My grades for **Mid-term I**

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1. **Spin** (30 points)

   (a) (2 points) What is the dimension of the relevant Hilbert space for a spin-2 particle?

   (b) (4 points) What is the dimension of the relevant Hilbert space for a two-particle system consisting of one spin-$\frac{3}{2}$ particle and one spin-1 particle?

   (c) (8 points) A spin-$\frac{1}{2}$ particle is in the state $|\psi\rangle = N\left[\sqrt{2}\,|\uparrow\rangle + i\,|\downarrow\rangle\right]$, where $|\uparrow\rangle$ and $|\uparrow\rangle$ denote the spin up and down eigenstates for $\hat{S}_z$, respectively. Calculate the normalization factor N, and then calculate the probabilities to measure this particle in $|+x\rangle$ and $|-y\rangle$ states.

   (d) (8 points) Calculate the expectation values of the three spin operators $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, $\langle \hat{S}_z \rangle$ for the spin-$\frac{1}{2}$ state $|\psi\rangle$ given in (c).

   (e) (8 points) Using the communication relation $\left[\hat{S}_x, \hat{S}_y\right] = i\hbar \hat{S}_z$ and the spin-$\frac{1}{2}$ state $|\psi\rangle$ given in (c), show that uncertainty principle holds.

Start writing your answers to question 1 here:

a) $Dim = (2S+1)$, $S=2 \Rightarrow Dim = 5$

b) $Dim = (2S_1+1)(2S_2+1) = 4\cdot 3 = 12$

c)

$$|\psi\rangle = N\left[\sqrt{2}|\uparrow\rangle + i|\downarrow\rangle\right] \Leftrightarrow \langle\psi|\psi\rangle = N^2\left[\langle\uparrow|2|\uparrow\rangle + \langle\downarrow|\downarrow\rangle\right]$$

$$= N^2(2+1) = N = 1/\sqrt{3}$$

$$P(+x) = \frac{1}{3}\left|\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} \\ i\end{pmatrix}\right|^2 = \frac{1}{3}\left|1 + \frac{i}{\sqrt{2}}\right|^2 = \frac{1}{3}\left(1+\frac{i}{\sqrt{2}}\right)\left(1-\frac{i}{\sqrt{2}}\right) = \frac{1}{3}\left(1+\frac{1}{2}\right) = \frac{1}{2}$$

$$P(-y) = \frac{1}{3}\frac{1}{2}\left|\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\sqrt{2} \\ i\end{pmatrix}\right|^2 = \frac{1}{6}\left|\sqrt{2}-1\right|^2 = \frac{1}{6}\left(2-2\sqrt{2}+1\right) = \frac{3-2\sqrt{2}}{6} = \frac{1}{2} - \frac{\sqrt{2}}{3}$$

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Continue your answers to Question 1 here:

d)

$$\langle S_x\rangle = \frac{1}{3}\frac{\hbar}{2}\begin{pmatrix}\sqrt{2} & -i\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ i\end{pmatrix} = \frac{\hbar}{6}\begin{pmatrix}\sqrt{2} & -i\end{pmatrix}\begin{pmatrix}i\\ \sqrt{2}\end{pmatrix} = i\sqrt{2}-i\sqrt{2} = 0$$

$$\langle S_y\rangle = \frac{\hbar}{6}\begin{pmatrix}\sqrt{2} & -i\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ i\end{pmatrix} = \frac{\hbar}{6}\begin{pmatrix}\sqrt{2} & -i\end{pmatrix}\begin{pmatrix}1\\ i\sqrt{2}\end{pmatrix} = \frac{\hbar}{6}\left(\sqrt{2}+\sqrt{2}\right) = \frac{\sqrt{2}\hbar}{3}$$

$$\langle S_z\rangle = \frac{\hbar}{6}\begin{pmatrix}\sqrt{2} & -i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ i\end{pmatrix} = \frac{\hbar}{6}\begin{pmatrix}\sqrt{2} & -i\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ -i\end{pmatrix} = \frac{\hbar}{6}(2-1) = \frac{\hbar}{6}$$

e)

$$\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}\cdot Id_{2\times 2}$$

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2},\quad \Delta S_y = \sqrt{\frac{\hbar^2}{4} - \frac{2\hbar^2}{9}} = \sqrt{\frac{9\hbar^2 - 8\hbar^2}{36}} = \frac{\hbar}{6}$$

$$\Delta S_x\cdot\Delta S_y \geq \frac{1}{2i}\, i\hbar S_z \iff \frac{\hbar^2}{12} \geq \frac{1}{2}\hbar\left(\frac{\hbar}{6}\right) \iff \frac{\hbar^2}{12} \geq \frac{\hbar^2}{12} \text{ holds}$$

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Continue your answers to Question 1 here:

2. **Step potential** (20 points + 2 bonus points)
Consider a particle of mass m moving in a one-dimensional step potential defined as follows:

$$V(x) = \begin{cases} 0 & \text{for } x < 0, \\ V_0 & \text{for } x \geq 0, \end{cases}$$

where V_0 is a positive constant.

(a) (4 points) Write down the time-independent Schrödinger equation for the particle in the regions $x < 0$ and $x \geq 0$.

(b) (2 points) Do bound state solutions exist for this potential? Briefly explain why.

(c) (6 points) For the energy of the particle E, in the case of $E \geq V_0$, write down the form of solutions to the time-independent Schrödinger equation for the wavefunctions $\psi(x)$ in both regions. Assume the particle is moving from the left to right, and there is no particle coming from the right side.

(d) (4 points) Apply the boundary conditions at $x = 0$ to find the relations of the coefficients defined in the solutions in (c).

(e) (4 points) Calculate the reflection coefficient R for $E \geq V_0$.

(f) (* Bonus 2 points) What will happen to the reflection if $E < V_0$?

Start writing your answers to Question 2 here:

a) For $x<0$, | For $x \geq 0$

$-\frac{\hbar^2}{2m}\psi = E\psi$ | $-\frac{\hbar^2}{2m}\psi + V_0\psi = E\psi$

b) V_0 is positive and so a bound state must exist

c) $E \geq V_0 \Rightarrow$ Scattering states

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & \text{for } x<0 \\ Ce^{ik_2x} & \text{for } x \geq 0 \end{cases}$$

And we define $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

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Continue your answers to Question 2 here:

At $x=0$, continuity of ψ

$A+B=C$ and continuity of ψ': $Ak_1 - Bk_1 = k_2 C$

$\Leftrightarrow k_1(A-B) = k_2(A+B) \Leftrightarrow A(k_1-k_2) = B(k_2+k_1)$

$$\Leftrightarrow \frac{B}{A} = \frac{(k_1-k_2)}{(k_1+k_2)}$$

e) $$R = \left|\frac{B}{A}\right|^2 = \frac{k_1^2 - 2k_1k_2 + k_2^2}{k_1^2 + 2k_1k_2 + k_2^2} = \frac{\frac{2mE}{\cancel{\hbar^2}} - \frac{2\sqrt{2mE}\sqrt{2m(E-V_0)}}{\cancel{\hbar^2}} + \frac{2m(E-V_0)}{\cancel{\hbar^2}}}{\frac{2mE}{\cancel{\hbar^2}} + \frac{2\sqrt{2mE}\sqrt{2m(E-V_0)}}{\cancel{\hbar^2}} + \frac{2mE-V_0}{\cancel{\hbar^2}}}$$

$$\Leftrightarrow \frac{(k_1-k_2)}{(k_1+k_2)} \cdot \frac{(k_1+k_2)}{(k_1+k_2)} = \frac{k_1^2-k_2^2}{(k_1+k_2)^2} = R$$

f) $R \to 1$ as everything is reflected

Continue your answers to Question 2 here:

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