Algorithms and Computability

Lecture 5: Parallel Algorithms

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Simonas Šaltenis

AALBORG UNIVERSITY
DENMARK

Parallel algorithms

- Goals of the lecture:
  - *to understand the model of **dynamic multithreading** (aka **fork-join parallelism**);*
  - *to understand **work**, **span**, and **parallelism** — the concepts necessary for the analysis of parallel algorithms;*
  - *to understand the main ideas used when parallelizing algorithms;*
  - *to understand and be able to analyze the **parallel merge sort** algorithm.*

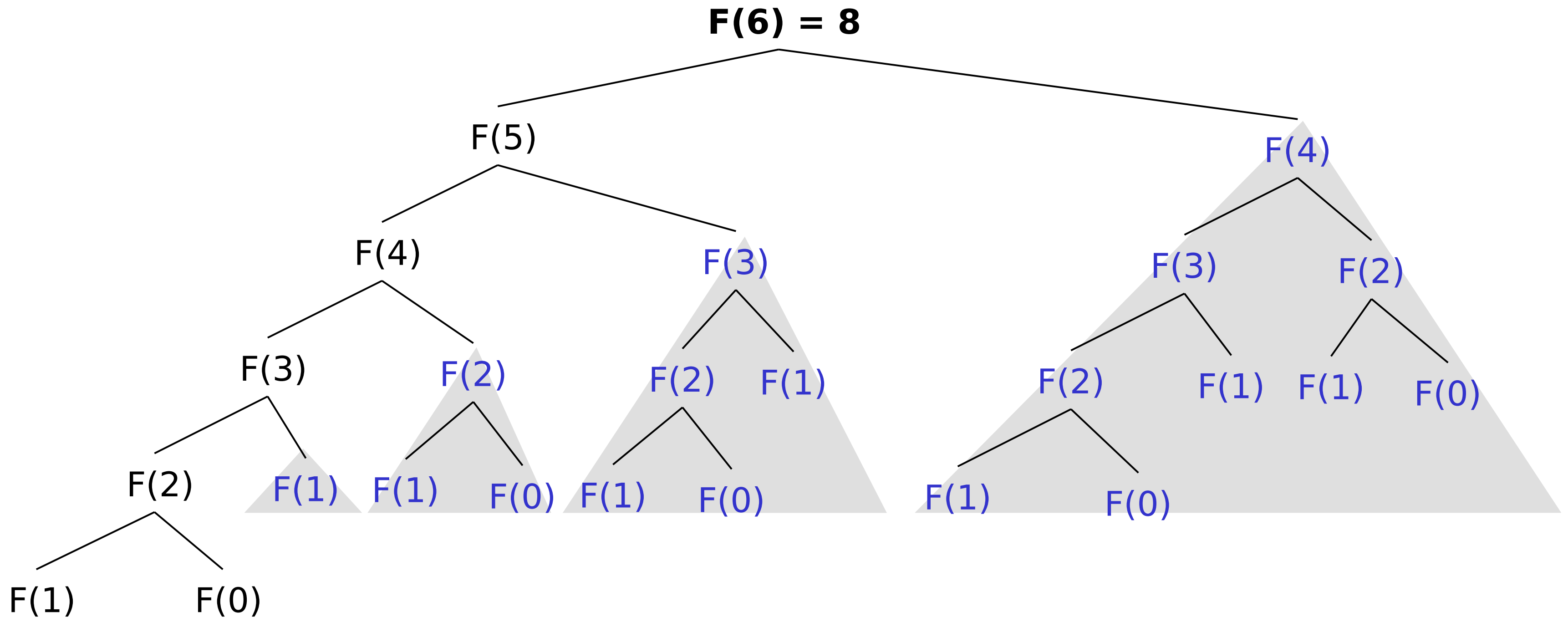
Fibonacci Numbers

- $F(n) = F(n-1) + F(n-2)$
- $F(0) = 0, F(1) = 1$
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 …

```
Fibonacci(n)
if n ≤ 1 then return n
else
  x = Fibonacci(n-1)
  y = Fibonacci(n-2)
return x + y
```

- Straightforward recursive procedure is slow!
- Why? How slow?
- Let’s draw the recursion tree

Fibonacci Numbers



Fibonacci Numbers

- How many summations are there $W(n)$?
  - $W(n) = W(n-1) + W(n-2) + 1$
  - $W(n) \geq \quad 2W(n-2) + 1$ and $W(1) = W(0) = 0$
  - *Is this*:
    - A: Arithmetic series?
    - B: Geometric series?
  - *Go to Socrative and vote.*

Fibonacci Numbers (4)

- How many summations are there $W(n)$?
  - $W(n) = W(n-1) + W(n-2) + 1$
  - $W(n) \geq 2W(n-2) + 1$ and $W(1) = W(0) = 0$
  - *Is this*:
    - A: Arithmetic series?
    - B: Geometric series?
  - *Go to Socrative and vote.*

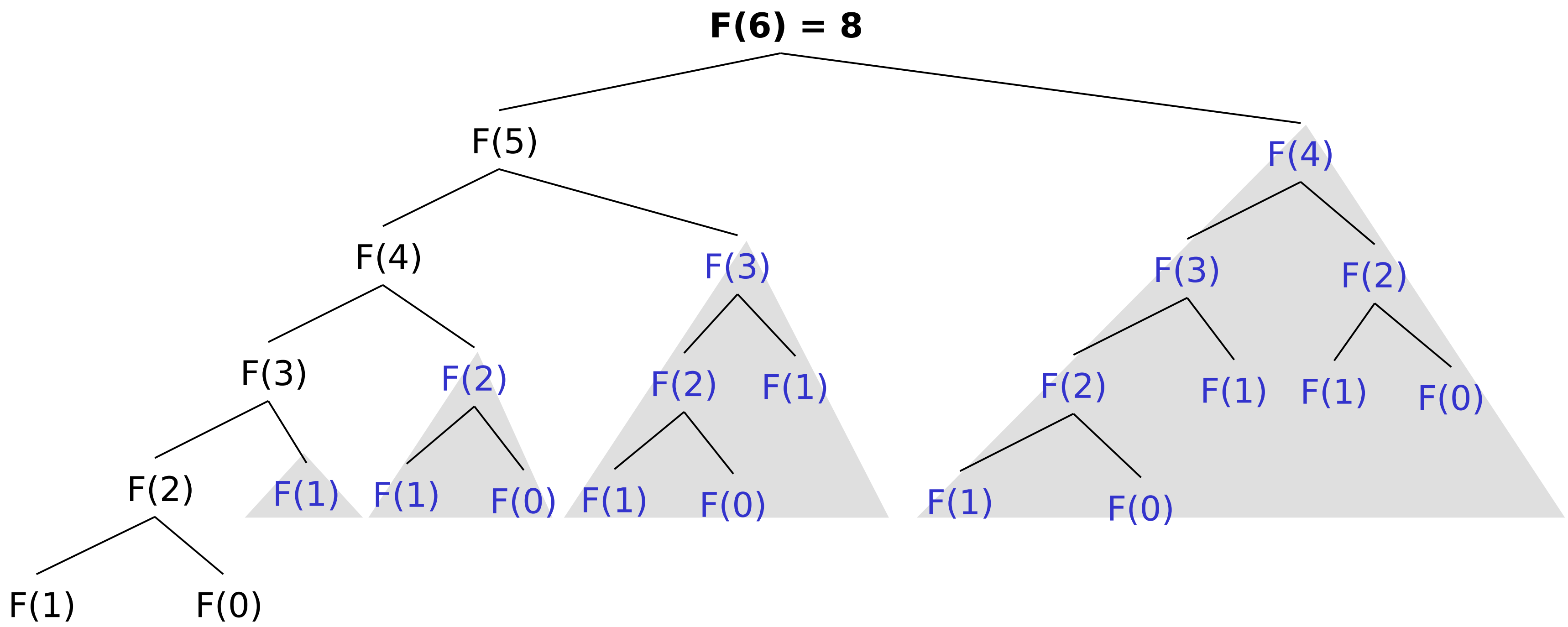
  - Solving the recurrence we get
    $W(n) \geq 2^{(n+2)/2} - 1 \approx 1.4^{n+2}$
  - Precisely $W(n) = \Theta(\varphi^n)$, were φ is the *golden ratio* $(1+\sqrt{5})/2 \approx 1.618$
- Running time is *exponential.*

Multithreaded version

- What if we can do the two recursive calls in parallel
  - Using the so-called *nested parallelism*

```
FibonacciP(n)
if n ≤ 1 then return n
else
  x = spawn FibonacciP(n-1)
  y = FibonacciP(n-2)
  sync
return x + y
```

FibonacciP analysis



- "Running time":
  - $S(n) = \max(S(n-1), S(n-2)) + 1 = S(n-1) + 1$
  - Thus $S(n) = \Theta(n)$

Work, Span, Parallelism

- Three main concepts (informally):
  - *Work*: the running time on a machine with one-processor (T_1).
    - Fibonacci: $\Theta(\varphi^n)$
  - *Span*: the running time on a machine with infinite processors (T_∞).
    - Fibonacci: $\Theta(n)$
  - *Parallelism* = *Work/Span* – how many processors on average are used by the algorithm.
    - Fibonacci: $\Theta(\varphi^n / n)$

- More formally:
  - *Computation log / trace*– a DAG of serial strands of instructions (vertices) and dependencies (edges) between them.
  - Work = the number of vertices in the computation log.
  - Span = the length of the *longest path (critical path)* in the computation log.

Computation DAG

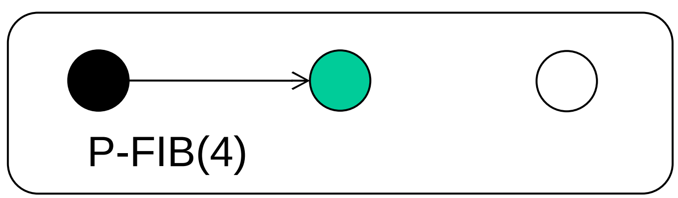
- Using an example of computing Fibonacci number of 4.

```
FibonacciP(n)
01 if n ≤ 1 then return n
02 else
03    x = spawn FibonacciP(n-1)
04    y = FibonacciP(n-2)
05    sync
06 return x + y
```

Lines 1-3

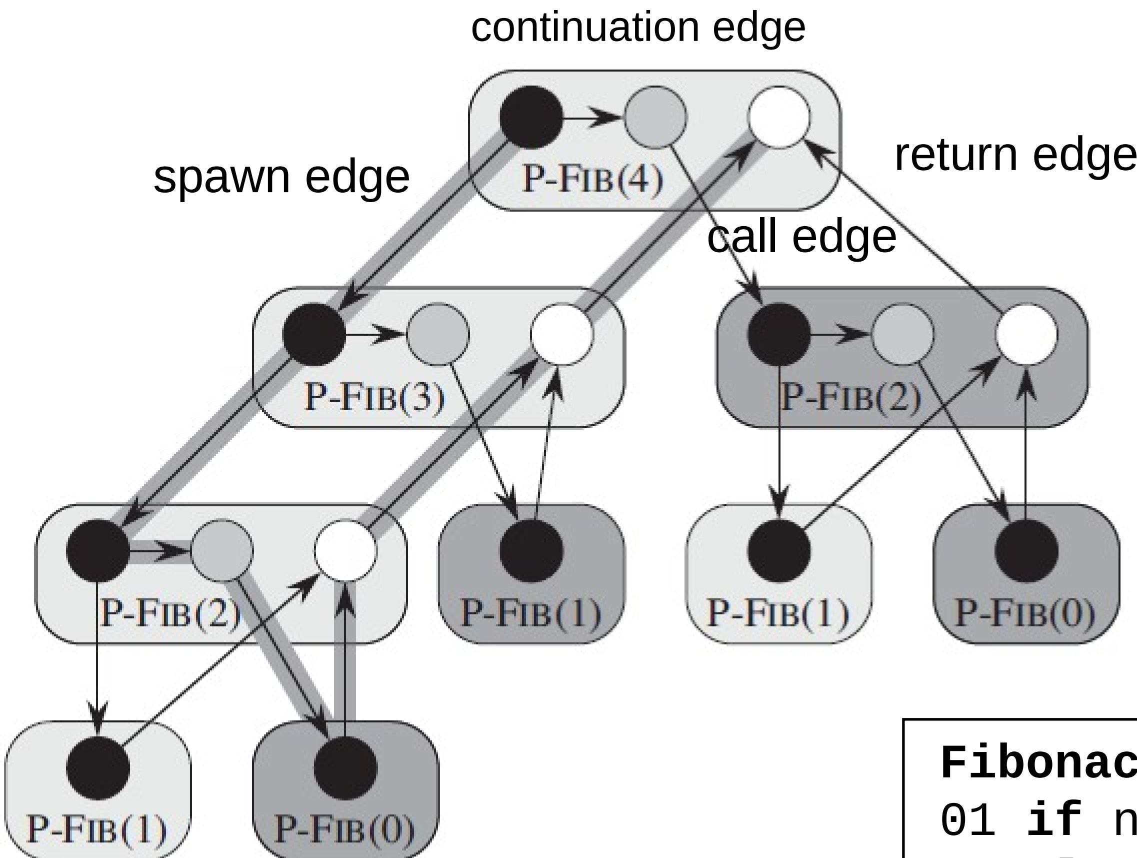
Lines 4-5

Line 6

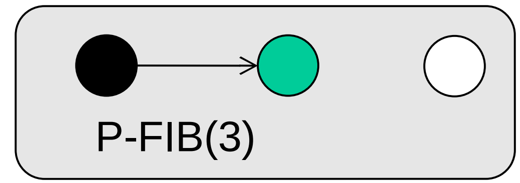


Computation DAG

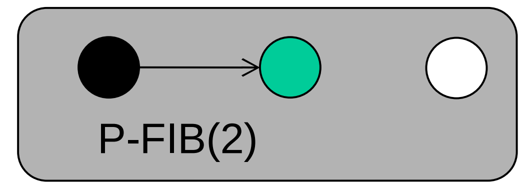
Edge(u, v) means that
u must execute before v.



Spawned procedure



Called procedure



Work: number of vertices, 17
Span: the length of the longest path (critical path), 8

```
FibonacciP(n)
if n ≤ 1 then return n
else
  x = spawn FibonacciP(n-1)
  y = FibonacciP(n-2)
  sync
return x + y
```

Work law and span law

- Notation
  - Work T_1
  - Span T_∞
  - Multithreaded computation on P processors: T_P

- Work law: $T_P \geq T_1 / P$
  - An ideal parallel computer with P processors can do at most P units of work.
- Span law: $T_P \geq T_\infty$
  - An ideal parallel computer with P processors cannot take less time than a machine with unlimited number of processors.

Assumptions

- The *fork-join parallelism (dynamic multithreading)* environment:
  - Shared-memory multi-core system
  - *Concurrency platform* – *task-prallel programming* :
    - Takes care of allocating work to physical threads (in other words: scheduling logical threads on physical threads)
    - Takes care of synchronization, consistent access to memory
  - Pseudocode keywords: *spawn, sync* and *parallel*
    - Indicates *potential* (or *logical*) parallelism: what *may* run in parallel.

  - We do not consider locking, race conditions, etc:
    - Parallel threads are *independent* – they work on separated items of data.
  - We abstract from actual physical scheduling:
    - It can be shown that simple greedy scheduling works well enough.

Speedup, Slackness

- When running on an actual system with *P* physical threads:
  - *Slackness* of a computation: *Parallelism / P.*
  - *What does it mean when slackness < 1? Slackness > 1?*

  - *Speedup* = T_1 / T_P.
  - *Perfect linear speedup*, when speedup = *P.*
  - Alternative way to think about parallelism:
    - *an upper bound on speedup*

Question

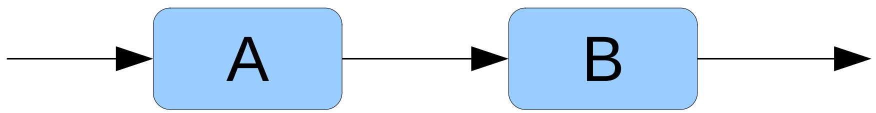
- Considering the case for computing P-Fib(4).
- We already know that the work T_1 =17 and the span T_∞ = 8
- Consider the two setups P = 2 and P = 3, each setup corresponding to a machine with P processing units. *Which setup is most likely to achieve the perfect linear speedup*?

To Summarize

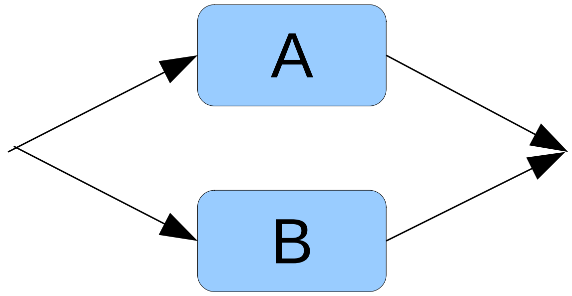
Notation	Meaning
T_1	Work, the running time on a machine with one processor.
T_∞	Span, the running time on a machine with infinite processors.
T_P	The running time on a machine with P processors.
$T_P \geq T_1 / P$	Work law
$T_P \geq T_\infty$	Span law
T_1 / T_P	Speedup. Speedup must be ≤ P according to the work law. When speedup is equal to P, it achieves ***perfect speed up***.
T_1 / T_∞	Parallelism. The maximum possible speedup that can be achieved on any number of processors
T_1 / PT_∞	Slackness = Parallelism/P. The larger the slackness, the more likely to achieve perfect speed up. When slackness is less than 1, it is impossible to achieve perfect speed up.

Computing span

- Sequential execution:
  - Work and span: $T(A\ followed\_by\ B) = T(A) + T(B)$



- Parallel execution:
  - Work: $T_1(A\ in\_parallel\_with\ B) = T_1(A) + T_1(B)$
  - Span: $T_\infty(A\ in\_parallel\_with\ B) = max(T_\infty(A), T_\infty(B))$



Goal of algorithm design

- Goal of the parallel algorithm design – *increase parallelism.*
  - Usually achieved by decreasing span (remember, parallelism = W/S)
  - It may pay off to slightly increase work, if span can be decreased significantly (in practice, relevant for highly parallel systems, such as supercomputers, but also GPUs)

- *What is this not*?
  - Task parallelism vs. data parallelism
  - Intra-operation parallelism vs. inter-operation parallelism.

Side Note: Efficient Fibonacci

- The efficient *O*(n) serial algorithm:
  - Simple application of “*dynamic programming*” (or *memoized* evaluation of the recursive version)

```
FibonacciImproved(n)
if n ≤ 1 then return n
Fim2 ←0
Fim1 ←1
for i ← 2 to n do
  Fi ← Fim1 + Fim2
  Fim2 ← Fim1
  Fim1 ← Fi
return Fi
```

  - Can be actually done in $O(\lg n)$ additions and multiplications.

Parallel loops

- Denoted by the **parallel** keyword

```
ArrayCopy(A, B)
parallel for i = 1 to sizeof(A) do
  B[i] = A[i]
```

- Analysis of span:
  - S(n) = O(lg n) + $\max_i$ S_iteration(i)
  - *Why*?
  - Parallel loop is implemented by divide-and-conquer

```
ArrayCopyRecursive(A, B, l, r)
if l > r then return
if l = r then B[l] = A[r]
else
  q = ⌈(l+r)/2⌉
  spawn ArrayCopyRecursive(A, B, l, q-1)
        ArrayCopyRecursive(A, B, q, r)
```

Examples

- Exchanging neighboring elements:

```
ArrayExchange(A)
parallel for i = 1 to ⌊sizeof(A)/2⌋ do
  tmp       = A[i*2]
  A[i*2]    = A[i*2-1]
  A[i*2-1]  = tmp
```

- Compute the largest stock price difference :

```
LargestSpike(A)
lspike = 0
parallel for i = 1 to sizeof(A)-1 do
  if |A[i+1] – A[i]| > lspike then
     lspike = |A[i+1] – A[i]|
return lspike
```

Examples

- Exchanging neighboring elements:

```
ArrayExchangeP(A)
parallel for i = 1 to ⌊sizeof(A)/2⌋ do
  tmp[i]      = A[i*2]
  A[i*2]   = A[i*2-1]
  A[i*2-1] = tmp[i]
```

- Compute the largest stock price difference:
  - Exercise

```
LargestSpikeP(A, l, r)
if trivial case then ...
else
  divide...
  ls = spawn LargestSpikeP(A, ...)
  rs = LargestSpikeP(A, ...)
  sync
  combine
```

Parallelization techniques

- To parallelize an algorithm:
  - Convert *loops to parallel*, if possible
  - Rewrite parts of algorithm as *divide&conquer*
  - Brake the algorithm into more *phases* (that can then be parallelized)
  - Use *additional memory* to parallelize loops.

Parallelization techniques

- To parallelize an algorithm:
  - Convert *loops to parallel*, if possible
  - Rewrite parts of algorithm as *divide&conquer*
  - Brake the algorithm into more *phases* (that can then be parallelized)
  - Use *additional memory* to parallelize loops.

- In the video example of removing duplicates:
  - Nested loop

    → *inner loop as D&C*

    → *factor out a compaction phase to parallelize the main loop*

    → *make compaction into D&C*

    → *make the combine loop in compaction parallel by using additional memory...*

Merge Sort

```
Merge-Sort(A, p, r)
if p < r then
    q = ⌊(p+r)/2⌋
    Merge-Sort(A, p, q)
    Merge-Sort(A, q+1, r)
    Merge(A, p, q, r)
```

Divide

Conquer

Combine

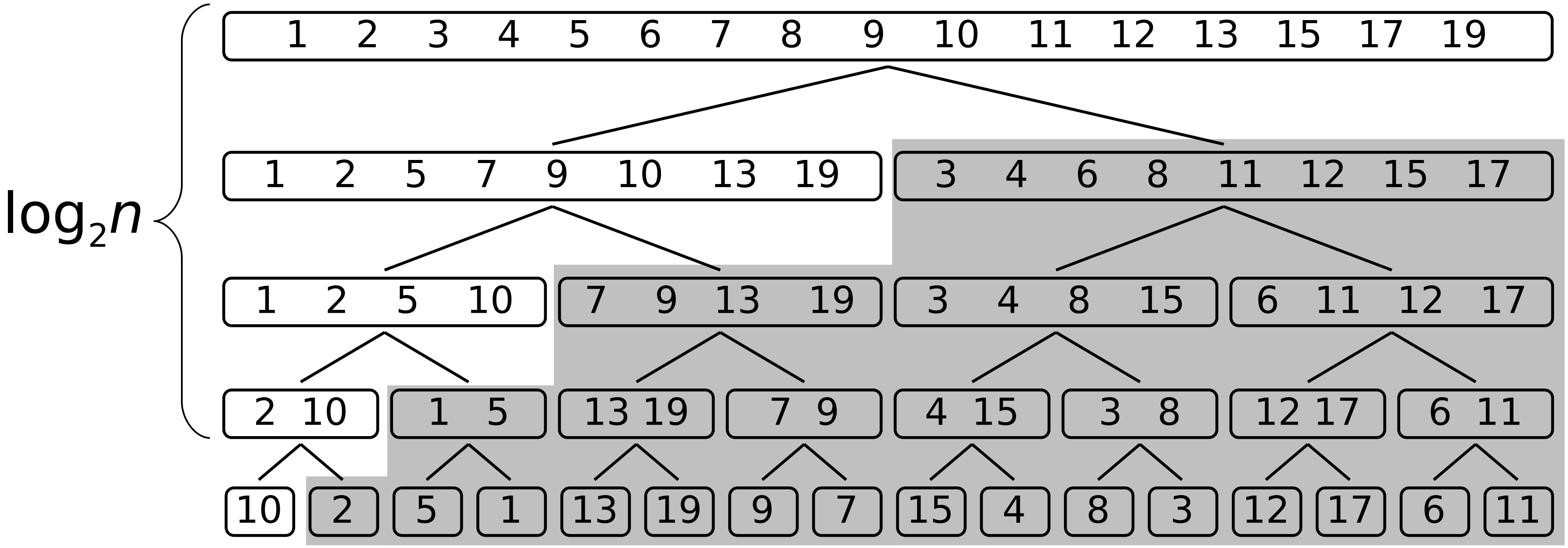
- Running time?

Parallelising Merge Sort

```
Merge-Sort'(A, p, r)
if p < r then
    q = ⌊(p+r)/2⌋
    spawn Merge-Sort'(A, p, q)
    Merge-Sort'(A, q+1, r)
    sync
    Merge(A, p, q, r)
```

- Work
  - $W(n) = 2W(n/2) + \Theta(n)$
  - $W(\mathrm{n}) = \Theta(n \lg n)$
- Span:
  - $S(n) = S(n/2) + \Theta(n)$
  - $S(\mathrm{n}) = \Theta(n)$
- Parallelism: $W(n)/S(n) = \Theta(\lg n)$. Rather low...

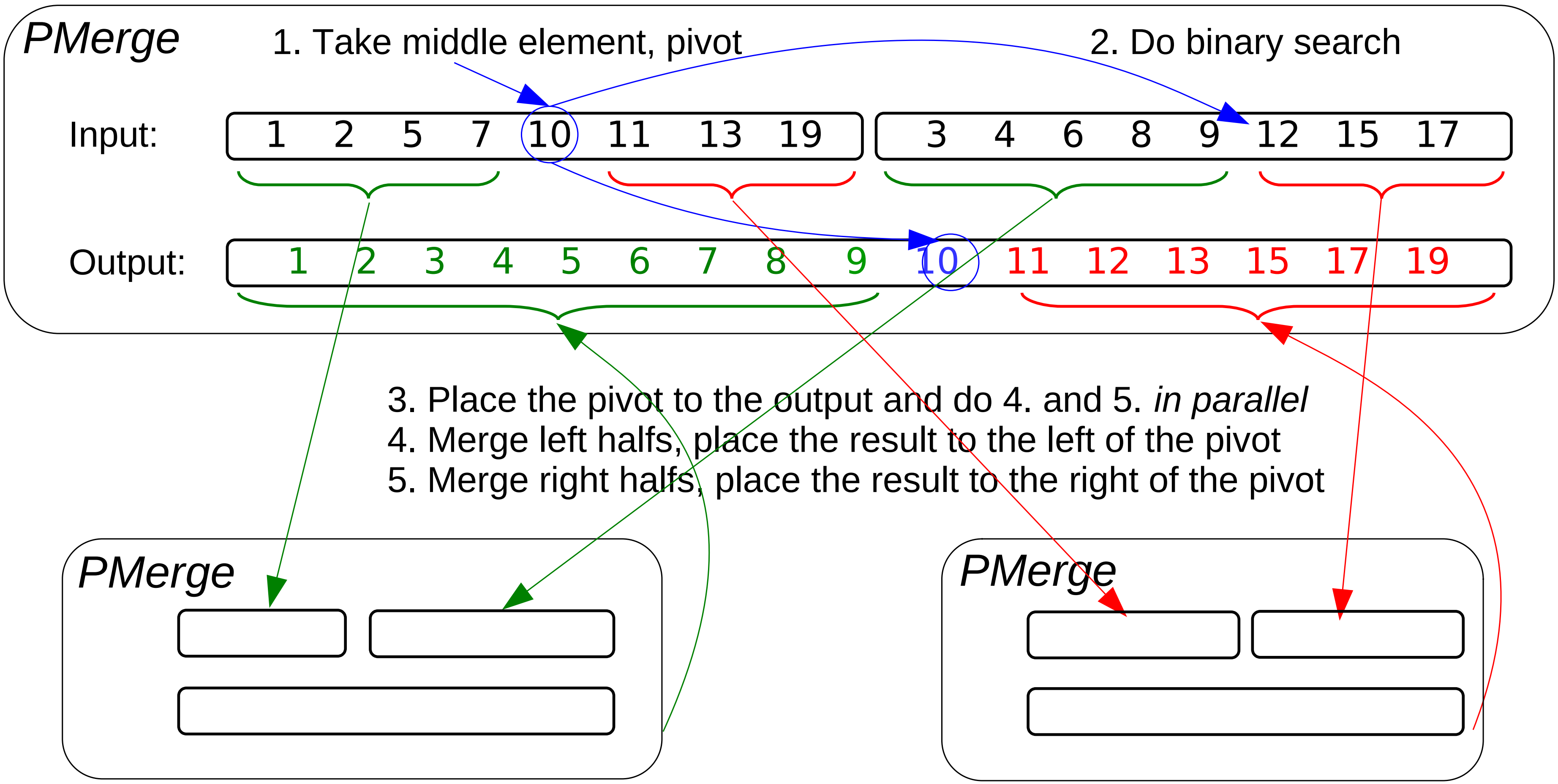
Merge-Sort Recursion Tree



- Problem – merging is very serial:
  - At the top level, only one processor does $\Theta(n)$ work in serial!
  - At the second level, only two processors do $\Theta(n)$ work.
  - …

Multithreaded merging

- Main idea – make the algorithm divide-and-conquer and use nested parallelism.



Multithreaded merging analysis

- One key idea: *do binary search in the smaller of the two arrays*!
  - This ensures that the *largest* of the two recursive calls works with *at most* $3n/4$ elements, where n = the sum of sizes of the two arrays.
  - *Why?*
- Span:
  - $S(n) = S(3n/4) + \Theta(\lg n)$
  - *What is the solution*?
  - $S(n) = \Theta(\lg^2 n)$
- Work:
  - Can be shown to be $\Theta(n)$.

Multithreaded merge sort

```
PMerge-Sort(A, p, r)
if p < r then
    q = ⌊(p+r)/2⌋
    spawn PMerge-Sort(A, p, q)
    PMerge-Sort(A, q+1, r)
    sync
    PMerge(A, p, q, r)
```

- Work:
  - The same recurrence and solution: $W(\mathrm{n}) = \Theta(n \lg n)$
- Span:
  - $S(n) = S(n/2) + \Theta(\lg^2 n)$
  - $S(n) = \Theta(\lg^3 n)$
- Parallelism:
  - $W(n) / S(n) = \Theta(n \lg n) / \Theta(\lg^3 n) = \Theta(n / \lg^2 n)$

To summarize

- Work: $\Theta(n \lg n)$

	Span of Merge	Total span	Parallelism
Naïve merge	$\Theta(n)$	$\Theta(n)$	$\Theta(\lg n)$
P-Merge	$\Theta(\lg^2 n)$	$\Theta(\lg^3 n)$	$\Theta(n / \lg^2 n)$

Goal of the parallel algorithm design

- Goal of the multi-threaded algorithm design – *increase parallelism.*
  - *Parallelism = work / span*
  - Usually achieved by decreasing span
    - MergeSort without P-Merge and with P-Merge
    - $\Theta(n)$ vs. $\Theta(lg^3 n)$
  - It may pay off to slightly increase work, if span can be decreased significantly (in practice, relevant for highly parallel systems, such as supercomputers, GPUs)