Homework sheet 2 - Tutorial 204 Electrodynamics 2023

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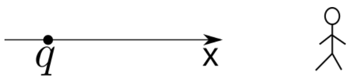
Arnaud Couairon - Centre de Physique Théorique, École Polytechnique
Tutorial organizers:
Erik Linnér (erik.linner@polytechnique.edu)
Sylvain Chaty (sylvain.chaty@polytechnique.edu)
Didina Serban (didina.serban@polytechnique.edu)
Max Gilljohann (max.gilljohann@polytechnique.edu)
https://moodle.polytechnique.fr/course/view.php?id=15612

The solutions are to be handed in on the 31st of May 2023 via Moodle. You can work in groups up to 2 students (i.e. 2 names per solution).

Exercise 1: Retarded potential for a point charge (Liénard-Wiechert potentials)

We will now consider a point charge moving along the x-direction towards an observer located at $\mathbf{r} = 0$. We want to find its effective electric and magnetic potentials V and $\mathbf{A}$.



a) Based on your naive expectations from the lectures, write down the expression for the retarded electric potential for the charge density ρ of a point charge q moving with velocity v.

b) Now consider the point charge to be an extended line charge of length $a > 0$ (parallel to the x-axis, with $a \to 0$. Determine the effective length a' of the line charge as the observer sees it at a given point in time, considering a finite speed c for the propagation of information.

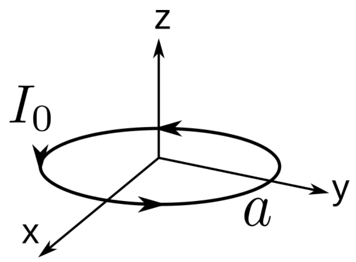
c) Now use this result to obtain the electric potential $V(\mathbf{r}, t)$ and then let $a \to 0$. (*Hint: l'Hopital*) You should obtain a retardation effect depending on v/c. How do you interpret this result?

d) Generalize your result to arbitrary directions of the point charge's velocity $\mathbf{v}$ and write it as a function of $\mathbf{v}$ and $\mathbf{u}_r$, where $\mathbf{u}_r$ is the unit vector pointing from the charge to the observer. (*Hint: A geometrical proof is sufficient*).

e) Derive the magnetic potential $\mathbf{A}$ and write it as a function of the electric potential V. (*Hint: What is the relation between* $\mathbf{j}$ *and* ρ*?*) V and $\mathbf{A}$ are called the Liénard-Wiechert potentials of a point charge. (They arise purely from geometrical considerations and do not take into account relativistic effects yet (like the Doppler effect)).

Exercise 2: Accurate radiation characteristics of an AC loop current

We now want to investigate the exact radiation profile of an oscillating ring current in the x-y-plane as shown in the following figure:



The ring has a radius a and it is charge-neutral. The current is given by

$$I(t) = \begin{cases} 0 & \text{for } t < 0 \\ I_0 \sin(\omega t) & \text{for } t \geq 0 \end{cases}$$

Here we will work with the electrostatic CGS units where $\frac{1}{4\pi\epsilon_0} = 1$, $\frac{\mu_0}{4\pi} = \frac{1}{c^2}$. (More information can be found on `https://en.wikipedia.org/wiki/Gaussian_units`)

a) Write down the retarded potentials $V(\mathbf{r}, t)$, $\mathbf{A}(\mathbf{r}, t)$ and the retarded time $t_r = t - \|\mathbf{r} - \mathbf{r}'\|/c$ for points $\mathbf{r}'$ on the ring. (*Hint: For this exercise it will be the easiest to work in cartesian coordinates and parametrize the points on the ring by the radius and an angle ϕ, but you are free to use another system*).

b) Use a programming language of your choice to write a code that evaluates the retarded potentials and the electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ for arbitrary points $\mathbf{r}$ and time t.

   Define all parameters in such a way that they an be easily modified later. You can start with $a = 1$, $\omega = 100$, $c = 10000$. For the numerical integrals you can use 40 grid points or more, and for the numerical derivative you can use the finite-difference method with step-size 0.001.

c) Plot your result for the components of the electric and magnetic fields at a point in the x-y-plane, for example $(10a, 0, 0)$, as a function of time up to $t_{max} = 2T = 2\frac{2\pi}{\omega}$. Check that your result is converged with respect to the size of the integration grid and step-size. Modify the position and time window if needed to show the relevant behaviour of your result.

d) How do you interpret your result? Which components of the fields are nonzero, and does it agree with what you expect? Is there a phase factor between the electric and magnetic field?

   Investigate the behaviour for different values of c and distances to the ring, and report your findings.