Minimisation du regret vs. Exploration pure: Deux critères de performance pour des algorithmes de bandit

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ANR Spadro, 9 avril 2014

1. Two bandit problems
2. Regret minimization: a well solved problem
3. Algorithms for pure-exploration
4. The complexity of m best arms identification

Bandit model

A **multi-armed bandit model** is a set of K arms where

- Arm a is an unknown probability distribution ν_a with mean μ_a
- Drawing arm a is observing a realization of ν_a
- Arms are assumed to be independent

In a **bandit game**, at round t, an agent

- chooses arm A_t to draw based on past observations, according to its **sampling strategy** (or **bandit algorithm**)
- observes a sample $X_t \sim \nu_{A_t}$

The agent wants to **learn which arm(s) have highest means**

$$a^* = \text{argmax}_a \ \mu_a$$

Bernoulli bandit model

A **multi-armed bandit model** is a set of K arms where

- Arm a is a Bernoulli distribution $\mathcal{B}(\mu_a)$ (with unknown mean μ_a)
- Drawing arm a is observing a realization of $\mathcal{B}(\mu_a)$ (0 or 1)
- Arms are assumed to be independent

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The (classical) bandit problem: regret minimization

Samples are seens as *rewards* (as in reinforcement learning)

The forecaster wants to **maximize the reward accumulated during learning** or equivalentely minimize its **regret**:

$$R_n = n\mu_{a^*} - \mathbb{E}\left[\sum_{t=1}^{n} X_t\right]$$

He has to find a sampling strategy (or bandit algorithm) that

- realizes a **tradeoff between exploration and exploitation**

Best arm identification (or pure exploration)

The forecaster has to **find the best arm(s)**, and does not suffer a loss when drawing 'bad arms'.

He has to find a sampling strategy that

- **optimaly explores** the environnement,

together with a stopping criterion, and then recommand a set $\mathcal{S}$ of m arms such that

$$\mathbb{P}(\mathcal{S} \text{ is the set of } m \text{ best arms}) \geq 1 - \delta.$$

Zoom on an application: Medical trials

A doctor can choose between K different treatments for a given symptom.

- treatment number a has unknown probability of sucess μ_a
- **Unknown** best treatment $a^* = \text{argmax}_a\ \mu_a$
- If treatment a is given to patient t, he is cured with probability p_a

The doctor:

- chooses treatment A_t to give to patient t
- observes whether the patient is healed : $X_t \sim \mathcal{B}(\mu_{A_t})$

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The doctor can ajust his strategy (A_t) so as to

Regret minimization	Pure exploration
Maximize the number of patient healed during a study involving n patients	Identify the best treatment with probability at least $1-\delta$ (and always give this one later)

Asymptotically optimal algorithms

$N_a(t)$ be the number of draws of arm a up to time t

$$R_T = \sum_{a=1}^{K} (\mu^* - \mu_a)\mathbb{E}[N_a(T)]$$

- [Lai and Robbins,1985]: every consistent policy satisfies

$$\mu_a < \mu^* \Rightarrow \liminf_{T\to\infty} \frac{\mathbb{E}[N_a(T)]}{\log T} \geq \frac{1}{\mathrm{KL}(\mathcal{B}(\mu_a), \mathcal{B}(\mu_{a^*}))}$$

- A bandit algorithm is **asymptotically optimal** if

$$\mu_a < \mu^* \Rightarrow \limsup_{n\to\infty} \frac{\mathbb{E}[N_a(T)]}{\log T} \leq \frac{1}{\mathrm{KL}(\mathcal{B}(\mu_a), \mathcal{B}(\mu_{a^*}))}$$

Algorithms: a family of optimistic index policies

- For each arm a, compute a confidence interval on μ_a:

$$\mu_a \leq UCB_a(t) \quad w.h.p$$

- Act as if the best possible model was the true model (*optimism-in-face-of-uncertainty*):

$$A_t = \underset{a}{\arg\max} \ UCB_a(t)$$

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Example UCB1 [Auer et al. 02] uses Hoeffding bounds:

$$UCB_a(t) = \frac{S_a(t)}{N_a(t)} + \sqrt{\frac{\alpha \log(t)}{2N_a(t)}}.$$

$S_a(t)$: sum of the rewards collected from arm a up to time t.

UCB1 is not asymptotically optimal, but one can show that

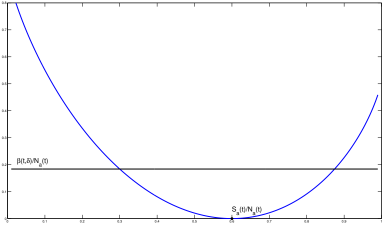
$$\mathbb{E}[N_a(T)] \leq \frac{K_1}{2(\mu_a - \mu^*)^2} \ln T + K_2, \quad \text{with } K_1 > 1.$$

KL-UCB: and asymptotically optimal frequentist algorithm

- KL-UCB [Cappé et al. 2013] uses the index:

$$u_a(t) = \underset{x > \frac{S_a(t)}{N_a(t)}}{\text{argmax}} \left\{ d\left(\frac{S_a(t)}{N_a(t)}, x\right) \leq \frac{\ln(t) + c \ln\ln(t)}{N_a(t)} \right\}$$

with $d(p,q) = \text{KL}\left(\mathcal{B}(p), \mathcal{B}(q)\right) = p \log\left(\frac{p}{q}\right) + (1-p)\log\left(\frac{1-p}{1-q}\right).$



$$\mathbb{E}[N_a(T)] \leq \frac{1}{d(\mu_a, \mu^*)} \ln T + C$$

Regret minimization: Summary

- An (asymptotic) lower bound on the regret of any good algorithm

$$\liminf_{T\to\infty} \frac{R_T}{\log T} \geq \sum_{a:\mu_a<\mu^*} \frac{\mu^* - \mu_a}{\mathrm{KL}(\mathcal{B}(\mu_a), \mathcal{B}(\mu^*))}$$

- An algorithm based on confidence intervals matching this lower bound: KL-UCB

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- An algorithm based on confidence intervals matching this lower bound: KL-UCB
- A Bayesian approach of the MAB problem can also lead to asymptotically optimal algorithms
(Thompson Sampling, Bayes-UCB)

m best arms identification

Assume $\mu_1 \geq \dots \geq \mu_m > \mu_{m+1} \geq \dots \mu_K$.

Parameters and notations

- m the number of arms to find
- $\delta \in]0,1[$ a risk parameter
- $\mathcal{S}_m^* = \{1, \dots, m\}$ the set of m optimal arms

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The forecaster

- chooses at time t one (or several) arms to draw
- decides to stop after a (possibly random) total number of samples from the arms τ
- recommends a set $\mathcal{S}$ of m arms

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His goal

- $\mathbb{P}(\mathcal{S} = \mathcal{S}_m^*) \geq 1 - \delta$, and $\mathbb{E}[\tau]$ is small (*fixed-confidence setting*)

Generic algorithms based on confidence intervals

Generic notations:

- confidence interval on the mean of arm a at round t:

$$\mathcal{I}_a(t) = [L_a(t), U_a(t)]$$

- $J(t)$ the set of estimated m best arms at round t
(m empirical best)

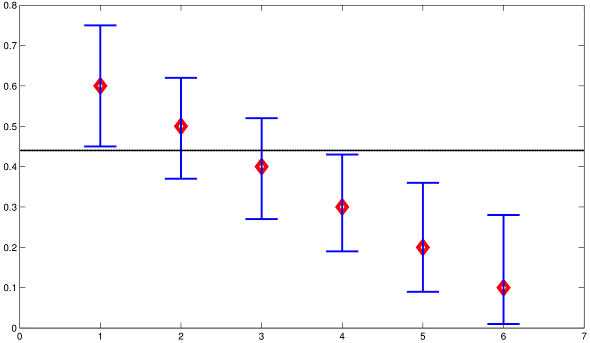
- $u_t \in J(t)^c$ and $l_t \in J(t)$ two 'critical' arms (likely to be misclassified)

$$u_t = \underset{a \notin J(t)}{\text{argmax}}\, U_a(t) \quad \text{and} \quad l_t = \underset{a \in J(t)}{\text{argmin}}\, L_a(t).$$

(KL)-Racing: uniform sampling and eliminations

The algorithm maintains a set of remaining arms $\mathcal{R}$ and at round t:

- draw all the arms in $\mathcal{R}$ (uniform sampling)
- possibly accept the empirical best or discard the empirical worst



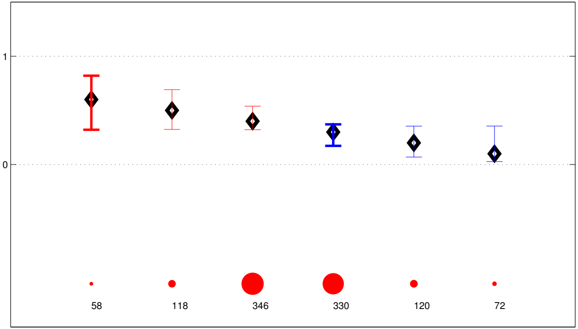
$$\mu = [0.6\ 0.5\ 0.4\ 0.3\ 0.2\ 0.1] \quad m = 3 \quad \delta = 0.1$$

In this situation, arm 1 is selected

(KL)-LUCB algorithm: adaptive sampling

At round t, the algorithm:

- draw only two well-chosen arms: u_t and l_t (adaptive sampling)
- stops when CI for arms in $J(t)$ and $J(t)^c$ are separated



Set $J(t)$, arm l_t in bold Set $J(t)^c$, arm u_t in bold

Two δ-PAC algorithms

$$\begin{aligned} L_a(t) &= \min\left\{q \in [0, \hat{\mu}_a(t)] : N_a(t) d(\hat{\mu}_a(t), q) \leq \beta(t,\delta)\right\}, \\ U_a(t) &= \max\left\{q \in [\hat{\mu}_a(t), 1] : N_a(t) d(\hat{\mu}_a(t), q) \leq \beta(t,\delta)\right\}. \end{aligned}$$

for $\beta(t,\delta)$ some exploration rate.

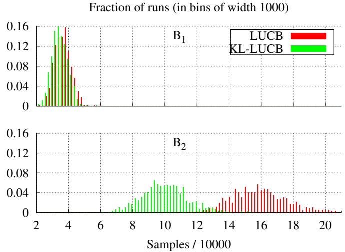
Theorem

The KL-Racing algorithm and KL-LUCB algorithm using

$$\beta(t,\delta) = \log\left(\frac{k_1 K t^\alpha}{\delta}\right), \tag{1}$$

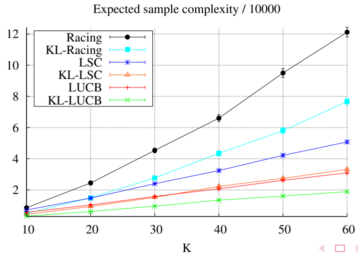
with $\alpha > 1$ *and* $k_1 > 1 + \frac{1}{\alpha - 1}$ *satisfy* $\mathbb{P}(\mathcal{S} = \mathcal{S}_m^*) \geq 1 - \delta$.

Confidence intervals based on KL are always better



$$B_1 : K = 15; \mu_1 = \frac{1}{2}; \mu_a = \frac{1}{2} - \frac{a}{40} \text{ for } a = 2, 3, \ldots, K. \quad B_2 = \frac{1}{2} B_1$$

Adaptive Sampling seems to do better than Uniform Sampling



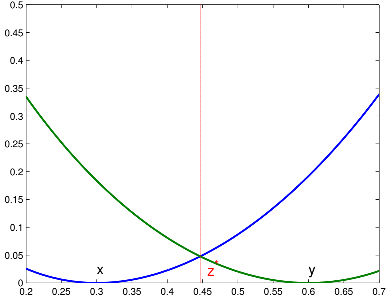
Sample complexity analysis

- **A new informational quantity: Chernoff information**

$$d^*(x,y) := d(z^*,x) = d(z^*,y),$$

where z^* is defined by the equality

$$d(z^*,x) = d(z^*,y).$$



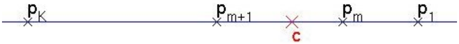
Sample Complexity analysis

KL-LUCB with $\beta(t,\delta) = \log\left(\frac{k_1 K t^\alpha}{\delta}\right)$ is δ-PAC and satisfies, for $\alpha > 2$,

$$\mathbb{E}[\tau] \leq 4\alpha H^* \left[\log\left(\frac{k_1 K (H^*)^\alpha}{\delta}\right) + \log\log\left(\frac{k_1 K (H^*)^\alpha}{\delta}\right)\right] + C_\alpha,$$

with

$$H^* = \min_{c \in [\mu_{m+1};\mu_m]} \sum_{a=1}^{K} \frac{1}{d^*(\mu_a, c)}.$$



Lower bound on the number of sample used complexity

For KL-LUCB, $\mathbb{E}[\tau] = O\left(H^* \log \frac{1}{\delta}\right)$.

Theorem

Any algorithm that is δ-PAC on every bandit model such that $\mu_m > \mu_{m+1}$ satisfies, for $\delta \leq 0.15$,

$$\mathbb{E}[\tau] \geq \left(\sum_{t=1}^{m} \frac{1}{d(\mu_a, \mu_{m+1})} + \sum_{t=m+1}^{K} \frac{1}{d(\mu_a, \mu_m)} \right) \log \frac{1}{2\delta}$$

The informational complexity of m best arm identification

For a bandit model ν, one can introduce the complexity term

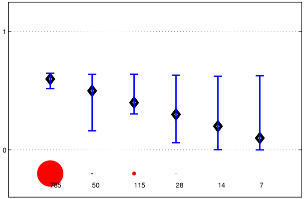
$$\kappa_C(\nu) = \inf_{\substack{\mathcal{A}\ \delta-\text{PAC} \\ \text{algorithm}}} \limsup_{\delta \to 0} \frac{\mathbb{E}_\nu[\tau]}{\log \frac{1}{\delta}}.$$

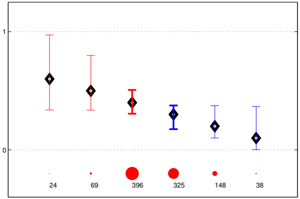
Our results rewrite

$$\sum_{t=1}^{m} \frac{1}{d(\mu_a, \mu_{m+1})} + \sum_{t=m+1}^{K} \frac{1}{d(\mu_a, \mu_m)} \leq \kappa_C(\nu) \leq 4 \min_{c \in [\mu_{m+1};\mu_m]} \sum_{a=1}^{K} \frac{1}{d^*(\mu_a, c)}$$

Regret minimization versus Best arms Identification

- KL-based confidence intervals are useful in both settings, altough KL-UCB and KL-LUCB draw the arms in a different fashion





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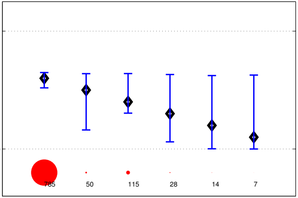

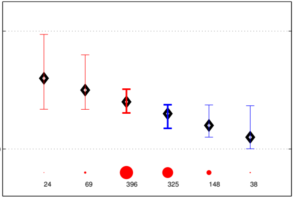

- Do the complexity of these two problems feature the same information-theoretic quantities?

$$\inf_{\substack{\text{constistent}\\ \text{algorithms}}} \limsup_{T\to\infty} \frac{R_T}{\log T} = \sum_{a=2}^{K} \frac{\mu_1 - \mu_a}{d(\mu_a, \mu_1)}$$

$$\inf_{\substack{\delta-PAC\\ \text{algorithms}}} \limsup_{\delta\to\infty} \frac{\mathbb{E}[\tau]}{\log(1/\delta)} \geq \sum_{a=1}^{K} \frac{1}{d(\mu_a, \mu_{m+1})} + \sum_{a=m+1}^{K} \frac{1}{d(\mu_a, \mu_m)}$$