On Bayesian index policies for sequential resource allocation

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Context: the multi-armed bandit model (MAB)

K arms $=$ K probability distributions (ν_a has mean μ_a)

ν_1 ν_2 ν_3 ν_4 ν_5

At round t, an agent

- chooses arm A_t
- observes reward $X_t \sim \nu_{A_t}$

$\mathcal{A} = (A_t)$ is his strategy or bandit algorithm :

$$A_{t+1} = F_t(A_1, X_1, \dots, A_t, X_t)$$

Goal: maximize the rewards obtained during T interactions

$\Leftrightarrow$ minimize regret:

$$\mathbb{E}\left[T(\max_a \mu_a) - \sum_{t=1}^{T} X_t\right] = \mathbb{E}\left[\sum_{t=1}^{T} (\mu^* - \mu_{A_t})\right]$$

Context: the multi-armed bandit model (MAB)

K arms = K probability distributions (ν_a has mean μ_a)





$\mathcal{B}(\mu_1)$ $\mathcal{B}(\mu_2)$ $\mathcal{B}(\mu_3)$ $\mathcal{B}(\mu_4)$ $\mathcal{B}(\mu_5)$

At round t, a doctor

- chooses treatment A_t
- observes response $X_t \in \{0,1\} : \mathbb{P}(X_t = 1) = \mu_{A_t}$

$\mathcal{A} = (A_t)$ is his strategy or bandit algorithm :

$$A_{t+1} = F_t(A_1, X_1, \dots, A_t, X_t)$$

Goal: maximize the number of patient healed within T patients

$\Leftrightarrow$ minimize regret:

$$\mathbb{E}\left[T(\max_a \mu_a) - \sum_{t=1}^{T} X_t\right] = \mathbb{E}\left[\sum_{t=1}^{T}(\mu^* - \mu_{A_t})\right]$$

Context: exponential family bandit model

ν_{θ_1}

ν_{θ_2}

ν_{θ_3}

ν_{θ_4}

ν_{θ_5}

$\nu_{\theta_1}, \dots, \nu_{\theta_K}$ belong to a one-dimensional exponential family:

$$\mathcal{P} = \{\nu_\theta, \theta \in \Theta : \nu_\theta \text{ has a density } f_\theta(x) = \exp(\theta x - b(\theta))\}$$

- ν_θ can be parametrized by its mean $\mu = \dot{b}(\theta)$: $\nu^\mu := \nu_{\dot{b}^{-1}(\mu)}$

For a given exponential family $\mathcal{P}$,

$$d_{\mathcal{P}}(\mu, \mu') := \mathrm{KL}(\nu^\mu, \nu^{\mu'}) = \mathbb{E}_{X \sim \nu^\mu}\left[\log \frac{d\nu^\mu}{d\nu^{\mu'}}(X)\right]$$

is the KL-divergence between the distributions of mean μ and μ'.

Bernoulli case: $(\theta = \log \frac{\mu}{1-\mu}, \;\; b(\theta) = \log(1 + e^\theta) \;)$

$$d(\mu, \mu') = \mathrm{KL}(\mathcal{B}(\mu), \mathcal{B}(\mu')) = \mu \log \frac{\mu}{\mu'} + (1 - \mu) \log \frac{1-\mu}{1-\mu'}.$$

A frequentist or a Bayesian model?

$$\nu_{\boldsymbol{\mu}} = (\nu^{\mu_1}, \dots, \nu^{\mu_K}) \in (\mathcal{P})^K.$$

- Two probabilistic modelings

Frequentist model	**Bayesian model**
$\mu_1, \dots, \mu_K$ unknown parameters	$\mu_1, \dots, \mu_K$ drawn from a prior distribution : $\mu_a \sim \pi_a$
arm a: $(Y_{a,s})_s \overset{\text{i.i.d.}}{\sim} \nu^{\mu_a}$	arm a: $(Y_{a,s})_s \vert \boldsymbol{\mu} \overset{\text{i.i.d.}}{\sim} \nu^{\mu_a}$

- The regret can be computed in each case

Frequentist regret (regret)	Bayesian regret (Bayes risk)
$R_T(\mathcal{A}, \boldsymbol{\mu}) = \mathbb{E}_{\boldsymbol{\mu}}\left[\sum_{t=1}^T (\mu^* - \mu_{A_t})\right]$	$\mathcal{R}_T(\mathcal{A}, \pi) = \mathbb{E}_{\boldsymbol{\mu}\sim\pi}\left[\sum_{t=1}^T (\mu^* - \mu_{A_t})\right]$ $= \int R_T(\mathcal{A}, \boldsymbol{\mu}) d\pi(\boldsymbol{\mu})$

Frequentist and Bayesian index policies

- An index policy is of the form

$$A_{t+1} = \underset{a=1\dots K}{\arg\max}\, I_a(t)$$

where the index $I_a(t)$ depends on the past observations from arm a.

- Examples:

Frequentist	**Bayesian**
popularized by [Auer et al. 02]...	... but the first index policy dates back to [Gittins 79]
index based on confidence intervals	index based on the posterior distribution $\pi_a^t = p(\mu_a \mid Y_{a,1}, \dots, Y_{a,N_a(t)})$

- Main message:

Index policies inspired by the Bayesian view on the MAB are efficient with respect to the (frequentist) regret

Outline

1. Baseline: a frequentist optimal index policy
2. Index policies inspired by the Bayesian optimal solution
3. Bayes-UCB, a simple Bayesian index policy

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Optimal algorithms for regret minimization

$\nu_{\boldsymbol{\mu}} = (\nu^{\mu_1}, \dots, \nu^{\mu_K}) \in (\mathcal{P})^K$.

$N_a(t)$: number of draws of arm a up to time t

$$R_T(\mathcal{A}, \boldsymbol{\mu}) = \sum_{a=1}^{K} (\mu^* - \mu_a) \mathbb{E}_{\boldsymbol{\mu}}[N_a(T)]$$

- Lai and Robbins lower bound:

$$\mu_a < \mu^* \Rightarrow \liminf_{T\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mu}}[N_a(T)]}{\log T} \geq \frac{1}{d(\mu_a, \mu^*)}$$

Definition

A bandit algorithm is **asymptotically optimal** if, for every $\boldsymbol{\mu}$,

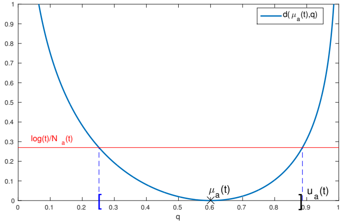
$$\mu_a < \mu^* \Rightarrow \limsup_{T\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mu}}[N_a(T)]}{\log T} \leq \frac{1}{d(\mu_a, \mu^*)}$$

The KL-UCB algorithm

- A UCB-type algorithm: $A_{t+1} = \arg\max_a \; u_a(t)$
- ... associated to the right upper confidence bounds:

$$u_a(t) = \max\left\{ q \geq \hat{\mu}_a(t) : d\left(\hat{\mu}_a(t), x\right) \leq \frac{\log(t) + c\log\log(t)}{N_a(t)} \right\},$$

$\hat{\mu}_a(t)$: empirical mean of rewards from arm a up to time t.



[Cappé et al. 13]: KL-UCB satisfies, for $c \geq 5$,

$$\mathbb{E}_{\boldsymbol{\mu}}[N_a(T)] \leq \frac{1}{d(\mu_a, \mu^*)} \log T + O(\sqrt{\log(T)}).$$

WANTED!

Index policies that are not only asymptotically optimal but also

- more efficient in practice
- with indices that are easier to compute
- easier to generalize beyond exponential family bandits

Our answer:

index policies inspired by the Bayesian MAB

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The Bayesian optimal solution

There exists an exact solution to Bayes risk minimization:

$$\underset{(A_t)}{\arg\max}\ \mathbb{E}_{\boldsymbol{\mu}\sim\pi}\left[\sum_{t=1}^{T} X_t\right].$$

Why? The history of the game can be summarized by a posterior matrix, that evolves in a Markov Decision Process.
$\Rightarrow$ optimal policy = solution to dynamic programming equations.

Example: Bernoulli bandit model $\nu^{\boldsymbol{\mu}} = (\mathcal{B}(\mu_1), \dots, \mathcal{B}(\mu_K))$

- $\mu_a \sim \mathcal{U}([0,1])$
- $\pi_a^t = \text{Beta}(\#|\text{ones observed}| + 1, \#|\text{zeros observed}| + 1)$

$$\begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{A_t=2} \begin{pmatrix} 1 & 2 \\ 6 & 1 \\ 0 & 2 \end{pmatrix} \textit{ if } X_t = 1$$

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INTRACTABLE !

Gittins' solution

[Gittins 79]: the solution of the discounted MAB,

$$\underset{(A_t)}{\arg\max}\ \mathbb{E}_{\boldsymbol{\mu}\sim\pi}\left[\sum_{t=1}^{\infty}\alpha^{t-1}X_t\right]$$

is an index policy:

$$A_{t+1} = \underset{a=1\dots K}{\text{argmax}}\ G_{\alpha}(\pi_a^t).$$

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In the undiscounted case: the Finite-Horizon Gittins algorithm

$$A_{t+1} = \underset{a=1\dots K}{\text{argmax}}\ G(\pi_a^t, T-t).$$

$G(p,r) = \inf\{\lambda \in \mathbb{R} : V_\lambda^*(p,r) = 0\}$, with

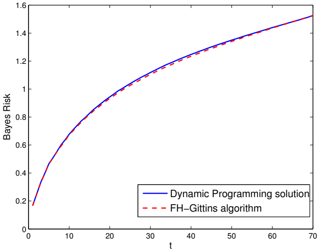
$$V_\lambda^*(p,r) = \sup_{0\leq\tau\leq r} \mathbb{E}_{\substack{Y_t \overset{\text{i.i.d}}{\sim} \nu^\mu \\ \mu\sim\pi}}\left[\sum_{t=1}^{\tau}(Y_t-\lambda)\right]$$

"price worth paying for playing arm $\mu \sim p$ for at most r rounds"

The FH-Gittins algorithm

FH-Gittins...

- does NOT coincide with the optimal solution of the undiscounted MAB ([Berry, Fristedt 1985]) but it is conjectured to be a good approximation



- displays good performance in terms of regret as well !

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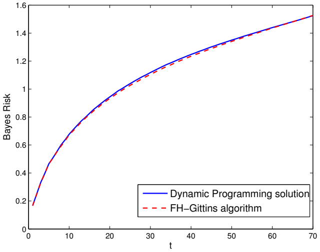

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INDICES ARE HARD TO COMPUTE...

Approximating the FH-Gittins indices

- [Burnetas and Katehakis, 03]: when n is large,

$$G(\pi_a^t, n) \simeq \max\left\{q \geq \hat{\mu}_a(t), N_a(t)d\left(\hat{\mu}_a(t), q\right) \leq \log\left(\frac{n}{N_a(t)}\right)\right\}$$

- [Lai, 87]: the index policy associated to

$$I_a(t) = \max\left\{q \geq \hat{\mu}_a(t), N_a(t)d\left(\hat{\mu}_a(t), q\right) \leq \log\left(\frac{T}{N_a(t)}\right)\right\}$$

is a good approximation of the Bayesian solution for large T.

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ASYMPTOTIC OPTIMALITY ?

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The Bayes-UCB algorithm

π_a^t the posterior distribution over μ_a at the end of round t.

Algorithm: Bayes-UCB [K., Cappé, Garivier 2012]

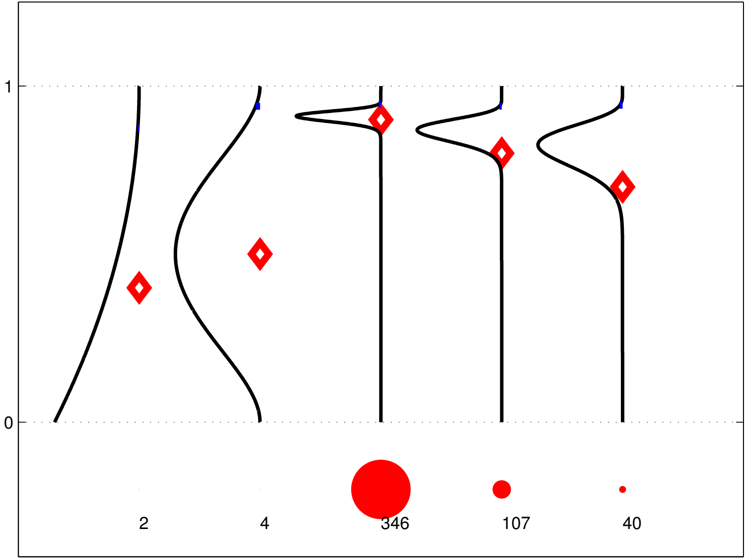
$$A_{t+1} = \underset{a}{\text{argmax}}\ Q\left(1 - \frac{1}{t(\log t)^c}, \pi_a^t\right)$$

where $Q(\alpha, p)$ is the quantile of order α of the distribution p.

Bernoulli reward with uniform prior:

- $\pi_a^0 \overset{i.i.d}{\sim} \mathcal{U}([0,1]) = \text{Beta}(1,1)$
- $\pi_a^t = \text{Beta}(S_a(t)+1, N_a(t)-S_a(t)+1)$

Bayes-UCB in practice



Theory

$\nu^{\mu_1}, \dots, \nu^{\mu_K}$ are such that $\mu_a \in \mathcal{J}$ ($\mathcal{J}$ open interval)

Assumptions

$\pi = \pi_1^0 \otimes \dots \otimes \pi_K^0$ is such that

- π_a^0 has a density h_a with respect to the Lebesgue measure
- $\forall u \in \mathcal{J}, \; h_a(u) > 0$

- The posterior distribution depends on two sufficient statistics:

$$\pi_a^t = \pi_{a, N_a(t), \hat{\mu}_a(t)}$$

An important rewriting of the posterior

$$\pi_{a,n,x}(\mathcal{I}) = \frac{\int_{\mathcal{I}} e^{-n d(x,u)} h_a(u) du}{\int_{\mathcal{J}} e^{-n d(x,u)} h_a(u) du}.$$

Theory

- Bayes-UCB rewrites

$$A_{t+1} = \underset{a}{\text{argmax}}\ Q\left(1 - \frac{1}{t(\log t)^c}, \pi_{a,N_a(t),\hat{\mu}_a(t)}\right)$$

Extra assumption

Bounds on the means of the arms are known: there exists μ^-, μ^+ in $\mathcal{J}$ such that for all a, $\mu_a \in [\mu^-, \mu^+]$

Theorem

Let $\overline{\mu}_a(t) = (\hat{\mu}_a(t) \vee \mu^-) \wedge \mu^+$. The index policy

$$A_{t+1} = \underset{a}{\text{argmax}}\ Q\left(1 - \frac{1}{t(\log t)^c}, \pi_{a,N_a(t),\overline{\mu}_a(t)}\right)$$

with parameter $c \geq 7$ is such that, for all $\epsilon > 0$,

$$\mathbb{E}_{\boldsymbol{\mu}}[N_a(T)] \leq \frac{1+\epsilon}{d(\mu_a, \mu^*)} \log(T) + O_\epsilon(\sqrt{\log(T)}).$$

A key element: Posterior bounds

Recall that $\pi_{a,n,x}(\mathcal{I}) = \frac{\int_{\mathcal{I}} e^{-nd(x,u)} h_a(u) du}{\int_{\mathcal{J}} e^{-nd(x,u)} h_a(u) du}$.

Bounds on the tail of the posterior distribution

The exists constants A, B, C such that, for all a, for all $n \in \mathbb{N}^*$ and $(x, v) \in [\mu^-, \mu^+]^2$,

1. if $v > x$, $An^{-1}e^{-nd(x,v)} \leq \pi_{a,n,x}([v, \mu^+[) \leq B\sqrt{n}e^{-nd(x,v)}$
2. if $v < x$, $\pi_{a,n,x}([v, \mu^+[) \geq 1/(C\sqrt{n} + 1)$

A key element: Posterior bounds

1. if $v > x$, $An^{-1}e^{-nd(x,v)} \leq \pi_{a,n,x}([v, \mu^+[) \leq B\sqrt{n}e^{-nd(x,v)}$
2. if $v < x$, $\pi_{a,n,x}([v, \mu^+[) \geq 1/(C\sqrt{n} + 1)$

Example of use:

$$
\begin{aligned}
\{\mu_1 \geq \overline{q}_1(t)\} &= \left\{\pi_{1,N_1(t),\overline{\mu}_1(t)}([\mu_1, \mu^+[) \leq \frac{1}{t \log^c t}\right\} \\
&\subset \left\{\frac{1}{C\sqrt{N_1(t)} + 1} \leq \frac{1}{t \log^c t}\right\} \bigcup \left\{\frac{Ae^{-N_1(t)d^+(\overline{\mu}_1(t),\mu_1)}}{N_1(t)} \leq \frac{1}{t \log^c t}\right\}, \\
&\subset \left\{N_1(t)d^+(\hat{\mu}_1(t), \mu_1) \geq \log\left(\frac{At \log^c t}{N_1(t)}\right)\right\},
\end{aligned}
$$

for t large enough.

An interesting by-product of our analysis

- We managed to handle alternative exploration rates !

Index policy: KL-UCB-H$^+$

$$u_a^{H,+}(t) = \max\left\{q \geq \hat{\mu}_a(t) : N_a(t) d\left(\hat{\mu}_a(t), x\right) \leq \log\left(\frac{T \log^c T}{N_a(t)}\right)\right\}$$

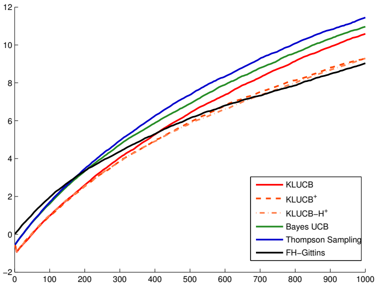
Index policy: KL-UCB$^+$

$$u_a^{+}(t) = \max\left\{q \geq \hat{\mu}_a(t) : N_a(t) d\left(\hat{\mu}_a(t), x\right) \leq \log\left(\frac{t \log^c t}{N_a(t)}\right)\right\}$$

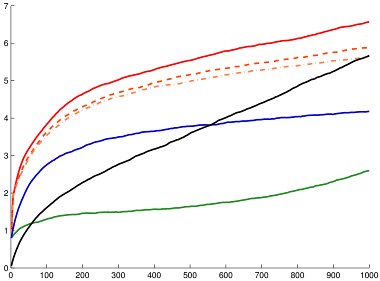
The index policy associated to the indices $u_a^{H,+}(t)$ and $u_a^{+}(t)$ satisfy, for all $\epsilon > 0$,

$$\mathbb{E}[N_a(T)] \leq \frac{1+\epsilon}{d(\mu_a, \mu^*)} \log(T) + O_\epsilon(\sqrt{\log(T)}).$$

- Short horizon, $T = 1000$ (average over $N = 10000$ runs)



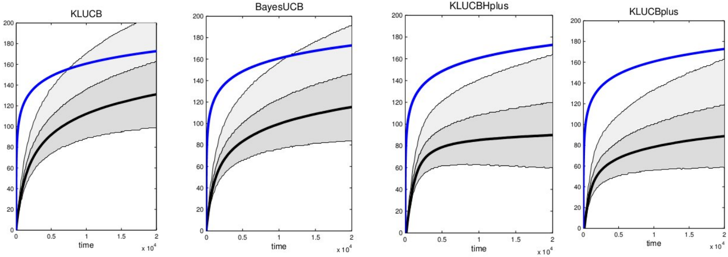
$\mu_1 = 0.2, \mu_2 = 0.25$



$\mu_1 = 0.85, \mu_2 = 0.95$

Numerical tour: regret curves

- Long horizon, $T = 20000$ (average over $N = 50000$ runs)



10 arms bandit problem

$$\mu = [0.1\ 0.05\ 0.05\ 0.05\ 0.02\ 0.02\ 0.02\ 0.01\ 0.01\ 0.01]$$

Conclusion

We presented several index policies inspired by the Bayesian MAB:

- FH-Gittins, based on the finite-horizon Gittins indices
- Bayes-UCB, based on posterior quantiles
- KL-UCB$^+$ and KL-UCB-H$^+$, two variants of KL-UCB using an alternative exploration rate, inspired by the Bayesian solution

We studied their performance in terms of (frequentist) regret:

- they compete with or even outperform KL-UCB
- Bayes-UCB, KL-UCB$^+$, KL-UCB-H$^+$ asymptotically optimal
- FH-Gittins may still be a good idea for short horizons

Among them:

- Bayes-UCB is the easiest to implement, and can be generalized to more complex bandit models

References

- E. Kaufmann, O. Cappé, A. Garivier, *On Bayesian Upper Confidence Bounds for Bandit Problems*, AISTATS 2012
- E. Kaufmann, *Analysis of Bayesian and frequentist strategies for sequential resource allocation*, PhD thesis, 2014
- E. Kaufmann, *On Bayesian index policies for sequential resource allocation* (work in progress !)