Project 1

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Introduction

The one-dimensional Poisson equation can be written as

$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

We will use the function $f(x) = 100e^{-10x}$, with $x \in [0, 1]$ and the boundry conditions $u(0) = u(1) = 0$.

Problem 1

We start by finding the analythic solution to the equation:

$$\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} = f(x) \implies u(x) &= -\iint f(x)\,\mathrm{d}x^2 \\
&= -\iint 100e^{-10x}\,\mathrm{d}x^2 \\
&= \int 10e^{-10x} + C_1\,\mathrm{d}x \\
&= -e^{-10x} + C_1 x + C_2 \\
u(0) = -1 + C_2 = 0 &\implies C_2 = 1 \\
u(1) = -e^{-10} + C_1 + 1 = 0 &\implies C_1 = e^{-10} - 1
\end{aligned}$$

This gives $u(x) = dunder 1 - (1 - e^{-10})x - e^{-10x}$

Problem 2

Problem 3

We want to discretize the one-dimensional Poisson equation. We start by discretizing the second order derivative. We will use the definition of the derivative,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to \infty} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to \infty} \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

This means that $\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$, where a smaller Δx gives a better approximation. We can then approximate the second order derivative:

$$\begin{aligned}
\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} &\approx \frac{\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}}{\Delta x} \\
&= \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2}
\end{aligned}$$

We can then discretize the one-dimensional Poisson equation.

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} &= f(x) \\
-\frac{v(x+\Delta x) - 2v(x) + v(x-\Delta x)}{\Delta x^2} &= f(x) \\
v(x+\Delta x) &= 2v(x) - v(x-\Delta x) - f(x)\Delta x^2
\end{aligned}
$$

Note that we have here distinguished the discretized approximation from the actual exact answer by calling the approximation $v(x)$ and the exact solution $u(x)$.