\right|\\
&=\left|\left(\eta_1^\alpha(O_1-O_1')-\nu_1^\alpha\left[\frac{A(O_1+O_2)}{(N-P)}-\frac{A'(O_1'+O_2')}{(N-P')}\right]+\xi^\alpha(R-R')\right)\right|\\
&=\left|\left(\eta_1^\alpha(O_1-O_1')-\nu_1^\alpha\left[\frac{A(N-P-R)}{(N-P)}-\frac{A'(N-P'-R')}{(N-P')}\right]+\xi^\alpha(R-R')\right)\right|\\
&\leq \eta_1^\alpha|O_1-O_1'|+\nu_1^\alpha|A-A'|+\xi^\alpha|R-R'|\\
&\leq \max\{\eta_1^\alpha,\nu_1^\alpha,\xi^\alpha\}\left(|A-A'|+|R-R'|+|O_1-O_1'|\right)\\
&\leq L\|X-X'\|
\end{aligned}\tag{10}$$

Thus, $f_1(t,A,O_1,P,R,O_2)$ satisfies Lipschitz condition. Similarly, other $f_i(t,A,O_1,P,R,O_2)$ are also Lipschitz continuous. Using Theorem 2.1 and [48], it can be easily concluded that the system (8) has unique continuous solution on $(0,\infty)$. □

4.2. Non-negativity and boundedness of solution

Theorem 4.2. *The solutions of system (8) are non-negative and bounded, if they start in Ω_+.*

Proof. From the model (8), we find that

$$\begin{aligned}
{}_0^C D_t^\alpha A(t)|_{A=0} &= \eta_1^\alpha O_1 + \xi^\alpha R \geq 0 \\
{}_0^C D_t^\alpha O_1(t)|_{O_1=0} &= \nu_1^\alpha \frac{AO_2}{(N-P)} \geq 0 \\
{}_0^C D_t^\alpha P(t)|_{P=0} &= \lambda_1^\alpha O_1 + \lambda_2^\alpha O_2 \geq 0 \\
{}_0^C D_t^\alpha R(t)|_{R=0} &= r^\alpha P + \eta_2^\alpha O_2 \geq 0 \\
{}_0^C D_t^\alpha O_2(t)|_{O_2=0} &= \nu_2^\alpha R \geq 0
\end{aligned} \tag{11}$$

Thus the sub-populations $A(t), O_1(t), P(t), R(t)$ and $O_2(t)$ are non-negative. As $A(t) + O_1(t) + P(t) + R(t) + O_2(t) = N$, where the total population N is considered to be constant, each sub-population lies in $[0, N]$. Hence the sub-populations $A(t), O_1(t), P(t), R(t)$ and $O_2(t)$ are bounded as well. □

5. Equilibrium points and their stability

In this section, crime-free equilibrium E_0 and positive equilibrium E^* are evaluated and their asymptotic stability has been analysed. The total population is assumed to be constant, thus the system (8) is reduced to a four-dimensional fractional-order system by putting $A = N - (O_1 + O_2 + P + R)$. Firstly, the stability of E_0 and E^* are analysed using Routh–Hurwitz criteria [49,50] with Matignon conditions [51]. Further, the extended Lyapunov function approach [45,46] has been employed on the obtained four-dimensional fractional system for verifying the threshold conditions.

5.1. Crime free equilibrium

In this section, threshold conditions have been derived for the global asymptotic stability of the crime-free equilibrium of the system using Matignon conditions [50,51] and Routh–Hurwitz Criteria [49] and then by using extended Lyapunov function method [45,46]. The crime-free equilibrium point E_0 for the system denotes the steady state solution when there is no criminal population

$$O_1 = P = O_2 = R = 0. \tag{12}$$

From the system (9), consider the following equations with $A = N - (O_1 + P + R + O_2)$

$$\begin{aligned}
\frac{d^\alpha O_1}{dt^\alpha} &= -\eta_1^\alpha O_1 + \nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)} - \lambda_1^\alpha O_1 = f_2(t, A(t), O_1(t), P(t), R(t), O_2(t)) \\
\frac{d^\alpha P}{dt^\alpha} &= \lambda_1^\alpha O_1 + \lambda_2^\alpha O_2 - r^\alpha P = f_3(t, A(t), O_1(t), P(t), R(t), O_2(t)) \\
\frac{d^\alpha R}{dt^\alpha} &= r^\alpha P + \eta_2^\alpha O_2 - \nu_2^\alpha R - \xi^\alpha R = f_4(t, A(t), O_1(t), P(t), R(t), O_2(t)) \\
\frac{d^\alpha O_2}{dt^\alpha} &= \nu_2^\alpha R - \lambda_2^\alpha O_2 - \eta_2^\alpha O_2 = f_5(t, A(t), O_1(t), P(t), R(t), O_2(t))
\end{aligned} \tag{13}$$

5.1.1. Stability analysis of E_0 using matignon criteria

To evaluate the stability of the steady-state E_0, we compute the Jacobian matrix $J(E_0)$ as below

$$J^0 = \begin{bmatrix} \frac{\partial f_2}{\partial O_1} & \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial O_2} \\ \frac{\partial f_3}{\partial O_1} & \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial R} & \frac{\partial f_3}{\partial O_2} \\ \frac{\partial f_4}{\partial O_1} & \frac{\partial f_4}{\partial P} & \frac{\partial f_4}{\partial R} & \frac{\partial f_4}{\partial O_2} \\ \frac{\partial f_5}{\partial O_1} & \frac{\partial f_5}{\partial P} & \frac{\partial f_5}{\partial R} & \frac{\partial f_5}{\partial O_2} \end{bmatrix}_{E_0} = \begin{bmatrix} -{\eta_1}^\alpha - {\lambda_1}^\alpha + {\nu_1}^\alpha & 0 & 0 & {\nu_1}^\alpha \\ {\lambda_1}^\alpha & -r^\alpha & 0 & {\lambda_2}^\alpha \\ 0 & r^\alpha & -{\nu_2}^\alpha - \xi^\alpha & {\eta_2}^\alpha \\ 0 & 0 & {\nu_2}^\alpha & -{\lambda_2}^\alpha - {\eta_2}^\alpha \end{bmatrix}$$

From row 1, first eigen value of J^0, $e_1 = -{\eta_1}^\alpha - {\lambda_1}^\alpha + {\nu_1}^\alpha$. For other eigen values, consider the following reduced forms of J^0 as follows:

$$\begin{bmatrix} -r^\alpha & 0 & {\lambda_2}^\alpha - \frac{{\lambda_1}^\alpha {\nu_1}^\alpha}{-{\eta_1}^\alpha - {\lambda_1}^\alpha + {\nu_1}^\alpha} \\ r^\alpha & -{\nu_2}^\alpha - \xi^\alpha & {\eta_2}^\alpha \\ 0 & {\nu_2}^\alpha & -{\lambda_2}^\alpha - {\eta_2}^\alpha \end{bmatrix} \rightarrow \begin{bmatrix} -r^\alpha & 0 & {\lambda_2}^\alpha - \frac{{\lambda_1}^\alpha {\nu_1}^\alpha}{-{\eta_1}^\alpha - {\lambda_1}^\alpha + {\nu_1}^\alpha} \\ 0 & -{\nu_2}^\alpha - \xi^\alpha & {\eta_2}^\alpha + {\lambda_2}^\alpha - \frac{{\lambda_1}^\alpha {\nu_1}^\alpha}{-{\eta_1}^\alpha - {\lambda_1}^\alpha + {\nu_1}^\alpha} \\ 0 & {\nu_2}^\alpha & -{\lambda_2}^\alpha - {\eta_2}^\alpha \end{bmatrix}$$

From here it can be seen that the second eigenvalue, $e_2 = -r^\alpha$ with real part less than 0. Now, consider the trace and determinant of the following reduced matrix

$$V = \begin{bmatrix} -{\nu_2}^\alpha - \xi^\alpha & {\eta_2}^\alpha + {\lambda_2}^\alpha - \frac{{\lambda_1}^\alpha {\nu_1}^\alpha}{-{\eta_1}^\alpha - {\lambda_1}^\alpha + {\nu_1}^\alpha} \\ {\nu_2}^\alpha & -{\lambda_2}^\alpha - {\eta_2}^\alpha \end{bmatrix}$$

$$e_3 + e_4 = \mathrm{Trace}(V) = -{\nu_2}^{\alpha} - \xi^{\alpha} - {\lambda_2}^{\alpha} - {\eta_2}^{\alpha} < 0 \tag{14}$$

$$\mathrm{Det}(V) = ({\nu_2}^{\alpha} + \xi^{\alpha})({\lambda_2}^{\alpha} + {\eta_2}^{\alpha}) - {\nu_2}^{\alpha}\left({\eta_2}^{\alpha} + {\lambda_2}^{\alpha} - \frac{{\lambda_1}^{\alpha}{\nu_1}^{\alpha}}{-{\eta_1}^{\alpha} - {\lambda_1}^{\alpha} + {\nu_1}^{\alpha}}\right) \tag{15}$$

For the eigen values with -ve real parts, their product will be positive, thus $e_3 \cdot e_4 = \mathrm{Det}(V) > 0$. Therefore from (15), we get the following:

$$({\nu_2}^{\alpha} + \xi^{\alpha})({\lambda_2}^{\alpha} + {\eta_2}^{\alpha}) > {\nu_2}^{\alpha}\left({\eta_2}^{\alpha} + {\lambda_2}^{\alpha} - \frac{{\lambda_1}^{\alpha}{\nu_1}^{\alpha}}{-{\eta_1}^{\alpha} - {\lambda_1}^{\alpha} + {\nu_1}^{\alpha}}\right) \tag{16}$$

$$\frac{{\nu_2}^{\alpha}{\lambda_1}^{\alpha}{\nu_1}^{\alpha}}{\xi^{\alpha}({\lambda_2}^{\alpha} + {\eta_2}^{\alpha})({\eta_1}^{\alpha} + {\lambda_1}^{\alpha} - {\nu_1}^{\alpha})} < 1 \implies \frac{\nu_1^{\alpha}}{\lambda_1^{\alpha} + \eta_1^{\alpha}} + \frac{\nu_1^{\alpha}\lambda_1^{\alpha}(\nu_2^{\alpha})}{(\lambda_2^{\alpha} + \eta_2^{\alpha})(\lambda_1^{\alpha} + \eta_1^{\alpha})\xi^{\alpha}} < 1 \tag{17}$$

From the above equation, we get $\frac{\nu_1^{\alpha}}{(\lambda_1^{\alpha} + \eta_1^{\alpha})} < 1 \implies \nu_1^{\alpha} < \lambda_1^{\alpha} + \eta_1^{\alpha} \implies \mathrm{Re}\{e_1\} < 0$. The expression in the above equation represents the threshold condition for the fractional-order system (13), which can be denoted by $\mathcal{R}_0$ as follows:

$$\mathcal{R}_0 = \left(\frac{\nu_1^{\alpha}}{\lambda_1^{\alpha} + \eta_1^{\alpha}}\right) + \frac{\nu_1^{\alpha}\lambda_1^{\alpha}(\nu_2^{\alpha})}{(\lambda_2^{\alpha} + \eta_2^{\alpha})(\lambda_1^{\alpha} + \eta_1^{\alpha})\xi^{\alpha}} < 1 \tag{18}$$

All the eigen values have negative real parts, *i.e.*, $|arg(e_i)| > \frac{\alpha\pi}{2}$ for $0 < \alpha < 1$ when $\mathcal{R}_0 < 1$. So the system satisfies Matignon conditions [51] and by Routh–Hurwitz criteria [49,50] it can be concluded that $E_0 = (0, 0, 0, 0)$ of the system (13) is locally asymptotically stable when $\mathcal{R}_0 < 1$. $(O_1^0, P^0, R^0, O_2^0) \to (0, 0, 0, 0)$ as $t \to \infty$. So the crime-free equilibrium point E_0 is globally asymptotically stable for $\mathcal{R}_0 < 1$. Thus, we have the following theorem.

Theorem 5.1. *If $\mathcal{R}_0 < 1$, equilibrium point $E_0 = (0, 0, 0, 0)$ of the system (13) is globally asymptotically stable, else it is unstable.*

5.1.2. Derivation of the threshold using Lyapunov method

Extended Lyapunov function approach [45,46] has been employed on four-dimensional fractional dynamical system (13) obtained by putting $A = N-(O_1+O_2+P+R)$ in (8). Consider the following Lyapunov function for the fractional-order system (13)

$$L = O_1 + C_1P + C_2R + C_3O_2 \tag{19}$$

where C_1, C_2, C_3 are positive constants and they will be chosen later such that the solutions of the fractional system propagate to lower level sets and hence the solution of the system tends to origin.

$$ {}_0^C D_t^{\alpha} L = {}_0^C D_t^{\alpha} O_1 + C_1 {}_0^C D_t^{\alpha} P + C_2 {}_0^C D_t^{\alpha} R + C_3 {}_0^C D_t^{\alpha} O_2 \tag{20}$$

$$\begin{aligned} &= \nu_1^{\alpha}\frac{A(O_1 + O_2)}{(N-P)} - \lambda_1^{\alpha}O_1 - \eta_1^{\alpha}O_1 \\ &\quad + C_1\lambda_1^{\alpha}O_1 + C_1\lambda_2^{\alpha}O_2 - C_1r^{\alpha}P \\ &\quad + C_2r^{\alpha}P + C_2\eta_2^{\alpha}O_2 - C_2\nu_2^{\alpha}R - C_2\xi^{\alpha}R \\ &\quad - C_3\lambda_2^{\alpha}O_2 + C_3\nu_2^{\alpha}R - C_3\eta_2^{\alpha}O_2 \end{aligned} \tag{21}$$

$$= O_1\left[\nu_1^{\alpha}\left(\frac{A}{N-P}\right) - (\lambda_1^{\alpha} + \eta_1^{\alpha}) + C_1\lambda_1^{\alpha}\right] \tag{22}$$

$$+ P\left[(C_2 - C_1)r^{\alpha}\right] \tag{23}$$

$$+ R\left[-C_2(\nu_{20}^{\alpha} + \xi^{\alpha}) + C_3\nu_2^{\alpha}\right] \tag{24}$$

$$+ O_2\left[\nu_1^{\alpha}\left(\frac{A}{N-P}\right) + (C_1 - C_3)\lambda_2^{\alpha} + (C_2 - C_3)\eta_2^{\alpha}\right] \tag{25}$$

We look for the positive constants C_1, C_2, C_3 in such a way that the values in the square brackets in Eqs. (22)–(25) remain negative, i.e., ${}_0^C D_t^{\alpha} L < 0$. Thus for term in square brackets of (22):

$$\nu_1^{\alpha}\left(\frac{A}{N-P}\right) - (\lambda_1^{\alpha} + \eta_1^{\alpha}) + C_1\lambda_1^{\alpha} \le \nu_1^{\alpha} - (\lambda_1^{\alpha} + \eta_1^{\alpha}) + C_1\lambda_1^{\alpha} \le 0 \tag{26}$$

requiring

$$0 < C_1 \le \frac{\lambda_1^{\alpha} + \eta_1^{\alpha} - \nu_1^{\alpha}}{\lambda_1^{\alpha}} \implies \nu_1^{\alpha} < \lambda_1^{\alpha} + \eta_1^{\alpha} \tag{27}$$

For the term in square brackets of (23) to be negative requires the following:

$$C_2 < C_1 \tag{28}$$

For the negativity of the term in (24), the following should hold:

$$\begin{aligned}-C_2(\nu_2^\alpha + \xi^\alpha) + C_3\nu_2^\alpha &< -C_2(\nu_{20}^\alpha + \xi^\alpha) + C_3\nu_{20}^\alpha \\ &= C_3(\nu_2^\alpha) - C_2(\nu_2^\alpha + \xi^\alpha) < 0\end{aligned} \tag{29}$$

$$\implies C_3 < C_2\left(\frac{\nu_2^\alpha + \xi^\alpha}{\nu_2^\alpha}\right) \tag{30}$$

And using (28), for the negativity of the term in square brackets of (25):

$$\begin{aligned}\nu_1^\alpha\left(\frac{A}{N-P}\right) + (C_1 - C_3)\lambda_2^\alpha + (C_2 - C_3)\eta_2^\alpha &\le \nu_1^\alpha - (C_1 - C_3)\lambda_2^\alpha + (C_2 - C_3)\eta_2^\alpha \\ &< \nu_1^\alpha + (C_1 - C_3)(\lambda_2^\alpha + \eta_2^\alpha) < 0\end{aligned} \tag{31}$$

$$C_1 + \frac{\nu_1^\alpha}{\lambda_2^\alpha + \eta_2^\alpha} < C_3 \tag{32}$$

Now, combining (28), (30) and (32)

$$C_1 + \frac{\nu_1^\alpha}{\lambda_2^\alpha + \eta_2^\alpha} < C_3 < C_2\left(\frac{\nu_2^\alpha + \xi^\alpha}{\nu_2^\alpha}\right) < C_1\left(\frac{\nu_2^\alpha + \xi^\alpha}{\nu_2^\alpha}\right) \tag{33}$$

$$\implies \frac{\nu_1^\alpha}{\lambda_2^\alpha + \eta_2^\alpha} < C_1 \cdot \frac{\xi^\alpha}{\nu_2^\alpha} \tag{34}$$

$$\implies \frac{\nu_1^\alpha(\nu_2^\alpha)}{\xi^\alpha(\lambda_2^\alpha + \eta_2^\alpha)} < C_1 \tag{35}$$

Combining (27) and (34), we get:

$$\frac{\nu_1^\alpha(\nu_2^\alpha)}{\xi^\alpha(\lambda_2^\alpha + \eta_2^\alpha)} < C_1 < \frac{\lambda_1^\alpha + \eta_1^\alpha - \nu_1^\alpha}{\lambda_1^\alpha} \tag{36}$$

There exists a positive constants C_1 and suitable C_2, C_3 which satisfy (36), if and only if the following holds:

$$\frac{\nu_1^\alpha(\nu_2^\alpha)}{\xi^\alpha(\lambda_2^\alpha + \eta_2^\alpha)} < \frac{\lambda_1^\alpha + \eta_1^\alpha - \nu_1^\alpha}{\lambda_1^\alpha} \tag{37}$$

$$\implies \frac{\nu_1^\alpha}{\lambda_1^\alpha + \eta_1^\alpha} + \frac{\nu_1^\alpha\lambda_1^\alpha(\nu_2^\alpha)}{(\lambda_2^\alpha + \eta_2^\alpha)(\lambda_1^\alpha + \eta_1^\alpha)\xi^\alpha} < 1 \tag{38}$$

The expression (38) is same as the threshold condition (18) obtained from the Routh–Hurwitz Criteria with Matignon conditions. For $\mathcal{R}_0 < 1$, the Lyapunov function L with suitable C_1, C_2&C_3 is bounded for the constant population N and the Caputo's fractional derivative of L is negative *i.e.*, ${}_0^C D_t^\alpha L < 0$ with ${}_0^C D_t^\alpha L = 0$ if and only if $O_1 = P = O_2 = R = 0$. Thus this makes the function L suitable as a Lyapunov candidate for the model and it can be concluded from Theorem 2.2 that our fractional crime transmission model is uniformly asymptotic stable [45,46]. Thus we have the following theorem.

Theorem 5.2. *If $\mathcal{R}_0 < 1$, equilibrium point $E_0 = (0, 0, 0, 0)$ of the system (13) is uniformly asymptotically stable, else it is unstable.*

Reproduction Number ($\mathcal{R}_0$). The expression in Eq. (38) represents the threshold condition for the fractional-order system (13), which is denoted by $\mathcal{R}_0$. The inequality $\mathcal{R}_0 < 1$ behaves as a separating border for crime-free equilibrium cases and the solution tending to endemic equilibrium point with the crime. The $\mathcal{R}_0$ is the reproduction number, representing the number of citizens a criminal can influence to indulge in crime. The first term of the expression (18) is dedicated to first-time offenders O_1. The numerator is the crime indulgence rate and the denominator comprises the sum of the law-enforcement rate and desistance rate. The numerator in the second term of the expression (18), is the product of the crime indulgence rate of first-time offenders, the crime indulgence rate of repeat offenders and the law-enforcement rate on first-time offenders. The second term of the expression (18) is dedicated to repeat offenders O_2. The denominator of the second term of the expression (18) is the product of three terms responsible for leaving the repeat-offender class O_2. It is the combination (sum/product) of the law-enforcement rate and desistance rate of repeat-offenders and the redemption rate of prison-released criminals. Hence for crime-free equilibrium, the numerator of $\mathcal{R}_0$ should be less than the denominator.

5.2. Endemic equilibrium in the system

The endemic equilibrium point $E^* = (O_1^*, P^*, R^*, O_2^*)$ is steady state solution of system (13) which is obtained when $O_1 > 0$ and, it yields the following equations:

$$\begin{aligned} -\eta_1^\alpha O_1 + \nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)} - \lambda_1^\alpha O_1 &= 0 \\ \lambda_1^\alpha O_1 + \lambda_2^\alpha O_2 - r^\alpha P &= 0 \\ r^\alpha P + \eta_2^\alpha O_2 - \nu_2^\alpha R - \xi^\alpha R &= 0 \\ \nu_2^\alpha R - \lambda_2^\alpha O_2 - \eta_2^\alpha O_2 &= 0 \end{aligned} \tag{39}$$

The above system can be simplified as:

$$\begin{aligned} \frac{A}{(N-P)} &= \frac{\eta_1^\alpha + \lambda_1^\alpha}{\nu_1^\alpha} \frac{O_1}{(O_1+O_2)} \\ r^\alpha P &= \lambda_1^\alpha O_1 + \lambda_2^\alpha O_2 \\ r^\alpha P + \eta_2^\alpha O_2 &= (\nu_2^\alpha + \xi^\alpha) R \\ \nu_2^\alpha R &= (\lambda_2^\alpha + \eta_2^\alpha) O_2 \end{aligned} \tag{40}$$

From the last three equations of the above simplified system, we obtain $O_1 = \frac{\xi^\alpha}{\lambda_1^\alpha} R$ and $O_2 = \frac{\nu_2^\alpha}{(\lambda_2^\alpha + \eta_2^\alpha)} R$. Putting this in equation first of system (39), we get

$$\frac{A}{(N-P)} = \frac{\eta_1^\alpha + \lambda_1^\alpha}{\nu_1^\alpha} \frac{1}{1 + \frac{\nu_2^\alpha \lambda_1^\alpha}{(\eta_2^\alpha + \lambda_2^\alpha)\xi^\alpha}} \tag{41}$$

Now, using Eq. (18), we get

$$\mathcal{R}_0 = \frac{\nu_1^\alpha}{\eta_1^\alpha + \lambda_1^\alpha} \left(1 + \frac{\nu_2^\alpha \lambda_1^\alpha}{(\eta_2^\alpha + \lambda_2^\alpha)\xi^\alpha}\right) = \frac{(N-P)}{A} \tag{42}$$

Thus at the endemic equilibrium of the system, we have

$$\frac{A}{(N-P)} = \frac{1}{\mathcal{R}_0} \tag{43}$$

$$\implies O_1 + O_2 + A + R = N - P = A \cdot \mathcal{R}_0 \implies O_1 + O_2 + R = A(\mathcal{R}_0 - 1) \tag{44}$$

As $O_1 > 0,\ O_2 > 0, A > 0 \& R > 0$, therefore $\mathcal{R}_0 > 1$. Moreover, the prevalence of equilibrium $\frac{1}{\mathcal{R}_0}$ of non-criminals can be interpreted as the ratio of the rate of discontinuing criminal activities and the rate of involvement in illegal activities. Thus E^* equilibrium occurs if $\mathcal{R}_0 > 1$. The endemic equilibrium $E^* = (O_1^*, P^*, R^*, O_2^*)$ where

$$O_1^* = \frac{\xi^\alpha}{\lambda_1^\alpha} R^*, \qquad O_2^* = \frac{\nu_2^\alpha}{(\lambda_2^\alpha + \eta_2^\alpha)} R^*, \qquad P^* = N - \left(\frac{\mathcal{R}_0}{\mathcal{R}_0 - 1}\right)\left[\frac{\xi^\alpha}{\lambda_1^\alpha} + \frac{\nu_2^\alpha}{\lambda_2^\alpha + \eta_2^\alpha} + 1\right] R^*,$$

$$R^* = N \frac{\lambda_1^\alpha \xi^\alpha (\lambda_2^\alpha + \eta_2^\alpha)(\mathcal{R}_0 - 1)}{(\mathcal{R}_0 - 1)\lambda_1^\alpha \left[\nu_2^\alpha \lambda_2^\alpha + \xi^\alpha(\eta_2^\alpha + \lambda_2^\alpha)\right] + \mathcal{R}_0 \xi^\alpha \left[(\lambda_2^\alpha + \eta_2^\alpha)\xi^\alpha + \nu_2^\alpha \lambda_1^\alpha + \lambda_1^\alpha(\lambda_2^\alpha + \eta_2^\alpha)\right]}.$$

5.2.1. Stability analysis of E^* using Routh–Hurwitz criteria

Theorem 5.3. *The endemic equilibrium point $E^* = (O_1^*, P^*, R^*, O_2^*)$ for system (13) exists and is locally asymptotically stable if and only if $\mathcal{R}_0 > 1$.*

Proof. To evaluate the stability of the endemic equilibrium E^*, we compute the Jacobian matrix $J(E^*)$ as below

$$J^* = \begin{bmatrix} \frac{\partial f_2}{\partial O_1} & \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial O_2} \\ \frac{\partial f_3}{\partial O_1} & \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial R} & \frac{\partial f_3}{\partial O_2} \\ \frac{\partial f_4}{\partial O_1} & \frac{\partial f_4}{\partial P} & \frac{\partial f_4}{\partial R} & \frac{\partial f_4}{\partial O_2} \\ \frac{\partial f_5}{\partial O_1} & \frac{\partial f_5}{\partial P} & \frac{\partial f_5}{\partial R} & \frac{\partial f_5}{\partial O_2} \end{bmatrix}_{E^*} = \begin{bmatrix} a & b & c & d \\ e & f & 0 & h \\ 0 & j & k & l \\ 0 & 0 & n & o \end{bmatrix}$$

where $a = -\eta_1^\alpha - \lambda_1^\alpha + \frac{\nu_1^\alpha}{\mathcal{R}_0} + c$, $b = -c\left[\frac{1}{\mathcal{R}_0} - 1\right]$, $c = -\nu_1^\alpha \left[\frac{\xi^\alpha}{\lambda_1^\alpha} + \frac{\nu_2^\alpha}{(\lambda_2^\alpha + \eta_2^\alpha)}\right] \frac{R^*}{N - P^*}$, $d = \frac{\nu_1^\alpha}{\mathcal{R}_0} + c$,
$e = {\lambda_1}^\alpha, f = -r^\alpha$, $h = {\lambda_2}^\alpha$, $j = r^\alpha$, $k = -{\nu_2}^\alpha - \xi^\alpha$, $l = {\eta_2}^\alpha$, $n = {\nu_2}^\alpha$, $o = -{\lambda_2}^\alpha - {\eta_2}^\alpha$.

The characteristic polynomial of the above Jacobian matrix is

$$m^4 + K_1 m^3 + K_2 m^2 + K_3\ m + K_4 = 0 \tag{45}$$

where

$$K_1 = -a - f - k - o$$

$$K_2 = af - be + ak + ao + fk + fo + ko - ln$$

$$K_3 = bek - afk - cej - afo + beo - ako + aln - fko + fln - hjn$$

$$K_4 = afko - afln + ahjn - beko + beln + cejo - dejn$$

Hence, by using the Routh–Hurwitz criterion [49,50], if the following conditions:

$$K_1 > 0;\ \ K_4 > 0;\ \ K_1K_2 - K_3 > 0;\ \ K_1K_2K_3 - K_3^2 - K_4K_1^2 > 0 \tag{46}$$

are met, the endemic equilibrium $E^* = (O_1^*, P^*, R^*, O_2^*)$ is locally asymptotically stable. □

6. Stability analysis using numerical simulations

6.1. Analysis of endemic equilibrium for the 5D model

In this section, the stability of endemic equilibrium points of the model (8) for different fractional orders α is shown with the help of numerical simulations. The values of parameters used for the simulations are presented in Table 2. Most of the values of parameters in S_2 are fetched from [17,18,38] to illustrate the dynamics of crime. For evaluating the numerical solution of the proposed fractional model, we used the Power Series Expansion (PSE) approach described in [52,53]. In this approach, Caputo's fractional derivative is approximated with the help of Grünwald–Letnikov's definition of fractional derivative in (4) because for a broad class of functions and for $t \to \infty$, these two definitions of fractional derivatives are equivalent [54]. The Adams–Bashforth–Moulton and PSE methods are the most frequently used methods for evaluating numerical solutions of fractional models. Both methods have approximately the same accuracy in terms of solutions [52]. In this work, the PSE method is used which gives a more straightforward numerical solution for the system ${}_0^C D_t^\alpha X(t) = f(X(t), t)$ of the form

$$X(t_k) = h^\alpha\left(f(X(t_k), t_k)\right) - \sum_{j=1}^{k} c_j^{(\alpha)} X(t_{k-j}) \tag{47}$$

where $t_k = kh$, h is the time step of calculation and $(-1)^j\binom{\alpha}{j}$ are binomial coefficients $c_j^{(\alpha)}$ (j = 0, 1, ...), computed using expression (48) given in [55]

$$c_0^{(\alpha)} = 1 \qquad c_j^{(\alpha)} = (1 - \frac{1+\alpha}{j})c_{j-1}^{(\alpha)}. \tag{48}$$

Thus we get the numerical solution of the system (8) by solving the following set of equations with parameters from the two sets S_1, S_2 which are chosen randomly such that $\mathcal{R}_0 > 1$.

$$\begin{aligned}
A(t_k) &= h^\alpha\left(\eta_1^\alpha O_1 - \nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)} + \xi^\alpha R\right) - \sum_{j=1}^{k} c_j^{(\alpha)} A(t_{k-j})\\
O_1(t_k) &= h^\alpha\left(-\eta_1^\alpha O_1 + \nu_1^\alpha \frac{A(O_1+O_2)}{(N-P)} - \lambda_1^\alpha O_1\right) - \sum_{j=1}^{k} c_j^{(\alpha)} O_1(t_{k-j})\\
P(t_k) &= h^\alpha\left(\ \lambda_1^\alpha O_1 + \lambda_2^\alpha O_2 - r^\alpha P\ \right) - \sum_{j=1}^{k} c_j^{(\alpha)} P(t_{k-j})\\
R(t_k) &= h^\alpha\left(\ r^\alpha P + \eta_2^\alpha O_2 - \nu_2^\alpha R - \xi^\alpha R\ \right) - \sum_{j=1}^{k} c_j^{(\alpha)} R(t_{k-j})\\
O_2(t_k) &= h^\alpha\left(\ \nu_2^\alpha R - \lambda_2^\alpha O_2 - \eta_2^\alpha O_2\ \right) - \sum_{j=1}^{k} c_j^{(\alpha)} O_2(t_{k-j})
\end{aligned} \tag{49}$$

Table 2
Set of parameter values used for showing numerical simulations such that $\mathcal{R}_0 > 1$. Most of the values of parameters in S_2 are fetched from [17,18,38].

Parameter	ν_1	η_1	λ_1	r	ν_2	η_2	λ_2	ξ
Value Set S_1	0.05	0.045	0.6	0.01	0.3	0.03	0.07	0.1
Value Set S_2	0.7	0.4	0.4	0.6	0.6	0.2	0.5	0.7

The methodology mentioned above is also presented in Algorithm 1. The resulting equations, input variables, set of parameters, initial conditions and output variables are also mentioned. Numerical results for fractional systems (8) where α is varying from 0.7 to 1 as seen in Figs. 2, 3, 4, and 5. The simulations are shown on two sets of parameters, S_1 has mostly realistic parametric values and S_2 has random parametric values such that $\mathcal{R}_0 > 1$. It is clear from the figures that the models for fractional derivative order α $(0.7 \leq \alpha \leq 1)$ are asymptotically stable. In the previous section, it has been theoretically proved that the endemic equilibrium point E^* is locally asymptotically stable if $\mathcal{R}_0 > 1$. Thus the simulations strengthen the theoretical results. The simulations are shown with two different time frames $T_{sim} = 100$ and $T_{sim} = 500$. The model with parameters from S_1 takes longer to reach the equilibrium point than the model with parameters from S_2. The significant difference between parameters of S_1 and S_2 are the values of $\eta_1,\ \eta_2$ and λ_2, which represents the ratio of non-incarcerated criminals (first-time offender/repeat offender) assimilating back into society and law enforcement on recidivists. The set S_1 has significantly lesser values of $\eta_1,\ \eta_2$ and λ_2. This means that society with a lower rate of assimilation back into society and less law enforcement on recidivists will achieve the equilibrium later.

Figs. 6, 7, 8, and 9 show the comparison between the performance of the basic differential model and fractional model by taking $\alpha = 0.99$ and $\alpha = 0.89$. It is visible from these figures that the fractional order crime model has richer dynamics than the basic model. The fractional-order model is even converging faster than the basic one. It has been observed that the fractional order has a significant effect on the dynamic behavior of all the components. Also, it has been noticed that when the derivative order α is reduced from 1, the memory effect of the system increases, and therefore the crime spreads slowly and the number of incarcerated criminals increases for a long time. Moreover, catching criminals in society takes time and depends on the rate of law enforcement in an area. This results in an increase in the non-incarcerated criminals, fast progress of crime and thus an increase in incarcerated criminals. On the other hand, the experience or knowledge of individuals about the punishment causes non-criminals and non-incarcerated criminals to take different precautions, such as behavioral change, avoiding contact with other criminals and joining rehabilitation programs against crime transmission. This leads to a slow growth of crime among the population. Therefore, from the numerical results in Figs. 2 to 5, we conclude that the order of Caputo's derivative α in the system (8) can play the role of precautionary measure against crime transmission, punishment of criminals and delay in catching criminals. Moreover, Figs. 6 to 9 infer that the differential equations with fractional order derivatives have rich dynamics and describe criminological systems better than classical integer-order models.

7. Analysing the effects of changing different parameters

7.1. Effect of increasing prison-length

It is intriguing to notice that the threshold $\mathcal{R}_0$ and the fractions of criminally active $\dfrac{A}{N-P}$ and criminally-inactive individuals $\dfrac{O_1+O_2+R}{N-P}$ are independent of prison-release rate r, hence of prison length $l = 1/r$. However, it can be seen from the second and third equations of the system (39) that the equilibrium values of all classes A, O_1, O_2, P, R depend on r. As the prison length parameter l increases (r decreases), the equilibrium of incarcerated criminals increases, while the equilibrium population of other classes decreases irrespective of order α. As we have $\dfrac{A}{N-P} = \dfrac{1}{\mathcal{R}_0}$, which implies that flows of A and I are in opposite directions when r increases. Moreover, $O_1 = \dfrac{\xi^\alpha}{\lambda_1^\alpha}R$ and $O_2 = \dfrac{\nu_2^\alpha}{(\lambda_2^\alpha + \eta_2^\alpha)}R$ which implies that flows of O_1, O_2 are in same directions when r increases. Hence, considering $R + O_1 + O_2 = kR$ for some $k > 0$ and using $A = \dfrac{1}{\mathcal{R}_0}(N-P)$ gives the following

$$\frac{1}{\mathcal{R}_0}(N-P) = N - (O_1+O_2+R) - P = N - kR - P \tag{50}$$

$$\implies \left(1 - \frac{1}{\mathcal{R}_0}\right)N = \left(1 - \frac{1}{\mathcal{R}_0}\right)P + kR \tag{51}$$

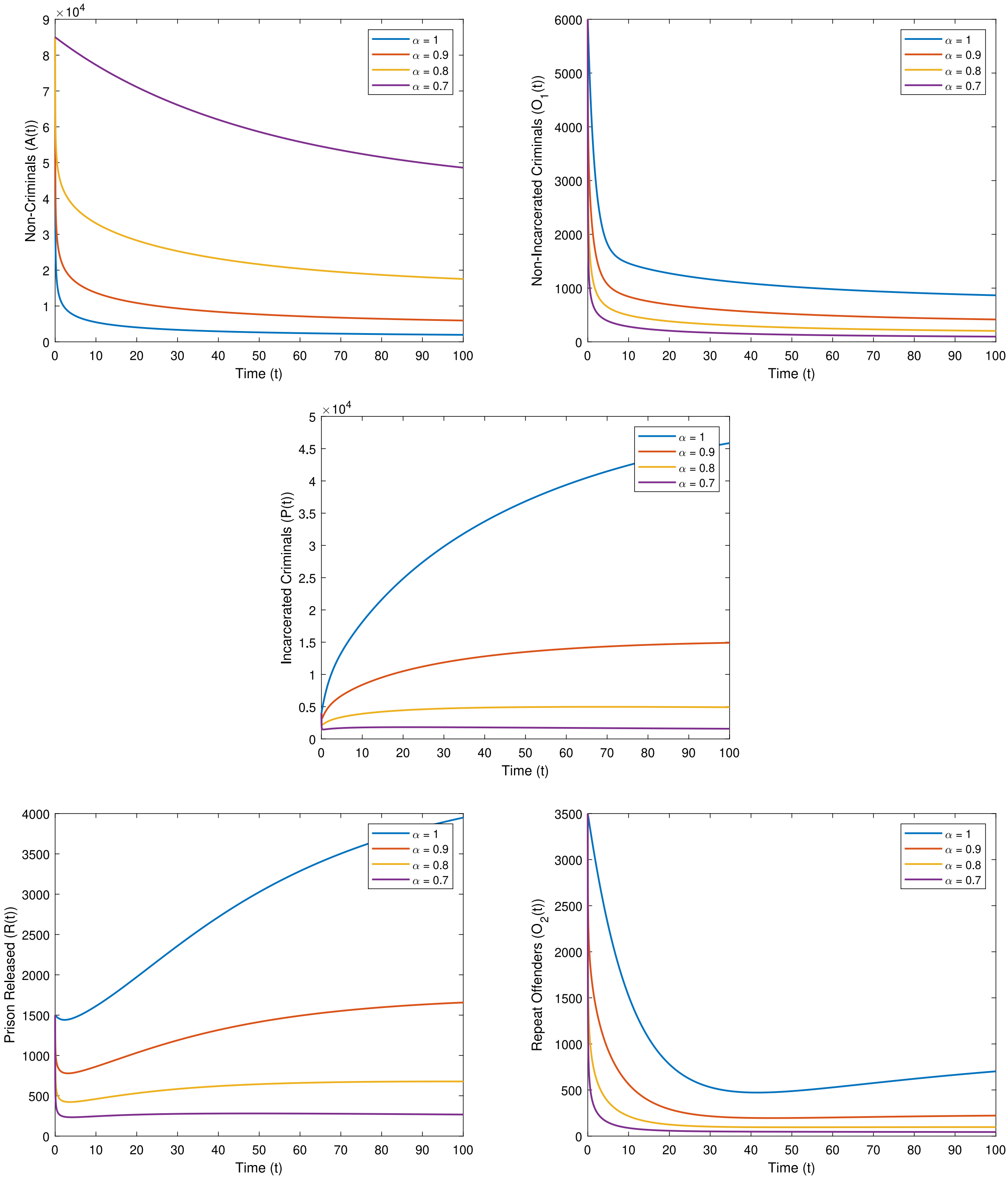

Fig. 2. Variations of Non-Criminals A, Non-Incarcerated Criminals O_1, Incarcerated Criminals P, Prison-Released population R and Repeat Offenders O_2 with time $T_{Sim} = 100$ for a set of parameters in S_1 and different order α. As the order decreases, the criminal population ($O_1(t)\&O_2(t)$) also decreases. The trajectories of all the population classes justify the stability of the proposed model irrespective of the order chosen.

As $\mathcal{R}_0 > 1$ for endemic equilibrium, the coefficients for variables on the right-hand side of the above equation are positive. Thus, P and R move in the opposite direction as r increases. In addition, the second equation of system (39), justifies that as r increases, R, O_1, O_2 increases and P decreases. Fig. 10 shows this dependence for varying prison lengths. In Fig. 10, one can see that as prison length l increases from 2 to 5, the equilibrium population of prisoners increases, and the rest of the equilibrium population decreases irrespective of order α. It can be concluded that increasing the prison length does not have much effect on the threshold condition, on the fractions of criminally active and criminally inactive

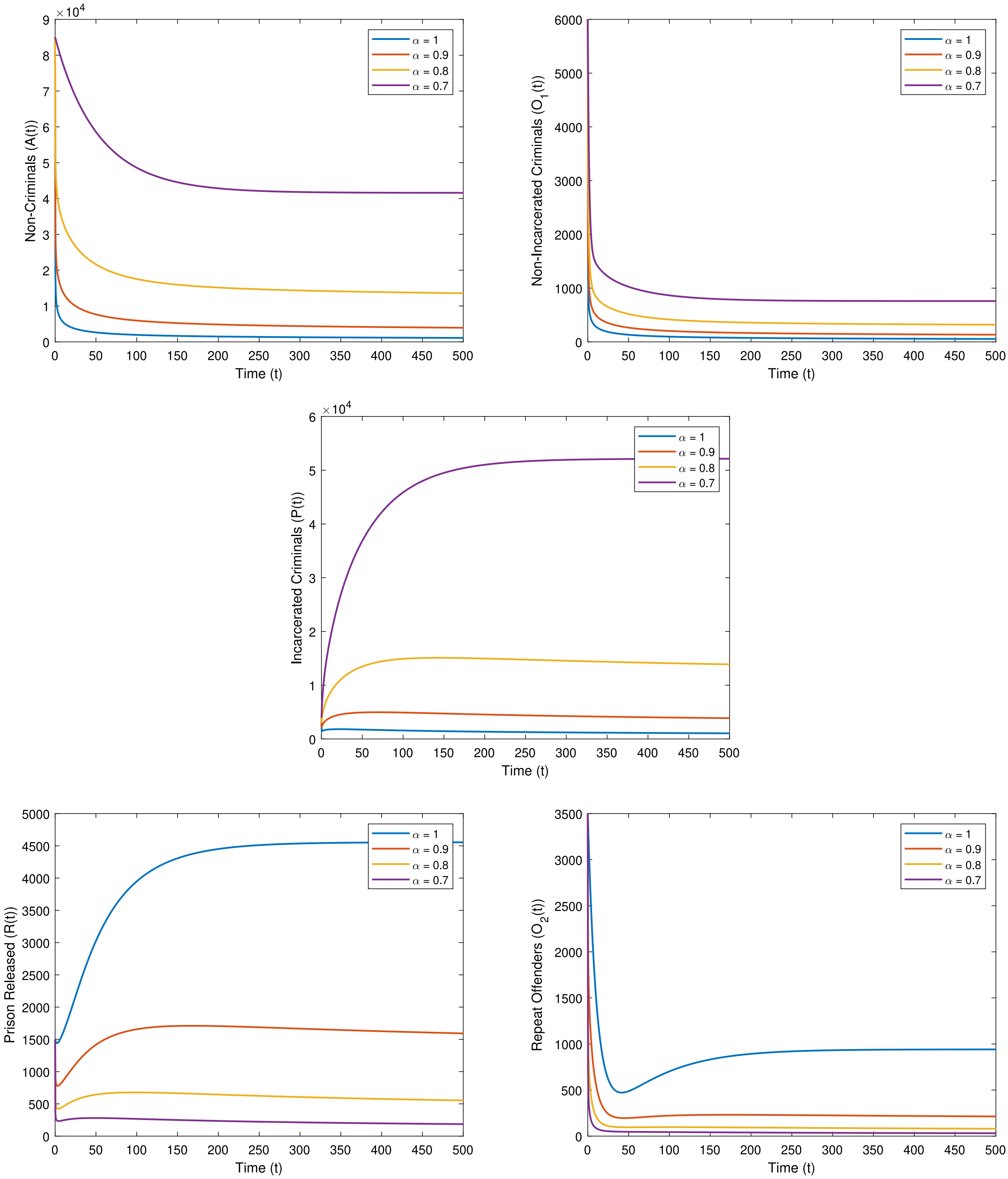

Fig. 3. Variations of Non-Criminals A, Non-Incarcerated Criminals O_1, Incarcerated Criminals P, Prison-Released population R and Repeat Offenders O_2 with time $T_{Sim} = 500$ for a set of parameters in S_1 and different order α. As order decreases, the criminal population $(O_1(t)\&O_2(t))$ also decreases. The stability of the proposed model is visible through trajectories of all the population classes irrespective of the order chosen for a longer time.

Algorithm 1 Numerical approximation of the true solution of proposed fractional model

Input Variables

```
α ← fractional order of the differential equation (0 < α ≤ 1)
X0 ← array of the initial conditions (A(1), O1(1), P(1), R(1), O2(1))
TSim ← simulation time
h ← time step size (0 < h ≤ 1)
n = round(TSim/h); % time step:
% parameters of the proposed model:
ν1 = 0.005;  η1 = 0.045;  λ1 = 0.6;  r = 0.01;  ν2 = 0.3;  η2 = 0.03;  λ2 = 0.7;  ξ = 0.1;

function [yo] = mem(r, c, p) % Memory Function
hist = 0;
for ii=1:p-1
  memory = memory + c(ii)*r(p-i);
end
yo = memory;
```

Discretization of Proposed Model using Power Series Expansion Method of Approximation:

```
% binomial coefficients calculation:
  cp1 = 1; cp2 = 1; cp3 = 1; cp4 = 1; cp5 = 1;
for j=1:n
  c1(j) = (1 − (1 + α)/j) ∗ cp1;
  c2(j) = (1 − (1 + α)/j) ∗ cp2;
  c3(j) = (1 − (1 + α)/j) ∗ cp3;
  c4(j) = (1 − (1 + α)/j) ∗ cp4;
  c5(j) = (1 − (1 + α)/j) ∗ cp5;
  cp1 = c1(j); cp2 = c2(j); cp3 = c3(j); cp4 = c4(j); cp5 = c5(j);
end
% initial conditions:
A(1) = X0(1);  O1(1) = X0(2);  P(1) = X0(3);  R(1) = X0(4);  O2(1) = X0(5);
% calculation of numerical solution of proposed model:
for i=2:n
  A(i) = (η1^α ∗ O1(i − 1) − ν1^α ∗ (O1(i − 1) + O2(i − 1)) ∗ (A(i−1)/(N−P(i−1))) + ξ^α ∗ R(i − 1)) ∗ h^α − mem(A, c1, i);
  O1(i) = (−η1^α ∗ O1(i − 1) + ν1^α ∗ (O1(i − 1) + O2(i − 1)) ∗ (A(i−1)/(N−P(i−1))) − λ1^α ∗ O1(i − 1)) ∗ h^α − mem(O1, c2, i);
  P(i) = (λ1^α ∗ O1(i) + λ2^α ∗ O2(i − 1) − r^α ∗ P(i − 1)) ∗ h^α − mem(P, c3, i);
  R(i) = (r^α ∗ P(i) + η2^α ∗ O2(i − 1) − ν2^α ∗ R(i − 1) − ξ^α ∗ R(i − 1)) ∗ h^α − mem(R, c4, i);
  O2(i) = (ν2^α ∗ R(i) − η2^α ∗ O2(i − 1) − η2^α ∗ O2(i − 1)) ∗ h^α − mem(O2, c5, i);
end
for j=1:n
  X(j, 1) = A(j);
  X(j, 2) = O1(j);
  X(j, 3) = P(j);
  X(j, 4) = R(j);
  X(j, 5) = O2(j);
end
T=h:h:TSim;
```

Output Variables

```
X ← arrays of n + 1 real numbers that contain the approximate solutions.
```

populations. It leads to a larger prison population along with a reduction of criminally active and inactive individuals. Thus it is advisable not to increase prison length, and it should be chosen according to the condition of society.

7.2. Impact of law-enforcement rate

It is evident from the expression (18) for threshold $\mathcal{R}_0$ that to achieve a crime-free society, low crime-indulgence rates (ν_1, ν_2) and high desistance rates (η_1, η_2, ξ) along with high law-enforcement rates (λ_1, λ_2) are required. However, there is a law-enforcement rate term in the numerator of $\mathcal{R}_0$, which means that over-incarceration of first-time offenders is not

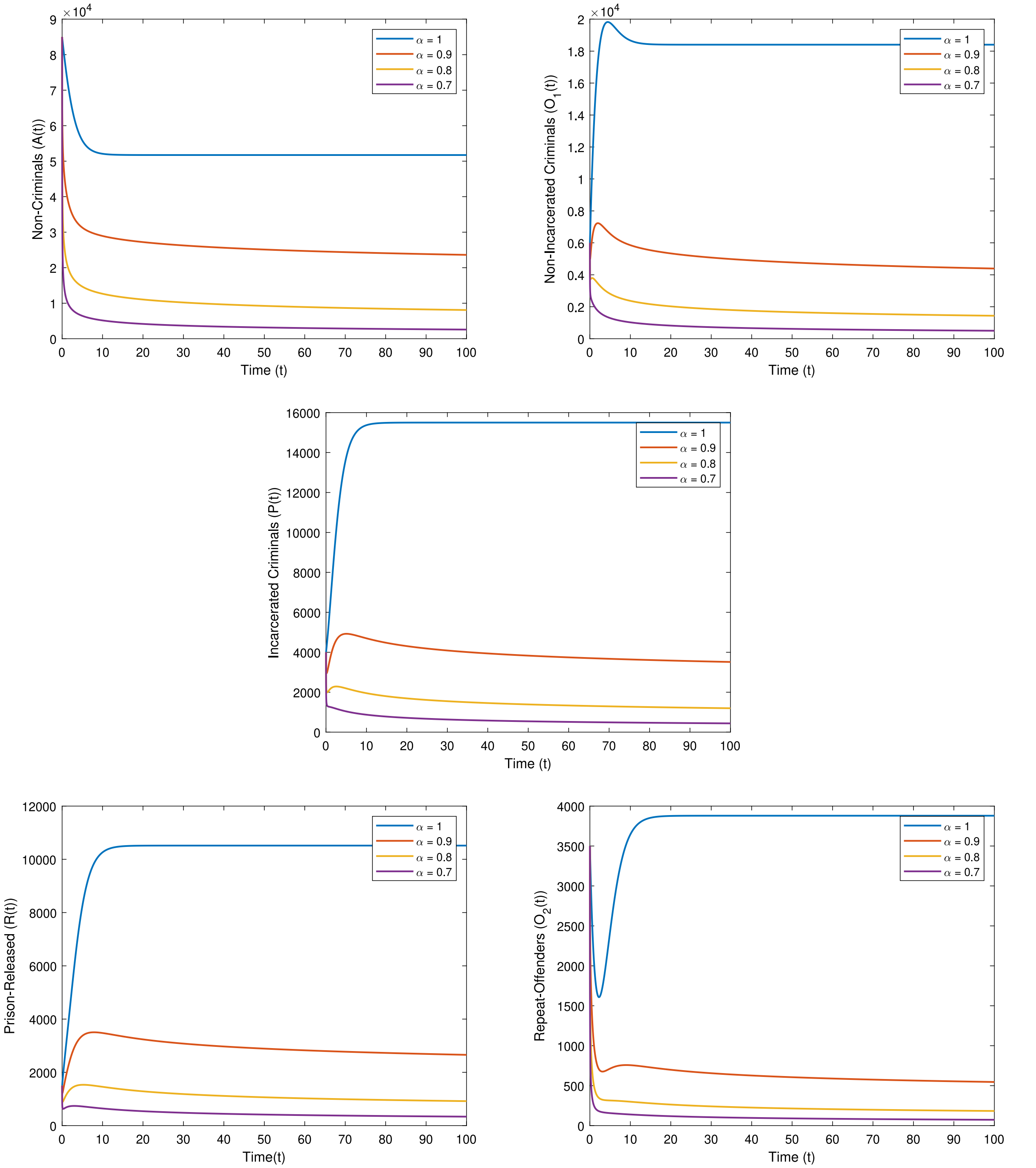

Fig. 4. Variations of Non-Criminals A, Non-Incarcerated Criminals O_1, Incarcerated Criminals P, Prison-Released population R and Repeat Offenders O_2 with time $T_{Sim} = 100$ for a set of parameters in set S_2 and different order α. As order decreases, the criminal population $(O_1(t)\&O_2(t))$ also decreases. The trajectories of all the population classes justify the stability of the proposed model, irrespective of the order chosen.

recommended and optimal values of law-enforcement rate exist. But in the denominator, the law-enforcement rate for recidivists balances the expression and suggests paying more attention to repeat-offenders than to first-time offenders. Therefore, the required imprisonment level can be evaluated by analyzing this fractional model. As we have

$$\frac{\nu_1^\alpha \nu_2^\alpha \lambda_1^\alpha}{\xi^\alpha(\eta_1^\alpha - \nu_1^\alpha + \lambda_1^\alpha)} - \eta_2^\alpha < \lambda_2^\alpha \tag{52}$$

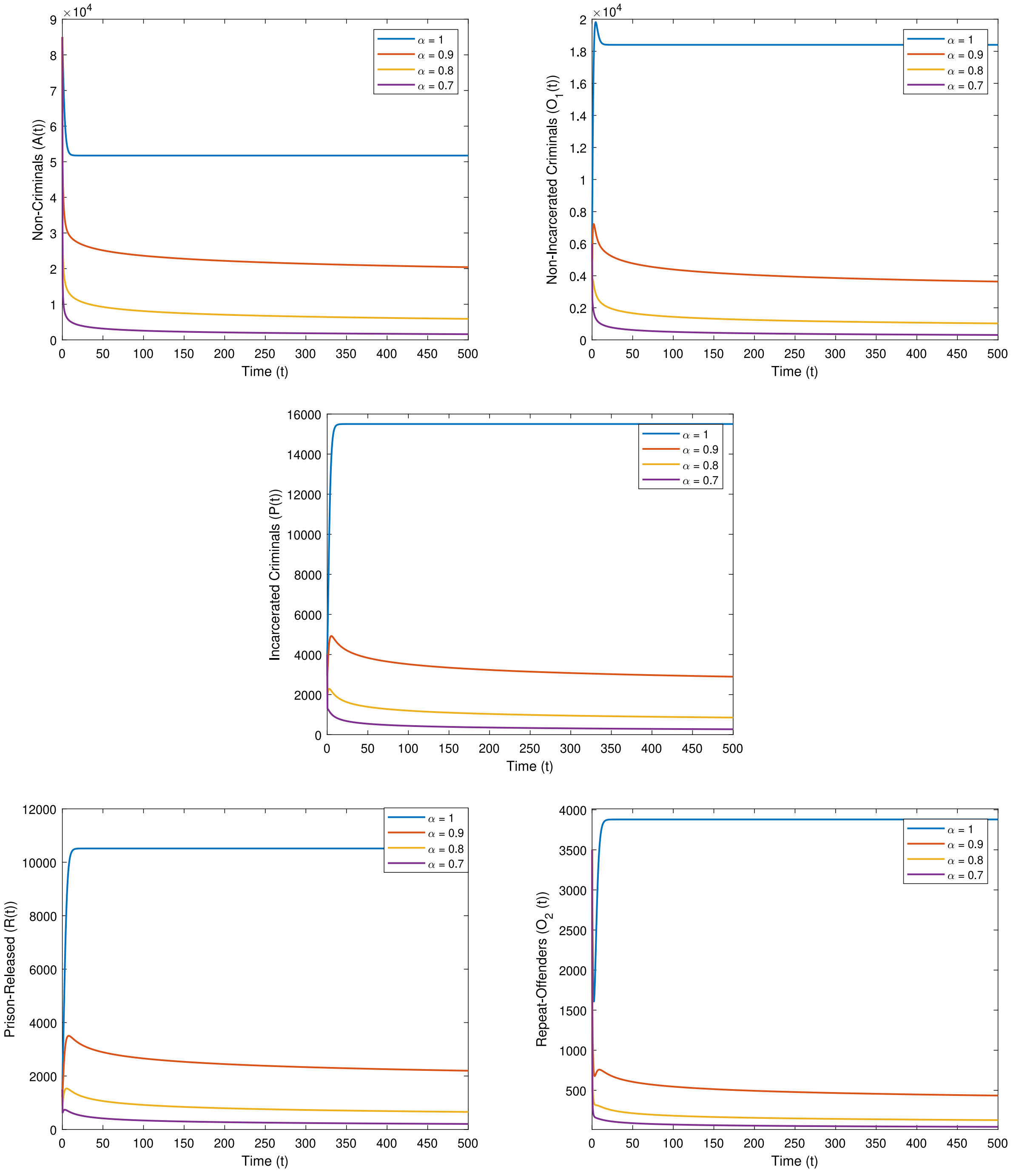

Fig. 5. Variations of Non-Criminals A, Non-Incarcerated Criminals O_1, Incarcerated Criminals P, Prison-Released population R and Repeat Offenders O_2 with time $T_{Sim} = 500$ for a set of parameters in set S_2 and different order α. As order decreases, the criminal population $(O_1(t)\&O_2(t))$ also decreases. The stability of the proposed model is visible through trajectories of all the population classes irrespective of the order chosen for a longer time.

Thus, if $\frac{\nu_1^\alpha \nu_2^\alpha \lambda_1^\alpha}{\xi^\alpha(\eta_1^\alpha - \nu_1^\alpha + \lambda_1^\alpha)} < \eta_2^\alpha$, i.e., if rate of crime indulgence is lesser than the rate of desistance of recidivists then there is no need of incarceration.

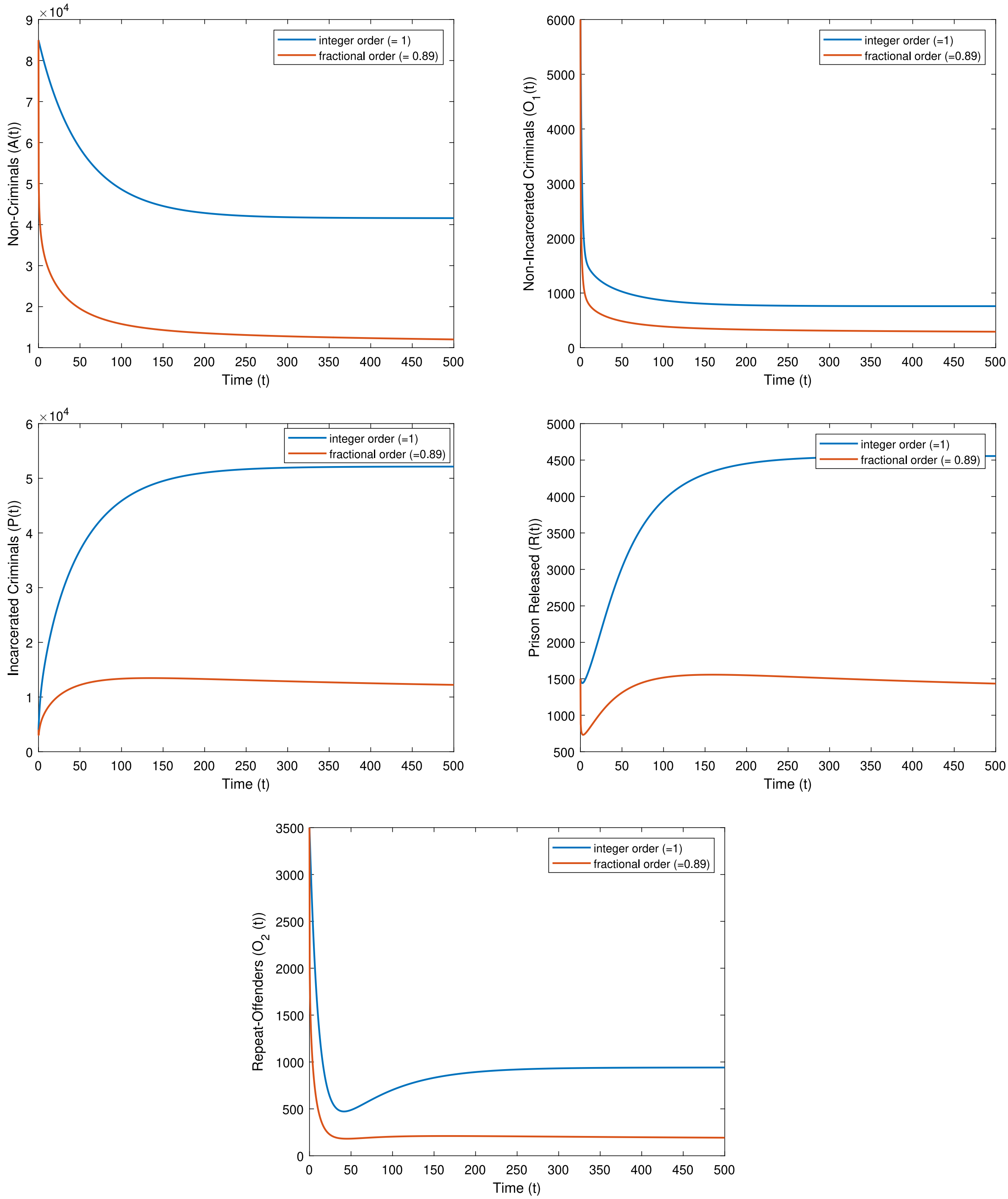

Fig. 6. Comparison between the basic differential model ($\alpha = 1$) and the fractional dynamics with $\alpha = 0.89$, when parameters are taken from set S_1. The fractional order model converges faster to the equilibrium point than the basic integer order model and has richer dynamics than the basic model.

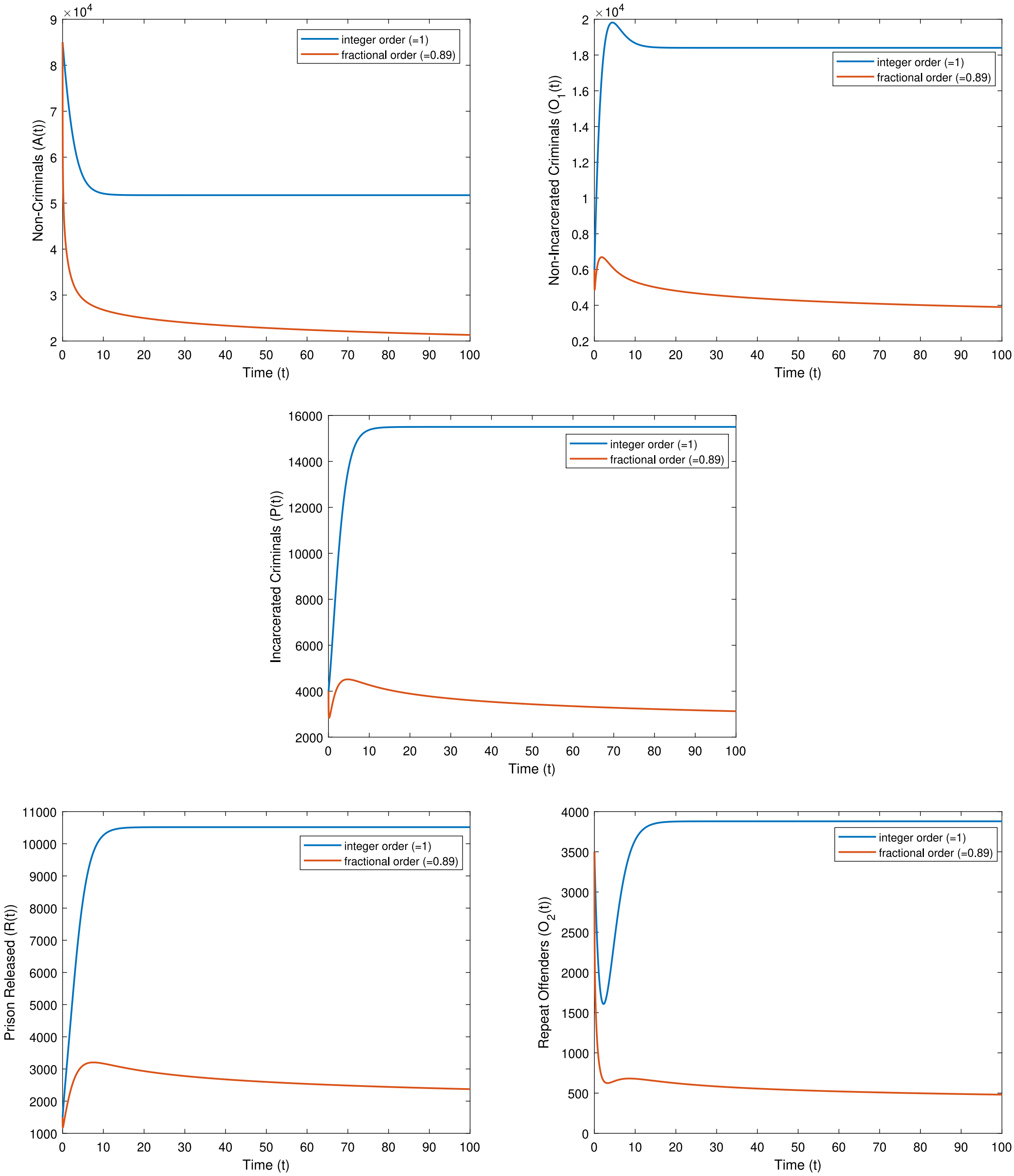

Fig. 7. Comparison between the basic differential model ($\alpha = 1$) and the fractional dynamics with $\alpha = 0.89$, when parameters are taken from set S_2. The fractional order model converges faster to the equilibrium point than the basic integer order model and has richer dynamics than the basic model.

7.3. Effects of desistance parameters

The rates for first-time offenders and repeat offenders of withdrawal from criminal activities without being punished are analyzed in this section. The partial derivatives of $\mathcal{R}_0$ with respect to η_1 and η_2 are found to be negative as shown in (53), which means that social interventions, rehabilitation programs for desistance of offenders decrease crime. For

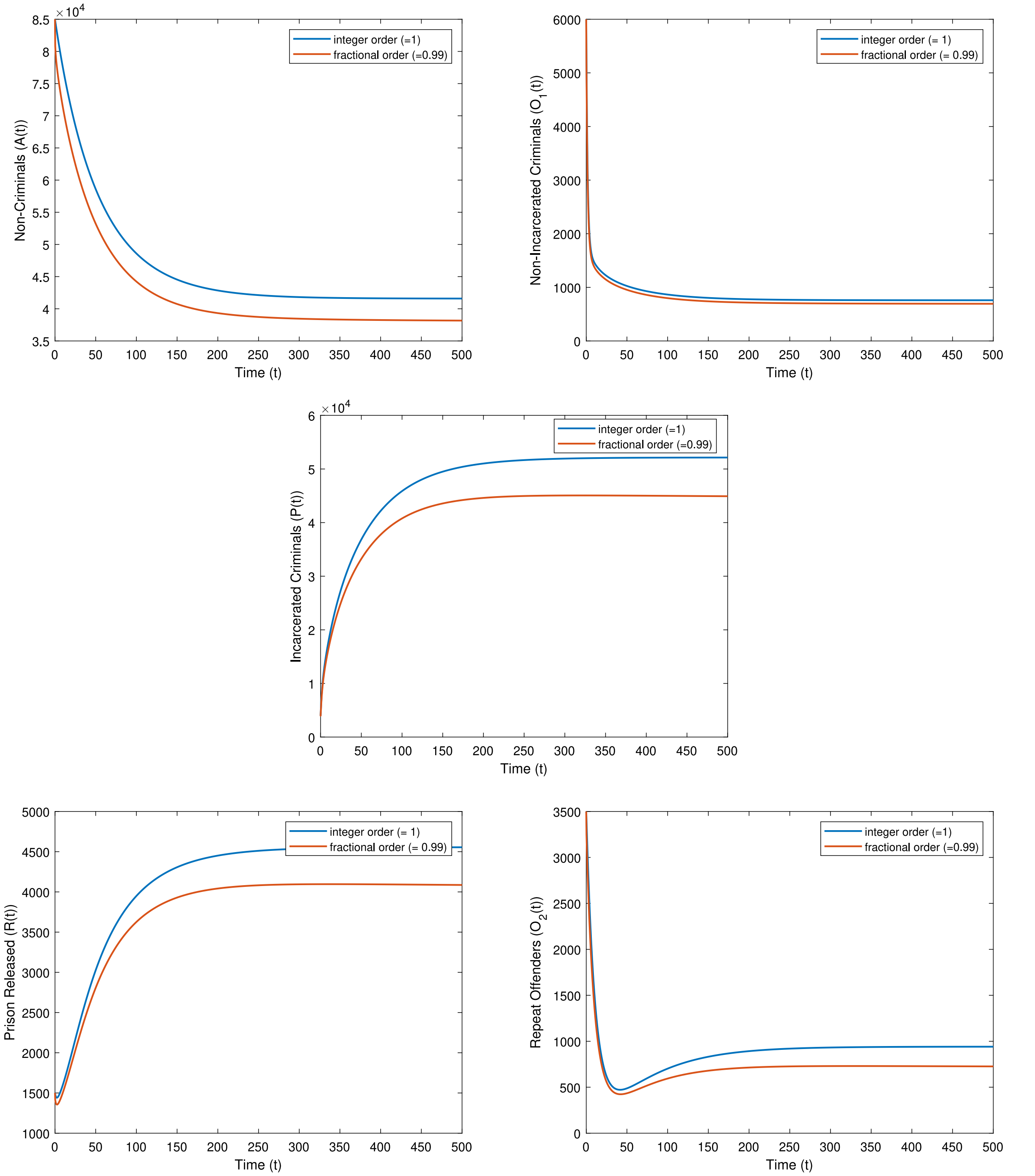

Fig. 8. Comparison between the classical model ($\alpha = 1$) and the fractional dynamics with $\alpha = 0.99$, when parameters are taken from set S_1. The Fractional order model converges faster to the equilibrium point as compared to the basic integer order model and also has richer dynamics than the basic model.

majority of judicious parameters, $\left|\frac{\partial \mathcal{R}_0}{\partial \eta_1}\right| > \left|\frac{\partial \mathcal{R}_0}{\partial \eta_2}\right|$, irrespective of order α. Consequently, first-time offenders require more observation as compared to repeat offenders. So it is required to increase the rate of desistance and decrease the reproduction number, hence decreasing the crime transmission in society. And therefore, punishments and rehabilitation programs should be designed separately for first-time offenders (O_1) and repeat offenders (O_2) with more focus on (O_1)

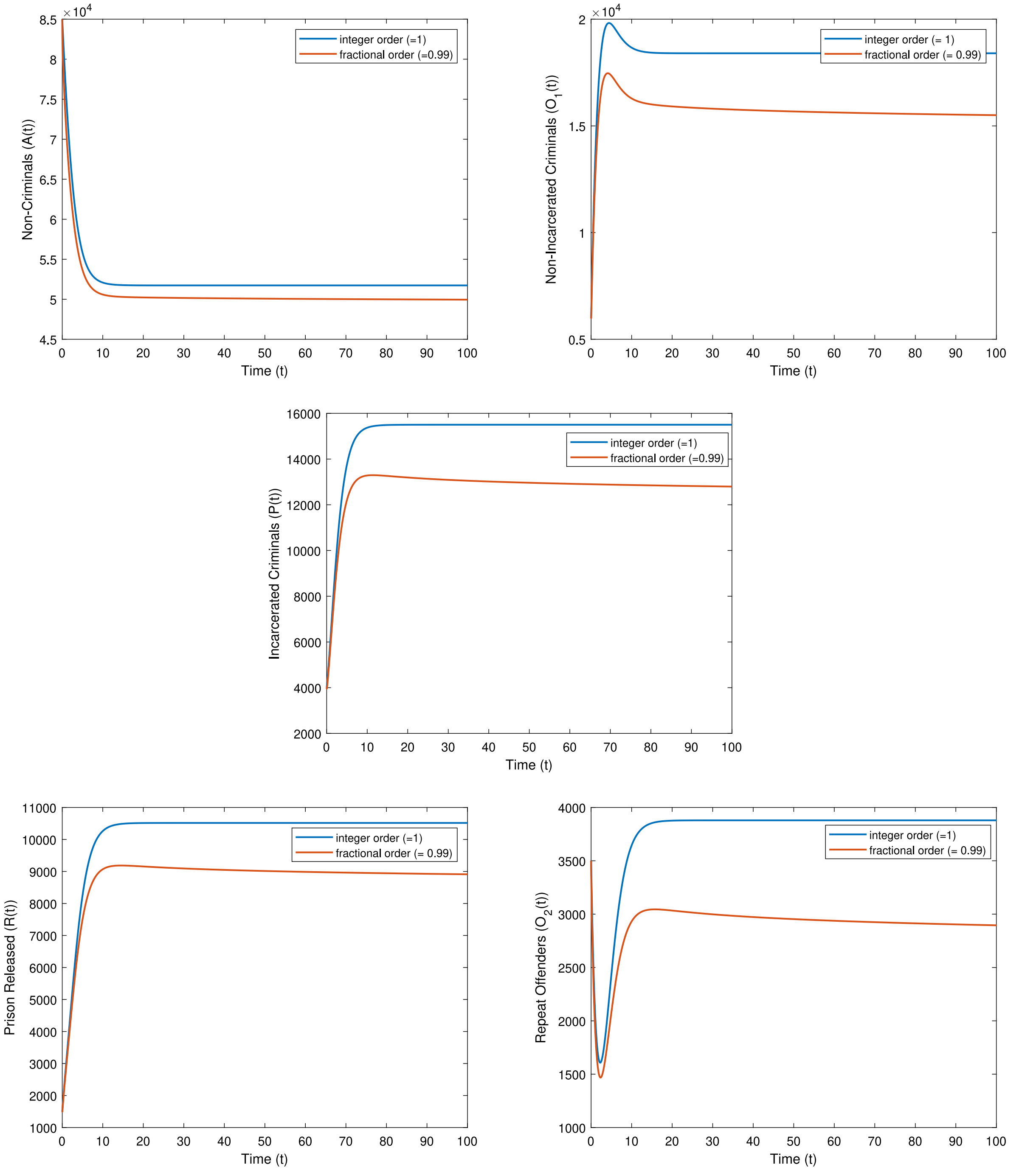

Fig. 9. Comparison between the classical model ($\alpha = 1$) and the fractional dynamics with $\alpha = 0.99$, when parameters are taken from set S_2. The Fractional order model converges faster to the equilibrium point as compared to the basic integer order model and also has richer dynamics than the basic model.

to efficiently control the crime spread due to the assimilation of criminals back into society.

$$\frac{\partial \mathcal{R}_0}{\partial \eta_1^\alpha} = \frac{-\,\alpha\ \eta_1^{\alpha-1}\nu_1^\alpha}{(\eta_1^\alpha + \lambda_1^\alpha)^2}\left[1 - \frac{\nu_2^\alpha\lambda_1^\alpha}{(\eta_2^\alpha + \lambda_2^\alpha)\xi^\alpha}\right] \qquad \frac{\partial \mathcal{R}_0}{\partial \eta_2^\alpha} = \frac{-\,\alpha\ \eta_2^{\alpha-1}\nu_1^\alpha}{(\eta_1^\alpha + \lambda_1^\alpha)}\left[\frac{\nu_2^\alpha\lambda_1^\alpha}{(\eta_2^\alpha + \lambda_2^\alpha)^2\xi^\alpha}\right] \tag{53}$$

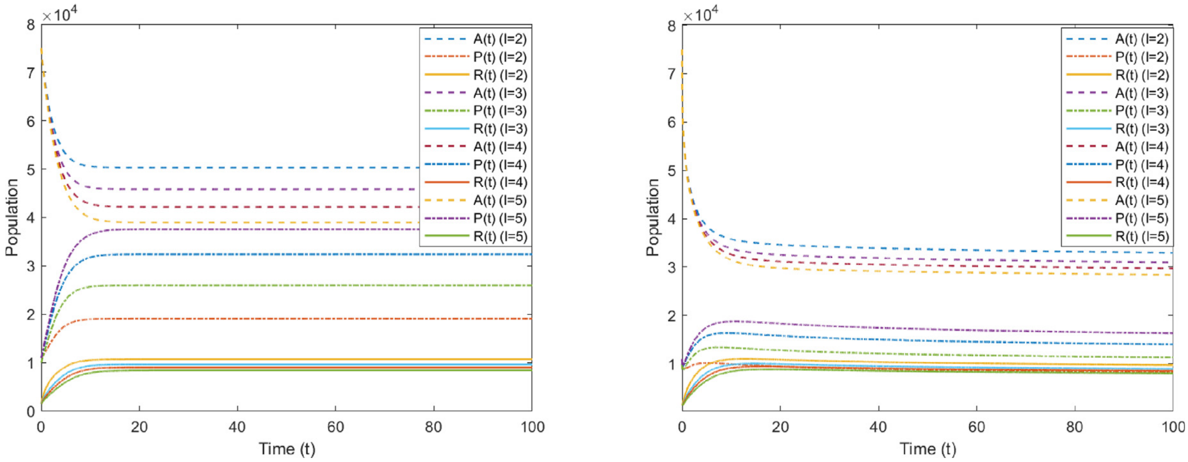

Fig. 10. Impact of Imprisonment shown for $\alpha = 1, 0.95$ and prison length (l = 1/r).

8. Conclusion

In this study, a non-linear fractional-order differential model has been introduced to study the dynamics of crime transmission using five population classes in society. It has been shown that the solution of the model not only exists but is unique, non-negative and bounded. The basic reproduction parameter $\mathcal{R}_0$ has been evaluated with non-locality indulgence, which behaves as a threshold for analyzing the crime status in society. The stability of the fractional order crime transmission model has been investigated concerning the values of $\mathcal{R}_0$. The crime-free equilibrium is uniform asymptotically stable for $\mathcal{R}_0 < 1$, and the endemic equilibrium is locally asymptotically stable for $\mathcal{R}_0 > 1$. Additionally, the threshold conditions have been derived for the uniform asymptotic stability of the crime-free equilibrium using an extended Lyapunov's function approach. For $\mathcal{R}_0 \geq 1$, the stability of the positive endemic equilibrium state E^* has been investigated. Numerical results are shown to strengthen the results of the proposed model. As when the order of derivative α is reduced from 1, the memory effect of the system increases, and it has been observed that crime spreads slowly and the number of incarcerated criminals increases for a long time. Meanwhile, it has been observed that increasing the prison length does not have much effect on the threshold condition, on the fractions of criminally active and criminally inactive populations. Moreover, this work recommends making more strict policies to catch repeat offenders than first-time offenders and focusing on first-time offenders by improving their desistance rate for eradicating crime from society.

9. Future scope

In this study, the whole population has been divided into five sub-classes to investigate crime propagation, but there can be several sub-classes in the actual scenario. Higher-dimensional models can be developed by considering more factors affecting the growth of crime in societies to model the crime dynamics more accurately. Further, the results can be validated using actual criminal data. For faster convergence to crime-free equilibrium, the optimal value of fractional order of derivative can be evaluated using optimization algorithms like PSO, genetic algorithm or other heuristic algorithms.

Data availability

Data will be made available on request.

Acknowledgments

Sugandha Arora is grateful to University Grants Commission, India for the financial assistance with UGC Ref No: 1034/(CSIR-UGC NET JUNE 2017).

References

[1] M. Costantini, I. Meco, A. Paradiso, Do inequality, unemployment and deterrence affect crime over the long run? Reg. Stud. 52 (4) (2018) 558–571.
[2] J.A. Gardiner, Research models in law enforcement and criminal justice, Law Soc. Rev. 6 (2) (1971) 223–230.
[3] J.Q. Wilson, G.L. Kelling, Broken windows, Atl. Mon. 249 (3) (1982) 29–38.
[4] A.R. Piquero, D.P. Farrington, A. Blumstein, Key Issues in Criminal Career Research: New Analyses of the Cambridge Study in Delinquent Development, Cambridge University Press, 2007.
[5] S.N. Durlauf, D.S. Nagin, Imprisonment and crime: Can both be reduced? Criminol. Public Policy 10 (1) (2011) 13–54.
[6] D. McMillon, C.P. Simon, J. Morenoff, Modeling the underlying dynamics of the spread of crime, PLoS One 9 (4) (2014) e88923.

[7] P. Brantingham, U. Glässer, P. Jackson, M. Vajihollahi, Modeling criminal activity in urban landscapes, in: Mathematical Methods in Counterterrorism, Springer, 2009, pp. 9–31.
[8] M. Felson, What every mathematician should know about modelling crime, European J. Appl. Math. 21 (4–5) (2010) 275–281.
[9] K. Jane White, E. Campillo-Funollet, F. Nyabadza, D. Cusseddu, C. Kasumo, N.M. Imbusi, V.O. Juma, A.J. Meir, T. Marijani, Towards understanding crime dynamics in a heterogeneous environment: A mathematical approach, J. Interdiscip. Math. (2021) 1–21.
[10] J. Sooknanan, B. Bhatt, D.M.G. Comissiong, Another way of thinking: A review of mathematical models of crime, Math. Today 131 (2013).
[11] D.J.B. Lloyd, H. O'Farrell, On localised hotspots of an urban crime model, Physica D 253 (2013) 23–39.
[12] M.B. Short, A.L. Bertozzi, P.J. Brantingham, Nonlinear patterns in urban crime: Hotspots, bifurcations, and suppression, SIAM J. Appl. Dyn. Syst. 9 (2) (2010) 462–483.
[13] Y.D. Abbasi, M. Short, A. Sinha, N. Sintov, C. Zhang, M. Tambe, Human adversaries in opportunistic crime security games: Evaluating competing bounded rationality models, in: Proceedings of the Third Annual Conference on Advances in Cognitive Systems ACS, 2015, p. 2.
[14] P. Buonanno, Crime, Education and Peer Pressure, Dipartimento di economia politica, 2003.
[15] M.B. Short, P.J. Brantingham, M.R. D'orsogna, Cooperation and punishment in an adversarial game: How defectors pave the way to a peaceful society, Phys. Rev. E 82 (6) (2010) 066114.
[16] A.A. Lacey, M.N. Tsardakas, A mathematical model of serious and minor criminal activity, European J. Appl. Math. 27 (3) (2016) 403–421.
[17] A.K. Srivastav, M. Ghosh, P. Chandra, Modeling dynamics of the spread of crime in a society, Stoch. Anal. Appl. 37 (6) (2019) 991–1011.
[18] S. Abbas, J.P. Tripathi, A.A. Neha, Dynamical analysis of a model of social behavior: Criminal vs non-criminal population, Chaos Solitons Fractals 98 (2017) 121–129.
[19] M.B. Short, M.R. D'orsogna, V.B. Pasour, G.E. Tita, P.J. Brantingham, A.L. Bertozzi, L.B. Chayes, A statistical model of criminal behaviour, Math. Models Methods Appl. Sci. 18 (2008) 1249–1267.
[20] N. Baloian, C.E. Bassaletti, M. Fernández, O. Figueroa, P. Fuentes, R. Manasevich, M. Orchard, S. Peñafiel, J.A. Pino, M. Vergara, Crime prediction using patterns and context, in: 2017 IEEE 21st International Conference on Computer Supported Cooperative Work in Design, CSCWD, IEEE, 2017, pp. 2–9.
[21] K.S. Pritam, S. Sugandha, T. Mathur, S. Agarwal, Underlying dynamics of crime transmission with memory, Chaos Solitons Fractals 146 (2021) 110838.
[22] K. Bansal, S. Arora, K.S. Pritam, T. Mathur, S. Agarwal, Dynamics of crime transmission using fractional-order differential equations, Fractals 30 (01) (2022) 2250012.
[23] N. Sene, SIR epidemic model with Mittag–Leffler fractional derivative, Chaos Solitons Fractals 137 (2020) 109833.
[24] M. Farman, A. Akgül, S.F. Aldosary, K.S. Nisar, A. Ahmad, Fractional order model for complex layla and majnun love story with chaotic behaviour, Alex. Eng. J. 61 (9) (2022) 6725–6738.
[25] H. Sun, W. Chen, Y. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, Physica A 388 (21) (2009) 4586–4592.
[26] M.R. D'Orsogna, M. Perc, Statistical physics of crime: A review, Phys. Life Rev. 12 (2015) 1–21.
[27] B. Berenji, T. Chou, M.R. D'Orsogna, Recidivism and rehabilitation of criminal offenders: A carrot and stick evolutionary game, PLoS One 9 (1) (2014) e85531.
[28] K. Diethelm, The Analysis of Fractional Differential Equations: An Application-Oriented Exposition using Differential Operators of Caputo Type, Springer Science & Business Media, 2010.
[29] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, 2012, arXiv preprint arXiv:1201.0863.
[30] V.P. Latha, F.A. Rihan, R. Rakkiyappan, G. Velmurugan, A fractional-order model for ebola virus infection with delayed immune response on heterogeneous complex networks, J. Comput. Appl. Math. 339 (2018) 134–146.
[31] S. Pooseh, H.S. Rodrigues, D.F.M. Torres, Fractional derivatives in dengue epidemics, in: AIP Conference Proceedings, volume 1389, American Institute of Physics, 2011, pp. 739–742.
[32] W. Wojtak, C.J. Silva, D.F.M. Torres, Uniform asymptotic stability of a fractional tuberculosis model, Math. Model. Nat. Phenom. 13 (1) (2018) 9.
[33] S. Ullah, M.A. Khan, J.F. Gómez-Aguilar, Mathematical formulation of hepatitis B virus with optimal control analysis, Optim. Control Appl. Methods 40 (3) (2019) 529–544.
[34] N. Sene, Analysis of the stochastic model for predicting the novel coronavirus disease, Adv. Difference Equ. 2020 (1) (2020) 1–19.
[35] S.-W. Yao, M. Farman, M. Amin, M. Inc, A. Akgül, A. Ahmad, Fractional order COVID 19 model with transmission rout infected through environment, AIMS Math. 7 (4) (2022) 5156–5174.
[36] M. Farman, A. Akgül, M.T. Tekin, M.M. Akram, A. Ahmad, E.E. Mahmoud, I.S. Yahia, Fractal fractional-order derivative for HIV/AIDS model with Mittag-Leffler kernel, Alex. Eng. J. 61 (12) (2022) 10965–10980.
[37] M. Farman, A. Ahmad, A. Akgül, M.U. Saleem, K.S. Nisar, V. Vijayakumar, Dynamical behavior of tumor-immune system with fractal-fractional operator, AIMS Math. 7 (5) (2022) 8751–8773.
[38] K. Bansal, T. Mathur, N.S.S. Singh, S. Agarwal, Analysis of illegal drug transmission model using fractional delay differential equations, AIMS Math. 7 (10) (2022) 18173–18193.
[39] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, 1993.
[40] M. Caputo, Linear models of dissipation whose Q is almost frequency independent—II, Geophys. J. Int. 13 (5) (1967) 529–539.
[41] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Volume 198, Academic press New York, 1999.
[42] J.-L. Wang, H.-F. Li, Surpassing the fractional derivative: Concept of the memory-dependent derivative, Comput. Math. Appl. 62 (3) (2011) 1562–1567.
[43] Y. Wei, Y. Chen, S. Cheng, Y. Wang, A note on short memory principle of fractional calculus, Fract. Calc. Appl. Anal. 20 (6) (2017) 1382–1404.
[44] I. Petráš, J. Terpák, Fractional calculus as a simple tool for modeling and analysis of long memory process in industry, Mathematics 7 (6) (2019) 511.
[45] Y. Li, Y. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability, Comput. Math. Appl. 59 (5) (2010) 1810–1821.
[46] H. Delavari, D. Baleanu, J. Sadati, Stability analysis of Caputo fractional-order nonlinear systems revisited, Nonlinear Dynam. 67 (4) (2012) 2433–2439.
[47] A. Dokoumetzidis, R. Magin, P. Macheras, A commentary on fractionalization of multi-compartmental models, J. Pharmacokinet. Pharmacodyn. 37 (2) (2010) 203–207.
[48] C.-S. Sin, Well-posedness of general Caputo-type fractional differential equations, Fract. Calc. Appl. Anal. 21 (3) (2018) 819–832.
[49] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, On some Routh–Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and chen systems, Phys. Lett. A 358 (1) (2006) 1–4.
[50] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies models, J. Math. Anal. Appl. 325 (1) (2007) 542–553.
[51] D. Matignon, Stability results for fractional differential equations with applications to control processing, in: Computational Engineering in Systems Applications, volume 2, Citeseer, 1996, pp. 963–968.
[52] I. Petráš, Chaos in the fractional-order Volta's system: Modeling and simulation, Nonlinear Dynam. 57 (1) (2009) 157–170.
[53] I. Petráš, Modeling and numerical analysis of fractional-order Bloch equations, Comput. Math. Appl. 61 (2) (2011) 341–356.
[54] I. Podlubny, Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers, IEEE Trans. Automat. Control 44 (1) (1999) 208–214.
[55] L. Dorcak, Numerical models for the simulation of the fractional-order control systems, 2002, ArXiv Preprint Math/0204108.