

Sequential Decision Making

Lecture 4 : Markov Decision Processes

Emilie Kaufmann
(emilie.kaufmann@univ-lille.fr)



M2 Data Science, 2021/2022

1 Introduction

2 Markov Decision Processes

3 Policies and Values

4 Warm-up : Computing values

What is Reinforcement Learning ?

- learning by “trial and error”
- learning to behave in an unknown, stochastic environment by maximizing some real-valued reward signal



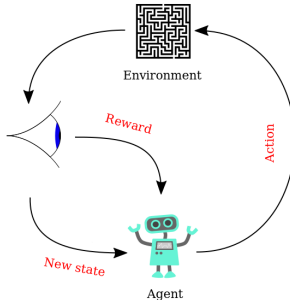
Example : learning to bike without a perfect knowledge of physics

Key RL concepts

A learning agent *sequentially* interacts with its environment by *performing actions*. Each action

- ▶ provides an *instantaneous* reward
- ▶ leads to an evolution of the agent's state

Agent's goal : act so as to maximize its *total* reward



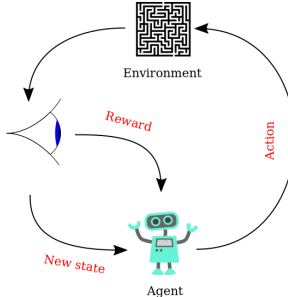
source : Wikipedia

Key RL concepts

Keywords (high-level) :

- ▶ **Reward** : instantaneous feedback received after acting
- ▶ **Policy** : strategy to choose an action in a given state
- ▶ **Value** : total reward the agent can get in some state by following some policy

Agent's goal : find a policy that maximizes the value in each state



source : Wikipedia

RL successes : Games (1/2)



From Backgammon...

1992, TD-gammon

... to Go

2015, AlphaGo

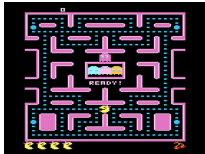
2017, AlphaGo Zero



→ RL agents learn **new types of strategies**

RL successes : Games (2/2)

- ▶ Learning to play from pixels (and rewards) : Atari Games
2010+ Deep Reinforcement Learning



- ▶ Recent challenges : multi-player / partial information games



OpenAI Five (2019)



Pluribus (2019)

RL sucessess : Content Optimization

► online advertisement



→ action : display an add / reward : click

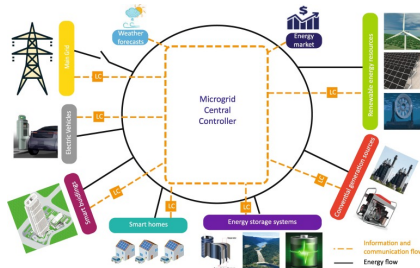
► (sequential) recommender systems



→ action : recommend a movie / reward : rating

RL : Many potential applications

► Smart grid / microgrid management



source : ScienceDirect.com

Actions :

- charge or discharge storage systems
- turn on or off renewable energy source
- buy energy from the market

...

Reward : - Cost

RL : Many potential applications

- ▶ Autonomous robotics



- ▶ Self-driving cars?



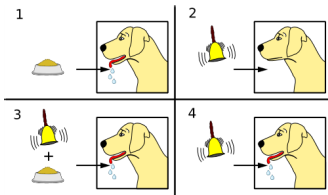
History of RL

- Learning to behave from rewards : an old idea from psychology

- ▶ 1900s : observation of animal behavior
(e.g. Thorndike 1911 “**Law of Effect**”)

Of several responses made to the same situation, those which are accompanied or closely followed by satisfaction to the animal will [...] be more likely to recur.

- ▶ 1920s : Pavlov work on **conditionnal reflexes**
first occurrence of “reinforcement” in animal learning



source : Wikipedia

History of RL

- Learning to behave from rewards : an inspiration from the brain ?
- ▶ 1950s : first experiments on electric brain stimuli for controlling mice behavior (Oak and Miller 1954)
- hypothesis that **dopamine** broadcast rewards signal to the brain

History of RL

- Some steps towards computational RL
 - ▶ 1950s, Shannon's machines : "Theseus", a mice finding how to get out of a maze, a chess player, a Rubik's cube solver
 - ▶ 1957, Bellmann : Dynamic Programming ([control of dynamical systems](#))
 - ▶ 1961, Minsky "Towards artificial intelligence"
 - ▶ 1978, Sutton : Temporal Difference Learning ([artificial intelligence](#))
 - ▶ 1989, Watkins : Q-Learning algorithm

Nowadays, reinforcement learning is mostly formalized as learning an optimal policy in an incompletely-known [Markov Decision Process](#).

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Markov Decision Process

A Markov Decision Process (MDP) models a situation in which **repeated decisions** (= choices of actions) are made. MDP provides models for the consequence of each decisions :

- ▶ in terms of **reward**
- ▶ in terms of the evolution of the system's **state**

In each (discrete) **decision time** $t = 1, 2, \dots$, a learning agent

- ▶ selects an **action** a_t based on his current **state** s_t (or possibly all the previous observations),
- ▶ gets a **reward** $r_t \in \mathbb{R}$ depending on his choice,
- ▶ transits to a **new state** s_{t+1} depending on his choice.

Markov Decision Process

A MDP is parameterized by a tuple $(\mathcal{S}, \mathcal{A}, R, P)$ where

- ▶ \mathcal{S} is the **state space**
- ▶ \mathcal{A} is the **action space**
- ▶ $R = (\nu_{(s,a)})_{(s,a) \in \mathcal{S} \times \mathcal{A}}$ where $\nu_{(s,a)} \in \Delta(\mathbb{R})$ is the **reward distribution** for the state-action pair (s, a)
- ▶ $P = (p(\cdot|s, a))_{(s,a) \in \mathcal{S} \times \mathcal{A}}$ where $p(\cdot|s, a) \in \Delta(\mathcal{S})$ is the **transition kernel** associated to the state-action pair (s, a)

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[Bellman 1957, Howard 1960, Blackwell 70s...]

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- ▶ \mathcal{A} is the **action space** (sometimes \mathcal{A}_s for each $s \in \mathcal{S}$)
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Goal : (made more precise later) select actions so as to maximize some notion of **expected cumulated rewards**

Mean reward of action a in state s

$$r(s, a) = \mathbb{E}_{R \sim \nu_{(s,a)}}[R]$$

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- **The tabular case** : finite state and action spaces

$$\mathcal{S} = \{1, \dots, S\}$$

$$\mathcal{A} = \{1, \dots, A\}$$

For every $s, s' \in \mathcal{S}$, $a \in \mathcal{A}$, $p(s'|s, a) = \mathbb{P}(s_{t+1} = s' | s_t = s, a_t = a)$.

Why Markov ?

In an MDP, the sequence of successive states / actions / rewards

$$s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}, r_{t-1}, s_t$$

satisfies some extension of the Markov property :

$$\begin{aligned}\mathbb{P}(s_t = s, r_{t-1} = r | s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}) \\ = \mathbb{P}(s_t = s, r_{t-1} = r | s_{t-1}, a_{t-1})\end{aligned}$$

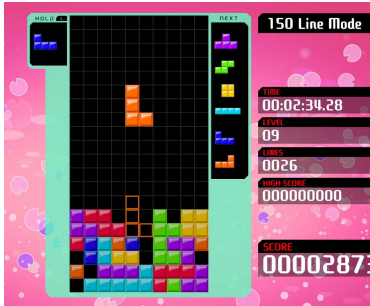
(discrete action and reward)

Definition

A Markov chain on a discrete space \mathcal{X} is a stochastic process $(X_t)_{t \in \mathbb{N}}$ that satisfies the **Markov property** :

$$\mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}, \dots, X_0 = x_0) = \mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}).$$

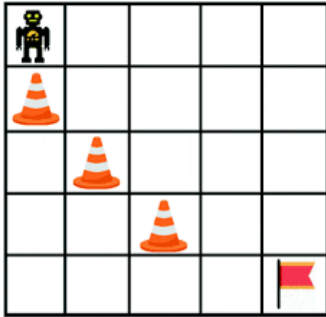
Example : Tetris



- **State** : current board and next blocks to add
- **Action** : orientation + position of the dropped block
- **Reward** : increment in the score/ number of lines
- **Transition** : new board + randomness in the new block

→ difficulty : large state space !

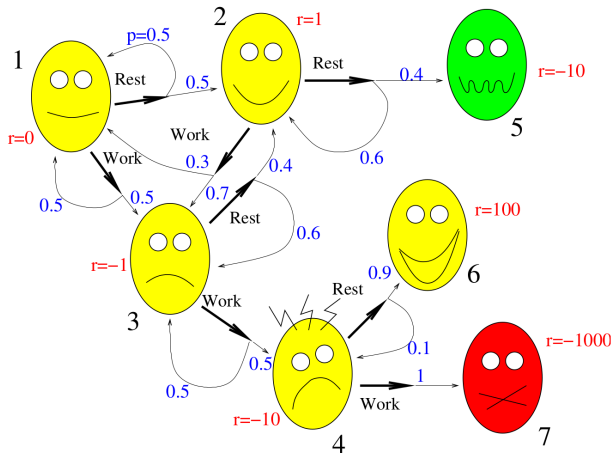
Example : Grid world



- **State** : position of the robot
- **Actions** : $\leftarrow, \uparrow, \rightarrow, \downarrow$
- **Transitions** : (quasi) deterministic
- **Rewards** : depends on the behavior to incentivise (positive or negative rewards on some states / -1 for each step before a goal...)

→ possible difficulty : sparse rewards

Example : The Student Dilemma



credit : Rémi Munos, Alessandro Lazaric

(running) Example : Retail Store Management

You own a bike store. During week t , the (random) demand is D_t units. On Monday morning you may choose to **command a_t additional units** : they are delivered immediately before the shop opens.

For each week :

- ▶ Maintenance cost : h per unit left in your stock
- ▶ Ordering cost : c per unit ordered + fix cost c_0 if an order is placed
- ▶ Sales profit : p per unit sold

Constraints :

- ▶ your warehouse has a maximal capacity of M bikes (any additional bike gets stolen)
- ▶ you cannot sell bikes that you don't have in stock

Exercise : Write down the underlying Markov Decision Process

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Policies

Definition

A (Markovian) **policy** is a sequence $\pi = (\pi_t)_{t \in \mathbb{N}^*}$ of mappings

$$\pi_t : \mathcal{S} \rightarrow \Delta(\mathcal{A}),$$

where $\Delta(\mathcal{A})$ is the set of probability distributions over the action space.

→ An agent acting under policy π selects at round t the action

$$a_t \sim \pi_t(s_t)$$

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► **Remark** : one could also consider *history-dependent* policies $\pi_t : \mathcal{H}_t \rightarrow \Delta(\mathcal{A})$, where the next action is chosen based on

$$h_t = (s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}, r_{t-1}, s_t)$$

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A policy may be

Deterministic	Stochastic
$\pi_t : \mathcal{S} \rightarrow \mathcal{A}$	$\pi_t : \mathcal{S} \rightarrow \Delta(\mathcal{A})$

► **Terminology** : policy = strategy = decision rule = control

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A policy may be

Stationary	Non-stationary
$\pi = (\pi, \pi, \pi, \dots)$	$\pi = (\pi_1, \pi_2, \dots)$

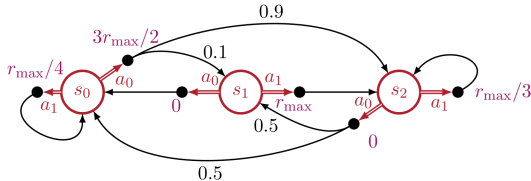
► **Terminology** : policy = strategy = decision rule = control

Policies

Under a stationary (deterministic) policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$, the random process $(s_t)_{t \in \mathbb{N}}$ is a **Markov chain**, with transition probability

$$\mathbb{P}^\pi(s_{t+1} = s' | s_t = s) = \mathbb{P}(s_{t+1} = s' | s_t = s, a_t = \pi(s)) = p(s' | s, \pi(s))$$

(can be extended to stochastic policies and continuous spaces)



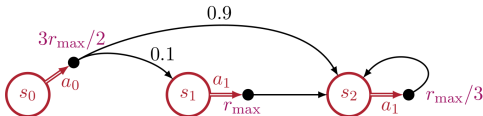
→ A MDP is sometimes referred to as a **controlled Markov chain**

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Value function

Value of a policy π in a state $s \in \mathcal{S}$

$V^\pi(s)$ measures the **expected cumulative reward** obtained by an agent starting from state s and applying **policy π** .

→ \neq notions of **cumulative reward** provide \neq definitions of the value

① Finite horizon

Given a known **horizon** $H \in \mathbb{N}^*$,

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starting from state s

→ **When is it used?** In the presence of a natural notion of duration of an episode (e.g. maximal number of steps in a game)

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② Infinite time horizon with a discount parameter

Given a known **discount parameter** $\gamma \in (0, 1)$,

$$V^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \mid s_1 = s \right]$$

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starting from state s

→ **When is it used?** To put more weight on short-term reward / when there is a natural notion of discount

Other possible definitions

(not discussed much in this class)

③ Infinite time horizon with a terminal state

Given τ the random time at which we first reach a terminal state.

$$V^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=1}^{\tau} r_t \mid s_1 = s \right]$$

→ **When ?** For tasks that have a natural notion of terminal state

④ Infinite time horizon with average reward

$$V^\pi(s) = \lim_{T \rightarrow \infty} \mathbb{E}^\pi \left[\frac{1}{T} \sum_{t=1}^T r_t \mid s_1 = s \right]$$

→ **When ?** The system should be controlled for a very long time

Optimal policy

Given a **value function** (①,②,③ or ④), one can define the following.

Definition

The **optimal value in a state s** is given by

$$V^*(s) = \max_{\pi} V^{\pi}(s).$$

Theorem [Puterman, 1994]

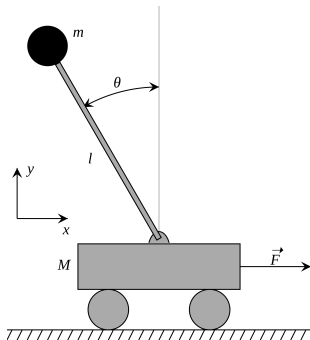
There exists an **optimal policy π^*** which satisfies

$$\forall s \in \mathcal{S}, \pi^* \in \operatorname{argmax}_{\pi} V^{\pi}(s)$$

Therefore, one can write **$V^* = V^{\pi^*}$** .

→ as we shall see, one of these optimal policies is **deterministic**.

Let's try a policy on Cart Pole



Task : maintain the pole as long as possible in a quasi-vertical position, by applying some force on the cart towards the left or right

one of the classical environment in **OpenAI Gym**

see the introductory notebook

Back to Retail Store Management

- ▶ State : number of bikes in stock on Sunday
State space : $\mathcal{S} = \{0, \dots, M\}$
- ▶ Action : number of bikes ordered at the beginning of the week
Action space : $\mathcal{A} = \{0, \dots, M\}$
- ▶ Reward = balance of the week : if your stock was s_t and you order a_t bikes, in week t you earn

$$r_t = -c_0 \mathbb{1}_{(a_t > 0)} - c \times a_t - h \times s_t + p \times \min(D_t, s_t + a_t, M)$$

- ▶ Transition : you end the week with

$$s_{t+1} = \max(0, \min(M, s_t + a_t) - D_t) \quad \text{bikes}$$

Goal : From an initial stock s , maximize the sum of **discounted** rewards

$$V^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \mid s_1 = s \right]$$

Possible policies

- Uniform policy :

$$\pi(s) \sim \mathcal{U}(\{0, \dots, M - s\})$$

- Constant policy : always buy m_0 bikes

$$\pi(s) = \max(M - s, m_0)$$

- Threshold policy : whenever there are less than m_1 bikes in stock, refill it up to m_2 bikes. Otherwise, do not order.

$$\pi(s) = \mathbb{1}_{(s \leq m_1)}(m_2 - s)$$

Simulations

Let's try out some policies !

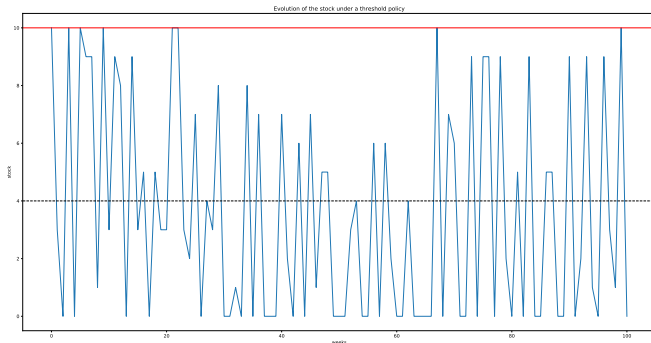


FIGURE – Evolution of the stock s_t under a threshold policy ($m_1 = 4, m_2 = 10$)

Questions

In an **known** Markov Decision Process

- ▶ can we compute an optimal policy?
(based on the explicit knowledge of $r(s, a)$ and $p(\cdot|s, a)$)
- ▶ ... even with very large (or infinite) state and/or action spaces?
(e.g. based on a *simulator* for transitions)

Beyond :

- ▶ Can we learn a good policy in an **unknown** MDP, only by selecting actions and performing transitions?
- ▶ ... and can we do it while maximizing reward?

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Broad goal of Reinforcement Learning

Learning an optimal **policy** in an **unknown** (or very large) MDP, by **acting** (=choosing action) and observing transitions.

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Policy evaluation

Given a policy $\pi = (\pi_t)_{t \in \mathbb{N}}$, how can we compute

- Finite horizon MDP :

$$V_h^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=h}^H r_t \middle| s_h = s \right]$$

and in particular $V^\pi(s) = V_1^\pi(s)$

- Discounted MDP :

$$V^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \middle| s_1 = s \right]$$

Intermezzo : Probability theory

We will need to compute several **conditional expectations**.

Recall that :

- ▶ $\mathbb{E}[X|Y = y]$ is a number :

$$\mathbb{E}[X|Y = y] = \sum_{x \in \mathcal{X}} x \mathbb{P}(X = x | Y = y) \text{ in the discrete case}$$

- ▶ $\mathbb{E}[X|Y]$ is a random variable that is $\sigma(Y)$ -measurable

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- ▶ more generally $\mathbb{E}[X|\mathcal{F}]$ is random variable that is \mathcal{F} -measurable

Useful properties

- ▶ Law of total expectation : $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
- ▶ $\mathbb{E}[X] = \sum_{y \in \mathcal{Y}} P(Y = y) \mathbb{E}[X|Y = y]$.

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Bellman equations (finite horizon)

$$V_h^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=h}^H r_t \mid s_h = s \right]$$

Proposition

The value functions of a deterministic policy π satisfies the following equations : for all $h \in \{1, \dots, H\}$,

$$V_h^\pi(s) = r(s, \pi_h(s)) + \sum_{s' \in \mathcal{S}} p(s'|s, \pi_h(s)) V_{h+1}^\pi(s'),$$

with the convention that $V_{H+1}^\pi(s) = 0$ for all $s \in \mathcal{S}$.

Exercise : Prove it !



Puterman, M. (1994).

Markov Decision Processes. Discrete Stochastic. Dynamic Programming.

Wiley.