Algorithms and Computability

Lecture 13: NP-Complete Problems

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slides courtesy of Martin Zimmermann

Last Time in Algorithms and Computability

We have seen:

- The complexity class NP: The class of languages accepted by polynomial-time NTMs
- Polynomial-time reductions

Recall: $\mathrm{P} \subseteq \mathrm{NP}$

The (literally) Million Dollar Question

Is verifying certificates easier than finding certificates or not:

$\mathrm{P} = \mathrm{NP}$ or $\mathrm{P} \subsetneq \mathrm{NP}$?

- One of the most challenging and most important questions of (theoretical) computer science

Polynomial-Time Reductions

Definition

Let $A \subseteq \Sigma_1^*$ and $B \subseteq \Sigma_2^*$ be languages. We say that A is polynomial-time reducible to B, written $A \leq_P B$, if and only if

1. there is a polynomial-time computable function $f : \Sigma_1^* \to \Sigma_2^*$ such that
2. for every $w \in \Sigma_1^*$: $w \in A \Leftrightarrow f(w) \in B$

Theorem

Let $A \leq_P B$

1. *If* $B \in \mathrm{P}$, *then* $A \in \mathrm{P}$
2. *If* $B \in \mathrm{NP}$, *then* $A \in \mathrm{NP}$

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This result can be used to prove **upper** bounds

What About Lower Bounds?

Recall:

Theorem

Let $A \leq_m B$

- *If* A *is not computable, then* B *is also not computable*
- *If* A *is not computably-enumerable, then* B *is also not computably-enumerable*

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- *If A is not computably-enumerable, then B is also not computably-enumerable*

- Today, we will study how to obtain analogous results for P and NP
- However, the fact that we have no problem that is proven to be in $\mathrm{NP} \setminus \mathrm{P}$ means we will identify candidates for such problems: the hardest problems in NP
- **Caveat:** These hardest problem might still be in P – we simply do not know
- We will discuss this in more detail later today

Agenda

We consider two types of graphs:

1. Directed Graphs: $G = (V, E)$ where
   - V is a finite set of vertices and
   - $E \subseteq V \times V$ is a set of (directed) edges

Example

(V, E) with $V = \{1, 2, 3, 4\}$ and
$E = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$:

$$\begin{tikzcd} 4 \arrow[d] & 3 \arrow[l] \\ 1 \arrow[r] & 2 \arrow[u] \end{tikzcd}$$

Reminder: Graphs

We consider two types of graphs:

1. Directed Graphs: $G = (V, E)$ where
   - V is a finite set of vertices and
   - $E \subseteq V \times V$ is a set of (directed) edges
2. Undirected Graphs: Directed graphs (V, E) such that $(v, v') \in E$ implies $(v', v) \in E$

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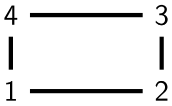

Note that we drop the arrow tips in undirected graphs

Reminder: The Clique Problem

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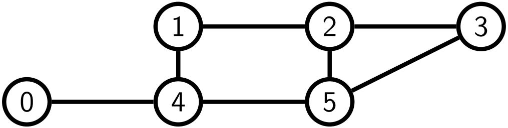

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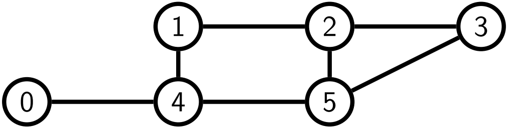

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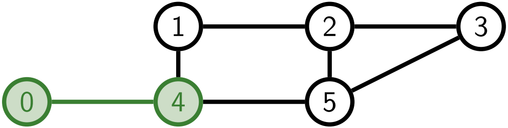

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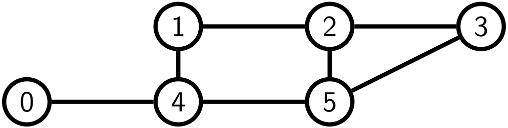

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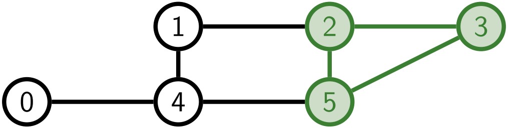

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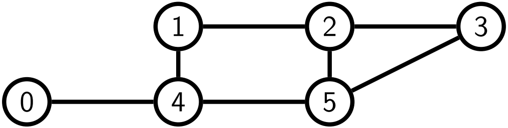

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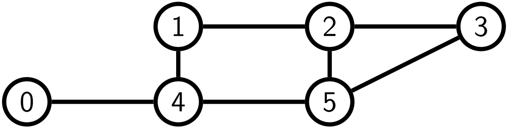

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Vertex Cover

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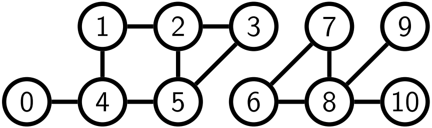

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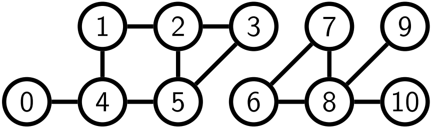

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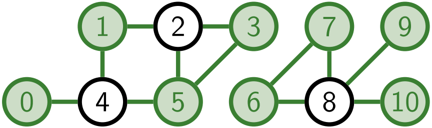

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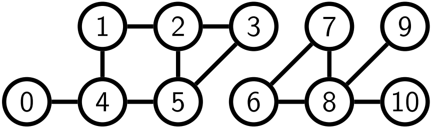

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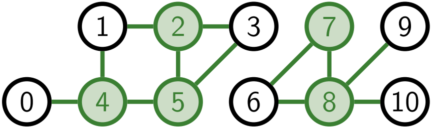

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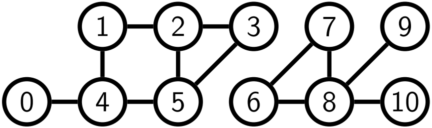

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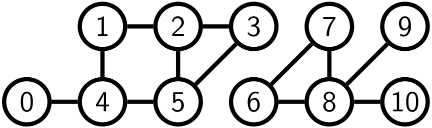

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- There are many other problems with the same status
- Is there a reason we fail to show that these problems are in P?

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- Is there a reason we fail to show that these problems are in P?

Yes, there is! They are some of the hardest problems in NP mentioned earlier

Intuition

If only one of these hardest problems is in P, then $\mathrm{P} = \mathrm{NP}$ (i.e., all of them are)

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- A single branch of the tree can be represented by a $(p(n) + 1) \times p(n)$ grid such that the entry at (i, j) represents (1) the content of the i-th tape cell in the j-th configuration, (2) whether the head is at that cell, and (3) the current state

Visualization

$p(n)$	$\$$									$\$$
$\vdots$	$\$$									$\$$
	$\$$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\$$
j	$\$$	X	1	$\#, q_\ell$	X	1	$\sqcup$	$\sqcup$	$\sqcup$	$\$$
	$\$$	X	1	$\#$	$1, q_1^\#$	1	$\sqcup$	$\sqcup$	$\sqcup$	$\$$
$\vdots$	$\$$	X	1	$\#, q_1$	1	1	$\sqcup$	$\sqcup$	$\sqcup$	$\$$
2	$\$$	X	$1, q_1$	$\#$	1	1	$\sqcup$	$\sqcup$	$\sqcup$	$\$$
1	$\$$	$1, s$	1	$\#$	1	1	$\sqcup$	$\sqcup$	$\sqcup$	$\$$
	0	1	$\cdots$					$\cdots$	$p(n)+1$	$p(n)+2$

i

$p(n)$	\$									\$
$\vdots$	\$									\$
	\$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	\$
j	\$	X	1	$\#,q_\ell$	X	1	␣	␣	␣	\$
	\$	X	1	$\#$	$1,q_1^\#$	1	␣	␣	␣	\$
$\vdots$	\$	X	1	$\#,q_1$	1	1	␣	␣	␣	\$
2	\$	X	$1,q_1$	$\#$	1	1	␣	␣	␣	\$
1	\$	$1,s$	1	$\#$	1	1	␣	␣	␣	\$
	0	1	$\cdots$		i			$\cdots$	$p(n)+1$	$p(n)+2$

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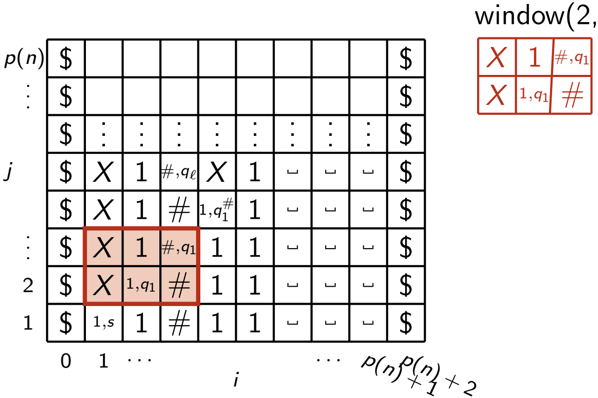

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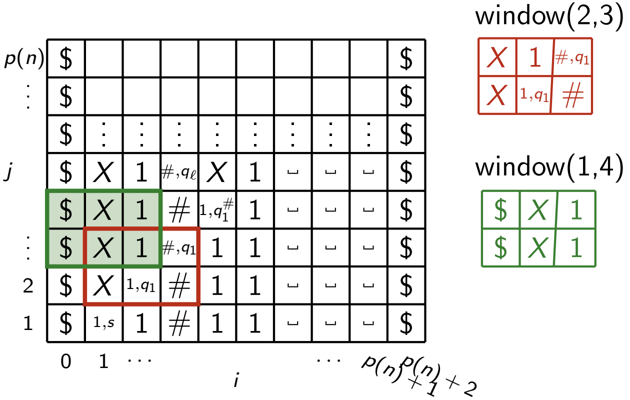

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window(2,3)

X	1	$\#,q_1$
X	$1,q_1$	$\#$

window(1,4)

$\$$	X	1
$\$$	X	1

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- Rely on locality: entry at (i,j) depends only on entries at $(i-1,j-1)$, $(i,j-1)$, and $(i+1,j-1)$: window of size 3×2
- How large is the grid? Polynomial in $p(n)$, and thus in n

Roadmap

$$w \in L(M) \Leftrightarrow \varphi_{M,w} \in \mathrm{SAT}$$

- $w \in L(M)$ is equivalent to the existence of a $(p(n)+3) \times p(n)$-grid encoding a branch of an accepting run
- We express the existence of such a grid by a formula:

  $\varphi_{M,w} = \varphi_{\text{setup}} \wedge \varphi_{\text{init}} \wedge \varphi_{\text{move}} \wedge \varphi_{\text{accept}}$

- Define $\Gamma_{\$} = \Gamma \cup \{\$\}$[1]
- We use the following propositions (for $i \in \{0, 1, \ldots, p(n)+2\}$, $j \in \{1, 2, \ldots, p(n)\}$, $a \in \Gamma_{\$}$, and $q \in Q$)
  - $x_{i,j,a}$: grid entry at (i,j) is a, and
  - $x_{i,j,(a,q)}$: grid entry at (i,j) is (a,q)

[1] Here we assume that $\$ \notin \Gamma$ is a fresh tape symbol

Setup

$$\varphi_{\text{setup}} = \bigwedge_{i=0}^{p(n)+2} \bigwedge_{j=1}^{p(n)} \bigvee_{e \in E} \left(x_{i,j,e} \wedge \bigwedge_{e' \in E \setminus \{e\}} \neg x_{i,j,e'} \right)$$

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Note

The size of φ_{setup} is $\mathcal{O}((p(n))^2 \cdot |E|^2)$ and therefore polynomial in $|\ulcorner M \urcorner| + n$

Init

$$\varphi_{\text{init}} = x_{0,1,\$} \wedge x_{1,1,(w_0,s)} \wedge \left(\bigwedge_{i=2}^{n} x_{i,1,w_{i-1}} \right) \wedge \left(\bigwedge_{i=n+1}^{p(n)+1} x_{i,1,\llcorner\lrcorner} \right) \wedge x_{p(n)+2,1,\$}$$

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This encodes:

- The bottom row in the grid is filled by $\$w{\sqcup}^{p(n)-n+1}\$$

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Move

$$\varphi_{\text{move}} = \bigwedge_{i=1}^{p(n)+1} \bigwedge_{j=2}^{p(n)} \psi_{i,j} \quad \text{with}$$

$$\psi_{i,j} = \bigvee_{\begin{array}{|c|c|c|}\hline e_3 & e_4 & e_5 \\ \hline e_0 & e_1 & e_2 \\ \hline \end{array}} x_{i-1,j-1,e_0} \wedge x_{i,j-1,e_1} \wedge x_{i+1,j-1,e_2} \wedge x_{i-1,j,e_3} \wedge x_{i,j,e_4} \wedge x_{i+1,j,e_5}$$

where the disjunction ranges over all legal windows

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- Every (3×2)-window is correctly filled
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Note

The size of φ_{accept} is $\mathcal{O}((p(n))^2)$ and therefore polynomial in $|\ulcorner M \urcorner| + n$

Wrapping Things Up

- The function mapping $\ulcorner M \urcorner$ and w to $\varphi_{M,w}$ is computable in polynomial time
- We have $w \in L(M) \Leftrightarrow \varphi_{M,w} \in \mathrm{SAT}$:

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- Hence, $L \leq_p \mathrm{SAT}$

Agenda

1. Some More Problems
2. SAT Can Simulate All Problems in NP
3. **NP-Hardness and NP-Completeness**
4. More NP-Complete Problems

Hardness and Completeness

We have seen that every language in NP can be reduced to SAT in polynomial time, and moreover that SAT is in NP

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Definition

Let L be a language

1. L is NP-hard if $L' \leq_p L$ for all $L' \in \mathrm{NP}$
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Corollary [Cook '61, Levin '63]

SAT is NP-complete

Quiz 1

If L is NP-hard, is L necessarily in NP?

Quiz 1

If L is NP-hard, is L necessarily in NP?

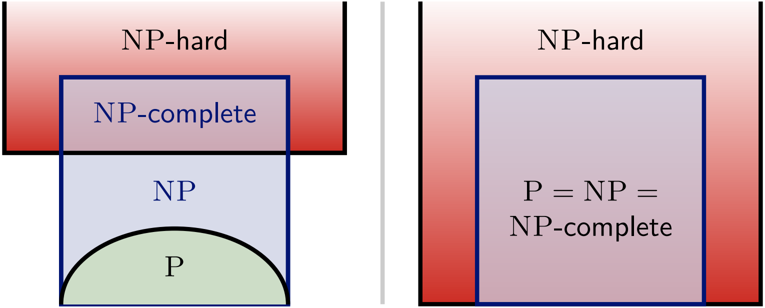
No, otherwise NP-hard and NP-complete would be identical

NP-hard really just means "very hard" – but there are still many different levels of "very hard"

For instance, the halting problem is NP-hard (not shown here) but not in NP (because it is not even computable)

Possible Worlds

We are in one of the following two worlds:



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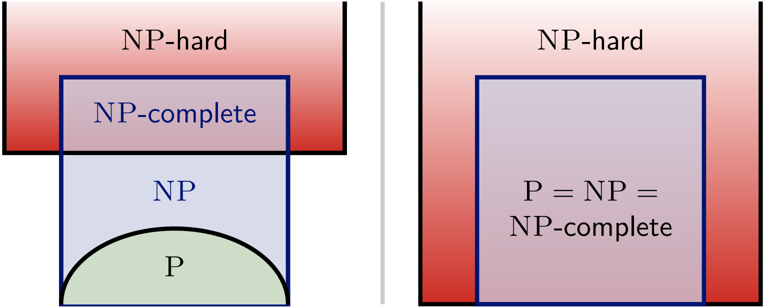

Theorem

Let L be NP-complete. If $L \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$

Proof

Exercise

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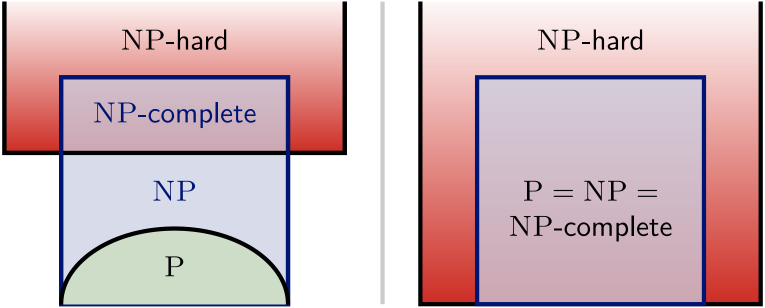

Theorem

Let L be NP*-complete. If* $L \in \mathrm{P}$, *then* $\mathrm{P} = \mathrm{NP}$

NP-complete problems are the hardest problems in NP:
a P-algorithm for L induces a P-algorithm for *all* $L' \in \mathrm{NP}$

P vs. NP

- In the 60 years since the Cook-Levin theorem, nobody has been able to show that any NP-complete problem is in P (and there are *many* candidates)
- Dually, nobody has shown that there is a problem in NP that is not in P

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- Dually, nobody has shown that there is a problem in NP that is not in P
- Most (but not all) computer scientists believe the latter to be the case
- But this has been hard to prove! In fact, we even have theorems that say such a proof must be hard (in some very technical sense: there are barriers to separations), e.g., a diagonalization proof cannot show $\mathrm{P} \neq \mathrm{NP}$

Exponential Time Hypothesis

Every halting DTM for SAT has exponential time complexity

Agenda

CNF-SAT

A Boolean formula φ is in conjunctive normal form (CNF) if it is of the form

$$\varphi = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m_i} \ell_{i,j}$$

such that each $\ell_{i,j}$ is a literal, i.e., either a variable x or a negated variable $\neg x$

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Decision problem:

$$\mathrm{CNF\text{-}SAT} = \{\varphi \mid \varphi \text{ is in CNF and has a satisfying assignment}\}$$

CNF-SAT is NP-Complete

Recall from the last exercise sheet: $\mathrm{DNF\text{-}SAT}$ is in P

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$\varphi_{M,w}$ can be transformed into CNF in polynomial time (see exercises)

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Note

- Every Boolean formula can be turned into an equivalent one in DNF and into an equivalent one in CNF
- Thus, the structure of the formula has direct consequences for the complexity of the satisfiability problem

Quiz 2

Does the following algorithm for SAT run in polynomial time?

On input φ:

1. Transform φ into an equivalent formula φ' in DNF
2. Check whether φ' (and therefore φ) is satisfiable

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No, the first step is generally exponential

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How can we prove that some L is NP-hard?

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Proof

Show that $\leq_p$ is transitive, i.e., $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$ implies $L_1 \leq_p L_3$

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$$\mathrm{3SAT} = \{\varphi \mid \varphi \text{ is in 3CNF and has a satisfying assignment}\}$$

Theorem

$\mathrm{3SAT}$ *is* NP*-complete*

Proof

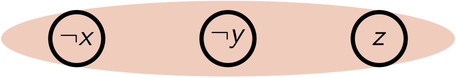
Exercise

Clique is NP-complete

- We show $3\mathrm{SAT} \leq_p \mathrm{CLIQUE}$
- First, we illustrate the reduction on an example

Clique is NP-complete

Let $\varphi = (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z)$



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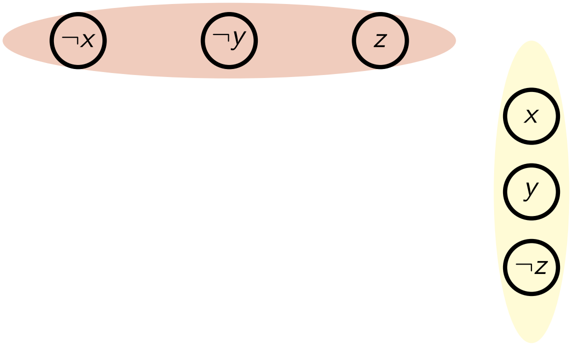

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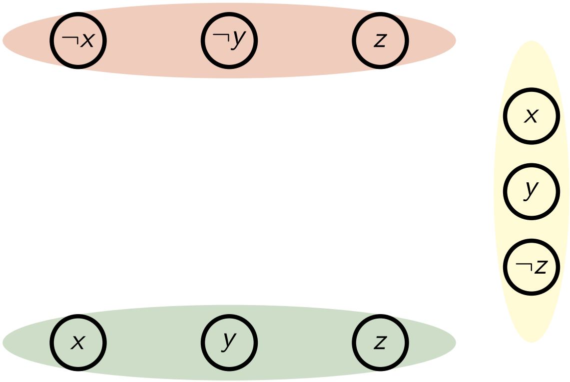

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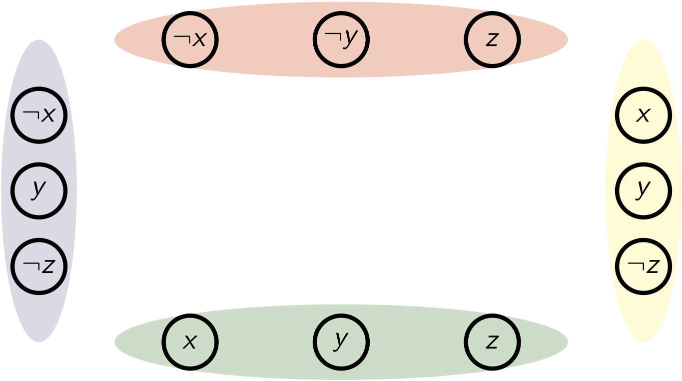

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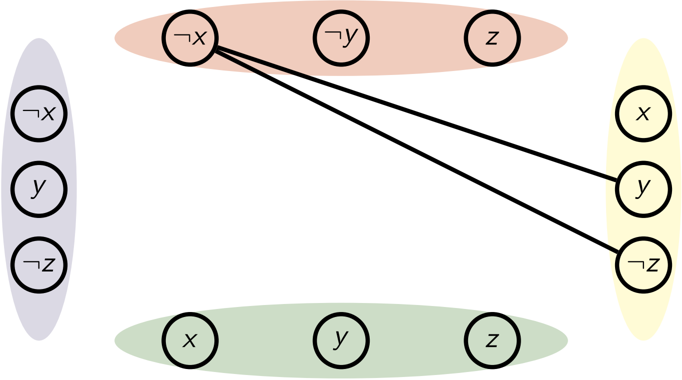

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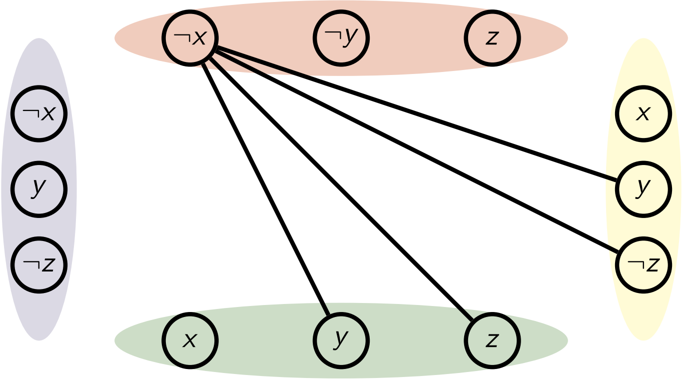

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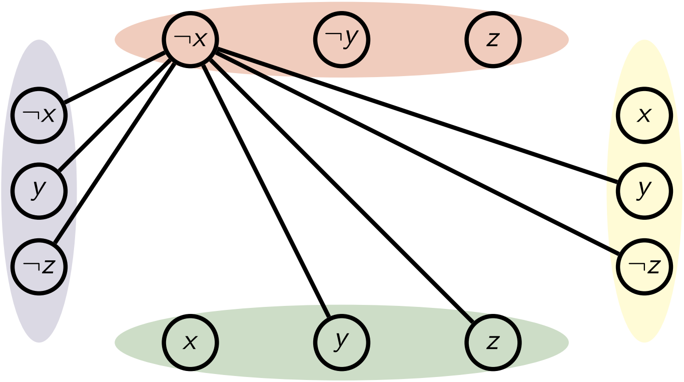

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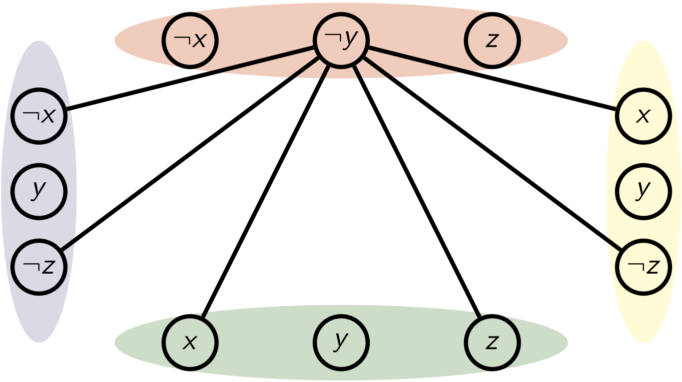

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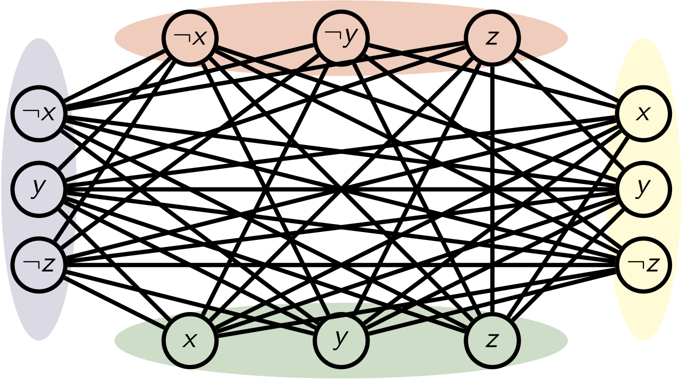

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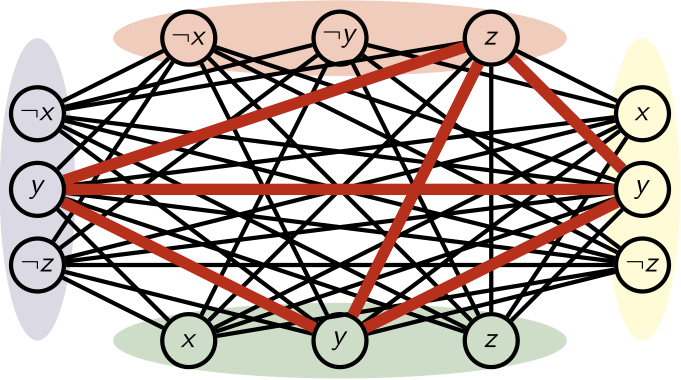
Let $\varphi = (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z)$

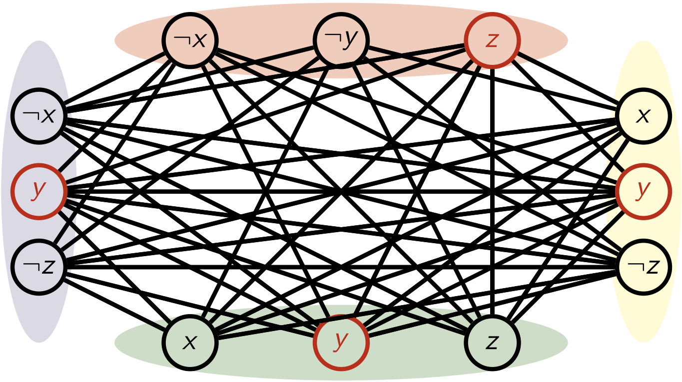
- A 4-clique contains exactly one vertex from each of the four clauses and no contradictory literals
- This can be extended to a satisfying assignment



Clique is NP-complete

Let $\varphi = (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z)$

- Vice versa: A satisfying assignment satisfies some literal in each clause
- Picking one from each clause gives a 4-clique



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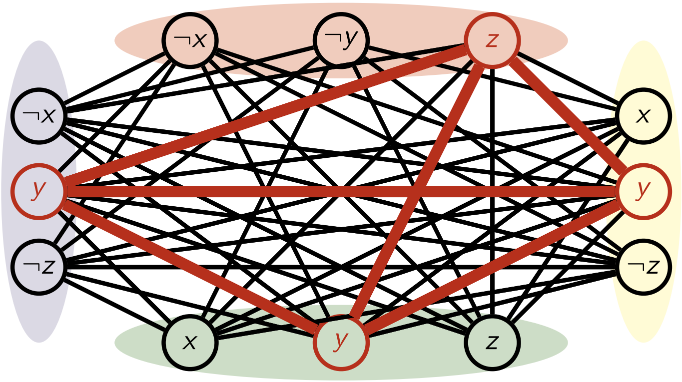

Clique is NP-complete

General construction for $\bigwedge_{i=1}^{n} \bigvee_{j=1}^{3} \ell_{i,j}$: Construct G_φ with

- One vertex for each literal $\ell_{i,j}$ ($3n$ nodes) and
- All edges but those between
  - Literals in the same clause and
  - Contradictory literals (e.g., x and $\neg x$)
- Then, we have $\varphi \in \mathrm{SAT} \Leftrightarrow (G_\varphi, n) \in \mathrm{CLIQUE}$

Vertex Cover is NP-complete

- We argue that $3\mathrm{SAT} \leq_p \mathrm{VC}$
- We only illustrate the reduction on an example

Vertex Cover is NP-complete

Let $\varphi = (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee y \vee z)$ with $v = 3$ variables and $c = 3$ clauses

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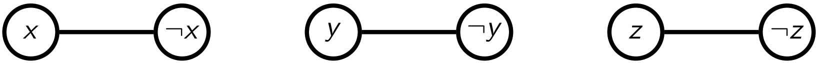

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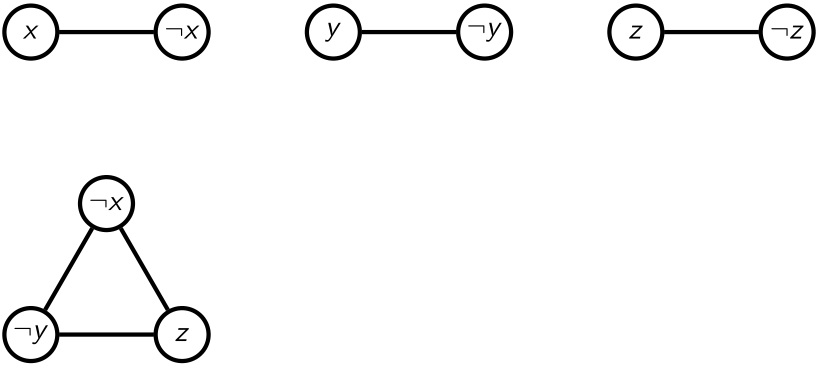

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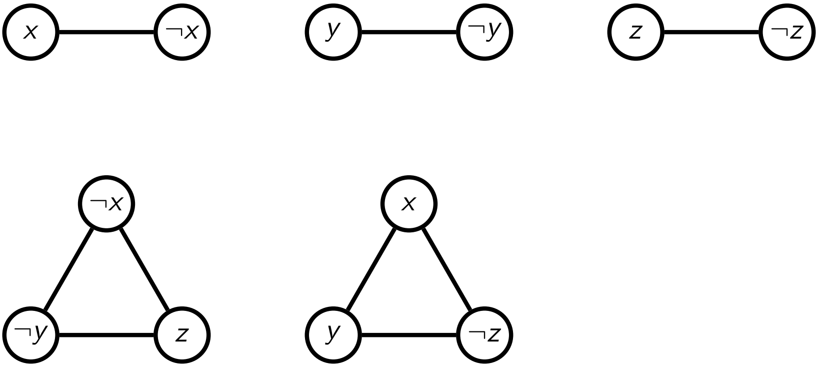

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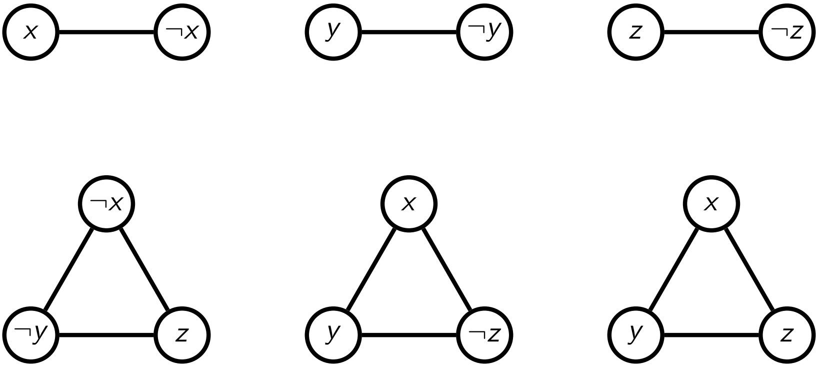

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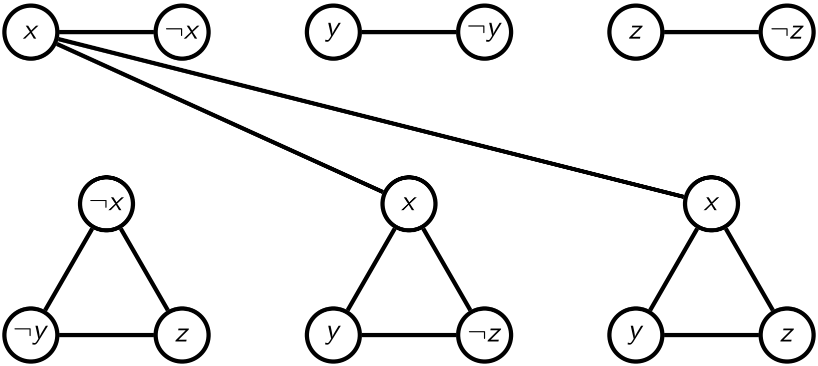

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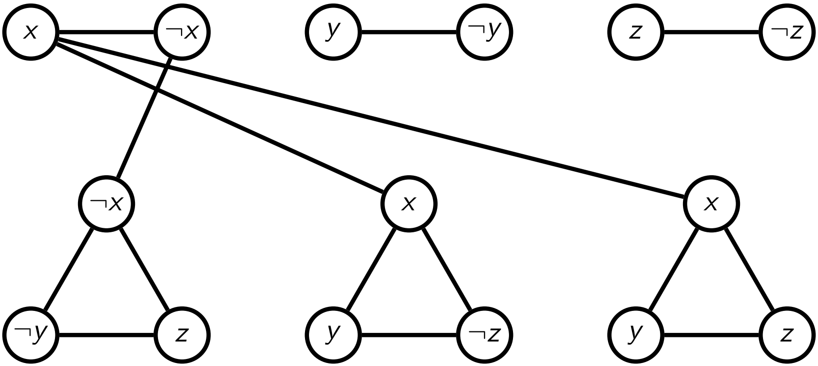

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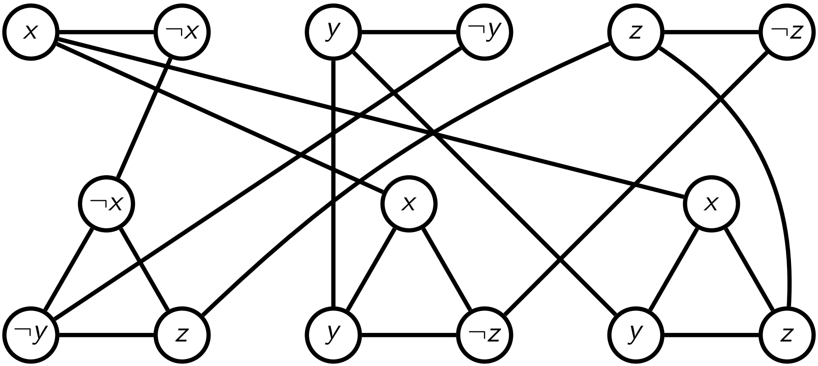

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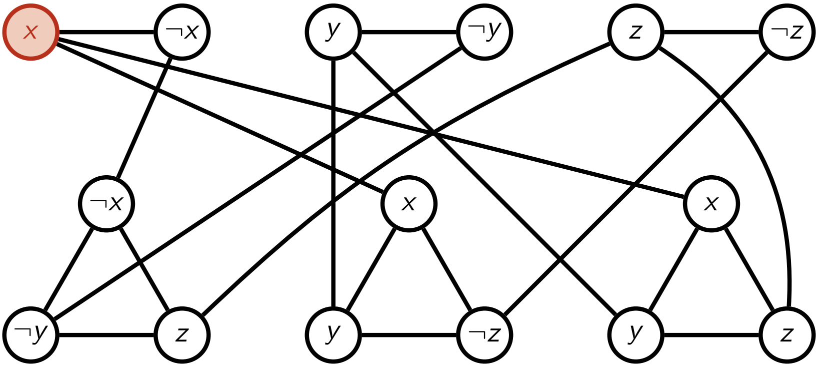
- The resulting graph has a $v + 2c = 9$-vertex cover if and only if φ is satisfiable



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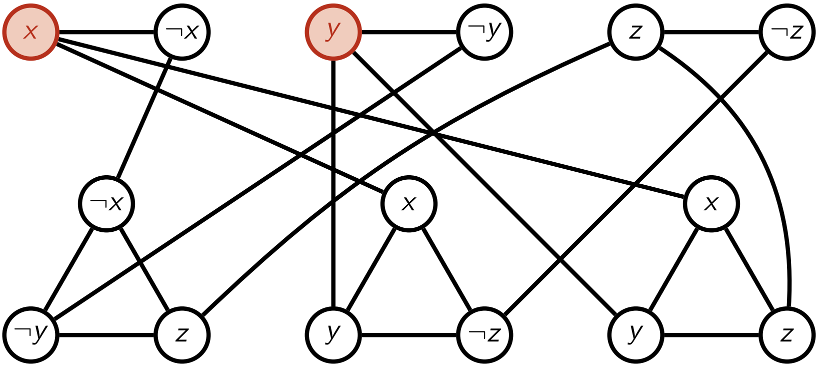

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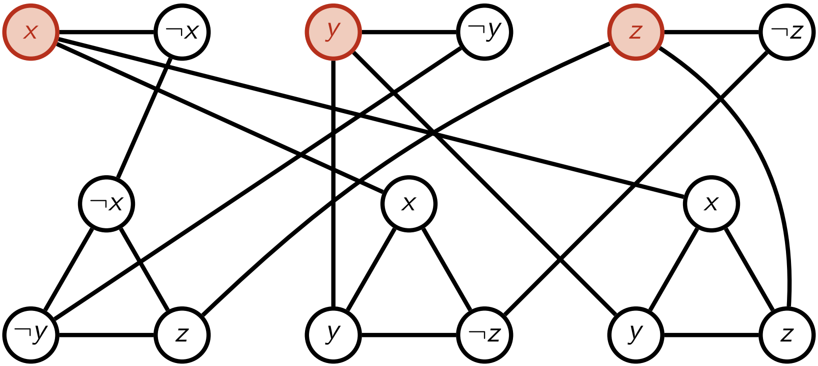

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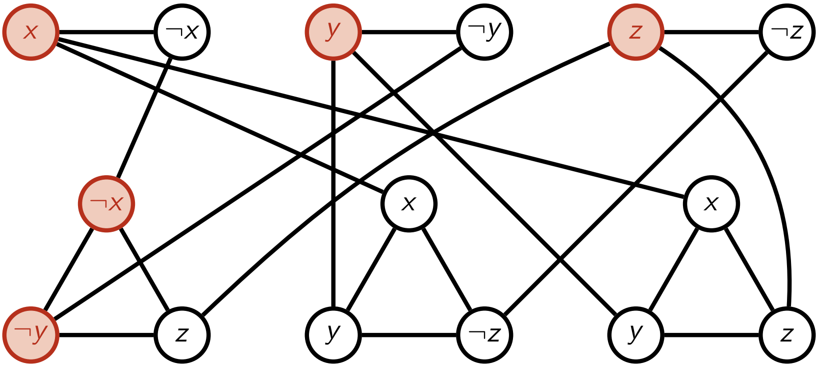

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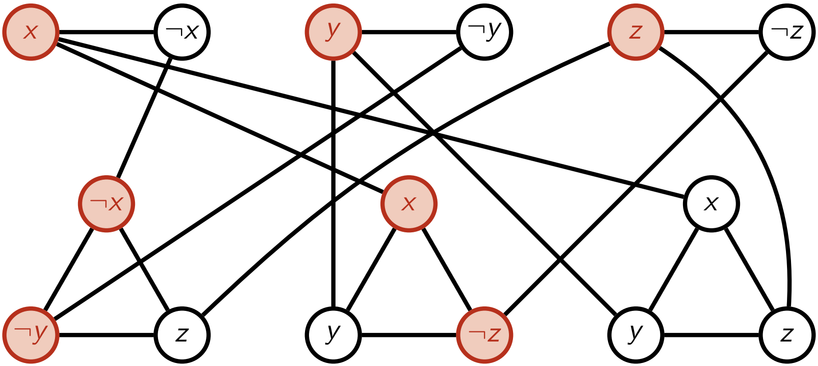

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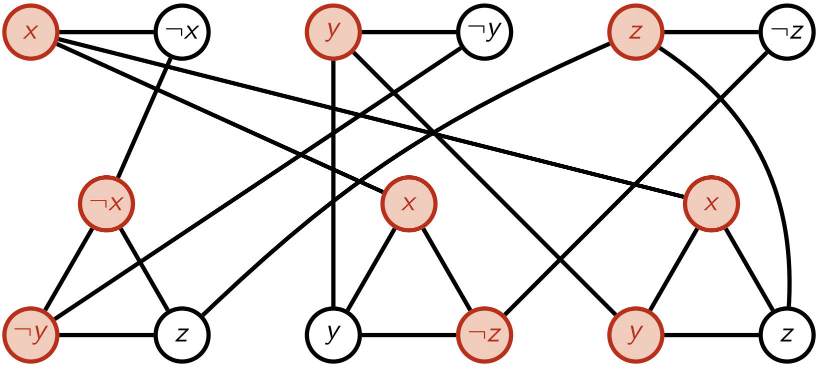

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- The resulting graph has a $v + 2c = 9$-vertex cover if and only if φ is satisfiable
- Upper part ("variable pairs"):
  - Cover at least one vertex per pair ($\geq v$)
  - Intuition: Select an assignment for each variable
- Lower part ("clause triangles"):
  - Cover at least two vertices per triangle ($\geq 2c$)
  - Can omit one vertex that is taken care of by assignment
  - Intuition: Satisfy all clauses
- Since we can only cover $\leq v + 2c$ vertices, we have no slack

There is Much More

Many more problems are NP-complete

- **[Karp '72]**: 21 NP-complete problems, among them SAT, $\mathrm{3SAT}$, CLIQUE, VC, Hamiltonian path...
- **[Garey & Johnson '79]**: a book with 300 NP-hard problems
- More modern list: `https://en.wikipedia.org/wiki/List_of_NP-complete_problems`
- Some of these are very relevant in practice

Conclusion

We have seen

- The hardest problems in NP, the NP-complete ones
- Many useful problems are NP-complete
- If we can find a polynomial-time algorithm for one such problem, then we have one for all problems in NP, i.e., $\mathrm{P} = \mathrm{NP}$
- Researchers have tried to resolve the P vs. NP question for more than 60 years, without success
- Note: Public-key cryptography depends on the assumption that some problems cannot be solved efficiently

Reading

- Section 3.5 of "Computability and Complexity"
- Unfortunately, the book uses a completely different approach to NP-completeness