Heterogeneous Agents Models & Algorithms

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October 26, 2022

Extend the neoclassical growth model to heterogeneous agents

- role of incomplete markets
- role of aggregate uncertainty
- solving the Hugget/Aiyagari model

Income fluctuation (many states) 1/2

- Consider the following two-period problem

$$\begin{aligned} \max &\left\{ u(c_0) + \beta \textstyle\sum_{s \in \mathscr{S}} u(c_1(s)) \Pr(s) \right\} \\ \text{s.t.} \quad & c_0 + \textstyle\sum_{s \in \mathscr{S}} q(s) a(s) = \mathscr{S}_0 \\ & c_1(s) = \mathscr{S}_1(s) + a(s) \end{aligned}$$

- Where s is idiosyncratic: $\sum_s \mathscr{S}_1(s) \Pr(s) = \mathscr{S}_1$

Income fluctuation (many states) 2/2

- First order conditions

$$u'(c_0)\, q(s) = \beta u'(c_1(s)) \Pr(s), \quad \forall s \in \mathscr{S}$$

- Then there exist equilibrium asset prices

$$q(s) = \frac{\beta u'(\mathscr{S}_1)}{u'(\mathscr{S}_0)} \Pr(s)$$

- Such that **in the aggregate**: $c_0 = \mathscr{S}_0$ and $c_1 = \mathscr{S}_1$.

Income fluctuation (single state) 1/2

- Consider the following two-period problem

$$\max \{u(c_0) + \beta u(c_1)\}$$

$$\text{s.t.} \quad c_0 + qa = \mathscr{S}_0$$

$$c_1 = \mathscr{S}_1 + a$$

Income fluctuation (single state) 2/2

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$$u'(c_0)\,q = \beta u'(c_1)$$

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$$q = \frac{\beta u'(\mathscr{S}_1)}{u'(\mathscr{S}_0)}$$

- Such that $c_0 = \mathscr{S}_0$ and $c_1 = \mathscr{S}_1$.

Contrasting the two problems

First problem:

- Many agents and many states
- Lots of trade in assets

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Representative agent models are justified by complete markets which relies on lots of trade

Building a model with heterogenous households

Key:

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* Agents are ex ante the same, but face different idiosyncratic shocks $\implies$ agents are different ex post
* Incomplete markets $\implies$ heterogeneity cannot be insured away

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Individual household-

- S.t. to employment shocks:
  - $\varepsilon_{i,t} \in \{0, 1\}$
- only way to save is through holding bonds (Hugget) and capital (Aiyagari) s.t. borrowing constraint

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Economy-

- S.t. to aggregate shock through productivity (Krusell-Smith)
  - $z_t \in \left\{z^b, z^g\right\}$ affects: (1) productivity, (2) $\mathbb{P}$ employed

The Hugget Model

* One-period obligation contracts is the only source of insurance (bonds).
* More goods (states) than markets $\implies$ incomplete markets
* No aggregate shocks to the economy, and therefore no aggregate risk

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$$v(b,s) = \max_{c,b'} \left\{ u(c) + \beta \sum_{s' \in \mathscr{S}} v\left(b', s'\right) p\left(s', s\right) \right\}$$

s.t $c + b' = (1+r)b + w(s)$

$$b' \geq \underline{b}$$

with associated policy function $b' = g(b,s)$

Hugget FOCs and Solution

$$u'\left((1+r)b + w(s) - b'\right) - \mu(b,s)$$
$$= \beta(1+r)\sum_{s' \in \mathscr{S}} u'\left((1+r)b' + w\left(s'\right) - b''\right) p\left(s', s\right)$$

where μ is the Lagrange mult. on borrowing constraint.

Hugget FOCs and Solution

$$u'\left((1+r)b + w(s) - b'\right) - \mu(b,s)$$
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where μ is the Lagrange mult. on borrowing constraint.

- Solution method: Find $\tilde{g}(b,s)$ as

$$\begin{aligned} & u'((1+r)b + w(s) - \tilde{g}(b,s)) \\ & = \beta(1+r) \sum_{s'\in\mathscr{S}} \left[u'\left((1+r)\tilde{g}(b,s) + w\left(s'\right)\right.\right. \\ & \quad \left.\left. - g_n\left(\tilde{g}(b,s), s'\right)\right)\right] p\left(s', s\right) \end{aligned}$$

Hugget FOCs and Solution

$$u'\left((1+r)b+w(s)-b'\right)-\mu(b,s)$$
$$=\beta(1+r)\textstyle\sum_{s'\in\mathscr{S}} u'\left((1+r)b'+w\left(s'\right)-b''\right)p\left(s',s\right)$$

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$$\begin{aligned}
&u'((1+r)b+w(s)-\tilde{g}(b,s)) \\
&=\beta(1+r)\sum_{s'\in\mathscr{S}}\left[u'\left((1+r)\tilde{g}(b,s)+w\left(s'\right)\right.\right. \\
&\qquad\left.\left.-g_n\left(\tilde{g}(b,s),s'\right)\right)\right]p\left(s',s\right)
\end{aligned}$$

- Then $g_{n+1}(b,s)=\max\{\tilde{g}(b,s),\underline{b}\}$.

Cross-sectional distribution of wealth

- Ok, so suppose that we have found $b' = g(b, s)$ conditional on some r, now what?
- As there are idiosyncratic risk, each individual will be exposed to different shocks in different periods
  - individual indexed by history of shocks

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Key object: cross-sectional distribution of wealth-holdings

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Key object: cross-sectional distribution of wealth-holdings

Together with (stochastic) law of motion for income, $p(s', s)$, calculate law of motion for ψ_t:

$$\psi_{t+1}(b', s') = \sum_{s \in \mathscr{S}} \sum_{\{b : b' = g(b,s)\}} \psi_t(b, s) p(s', s)$$

which converges to invariant distribution.

Definition of equilibrium in Hugget

A competitive equilibrium consists of an interest-rate r such that:

- Given r, the policy function $g(b, s)$ solves the household's optimization problem.
- The stationary distribution satisfies.

$$\psi\left(b', s'\right) = \sum_{s \in \mathscr{S}} \sum_{\{b : b' = g(b,s)\}} \psi(b, s) p\left(s', s\right)$$

- Markets clear. That is $B(r) = \sum_{s \in \mathscr{S}} \sum_{b} b\psi(b, s) = 0$.

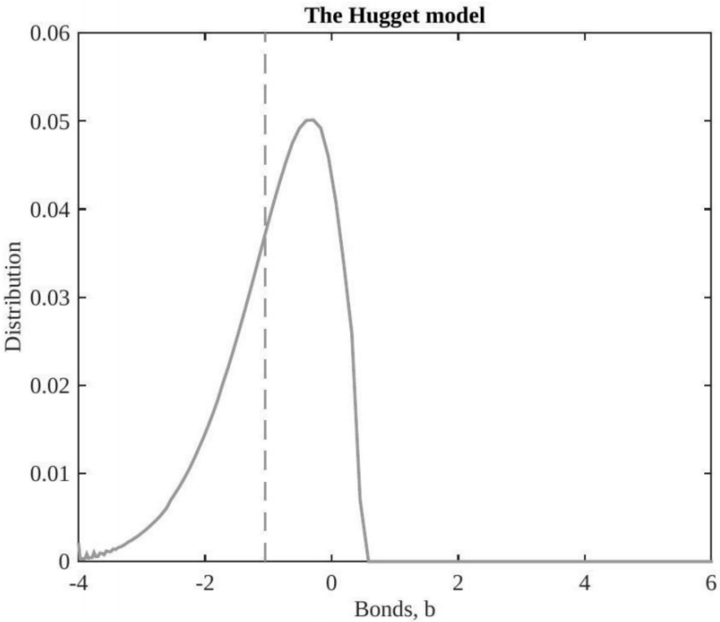
A note of the distribution ψ

What is the probability that the agent will have some arbitrary wealth b and arbitrary s in t-periods?

- We can calculate this using the exact same recursive formula as for the cross-sectional distribution!
- But as for any $\psi_0, \psi_t \to \psi$, the probability of (b, s) occurring when $t \to \infty$ must be $\psi(b, s)$.
- So ψ is not only the cross-sectional distribution of wealth, but also the unconditional distribution of wealth for an individual

Hugget Algorithm

1. Guess for an interest-rate, r.
2. Solve the consumer's problem (*VFI, PF iteration*) $\implies$ policy functions $b' = g(b, s; r)$
3. Calculate the cross-sectional distribution $\psi(b, s)$.
4. If excess demand is positive, adjust the interest-rate downwards. If negative upwards.


The Hugget model
Distribution
Bonds, b
0.06
0.05
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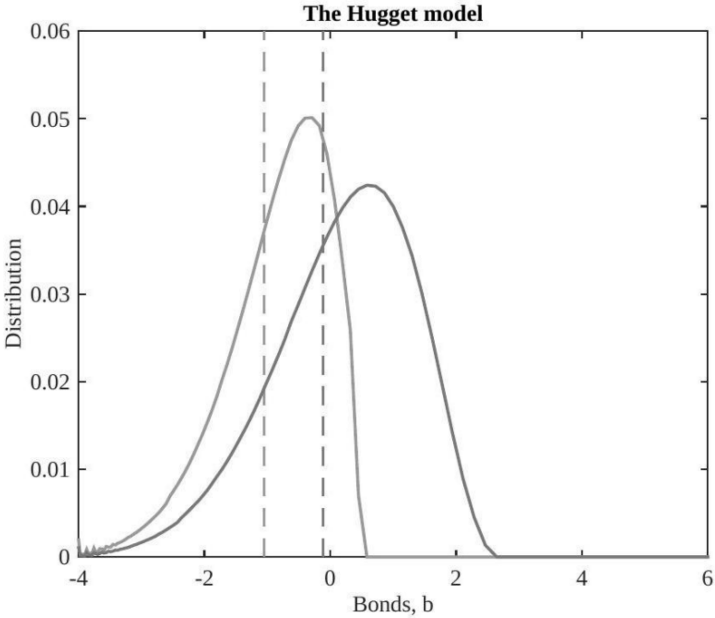
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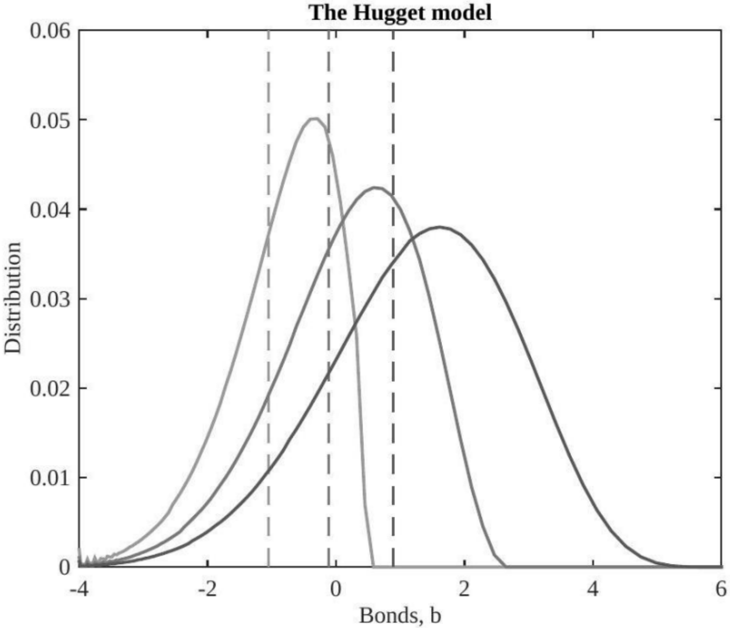

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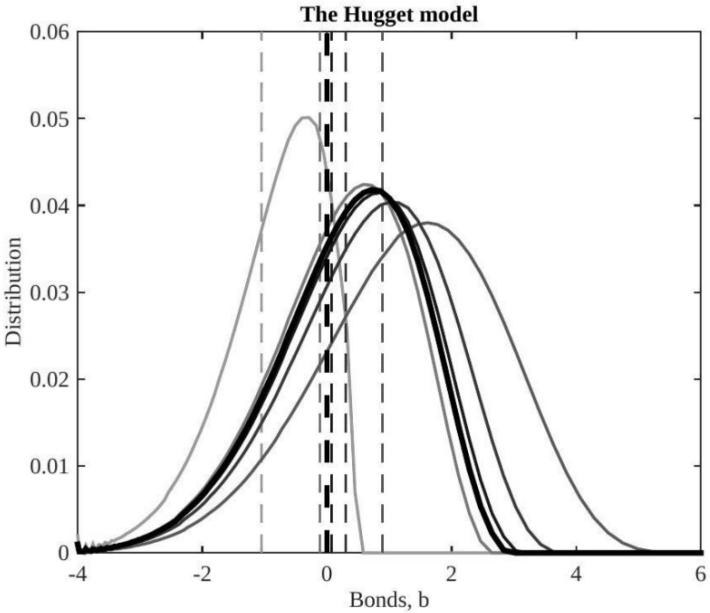

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Bonds, b

How to adjust r? Bisection

- Suppose that $f(x)$ is continuous and monotone and $f(\bar{x}) > 0$ but $f(\underline{x}) < 0$.
- Then pick $x = \frac{\bar{x}+\underline{x}}{2}$, and evaluate $f(x)$.
- If $f(x) > 0$, set $\bar{x} = x$ and repeat. Else set $\underline{x} = x$ and repeat.
- Eventually you will find an x such that $f(x) = 0$.
- In our case, $f(x) = B(r)$, and $\bar{x} = \frac{1}{\beta} - 1 - \varepsilon$ and $\underline{x} = 0$.

The Aiyagari Model

- **Main difference:** households both underwrite debt contracts to each other (bonds) but also lend out resources to firms which are then used as investments
  - positive savings in the economy which determines the capital stock
- Wages are not simple endowments, but paid by firms in a competitive market

Given w and r, the household's problem is given by,

$$v(a,s) = \max_{c,a'} \{ u(c) + \beta \sum_{s' \in \mathscr{S}} v(a', s') \, p(s', s) \Bigg\}$$

$$\text{s.t} \quad c + a' = (1+r)a + ws + \mu w(1-s)$$

$$a' \geq \underline{a}$$

Notice too that v (the stationary distribution of the transition matrix) gives us the employment rate, $(1-u)$, and the unemployment-rate, u.

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Notice too that v (the stationary distribution of the transition matrix) gives us the employment rate, $(1-u)$, and the unemployment-rate, u.

- Firms can hire workers on a labor sport market at wage rate w, and rent capital at the interest rate $\tilde{r}$.
- CRS technology $F(k,n)$, representative firm

$$\max_{k,n} \{F(k,n) - nw - k\tilde{r}\}$$

The Aiyagari FOC

$$\tilde{r} = F_k(k, n), \quad w = F_n(k, n)$$

The Aiyagari FOC

$$\tilde{r} = F_k(k, n), \quad w = F_n(k, n)$$

1. net interest rate : $r = \tilde{r} - \delta$.
2. Let $A(r) = \sum_{s \in \mathscr{S}} \sum_a a\psi(a, s) \implies$ market clearing $k = A$ and $n = (1 - u)$.

Aiyagari equilibrium

A competitive equilibrium is an interest-rate r and a wage-rate w, such that:

- Given r and $w, g(a, s)$ solves the households problem.
- Given r and w, k and n solves the firms problem.
- The stationary distribution ψ satisfies

$$\psi\left(a', s'\right) = \sum_{s \in \mathscr{S}} \sum_{\{a : a' = g(a,s)\}} \psi(a, s) p\left(s', s\right)$$

- Markets clear: $k = \sum_{s \in S} \sum_{a} a\psi(a, s)$ and $n = (1 - u)$

Aiyagari Algorithm (By Simulation)

1. Set an initial guess for the interest rate $r^0 \in (-\delta, 1/\beta - 1)$ such that $r_{\text{min}} = -\delta$ and $r_{\text{max}} = 1/\beta - 1$
2. I know that the equilibrium interest rate must lie within this bracket, so I set:

$$r^0 = \frac{r_{\text{min}} + r_{\text{max}}}{2}$$

The interest rate r^0 is the first candidate for the equilibrium (the superscript denotes the iteration number).

3. Compute the aggregate demand for capital implied by the interest rate $r^0, K^d\left(r^0\right)$, using:

$$K^d\left(r^0\right) = N\left(\frac{r^0 + \delta}{\theta}\right)^{-\frac{1}{1-\theta}}$$

where N is the stationary distribution of labour, defined as the eigenvector of Π' associated with its unit eigenvalue.

4. Given the interest rate r^0, solve for the implied wage rate $w\left(r^0\right)$ using:

$$w\left(r^0\right) = (1-\theta)\left(\frac{\theta}{r^0+\delta}\right)^{\frac{\theta}{1-\theta}}$$

which is implied by (6) and (7).

5. Given prices $\left(r^0, w\left(r^0\right)\right)$, solve household problem to obstain decision rules $a' = g\left(a, l; r^0\right)$ and $c = h\left(a, l; r^0\right)$.

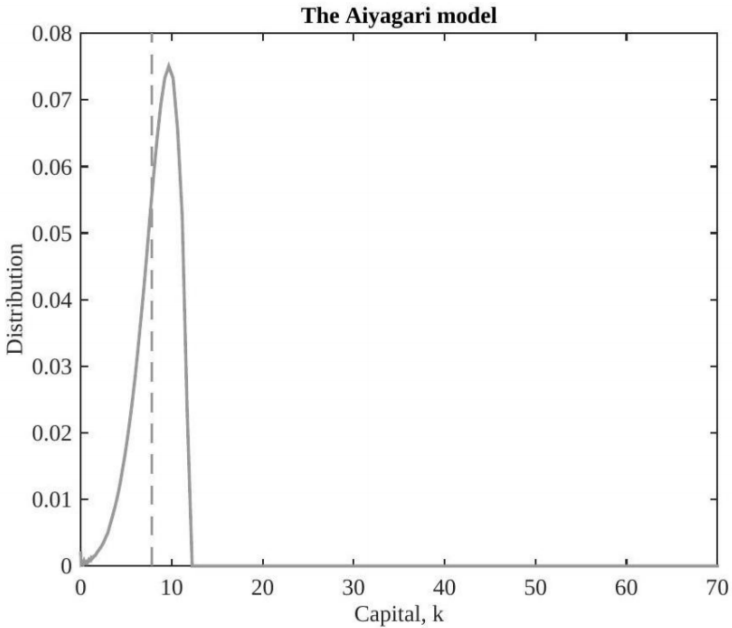
6. Compute the stationary distribution of asset holdings by simulating a sample of N_h households for T periods.

- Choose a sample size of N_i households and a sample length T. To prevent initial values from influencing results, set a burn-in period T_{burn} such that the first T_{burn} observations are discarded when calculating the statistics of the distribution.
- Simulate the T-period discretised Markov chain for the labour endowment shock for each individual $i = 1, \ldots, N_i$. Also a initialise each household i with an initial asset holding $a_{i,0}$.
- Compute $a_{i,t} = g\left(a_{i,t-1}, l_{i,t-1}\right)$ for all individuals $i = 1, \ldots, N_i$ and all time periods $t = 1, \ldots, T$
- Calculate the mean asset holdings for each household $i = 1, \ldots, N_i$ from $t = T_{\text{burn}} + 1$ to $t = T$. This gives a measure of the capital supply implied by the interest rate $r^0, K^s\left(r^0\right)$.

7. If $K^d\left(r^0\right) > K^s\left(r^0\right)$ (i.e. there is excess demand for capital), then the equilibrium interest rate must exceed r^0. Then, replace $r_{\min}$ with r^0. Alternatively, if $K^d\left(r^0\right) < K^s\left(r^0\right)$ (i.e. there is excess supply of capital), then the equilibrium interest rate must be below r^0. Then replace $r_{\max}$ with r^0.
8. Return to step 2 , or end the algorithm when:

$$\left|r_{\max} - r_{\min}\right| < \epsilon$$

where ϵ is a pre-specified tolerance level.


The Aiyagari model
Distribution
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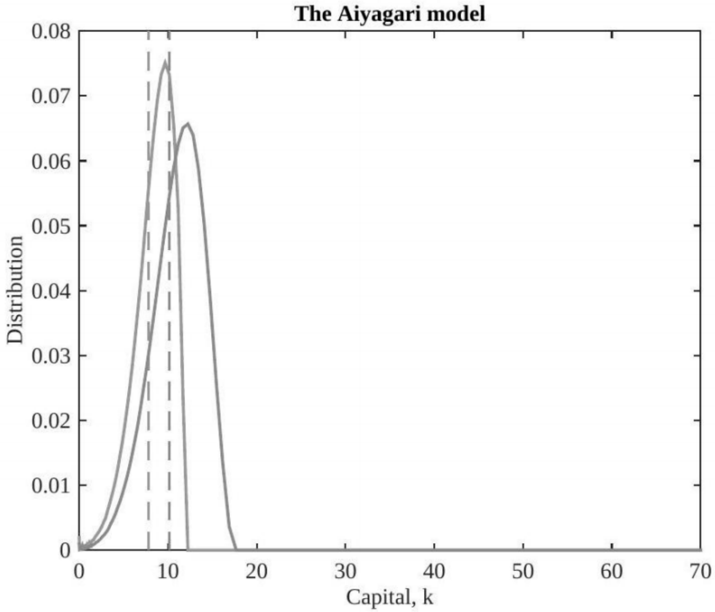
Capital, k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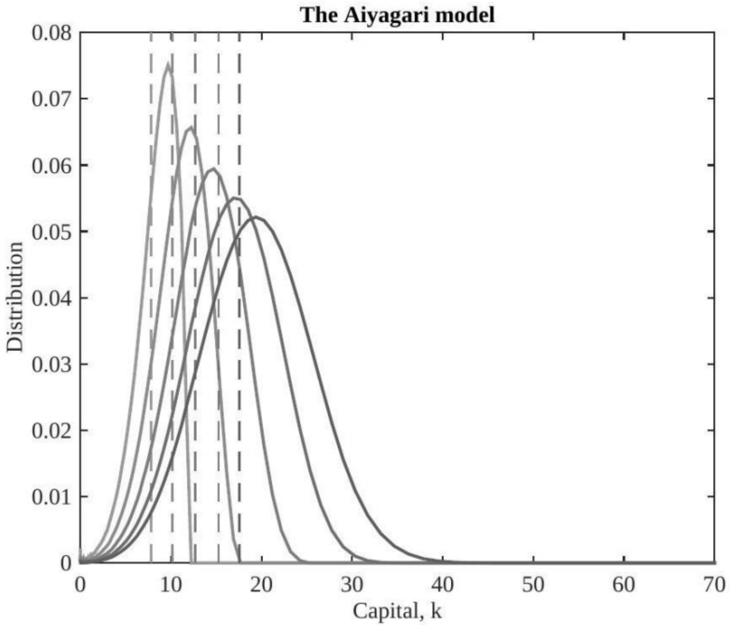
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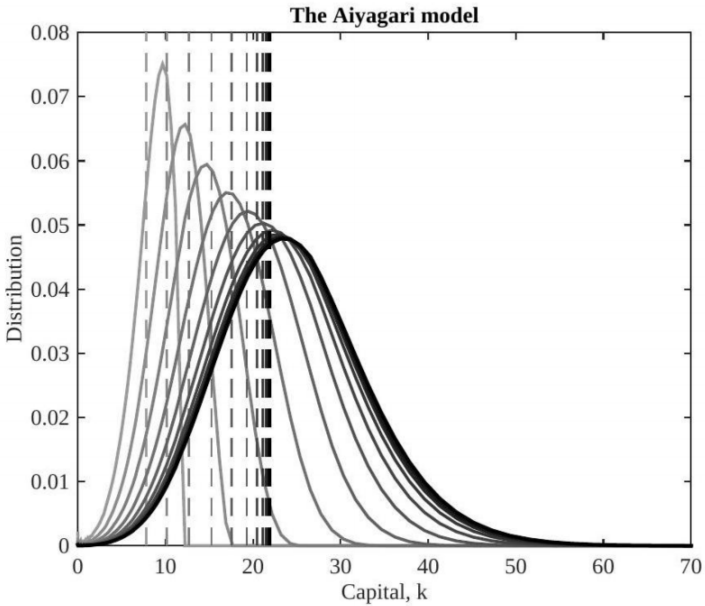
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Aiyagari Algorithm (By iterating on the distr.)

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2. I know that the equilibrium interest rate must lie within this bracket, so I set:

$$r^0 = \frac{r_{\text{min}} + r_{\text{max}}}{2}$$

The interest rate r^0 is the first candidate for the equilibrium (the superscript denotes the iteration number).

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where N is the stationary distribution of labour defined as the eigenvector of Π' associated with its unit eigenvalue.

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5. Given prices $\left(r^0, w\left(r^0\right)\right)$, solve the household problem to obtain decision rules $a' = g\left(a, l; r^0\right)$ and $c = h\left(a, l; r^0\right)$.

6. Given the policy function $a' = g\left(a, l; r^0\right)$ and the transition function Π for l_t, use (8)

$$f^*(A \times B) = \int \mathbf{1}_{g(a,l)\in A} \Pi(B, l) \mathrm{d} f^*(a, l)$$

to solve for a stationary distribution $f^*\left(r^0\right)$. Compute the aggregate supply of capital $K^s\left(r^0\right)$:

$$K^s\left(r^0\right) = \int g(a, l) \mathrm{d} f^*\left(a, l; r^0\right)$$

7. If $K^d(r^0) > K^s(r^0)$ (i.e. there is excess demand for capital), then the equilibrium interest rate must exceed r^0. Then, replace $r_{\min}$ with r^0. Alternatively, if $K^d(r^0) < K^s(r^0)$ (i.e. there is excess supply of capital), then the equilibrium interest rate must be below r^0. Then replace $r_{\max}$ with r^0.

8. Return to step 2 , or end the algorithm when:

$$|r_{\max} - r_{\min}| < \epsilon$$

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A quick look at aggregate uncertainty

Main drawback of Aiyagari: Aggregate variables (Y, C, K) constant - no business cycle properties

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Krusell and Smith (1998)

- $*$ Aiyagari + aggregate shocks to TFP.

$\Longrightarrow$ No stationary distribution. No constant prices.

- $*$ main contribution of K&S is work around this problem in a very accurate way $\equiv$ "approximate aggregation".

Household problem

- Let $s_t = (e_t, z_t)$, and $s^t = ((e_0, z_0), (e_1, z_1), \ldots (e_t, z_t))$.
- Then taking price processes $r_t(s^t)$ and $w_t(s^t)$ as given, the household's optimization problem is given by,

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s_t \in \mathscr{S}^{t+1}} \beta^t u\left(c_t\left(s^t\right)\right) f_t\left(s^t\right)$$

$$c_t\left(s^t\right) + a_{t+1}\left(s^{t+1}\right) = (1 + r_t)\, a_t\left(s^{t-1}\right) + w\left(s_t\right)$$

$$a_{t+1}\left(s^t\right) \geq \underline{a}$$

$$a_0, s_0, \text{ given}$$

- Where, of course, f_t (and p), is defined as previously using the transition matrix P over s^t

Krusell-Smith equilibrium.

- A competitive equilibrium are prices $\{r_t(s^t), w_t(s^t)\}_{t=0}^{\infty}$ such that
- Given prices, $g(a, e, z, x)$ solves the households problem.
- Given prices, $\{k_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ solves the firms problem.
- Markets clear: $k_t = \sum_{e \in \{0,1\}} \sum_a a\psi_t(a, e)$ and
$$n_t = (1 - u_t).$$
- **Rationality**: $r_{t+1} = r(z_{t+1}, \phi(z_{t+1}, z_t, x_t))$ and
$$w_{t+1} = w(z_{t+1}, \phi(z_{t+1}, z_t, x_t)).$$

“Limited Rationality” approximation

What if only mean K matters and: $K' = a_z + \beta_z K$:

$$\approx \sum_{e'} \sum_{a'} a' \psi_{t+1}\left(a', e'\right)$$

"Limited Rationality" approximation

What if only mean K matters and: $K' = a_z + \beta_z K$:

$$\approx \sum_{e'} \sum_{a'} a' \psi_{t+1} \left(a', e'\right)$$

Now, much easier problem, find a_z, β_z. Replace rationality in equilibrium with:

- Approximate Aggregation:

$$K' = \alpha_z + \beta_z K \approx \sum_{e'} \sum_{a'} a' \psi_{t+1} \left(a', e'\right).$$

Krusell & Smith algorithm

1. Guess for α_z and β_z (e.g. $\alpha_z = 0$ and $\beta_z = 1$).
2. Solve the household's problem to get the policy function

$$a' = g(a, e, z, K).$$

3. Use this policy rule and simulate the savings behavior of a long ($\mathrm{T} = 6000$) panel of many individuals (a continuum).
4. Regress K_{t+1} on K_t and a constant, conditional on z_t.
5. If your coefficient from the regression matches your guessed α_z and β_z you're done!
6. Otherwise update α_z and β_z, and repeat.
7. Once convergence is obtained, check for accuracy of your laws of motion for aggregate capital.