Inverse Problems - Assignments 1

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The full code is found in the `.ipynb` file alongside similar explanations/answers to the questions as in this pdf.

The direct problem

1. Arrival-time anomalies

The arrival-time anomalies are calculated with the function: `time_anomalies`.
The function takes a matrix as input, which is the area of interest discretized into 1m x 1m squares and has the value 1 in the grey zone and 0 elsewhere. For each ray the function sums the value of squares, that the ray passes through. The sum is then multiplied by a constant, which is the difference in the inverse velocities and the square root of 2: $k = \sqrt{2}(\frac{1}{v_0} - \frac{1}{v_G})$. The result is the time anomaly for each ray.

Rays from source 1:

Detector	1	2	3	4	5	6	7	8	9	10	11	12
Time anomaly	0.	0.	0.	0.	0.	0.011	0.022	0.033	0.033	0.033	0.033	0.033

Rays from source 2:

Detector	1	2	3	4	5	6	7	8	9	10	11	12
Time anomaly	0.033	0.033	0.033	0.022	0.011	0.	0.	0.	0.	0.	0.	0.

2. Discretizing equation 1

Equation 1 from the assignment is discretized in the following way:

$$t_\gamma = \sqrt{2}(\frac{1}{v_0} - \frac{1}{v_G}) \sum_j G_{\gamma,j} \cdot m_j$$

where

- t_γ: the arrival time anomaly for a wave propagating along a ray γ
- v_0: the wave propagation velocity outside the grey zone.
- v_G: the wave propagation velocity inside the grey zone.
- G_γ: matrix describing the path of the rays. This matrix is constructed by creating a 13x11 matrix for each of the rays, where the value 1 is assigned to all the squares where the ray passes through and 0 to the others. The matrices for each of the rays are then flattened and stacked vertically to create the final matrix G. In this way, each row in G will describe one ray γ path through the area. The factor of $\sqrt{2}$ is to get the correct distance.

- m: the vector containing the information on the location of the grey zone. The vector is obtained by flattening the 13x11 area matrix, which has the value 1 assigned to the squares included in the grey zone and the value 0 to the squares not included in the grey zone.

The inverse problem

3. Formulation of the inverse problem

This is done by constructing the matrix for the path of the rays, $\mathbf{G}$ and the vector $\mathbf{m}$ with the information on the location of the grey area. Figure 1 shows a visual representation of the matrix $\mathbf{G}$. Ray 1-12 is from source 1, and ray 13-24 is from source 2.

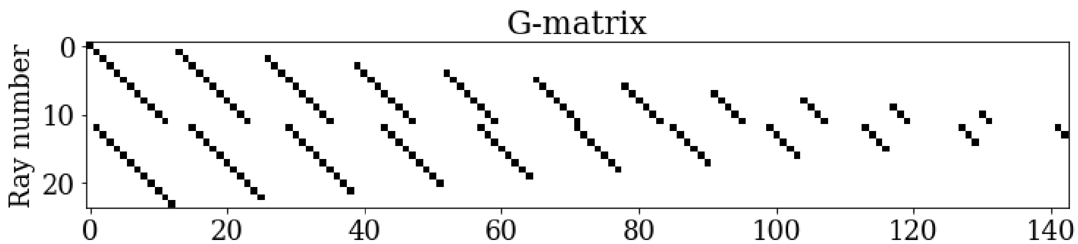

Figure 1: Visual representation of the matrix $\mathbf{G}$. Ray 1-12 is from source 1, and ray 13-24 is from source 2.

4. Show that the problem is linear

The problem can be described as a matrix equation of the form (a forward relation):

$$\mathbf{d_{obs}} = \mathbf{Gm}$$

Whereby it is shown that the problem is linear.

5. Uniqueness of the solution

The solution to the problem is not unique, as there are 143 model parameters and 24 data points. The system is under-determined.

6. Solution with Tikhonov regularization

At first the function `get_data` is used to simulate the observed data vector t_{obs} by adding noise to the pure data vector t_{pure} as described in the assignment. The noise distribution is scaled to satisfy eq. 3 in the assignment by dividing the initial noise distribution by its own norm to normalize it, then multiplying it by the result of eq. 3 from the assignment.

Then the function `Tikhonov` is used to estimate the model $\tilde{\mathbf{m}}$, i.e. the position of the grey area. Tikhonov regularization can be used on mixed-determined systems, i.e. systems, which are both over- and under-determined or it can be used on under-determined problems, as in this case. Tikhonov regularization is based on the minimization of (for appropriate $\epsilon > 0$):

$$||\mathbf{Gm} - \mathbf{d_{obs}}||^2 - \epsilon^2 ||\mathbf{m}||^2$$

This can be done by solving the normal equations for different values of ϵ:

$$\tilde{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \epsilon^2 \mathbf{I})^{-1} \mathbf{G}^T \mathbf{d_{obs}}$$

The optimal ϵ-value is the one satisfying:

$$||\mathbf{G}\tilde{\mathbf{m}} - \mathbf{d_{obs}}|| \approx N\sigma^2$$

where N is the number of data point (24) and σ is the norm of the noise distribution described in eq. 3 in the report.

The function `optimum` is used to find the best epsilon value that satisfies the criterion above. The function returns the index and value of the epsilon that is closest to satisfy the criterion and the corresponding model parameters.

The solution obtained with this approach is shown in Fig. 2 and the optimal ϵ-value is 2.03.

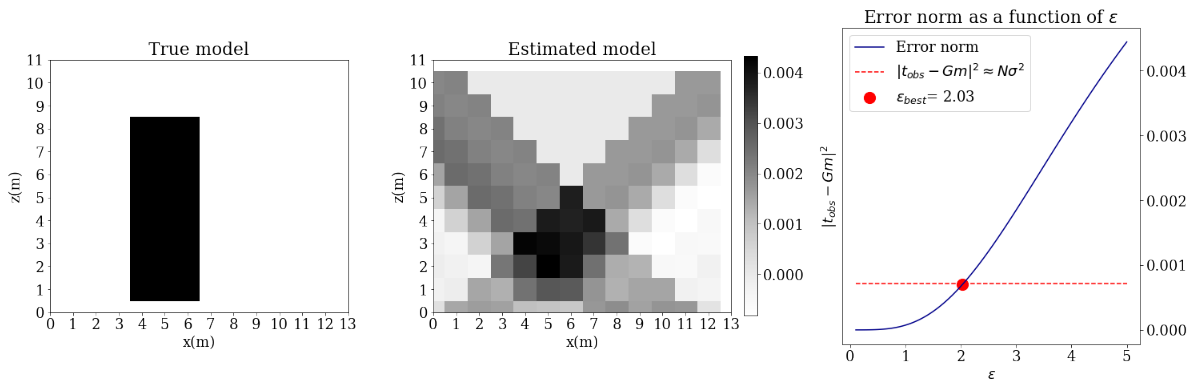

Figure 2: The solution obtained using Tikhonov regularization. The optimal ϵ-value is 2.03.

The estimated model (inverse operator) is able to figure out, that a zone (the black squares in the estimated model) with a higher wave propagation velocity than the surrounding area exist. The estimated model locates that part of the zone with most rays passing through in the true model with the highest accuracy. The estimated model also assigns a higher wave propagation velocity than the surroundings to the squares (the grey diagonals in the estimated model), where the rays in the true model are travelling the longest distance in the grey zone. The estimated model is not able to capture the hard boundary of the grey zone in the true model. The true model has an absence of noise, which is visible in the estimated model, as the surrounding area has varying wave propagation velocities.

7. Quality of the inverse operator

The same method is used on a delta-function as the true model. Two different locations for the single square was used. One location where two rays passed through the square - see Fig. 3, and one location where only one ray passed through it - see Fig. 4.

In the first case, two rays passes through the square and the estimated model is doing a fine job of locating the square. The estimated model is assigning a higher wave propagation velocity along the path of the two rays (dark grey squares along diagonals), as we also saw for the full problem. In the surrounding area the wave propagation velocity is differing a little which is due to noise.

In the second case, only one ray passes through the square in the true model, and the estimated model therefore assigns the same wave propagation velocity to all the squares along the path of the ray and is unable to locate a single position for the square in the true model.

8. Differences between the solution and the true model

See comments below Fig. 2.

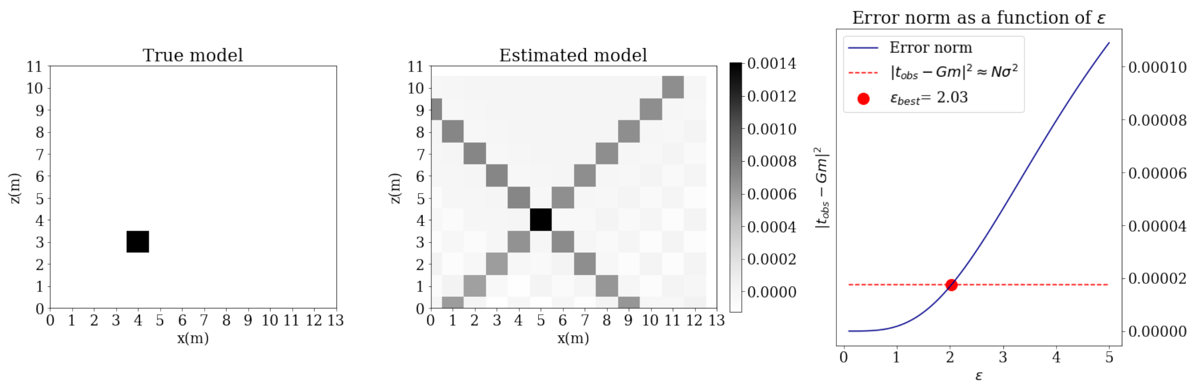

Figure 3: Solution to the true model being a delta function, where two rays pass through the single square.

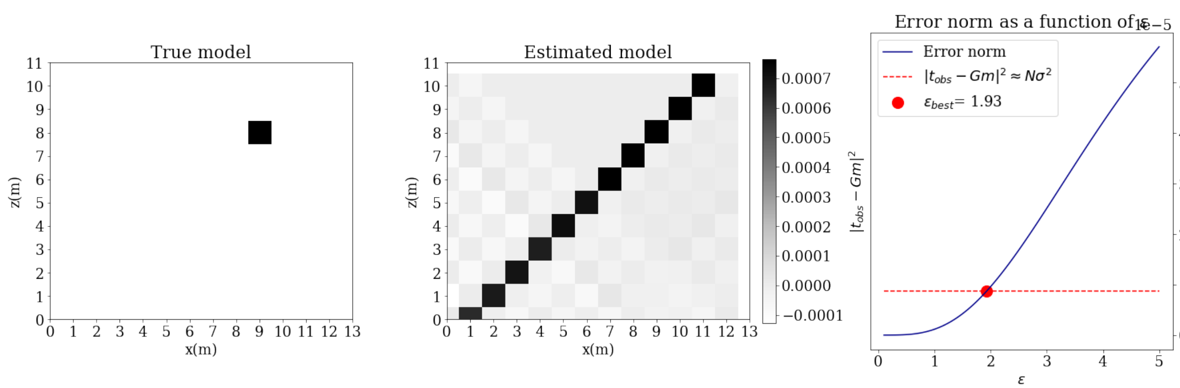

Figure 4: Solution to the true model being a delta function, where one ray pass through the single square.