University of Lille 2024-2025
M1 Data Science - Stats 2

TD1 - Estimation

Exercise 1 Let $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma_0^2)$ be i.i.d Gaussian random variables for an unknown $\mu \in \mathbb{R}$ but known σ_0.

1. Derive the maximum likelihood estimator $\widetilde{\mu}_n$ of μ.
2. Using $\widetilde{\mu}_n$, derive a confidence interval for μ at the confidence level $1 - \alpha$.

Exercise 2 A Poisson distribution with parameter $\lambda > 0$, denoted by $\mathcal{P}(\lambda)$, is a discrete distribution supported on $\mathbb{N}$ defined as

$$\mathbb{P}_{Z\sim\mathcal{P}(\lambda)}(Z = k) = \frac{\lambda^k}{k!}e^{-\lambda}.$$

1. Compute the maximum likelihood estimator of λ given iid observations $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{P}(\lambda)$.
2. What other method(s) could you use to obtain the same estimator?
3. Compute its biais and its mean square error.

Exercise 3 Let $(X_1, \dots, X_n)$ be a n-sample drawn from the uniform distribution $\mathcal{U}(0, \theta)$, for an unknown parameter value $\theta > 0$.

1. Calculate $E_\theta[X_1]$ and deduce the moment estimator $\widehat{\theta}_n$ of θ.
2. Calculate the maximum likelihood estimator $\widetilde{\theta}_n$ of θ.
3. Are these estimators biased?
4. Compare the quadratic risks of these estimators.

Exercise 4 Let $(X_1, \dots, X_n)$ be a n-sample drawn from the uniform distribution over $[\theta - 1/2; \theta + 1/2]$, where $\theta \in \mathbb{R}$ is unknown. What is the MLE of θ?

Exercise 5 We are given an i.i.d sample $X_1, \dots, X_n \sim f_\theta$ where $\theta > 0$ is an unknown parameter, and

$$f_\theta(x) = \frac{2\theta^2}{x^3}\mathbb{1}_{[\theta, +\infty[}(x).$$

1. Show that f_θ is a density and find $\mathbb{E}_\theta[X]$.
2. Using the moment method find an unbiased estimator $\widetilde{\theta}_n$ of θ.
3. Show that the MLE $\widehat{\theta}_n$ is given by $\widehat{\theta}_n = \min_i X_i$. Evaluate the bias of $\widehat{\theta}_n$.

Exercise 6 (exam 2024) On the figure below the densities of the distribution of three estimators $\hat{\mu}_1 = h_1(X)$, $\hat{\mu}_2 = h_2(X)$ and $\hat{\mu}_3 = h_3(X)$ of a parameter μ are displayed, where X is generated from the distribution P_μ with $\mu = 0.5$. Discuss the relative merits of the three estimators.