Value Function Iteration & Stochastic Growth Model

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Dynamic Models: features and solution approaches

- Value function iteration: deterministic dynamic programming;
  - Discretization;
  - Finite element method;
  - Chebyshev polynomials (time permitting);
- Value function iteration: how to deal with uncertainty.

Deterministic Growth Model

The model's characteristics:

1. time is discrete and starts from 0
2. infinite horizon
3. no uncertainty
4. no productivity growth
5. only 1 consumer

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Two methods to solve the problem:

- Optimal control
- Dynamic programming

Optimal control

You need the sequential formulation of the problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + k_{t+1} = F(k_t) + (1-\delta)k_t$$

$$k_{t+1}, c_t > 0 \text{ and } k_0 \text{ given}$$

$$\beta \in (0,1)$$

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3. Collect the FOCs and other conditions that must hold in equilibrium (e.g. from imposing market equilibrium)

The solution is an infinite sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ *which solves the system of equations.*

Dynamic programming

You need the recursive representation of the problem

$$V(k) = \max_{c,k'} \left\{ u(c) + \beta V\left(k'\right) \right\}$$
$$c + k' = F(k) + (1-\delta)k$$
$$c \geq 0$$
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\end{aligned}
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but lets take a step back first!

From Optimal Control to Dynamic Programming

We can rewrite the sequential problem as:

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* $\overrightarrow{V}_1 = V(\hat{k}_{t+1})$ (prove, go through the cases)

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to

$$V(k_0) = \max_{c_0, k_1} \{u(c_0) + \beta V(k_1)\}$$

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- and time subscripts redundant, only state k matters

Bellman Operator and Policy Functions

We are interested in $c = h(k)$ and $k' = g(k)$:

$$\begin{aligned} V(k) &= u(h(k) + \beta V(g(k))\} \\ s.t. \quad h(k) + g(k) &= F(k) + (1-\delta)k \end{aligned}$$

- To find policy functions, need $V(k)$
  - functional equation in one unknown function)
- Since everything depends on k, make grid and guess and verify
- Once we have h and g we can simulate model

VFI: Discretization 1/2

1. Set:
   - n_k (number of grid points);
   - $[k_{\min}, k_{\max}]$ (support of k);
   - ε (the tolerance error);
2. Define the grid $\{k_1, k_2, \dots, k_{n_k}\}$
3. Choose the initial guess of the value function: $V^0 = \{V_i^0\}_{i=1}^{n_k}$
4. Find $V^1 = \{V_i^1\}_{i=1}^{n_k}$ following the next steps:

a) For every $j = 1, \dots, n_k$ compute:

$$V_{i,j}^1 = u\left(F\left(k_i\right) + (1-\delta)k_i - k_j\right) + \beta V_j^0$$

b) Choose j which gives the highest $V_{i,j}^1$ and set $V_i^1 = V_{i,j}^1$. Store the optimal decision rule as $j = g_i \in \{1, 2, \dots, n_k\}$.

VFI: Discretization 2/2

5. Do a) and b) for every $i = 1, \dots, n_k$ until you get the entire vector $V^1 = \{V_i^1\}_{i=1}^{n_k}$
6. Compare V^0 and V^1. Compute:

$$d = \max_{i=1,2,\dots,n_k} \left| V_i^0 - V_i^1 \right|$$

   a) if $d > \varepsilon$ go to step 4 and set $V^0 = V^1$;
   b) if $d \leq \varepsilon$ you are done: $V^0 = V^*$ and the optimal decision rule is $g^* = \{g_i\}_{i=1}^{n_k}$

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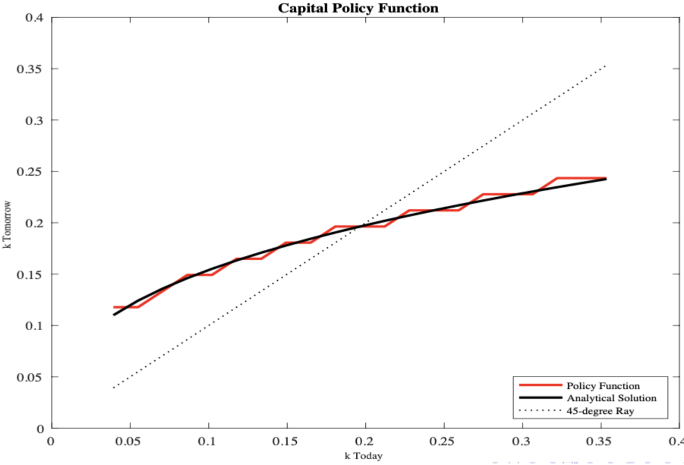
CAVEATS:
a) Check the bounds of $\{k_1, \dots, k_{n_k}\}$;
b) Redo 1)- 6) with a smaller ε.
c) Increase n_k.

$\varepsilon = 0.0001$
$\mathbf{n}_k = \mathbf{21}$
$k_{\min} = 0.1k_{ss}$ and $k_{\min} = 1.9k_{ss}$



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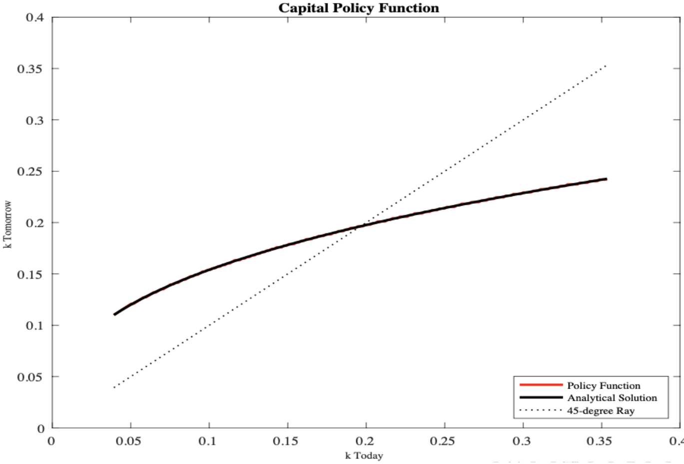

Improvement algorithms (1/3)

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4. Find $V^1 = \{V_i^1\}_{i=1}^{n_k}$ following the next steps. For every $i = 1, \ldots, n_k$:

   a) Set $j = 1$;

   b) Compute $V_{i,j}^1$:

$$V_{i,j}^1 = u\left(F\left(k_i\right) + (1-\delta)k_i - k_j\right) + \beta V_j^0$$

c) Compute $V_{i,j+1}^1$:

$$V_{i,j+1}^1 = u\left(F\left(k_i\right) + (1-\delta)k_i - k_{j+1}\right) + \beta V_{j+1}^0$$

d) Compare $V_{i,j}^1$ and $V_{i,j+1}^1$

   i. If $V_{i,j}^1, < V_{i,j+1}^1$ do $j = j + 1$ and go to step b)
   ii. If $V_{i,j}^1 \geq V_{i,j+1}^1$you are done and $V_i^1 = V_{i,j}^1,\ and g_i = j$.

Improvement algorithms (2/3)

If the optimal control is also monotone:

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If the optimal control is also monotone:

4. Find $V^1 = \{V_i^1\}_{i=1}^{n_k}$ following the next steps. For every $i = 1, \ldots, n_k$:
   a) Find the lower bound of the optimal choice $j_{\min}$. For $i = 1$, $j_{\min} = 1$. For $i > 1, j_{\min} = g_{i-1}$
   b) For every $j = j_{\min}, j_{\min} + 1, \ldots, n_k$ compute:

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c) Choose j which gives the highest $V_{i,j}^1$ and set $V_i^1 = V_{i,j}^1$. Store the optimal decision rule as $j = g_i \in \{1, 2, \ldots, n_k\}$

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Choose n_h : # times to update the VF using current PF

4. Set $V^{1,1} = V^1$. Then update the value function $V^{1,t}$ by applying the following steps n_h times and obtain V^{1,n_k}. Replace V^1 by V^{1,n_k} and go to step 5

a) For each $i = 1, \ldots, n_h$ we have the optimal decision $g_i \in \{1, 2, \ldots, n_k\}$ associated for each i.

b) Update the value function from $V^{1,t}$ to $V^{1,t+1}$ using the following modified Bellman equation for each $i = 1, \ldots, n_k$:

$$V_i^{1,t+1} = u\left(F\left(k_i\right) + (1-\delta)k_i - k_{g_i}\right) + \beta V_{g_i}^{1,t}$$

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What if $n_h \to \infty$?

Intuition to speed up VFI

+ VFI is a stable and reliable convergence algorithm
+ Can easily handle occasionally binding constraints (global)
− Slow and suffers from curse of dimensionality

Do not do any operations that are not necessary!

e.g. : Calculating $F(k_i) + (1-\delta)k_i - k_j$ does not depend on V or PF ! Calculate once at the beginning and retrieve values as needed!

VFI: Finite element method

- Likely, your V^{i+1} will not lie on the grid $\implies$ error

SO approximate V with a continuous function

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SO approximate V with a continuous function

We split the domain of x into different regions and use different polynomial for each region.

* Piece-wise linear for $x \in [x_i, x_{i+1}]$

$$f(x) \approx f_i + \left(\frac{f_{i+1} - f_i}{x_{i+1} - x_i}\right)(x - x_i)$$

CAVEAT: it is not differentiable in the nodes.

* n^{th}-order spline for $x \in [x_{i-1}, x_i]$

$$f(x) \approx a_i + b_i x + c_i x^2 + d_i x^3 + .. + z_i x^n$$

This is differentiable & shapre preserving

VFI: Finite element method (piece-wise linear)

1. Set:
   - n_k (number of grid points);
   - $[k_{\min}, k_{\max}]$ (support of k);
   - ε (the tolerance error);
2. Define the grid $\{k_1, k_2, \ldots, k_{n_k}\}$
3. Choose the initial guess of the value function at the nodes: $V^0 = \{V_i^0\}_{i=1}^{n_k}$
4. Call $\widetilde{V}(k)$ the approximation of the value function implied by the piece-wise linear interpolation with $\{V_i^0\}_{i=1}^{n_k}$ (in MATLAB interp1 (., ., ., 'linear')
5. $\forall i = 1, 2, \ldots, n_k$ solve the following problem

$$\max_{k' \in [k_{\min}, k_{\max}]} \{u(F(k_i) + (1-\delta)k_i - k')\} + \beta \widetilde{V}^0(k')$$

(CAVEAT: one dimension optimization algorithm needed)

6. Set g_i the argmax of the previous problem.
7. Update the value function, such as:

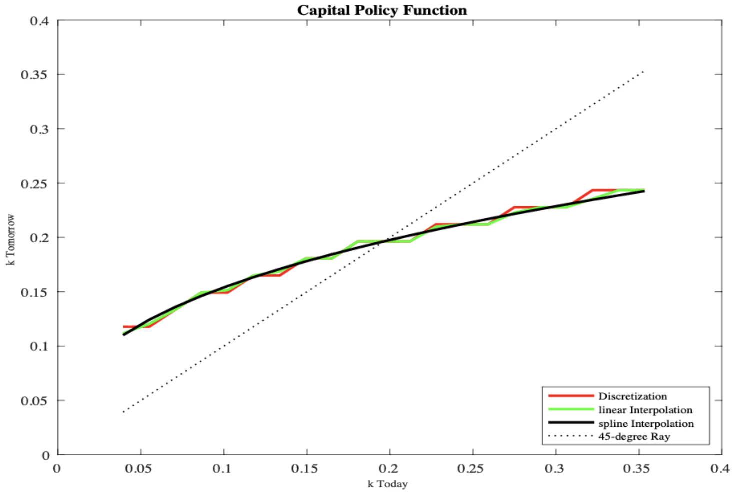
$$V_i^1 = \{u\left(F\left(k_i\right) + (1-\delta)k_i - g_i\right)\} + \beta\widetilde{V}^0\left(g_i\right)$$

9. Compare V^0 and V^1. Compute:

$$d = \max_{i=1,2,\ldots,n_k} \left| V_i^0 - V_i^1 \right|$$

   a) if $d > \varepsilon$ go to step 4 and set $V^0 = V^1$
   b) if $d \leq \varepsilon$ you are done: $V^0 = V^*$ and the optimal decision rule is $g^* = \{g_i\}_{i=1}^{n_k}$

CAVEAT: This approximation is not differentiable in the nodes. A spline is more advisable, but has more conditions to satisfy.


Capital Policy Function
k Tomorrow
k Today
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Discretization
linear Interpolation
spline Interpolation
45-degree Ray

VFI: Chebyshev

This time we approximate the value function using Chebyshev polynomials of order n.

Chebyshev polynomial of order j defined in $[-1, 1]$:

$$T_j(x) = \cos\left(j \cos^{-1} x\right)$$

The polynomials are:

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_j(x) &= 2xT_{j-1}(x) - T_{j-2}(x) \quad \text{ for } j > 1
\end{aligned}$$

The approximation of a function $f(x)$ defined in $[-1, 1]$ with Chebyshev polynomials of order n is

$$\widetilde{f}(x) = \sum_{j=0}^{n} \theta_j T_j(x)$$

The problem is to choose $n + 1$ coefficients $\theta = \{\theta_j\}_{j=0}^{n}$ such as the error between $f(x)$ and $\widetilde{f}(x)$ is minimized.

VFI: Chebyshev

1. Choose the number of nodes m :

$$\text{(a) } m = n + 1 \Longrightarrow \text{ Chebyshev collocation}$$
$$\text{(b) } m > n + 1 \Longrightarrow \text{ Chebyshev regression}$$

3. Find the nodes $\{z_i\}_{i=1}^{m}$

$$z_i = -\cos\left(\frac{(2i-1)\pi}{2m}\right)$$

4. Find the coefficients minimizing the sum of the squared errors between $f(x)$ and $\widetilde{f}(x)$ evaluated in $\{z_i\}_{i=1}^{m}$

$$\theta_0 = \frac{1}{m}\sum_{i=1}^{m} f(z_i)$$
$$\theta_j = \frac{2}{m}\sum_{i=1}^{m} f(z_i)\, T_j(z_i)$$

VFI: Chebyshev Cookbook

1. Set:

   - order of the polynomial n; number of collocation points m such that $m \geq n+1$;
   - ε (the tolerance error);
   - $[k_{\min}, k_{\max}]$ (support of k)

3. Compute the Chebyshev roots $\{z_i\}_{i=1}^{m}$

$$z_i = -\cos\left(\frac{(2i-1)\pi}{2m}\right)$$

4. Compute the collocation points implied by $\{z_i\}_{i=1}^{m}$

$$k_i = (z_i+1)\left(\frac{k_{\max}-k_{\min}}{2}\right) + k_{\min}$$

5. We want to approximate the value function $V(k)$ by

$$\widetilde{V}(k) = \sum_{j=0}^{n} \theta_j T_j t(k)$$

VFI: Chebyshev

6. Implicitly guess $\left\{\theta_j^0\right\}_{j=0}^n$:

   (a) Choose a guess of the value function at the collocation points $\{k_i\}_{i=1}^m$ and call them $\{y_i^0\}_{i=1}^m$;

   (b) back up the values of $\left\{\theta_j^0\right\}_{j=0}^n$ using:

$$\theta_0^0 = \frac{1}{m}\sum_{i=1}^m y_i^0$$

$$\theta_j^0 = \frac{2}{m}\sum_{i=1}^m y_i^0 T_j\left(z_i\right)$$

7. Denote the guessed value function as $\widetilde{V}^0(k)$;

8. For each $i = 1, 2, \ldots, m$ solve the following problem

$$\max_{k' \in [k_{\min}, k_{\max}]} \left\{u\left(F\left(k_i\right) + (1-\delta)k_i - k'\right)\right\} + \beta \widetilde{V}^0\left(k'\right)$$

using a one dimension optimization algorithm.

9. Set g_i the argmax of the previous problem
10. Use g_i to update the value function:

$$y_i^1 = \{u\left(F\left(k_i\right) + (1-\delta)k_i - g_i\right)\} + \beta \widetilde{V}^0\left(g_i\right)$$

11. Obtain an updated guess of the coefficients:

$$\theta_0^1 = \frac{1}{m}\sum_{i=1}^{m} y_i^1$$

$$\theta_j^1 = \frac{2}{m}\sum_{i=1}^{m} y_i^1 T_j\left(z_i\right)$$

12. Compare $\left\{\theta_j^0\right\}_{j=0}^{n}$ and $\left\{\theta_j^1\right\}_{j=0}^{n}$:

$$d = \max_{j=1,\ldots,n_k} \left|\theta_j^0 - \theta_j^1\right|$$

(a) if $d > \varepsilon$ set $\theta^0 = \theta^1$ and go to step 6
(b) if $d \leq \varepsilon$ you are done: $V^0 = V^*$ and the optimal decision rule is $g^* = \{g_i\}_{i=1}^{m}$

Moving to Stochastics

Suppose now that the model is the following:

$$
\begin{aligned}
V(k,z) &= \max_{c,k'} \mathbb{E}\left\{u(c) + \beta V\left(k', z'\right)\right\} \\
c + k' &= zF(k) + (1-\delta)k \\
c &\geq 0 \\
k' &\geq 0
\end{aligned}
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where z is a stochastic shock.

- discrete valued iid shock (very easy)
- a (discrete valued) Markov chain (easy)
- a (continuous valued) autoregressive process (not so easy - need to discretise/approximate)

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Unconditional distribution (Transition Matrices)

Consider a state vector $v = [01]'$ and transition matrix
$T = \begin{bmatrix} \pi_{11} & \pi_{1,2} \\ \pi_{21} & \pi_{22} \end{bmatrix}$

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$T = \begin{bmatrix} \pi_{11} & \pi_{1,2} \\ \pi_{21} & \pi_{22} \end{bmatrix}$

Probability of the state in t years: $v_t = (T')^t v_0 = T' v_{t-1}$

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Probability of the state in t years: $v_t = (T')^t v_0 = T' v_{t-1}$
The unconditional distribution is given by $v = T'v$

$$\implies (I - T')v = 0$$

v is the eigenvector associated with a unit eigenvalue of matrix T' and normalised to sum to one

Stochastic Growth Model

With a discrete iid shock:

- Assume

$$z_t = \begin{cases} z_1 & \text{w.p. } \pi_1 \\ z_2 & \text{w.p. } \pi_2 \end{cases}, \text{ for all } t$$

- The Bellman equation is then

$$\begin{aligned} V(k,z) &= \max\left\{U + \beta EV\left(k', z'\right)\right\} \\ &= \max\left\{U + \beta \sum_{i=1}^{2} \pi_i V_i\left(k', z_i'\right)\right\} \end{aligned}$$

- Steps are like for the deterministic case but repeat twice, one for each state z

Stochastic VFI algorithm 1/2

1. Set:
   - n_k (number of grid points);
   - $[k_{\min}, k_{\max}]$ (support of k);
   - ε (the tolerance error);
2. Define the grid $\{k_1, k_2, \dots, k_{n_k}\}$
3. Choose the initial guess of the value functions: $V_1^0 = \left\{V_{1,i}^0\right\}_{i=1}^{n_k}$ and $V_2^0 = \left\{V_{2,i}^0\right\}_{i=1}^{n_k}$
4. Find $V_1^1 = \left\{V_{1,i}^1\right\}_{i=1}^{n_k}$ and $V_2^1 = \left\{V_{2,i}^1\right\}_{i=1}^{n_k}$:
   a. For every $j = 1, \dots, n_k$ compute:

$$
\begin{aligned}
V_{1,i,j_1}^1 =& u\left(z_1 F\left(k_i\right) + (1-\delta)k_i - k_{j_1}\right) \\
&+ \beta\left(\pi_1 V_{1,j_1}^0 + \pi_2 V_{2,j_1}^0\right) \\
V_{2,i,j_2}^1 =& u\left(z_2 F\left(k_i\right) + (1-\delta)k_i - k_{j_2}\right) \\
&+ \beta\left(\pi_1 V_{1,j_2}^0 + \pi_2 V_{2,j_2}^0\right)
\end{aligned}
$$

Stochastic VFI algorithm 1/2

b. Choose j_1 and j_2 which gives the highest V^1_{1,i,j_1} and V^1_{2,i,j_2} and set $V^1_{1,i} = V^1_{1,i,j}$ and $V^1_{2,i} = V^1_{2,i,j_2}$. Store the optimal decision rule as $j_1 = g_{1,i} \in \{1, 2, \ldots, n_k\}$ and $j_2 = g_{2,i} \in \{1, 2, \ldots, n_k\}$

5. Do a) and b) for every $i = 1, \ldots, n_k$ until you get the entire vector

$$V^1_1 = \left\{ V^1_{1,i} \right\}_{i=1}^{n_k} \text{ and } V^1_2 = \left\{ V^1_{2,i} \right\}_{i=1}^{n_k}$$

7. Compare V^0 and V^1. Compute:

$$d_1 = \max_{i=1,2,\ldots,n_k} \left| V^0_{1,i} - V^1_{1,i} \right|$$

$$d_2 = \max_{i=1,2,\ldots,n_k} \left| V^0_{2,i} - V^1_{2,i} \right|$$

a) if $d_1 > \varepsilon$ or $d_2 > \varepsilon$ go to step 4 and set $V^0_1 = V^1_1$ and $V^0_2 = V^1_2$;

b) if $d_1 < \varepsilon$ and $d_2 < \varepsilon$ you are done: $V^0_1 = V^*_1$ and $V^0_2 = V^*_2$ and the optimal decision rules is $g^*_1 = \{g_{1,i}\}_{i=1}^{n_k}$ and

Stochastic VFI (Markov Chain)

- Assume

$$z_t = \begin{cases} z_1 \\ z_2 \end{cases}$$

with transition matrix

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

where $\pi_{i1} + \pi_{i2} = 1$ (rows sum up to one)

- The steps are the same as with iid shocks - step 4 is now

$$V^1_{1,i,j_1} = u\left(z_1 F\left(k_i\right) + (1-\delta)k_i - k_{j_1}\right) + \beta\left(\pi_{1,1} V^0_{1,j_1} + \pi_{1,2} V^0_{2,j_1}\right)$$
$$V^1_{2,i,j_2} = u\left(z_2 F\left(k_i\right) + (1-\delta)k_i - k_{j_2}\right) + \beta\left(\pi_{2,1} V^0_{1,j_2} + \pi_{2,2} V^0_{2,j_2}\right)$$

Stochastic VFI (Stochastic Process)

Assume now that

$$\log z_t = \rho \log z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid}\left(0, \sigma^2\right)$$

- Problem: The (stochastic) state variable z_t is continuous. Like for capital, we need to discretise the space that z_t takes values from
- Solution: we approximate the Markov process with a Markov chain
- Method: Tauchen (see Heer and Maussner pp 497-499 for the algorithm)
- Then use the approximate Markov chain as above

Readings

- Heer and Maussner,2008, "Value Function Itereration as a Solution Method for the Ramsey Model", CESifo Wp n.2278
- Heer and Maussner, chapters 8-9, relevant sections
- Heer and Maussner, 1.2 and 1.3
- Ljunqvist and Sargent, p. 29 - 36 and p. 39 – 41
- Tauchen, 1986, Finite state markov-chain approximations to univariate and vector autoregressions, Economics Letters
- Wouter den Haan Notes *http://www.wouterdenhaan.com/notes*