Sequential Decision Making

Lecture 5 : Dynamic Programming

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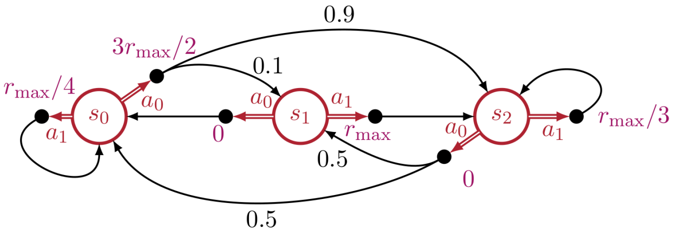
M2 Data Science, 2022/2023

Reminder : Markov Decision Process

A MDP is parameterized by a tuple $(\mathcal{S}, \mathcal{A}, R, P)$ where

- $\mathcal{S}$ is the state space
- $\mathcal{A}$ is the action space (or $\mathcal{A}_s$ for each $s \in \mathcal{S}$)
- $R = (\nu_{(s,a)})_{(s,a)\in\mathcal{S}\times\mathcal{A}}$ where $\nu_{(s,a)} \in \Delta(\mathbb{R})$ is the reward distribution for the state-action pair (s,a) $\rightarrow r(s,a) = \mathbb{E}_{R\sim\nu_{(s,a)}}[R]$
- $P = (p(\cdot|s,a))_{(s,a)\in\mathcal{S}\times\mathcal{A}}$ where $p(\cdot|s,a) \in \Delta(\mathcal{S})$ is the transition kernel associated to the state-action pair (s,a)



Reminder : Policy

A policy $\pi = (\pi_0, \pi_1, \dots)$ is a sequence of mapping $\pi_t : \mathcal{S} \to \Delta(\mathcal{A})$ that maps a state to a distribution over actions.

Under policy $\boldsymbol{\pi}$, at time t, the agent in state s_t selects

$$a_t \sim \pi_t(s_t),$$

receives the instantaneous reward

$$r_t \sim \nu_{(s_t,a_t)} \text{ such that } \mathbb{E}[r_t | s_t, a_t] = r(s_t, a_t)$$

and transits to the new state $s_{t+1} \sim p(\cdot | s_t, a_t)$.

➜ a policy defines a probability model $\mathbb{P}^\pi, \mathbb{E}^\pi$ over sequences of observations :

$$s_1, a_1, r_1, s_2, a_2, r_2 \dots$$

Reminder : Value Function

Definition

The value function of a policy $\pi = (\pi_1, \pi_2, \dots)$ is $V^\pi : \mathcal{S} \to \mathbb{R}$

① Finite-horizon criterion

$$V^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=1}^{H} r_t \middle| s_1 = s\right]$$

➜ We want to compute the optimal value $V^\star(s) = \max_\pi V^\pi(s)$ and an optimal policy $\pi_\star$ such that $V^\star = V^{\pi_\star}$.

➜ We will be able to do so when $\mathcal{S}$ and $\mathcal{A}$ are finite

$$S := |\mathcal{S}| < \infty \quad \text{and} \quad A := |\mathcal{A}| < \infty$$

(some optimality equation may extend to continuous state spaces)

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② Infinite horizon with a discount γ

$$V^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \middle| s_1 = s\right]$$

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(some optimality equation may extend to continuous state spaces)

Outline

Value functions

Let H be the known time horizon.

Value functions at step h

For a policy $\pi = (\pi_1, \dots, \pi_H)$,

$$V_h^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=h}^{H} r_t \middle| s_h = s\right]$$

and

$$V_h^\star(s) = \max_{\pi_h,\dots,\pi_H} \mathbb{E}^\pi\left[\sum_{t=h}^{H} r_t \middle| s_h = s\right]$$

Goal : compute

$$V^\pi(s) = V_1^\pi(s), V^\star(s) = V_1^\star(s) \quad \text{and} \quad \pi^\star = (\pi_1^\star, \dots, \pi_H^\star).$$

➜ we will actually compute $V_h^\pi(s)$ and $V_h^\star(s)$ for all $h \leq H$.

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For a policy $\pi = (\pi_1, \ldots, \pi_H)$,

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Value functions

Let H be the known time horizon.

Value functions at step h

For a deterministic policy $\pi = (\pi_1, \dots, \pi_H)$,

$$V_h^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=h}^{H} r(s_t, \pi_t(s_t)) \middle| s_h = s\right]$$

and

$$V_h^\star(s) = \max_{\pi_h,\dots,\pi_H} \mathbb{E}^\pi\left[\sum_{t=h}^{H} r(s_t, \pi_t(s_t)) \middle| s_h = s\right]$$

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$$V_h^\star(s) = \max_{\pi_h, \dots, \pi_H} \mathbb{E}^\pi \left[\sum_{t=h}^{H} r_t \middle| s_h = s \right]$$

How ?

- ➜ Monte-Carlo estimation ? only approximate
- ➜ Develop the tree of all possible realizations ? too complex

Bellman equations

Proposition

The value functions of a deterministic policy π satisfies the following equations : for all $h \in \{1, \dots, H\}$,

$$V_h^\pi(s) \quad = \quad r(s, \pi_h(s)) + \sum_{s' \in \mathcal{S}} p(s'|s, \pi_h(s)) V_{h+1}^\pi(s'),$$

with the convention that $V_{H+1}^\pi(s) = 0$ for all $s \in \mathcal{S}$.

Consequence : for a finite state space $\mathcal{S}$ such that $|\mathcal{S}| = S$

- ➜ $V_1^\pi(s)$ can be computed using backwards induction
- ➜ space complexity : $S \times H$
- ➜ time complexity : $S \times (S+1) \times H$

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Proof.

$$\begin{aligned}
V_h^\pi(s) &= \mathbb{E}^\pi\left[r(s_h, \pi_h(s_h)) + \sum_{t=h+1}^{H} r(s_t, \pi_t(s_t)) \middle| s_h = s \right] \\
&= r(s, \pi_h(s)) + \mathbb{E}^\pi\left[\sum_{t=h+1}^{H} r(s_t, \pi_t(s_t)) \middle| s_h = s, a_h = \pi_h(s) \right] \\
&= r(s, \pi_h(s)) + \sum_{s' \in \mathcal{S}} \mathbb{P}(s_{h+1} = s' | s_h = s, a_h = \pi_h(s)) \mathbb{E}^\pi\left[\sum_{t=h+1}^{H} r(s_t, \pi_t(s_t)) \middle| s_{h+1} = s' \right] \\
&= r(s, \pi_h(s)) + \sum_{s' \in \mathcal{S}} p(s'|s, \pi_h(s)) V_{h+1}^\pi(s')
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These equations may be generalized :

- to a possibly infinite state space

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with the convention that $V_{H+1}^\pi(s) = 0$ for all $s \in \mathcal{S}$.

These equations may be generalized :

- to a possibly infinite state space
- to randomized policies

Bellman equations for the optimal values

Proposition

The optimal values $V_h^\star$ satisfy the **Bellman equations** :

$$V_h^\star(s) = \max_{a\in\mathcal{A}} \left[r(s,a) + \sum_{s'\in\mathcal{S}} p(s'|s,a) V_{h+1}^\star(s') \right] \quad \text{for all } h \leq H,$$

with the convention that $V_{H+1}^\star(s) = 0$ for all $s \in \mathcal{S}$.
Moreover, an optimal policy is given by

$$\pi_h^\star(s) \in \operatorname*{argmax}_{a\in\mathcal{A}} \left[r(s,a) + \sum_{s'=1}^{S} p(s'|s,a) V_{h+1}^\star(s') \right].$$

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Consequence : for finite $\mathcal{S}, \mathcal{A}$ such that $|\mathcal{S}| = S$, $|\mathcal{A}| = A$

- ➜ $\pi^\star = (\pi_1^\star, \dots, \pi_H^\star)$ can be computed using backwards induction
- ➜ space complexity : $S \times H$
- ➜ time complexity : $O(S^2 AH)$

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This technique is known as **Dynamic Programming**

- term invented in the 50s by Bellman : an algorithmic principle for optimization in which solving an optimization problem of a given size reduces to solving (several) of the same optimization problem but of smaller size

Proof

$$V_h^\star(s) = \max_{\pi_h,\pi_{h+1},\dots} \mathbb{E}^\pi\left[\sum_{t=h}^{H} r(s_t,a_t)\middle| s_h = s\right]$$

$$= \max_{\pi_h,\pi_{h+1},\dots} \sum_{a\in\mathcal{A}} \pi_h(a_h = a|s_h = s)\mathbb{E}^\pi\left[r(s,a) + \sum_{t=h+1}^{H} r(s_t,a_t)\middle| s_h = s, a_h = a\right]$$

$$= \max_{\pi_h,\pi_{h+1},\dots} \sum_{a\in\mathcal{A}} \pi_h(a_h = a|s_h = s)\left[r(s,a) + \sum_{s'\in\mathcal{S}} p(s'|s,a)\mathbb{E}^{\pi_{h+1},\dots}\left[\sum_{t=h+1}^{H} r(s_t,a_t)\middle| s_{h+1} = s'\right]\right]$$

$$= \max_{a\in\mathcal{A}}\left[r(s,a) + \sum_{s'\in\mathcal{S}} p(s'|s,a)\max_{\pi_{h+1},\dots} V_{h+1}^{\pi_{h+1},\dots}(s')\right]$$

$$= \max_{a\in\mathcal{A}}\left[r(s,a) + \sum_{s'\in\mathcal{S}} p(s'|s,a)V_{h+1}^\star(s')\right]$$

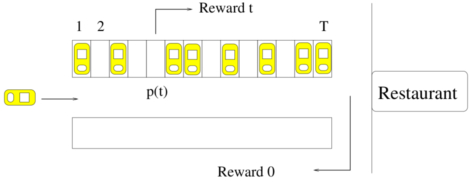
The maximizing policy is $\pi_h^\star, \pi_{h+1}^\star, \dots$ with

$$V_{h+1}^{\pi_{h+1}^\star,\dots} = V_{h+1}^\star = \operatorname*{argmax}_\pi V_{h+1}^\pi$$

and a deterministic mapping $\pi_h^\star : \mathcal{S} \to \mathcal{A}$ given by

$$\pi_h^\star(s) \in \operatorname*{argmax}_{a\in\mathcal{A}}\left[r(s,a) + \sum_{s'\in\mathcal{S}} p(s'|s,a)V_{h+1}^\star(s')\right].$$

Example : Optimal Parking



Exercise :

- ➜ model optimal parking as solving a MDP with a finite horizon
- ➜ write the optimal policy

Outline

Values functions

Let $\gamma \in (0,1)$ be a known discount factor

Value functions

For a policy $\pi = (\pi_1, \pi_2, \dots)$,

$$V^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1 = s\right]$$

and

$$V^\star(s) = \max_\pi \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1 = s\right]$$

How to compute them ?

➜ We need to generalize Dynamic Programming to infinite horizon...

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Bellman equation for a stationary policy

Proposition

Any stationary deterministic policy π satisfies, for all $s \in \mathcal{S}$,

$$V^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, \pi(s)) V^\pi(s')$$

Proof.

$$\begin{aligned}
V^\pi(s) &= \mathbb{E}^\pi \left[r(s, \pi(s)) + \sum_{t=2}^{\infty} \gamma^{t-1} r(s_t, \pi(s_t)) \middle| s_1 = s \right] \\
&= r(s, \pi(s)) + \gamma \mathbb{E}^\pi \left[\sum_{t=2}^{\infty} \gamma^{t-2} r(s_t, \pi(s_t)) \middle| s_1 = s, a_1 = \pi(s) \right] \\
&= r(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} \mathbb{P}(s_2 = s' | s_1 = s, a_1 = \pi(s)) \mathbb{E}^\pi \left[\sum_{t=2}^{\infty} \gamma^{t-2} r(s_t, \pi(s_t)) \middle| s_2 = s' \right] \\
&= r(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, \pi(s)) \underbrace{\mathbb{E}^\pi \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, \pi(s_t)) \middle| s_1 = s' \right]}_{V^\pi(s')}
\end{aligned}$$

Bellman equation for a stationary policy

Proposition

Any stationary deterministic policy π satisfies, for all $s \in \mathcal{S}$,

$$V^{\pi}(s) = r(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, \pi(s)) V^{\pi}(s')$$

More general statement :

$$V^{\pi}(s) = \mathbb{E}_{a \sim \pi(s)} \left[r(s, a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[V^{\pi}(s') \right] \right]$$

(also applies to infinite state space and randomized policies)

Solving the Bellman equations

Fix a stationary, deterministic policy π.

Proposition

$$\forall s \in \mathcal{S}, \quad V^{\pi}(s) = r(s,\pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,\pi(s)) V^{\pi}(s')$$

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Introducing the vectors

$$\begin{aligned} V^{\pi} &= (V^{\pi}(s))_{s=1}^{S} \in \mathbb{R}^{S} \\ r^{\pi} &= (r(s, \pi(s)))_{s=1}^{S} \in \mathbb{R}^{S} \end{aligned}$$

and the matrix

$$P^{\pi} = \Big(p(s'|s, \pi(s))\Big)_{\substack{1 \leq s \leq S \\ 1 \leq s' \leq S}} \in \mathbb{R}^{S \times S},$$

the Bellman equations rewrite

$$\boldsymbol{V^{\pi} = r^{\pi} + \gamma P^{\pi} V^{\pi}}$$

Solving the Bellman equations

$$V^{\pi} = r^{\pi} + \gamma P^{\pi} V^{\pi}$$

The vector $V^{\pi} \in \mathbb{R}^{S}$ satisfies

$$\begin{aligned} (I - \gamma P^{\pi}) V^{\pi} &= r^{\pi} \\ V^{\pi} &= (I - \gamma P^{\pi})^{-1} r^{\pi} \end{aligned}$$

provided that the matrix $I - \gamma P^{\pi}$ is invertible.

Proposition

The eigenvalues of the *stochastic*[a] matrix P^{π} all belong to $[0, 1]$. As a consequence, $-\gamma \notin \mathrm{sp}(P^{\pi})$ thus $I - \gamma P^{\pi}$ is invertible.

a. the entries in its rows sum to 1

➜ V^{π} can be computed by inverting a $S \times S$ matrix

An alternative : Exploiting the Bellman operator

Definition

The **Bellman operator** associated to a policy π is defined by

$$\begin{array}{rcl} T^\pi : \mathbb{R}^S & \longrightarrow & \mathbb{R}^S \\ V & \mapsto & T^\pi(V) \end{array}$$

where
$$T^\pi(V)(s) = r(s,\pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,\pi(s))V(s')$$

The Bellman equation for policy π rewrites

$$\boldsymbol{V^\pi = T^\pi V^\pi}$$

➜ the vector V^π is a fixed point of the Bellman operator T^π

Intermezzo : Fixed Point Theorem

Definition

Let $(\mathcal{X}, |\cdot|)$ be a normed vector space.
A mapping $f : \mathcal{X} \to \mathcal{X}$ is called L-Lipschitz is

$$\forall (x,y) \in \mathcal{X}^2, |f(x) - f(y)| \leq L|x-y|.$$

If $L < 1$, f is called a contraction.

Banach's fixed point theorem

Let $(\mathcal{X}, |\cdot|)$ be a *complete* normed vector space and $f : \mathcal{X} \to \mathcal{X}$ be a contraction. Then

- there exists a unique fixed point $x^\star$ satisfying $f(x^\star) = x^\star$
- for every $x_0 \in \mathcal{X}$, the sequence defined by $x_{n+1} = f(x_n)$ for all $n \geq 0$ satisfies
$$\lim_{n\to\infty} x_n = x_\star$$

Exploiting the Bellman Operator

Proposition

The operator

$$\begin{array}{rcl} T^\pi : \left(\mathbb{R}^S, ||\cdot||_\infty\right) & \longrightarrow & \left(\mathbb{R}^S, ||\cdot||_\infty\right) \\ V & \mapsto & T^\pi(V)(s) = r(s,\pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,\pi(s)) V(s') \end{array}$$

is a γ-contraction.

Proof.

$$\begin{array}{rcl} ||T^\pi(V) - T^\pi(V')||_\infty & = & \sup_{s\in\mathcal{S}} |T^\pi(V)(s) - T^\pi(V')(s)| \\ & = & \sup_{s\in\mathcal{S}} |\gamma \sum_{s'\in\mathcal{S}} p(s'|s,\pi(s))(V(s') - V'(s'))| \\ & \leq & \gamma \sum_{s'\in\mathcal{S}} p(s'|s,\pi(s)) ||V - V'||_\infty \\ & = & \gamma ||V - V'||_\infty . \end{array}$$

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The operator

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is a γ-contraction.

Consequence :

- V^π is the unique fixed point of T^π
- V^π can be approximated by an iterative scheme :

$$V^\pi = \lim_{k\to\infty} V_k$$

where

$$\begin{cases} V_0 & \in \mathbb{R} \\ V_{k+1} & = T^\pi(V_k) \text{ for all } k \geq 0. \end{cases}$$

Summary : Policy Evaluation

Two methods for computing $V^\pi(s)$ for all s :

- solving linear equations (matrix inversion)
- iterating the Bellman operator T^π

Other possibility : Monte-Carlo simulation

➜ provides only an approximation

➜ ... but doesn't require the knowledge of $r(s,a)$ and $p(\cdot|s,a)$...

Back to Retail Store Management

- Constant policy : $\pi(s) = \max(M - s, c)$
- Threshold policy : $\pi(s) = \mathbb{1}_{(s \leq m_1)}(m_2 - s)$

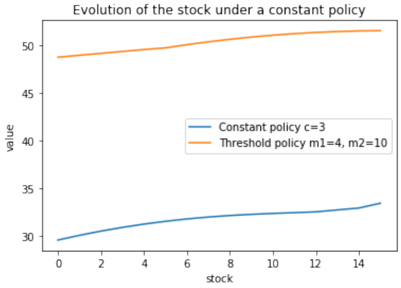

FIGURE – V^π for two different policies, $\gamma = 0.97$

Outline

Bellman equations for the optimal values

Proposition

$V^\star(s) = \max_\pi \; V^\pi(s)$ satisfy the **Bellman equations** :

$$V^\star(s) \quad = \quad \max_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^\star(s') \right]$$

Moreover, an optimal policy is given by $\pi^\star = (\pi^\star, \pi^\star, \dots)$ where

$$\pi^\star(s) \in \operatorname*{argmax}_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^\star(s') \right].$$

➜ $\pi^\star$ is the greedy policy with respect to $V^\star$:

Definition

The **greedy policy** with respect to a value V, $\pi = \text{greedy}(V)$ is

$$\pi(x) = \operatorname*{argmax}_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s') \right]$$

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Moreover, an optimal policy is given by $\pi^\star = (\pi^\star, \pi^\star, \dots)$ where

$$\pi^\star(s) \in \underset{a \in \mathcal{A}}{\operatorname{argmax}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^\star(s') \right].$$

➜ $\pi^\star$ is the greedy policy with respect to $V^\star$:

Intuition : greedy(V) is the policy that "improves" a policy with value V by taking the best possible first action and then following the policy

Solving the Bellman equations

Proposition

The optimal value function $V^\star$ satisfies

$$V^\star(s) = \max_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^\star(s') \right]$$

- a system of S non-linear equations for computing $(V^\star(s))_{s \in \mathcal{S}}$.

➜ no hope for a simple “matrix inversion” technique...

Bellman operator to the rescue

Optimal Bellman operator

The optimal Bellman operator (or dynamic programming operator) is

$$\begin{array}{rcl} T^\star : \mathbb{R}^S & \longrightarrow & \mathbb{R}^S \\ V & \mapsto & T^\star(V) \end{array}$$

where

$$T^\star(V)(s) = \max_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s') \right]$$

The optimal value function satisfies

$$\boldsymbol{V^\star = T^\star(V^\star)}$$

➜ $V^\star$ is a fixed point of the optimal Bellman operator $T^\star$.

Optimal Bellman Operator

Properties

The optimal Bellman operator is a γ-contraction :

$$\forall\ V, V' \in \mathbb{R}^S, \quad ||T^\star(V) - T^\star(V')||_\infty < \gamma\, ||V - V'||_\infty .$$

As a consequence :

- $T^\star$ admits a unique fixed point, $V^\star$
- for every V_0, the sequence $V_{n+1} = T^\star(V_n)$ converges to $V^\star$

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Proof. Uses that for all functions $|\max f - \max g| \leq \max |f - g|$.

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➜ provides a method for approximating $V^\star$

Outline

Value Iteration

- **Idea :** Iterate the operator $T^\star$ until V doesn't change much

Algorithm 1: Value Iteration

```
Input  : $\epsilon$ = stopping parameter
         $V_0$ = any function (e.g. $V_0 \leftarrow 0_S$)
1 $V \leftarrow V_0$
2 while $\|V - T^\star(V)\| \geq \epsilon$ do
3     $V \leftarrow T^\star(V)$
4 end
Return: $\pi$ = greedy($V$)
```

Theorem

Value iteration converges in at most $\log\left(\frac{||T^\star(V_0) - V_0||_\infty}{\epsilon(1-\gamma)}\right)/\log(1/\gamma)$ iterations and outputs a policy π satisfying $||V^\pi - V^\star|| \leq \varepsilon$.

Proof. contraction property + triangular inequality

$$||V^\star - V_k|| \leq ||V^\star - T^\star(V_k)|| + ||T^\star(V_k) - V_k||$$

Policy Iteration

- **Idea :** alternate between policy evaluation and **policy improvement**

Greedy policy

Recall that $\pi' = \text{greedy}(V)$ is the policy defined as

$$\pi'(s) \in \operatorname*{argmax}_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s') \right]$$

Policy improvement theorem

For any policy π, $\pi' = \text{greedy}(V^\pi)$ improves over π : $V^{\pi'} \geq V^\pi$.

Proof.

Policy Iteration

- **Idea :** alternate between policy evaluation and policy improvement .

Algorithm 2: Policy Iteration

```
Input : $\pi_0$ = any policy (e.g. chosen at random)
1 $\pi \leftarrow \pi_0$
2 $\pi' \leftarrow \mathrm{NULL}$
3 while $\pi \neq \pi'$ do
4     $\pi' \leftarrow \pi$
5     Evaluate policy $\pi$ : compute $V^\pi$
6     Improve policy $\pi$ : $\pi \leftarrow$ greedy($V^\pi$)
7 end
Return: $\pi$
```

Theorem

Policy iteration terminates after a finite number of steps and outputs the optimal policy $\pi^\star$.

Policy Iteration

Why is that ?

Policy iteration generates a sequence of policies $\pi_0, \pi_1, \dots$ such that

$$\pi_{k+1} = \text{greedy}(V^{\pi_k}).$$

By the policy improvement theorem,

$$V^{\pi_{k+1}} \geq V^{\pi_k}$$

and if $\pi_{k+1} \neq \pi_k$ there must exists s such that $V^{\pi_{k+1}}(s) > V^{\pi_k}(s)$
(otherwise $V^{\pi_k} = V^{\pi_{k+1}} = T^{\pi_{k+1}}(V^{\pi_{k+1}}) = T^\star(V^{\pi_k})$ thus $\pi_k = \pi^\star$)

➜ as there is a finite number of possible values of V^π (finite number of policies), the sequence must be stationary at some point.

Implementation of VI and PI

Both algorithm can be implemented using Q-Values .

Definitions

The ***Q*-value** of a stationary policy is the expected cumulative reward when performing action a in state s and then following policy π :

$$Q^\pi(s,a) = \mathbb{E}^\pi\left[\sum_{t=1}^{\infty}\gamma^{t-1}r(s_t,a_t)\middle| s_1 = s, a_1 = a\right]$$

The **optimal Q-value** is $Q^\star(s,a) = \max_\pi Q^\pi(s,a) = Q^{\pi^\star}(s,a)$.

Properties :

- $V^\pi(s) = Q^\pi(s,\pi(s))$
- $V^\star(s) = Q^\star(s,\pi^\star(s))$

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- $V^{\star}(s) = Q^{\star}(s,\pi^{\star}(s))$

Implementation of VI and PI

Q-Values are convenient for policy improvement.

Q-value associated to a value V

To each value function V, we can associate the corresponding Q-value

$$Q(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s')$$

Property : $\pi' = \text{greedy}(V)$ can be rewritten $\pi'(s) = \text{argmax}_a\, Q(s,a)$.

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Remark : $\pi^\star = \text{greedy}(Q^\star)$

Value Iteration versus Policy Iteration

In these implementations, we propose to store Q-values.

Value Iteration

Initialize Q_0.

At iteration k :

$$V_k(s) = \max_a Q_{k-1}(s,a) \quad \text{(apply } T^\star\text{)}$$

$$Q_k(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V_k(s)$$

Output : $\pi_K(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\, Q_K(s,a)$

Policy iteration

Initialize π_0

At iteration k :

$$Q_{k-1}(s,a) = Q^{\pi_{k-1}}(s,a) \text{ (policy evaluation)}$$

$$\pi_k(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\, Q_{k-1}(s,a)$$

Output : π_K

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$$\pi_k(s) = \operatorname{argmax}_{a \in \mathcal{A}} Q_{k-1}(s,a)$$

Output : π_K

Space Complexity : $O(SA)$ in both cases

- VI : Storing Q Values + Values
- PI : Storing Q values + Policy

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Value Iteration

Initialize Q_0.

At iteration k :

$$V_k(s) = \max_a Q_{k-1}(s,a) \quad (\text{apply } T^\star)$$

$$Q_k(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V_k(s)$$

Output : $\pi_K(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\, Q_K(s,a)$

Policy iteration

Initialize π_0

At iteration k :

$$Q_{k-1}(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^{\pi_{k-1}}(s')$$

$$\pi_k(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\, Q_{k-1}(s,a)$$

Output : π_K

Per Iteration Time Complexity : $O(S^2 A)$ $+O(S^3)$

- VI : Compute Q values + compute S max
- PI : Compute Q values + compute S argmax + Policy Evaluation

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$$\pi_k(s) = \underset{a\in\mathcal{A}}{\text{argmax}}\, Q_{k-1}(s,a)$$

Output : π_K

Number of Iterations ? : PI often requires few iterations

- VI : wait for $V_{k+1} \simeq V_k$ (requires a termination criterion)
- PI : wait for $\pi_{k+1} = \pi_k$ (finite number of iterations)

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Output : π_K

Guarantees :

- VI : a policy with value very close to $V^\star$ (often $\pi^\star$)
- PI : an optimal policy $\pi^\star$

Summary

We learned how to find the optimal policy in an MDP with finite state and action spaces :

- using backwards induction for a finite horizon H
- using Policy and Value iteration for an infinite horizon with a discount $\gamma \in (0, 1)$

Those two types of techniques are often indifferently referred to as

Dynamic Programming.

We are now ready to propose **reinforcement learning algorithms**, that :

- operate without the knowledge of $r(s, a)$ and $p(\cdot|s, a)$
- or in very large state spaces in which standard Dynamic Programming is intractable