Algorithms and Computability

Lecture 12: Nondeterministic Polynomial Time

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slides courtesy of Martin Zimmermann

Yesterday in Algorithms and Computability

We have compared algorithmic complexity:

- Study of concrete algorithms and precise running times
- Difference between $\mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$ is huge
- Running times depend on model of computation

and complexity theory:

- Study of problems (languages) rather than algorithms
- Difference between $\mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$ may just depend on choice of model (but people still try to find optimal (for fixed model) algorithms)
- Results should be valid for most models of computation

Run Time of a Deterministic Turing Machine

Definition

Let M be a halting DTM

- Let $time_M(w)$ denote the number of configurations in the unique run of M on input w
- Let $T\colon \mathbb{N} \to \mathbb{R}_{>0}$. We say that M runs within time T if $time_M(w) \leq T(|w|)$ for all inputs w

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We are interested in classifying problems (languages) according to their asymptotic time complexity

Definition

Let $T \colon \mathbb{N} \to \mathbb{R}_{>0}$ be a function. The complexity class $\textsc{Time}(T)$ is defined as

$$\textsc{Time}(T) = \{L(M) \mid M \text{ is a halting DTM that runs within time } \mathcal{O}(T)\}$$

The Complexity Class P

Definition

The complexity class P (polynomial time) is defined as

$$\mathrm{P} = \bigcup_{k \geq 0} \mathrm{TIME}(n^k)$$

- Robust definition (can use other deterministic (!) models of computation, e.g., multi-tape Turing machines)
- Cobham's thesis: A problem can be efficiently computed if and only if it is in P

Agenda

1. Motivation
2. Nondeterministic Time Complexity
3. Polynomial-Time Reductions

Knapsack

- Recall the Knapsack problem:

  A thief has a knapsack holding at most W kg of loot. The thief robs a store that has items $1, \ldots, n$ of weight w_j and value c_j (each item only once). What is the maximal value the thief can put in the knapsack?

- This is an optimization problem

  But we only focus on decision problems here

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- This is an optimization problem

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- As a decision problem:

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t. } \sum_{j=1}^{n} b_j \cdot w_j \leq W \text{ and } \sum_{j=1}^{n} b_j \cdot c_j \geq T\}$$

Knapsack

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Consider a backpack of size $W = 100$, the threshold $T = 200$, and the following items:

item j	1	2	3	4	5	6
weight w_j	10	20	60	25	50	35
value c_j	5	70	20	105	30	35

Is $b_1 = 0,\ b_2 = 1,\ b_3 = 0,\ b_4 = 1,\ b_5 = 1,\ b_6 = 0$ a valid solution?

Knapsack

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t.}$$
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Is $b_1 = 0,\ b_2 = 1,\ b_3 = 0,\ b_4 = 1,\ b_5 = 1,\ b_6 = 0$ a valid solution?
Yes, because

$$w_2 + w_4 + w_5 = 20 + 25 + 50 = 95 \leq 100 = W$$

and

$$c_2 + c_4 + c_5 = 70 + 105 + 30 = 205 \geq 200 = T$$

Knapsack

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t. } \sum\nolimits_{j=1}^{n} b_j \cdot w_j \leq W \text{ and } \sum\nolimits_{j=1}^{n} b_j \cdot c_j \geq T\}$$

Now, consider a backpack of size 15, the threshold $T = 50$, and the following items:

item j	1	2	3	4	5	6	7
weight w_j	2	3	5	7	1	4	1
value c_j	10	5	15	7	6	18	3

Is there a valid solution?

Knapsack

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t.} \\ \sum_{j=1}^{n} b_j \cdot w_j \leq W \text{ and } \sum_{j=1}^{n} b_j \cdot c_j \geq T\}$$

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Yes, e.g., $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 0$, $b_5 = 1$, $b_6 = 1$, $b_6 = 0$

Knapsack

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t.}$$
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Is there a valid solution?

Yes, e.g., $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 0$, $b_5 = 1$, $b_6 = 1$, $b_6 = 0$

Algorithm to compute a solution (or conclude that there is none)?

Satisfiability

$$\varphi = (x_0 \vee x_2) \wedge (x_0 \vee \neg x_3) \wedge (x_1 \vee \neg x_3) \wedge (x_1 \vee \neg x_4) \wedge (x_2 \vee \neg x_4) \wedge \\ (x_0 \vee \neg x_5) \wedge (x_1 \vee \neg x_5) \wedge (x_2 \vee \neg x_5) \wedge (x_3 \vee x_6) \wedge (x_4 \vee x_6) \wedge (x_5 \vee x_6)$$

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Is $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$ a satisfying assignment?

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Is $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$ a satisfying assignment?

Satisfiability

$$\varphi = (1) \wedge (1 \vee 0) \wedge (1 \vee 0) \wedge (1 \vee 0) \wedge (1 \vee 0) \wedge$$
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Satisfiability

$$\begin{aligned}\varphi = &(1) \wedge (1) \wedge (1) \wedge (1) \wedge (1 \vee 0) \wedge \\ &(1 \vee 0) \wedge (1 \vee 0) \wedge (1 \vee 0) \wedge (1 \vee 1) \wedge (1 \vee 1) \wedge (1 \vee 1)\end{aligned}$$

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Satisfiability

$$\varphi = (1) \wedge (1) \wedge (1) \wedge (1) \wedge (1) \wedge$$
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Is $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$ a satisfying assignment?

$\varphi = 1$ **Yes**!

Is $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$ a satisfying assignment?

Satisfiability

$\varphi = 1$ **Yes**!

Does

$$\varphi = (x_0 \vee x_2 \vee \neg x_4) \wedge (\neg x_0 \vee \neg x_3 \vee \neg x_1) \wedge (x_1 \vee \neg x_3 \vee \neg x_2) \wedge$$
$$(x_0 \vee \neg x_2 \vee x_1) \wedge (x_1 \vee \neg x_5 \vee x_2) \wedge (x_2 \vee x_5 \vee x_4) \wedge$$
$$(\neg x_1 \vee x_2 \vee x_3) \wedge (x_5 \vee \neg x_3 \vee \neg x_1) \wedge (x_1 \vee \neg x_2 \vee x_4)$$

have a satisfying assignment?

Similarities

- Knapsack:

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t. } \sum\nolimits_{j=1}^{n} b_j \cdot w_j \leq W \text{ and } \sum\nolimits_{j=1}^{n} b_j \cdot c_j \geq T\}$$

- Boolean satisfiability:

$$\mathrm{SAT} = \{\varphi \mid \text{there exists a satisfying assignment for } \varphi\}$$

- Knapsack:

$$\{W, T, w_1, \ldots, w_n, c_1, \ldots, c_n \in \mathbb{N} \mid \exists b_1, \ldots, b_n \in \{0,1\} \text{ s.t. } \sum\nolimits_{j=1}^{n} b_j \cdot w_j \leq W \text{ and } \sum\nolimits_{j=1}^{n} b_j \cdot c_j \geq T\}$$

- Boolean satisfiability:

$$\mathrm{SAT} = \{\varphi \mid \text{there exists a satisfying assignment for } \varphi\}$$

Similarities

- Both problems ask for the existence of a certificate "proving" that the input is in the language
  (a bit vector $b_1 \cdots b_n$ resp. an assignment)
- Certificates are easy to verify, i.e, in polynomial time
- Certificates are short, i.e., polynomial in the input length

More

Many other important problems share these traits:

- Nonprimality: $\{n \in \mathbb{N} \mid n = a \cdot b \text{ for } a, b > 1\}$

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Side note: some problems (are believed to) have different complexity

Quiz 1

Is verifying a certificate easier than finding a certificate?

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Generally: widely believed to be true (but no proof yet)

Agenda

Nondeterminism to the Rescue

Nondeterminism allows a Turing machine to find a certificate: For example, to turn any sequence of n consecutive X's into all $0/1$ vectors of length n, use the following transitions:

$$\delta(q, X) \to \{(q, 0, +1), (q, 1, +1)\}$$

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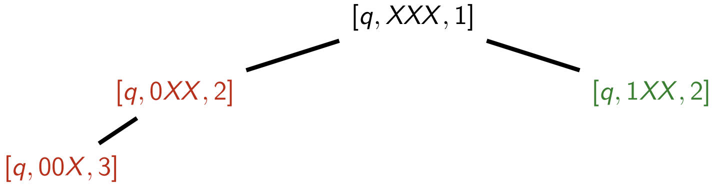

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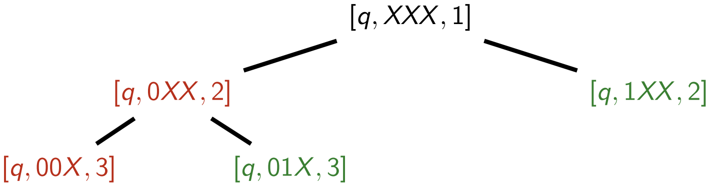

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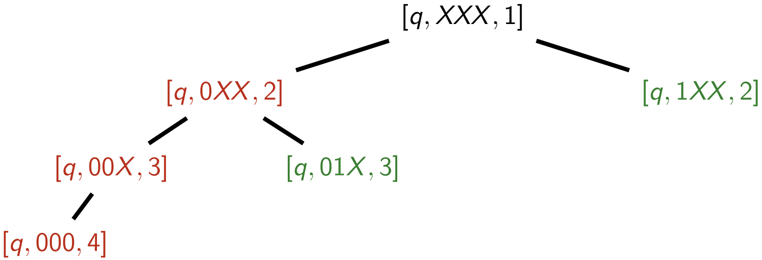

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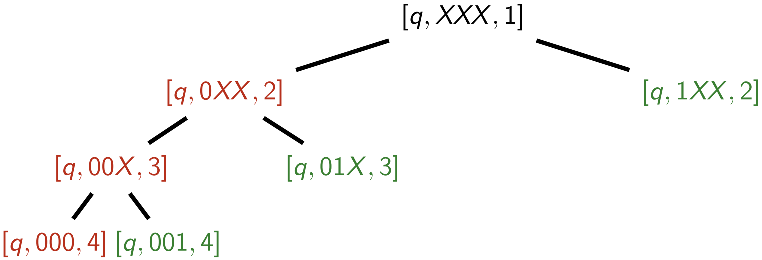

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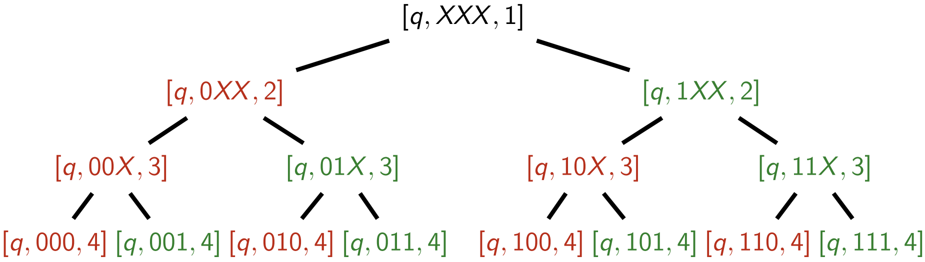

A Nondeterministic Algorithm for SAT

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Definition

Let M be a halting DTM

- Let $time_M(w)$ denote the number of configurations in the unique run of M on input w

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We are still interested in classifying problems (languages) according to their asymptotic time complexity

Definition

Let $T \colon \mathbb{N} \to \mathbb{R}_{>0}$ be a function. The complexity class $\mathrm{NT{\scriptsize IME}}(T)$ is defined as

$$\mathrm{NT{\scriptsize IME}}(T) = \{L(M) \mid M \text{ is a halting NTM that runs within time } \mathcal{O}(T)\}$$

The Complexity Class NP

Definition

The complexity class NP (nondeterministic polynomial time) is defined as

$$\mathrm{NP} = \bigcup_{k \geq 0} \mathrm{NT\scriptstyle{IME}}(n^k) \qquad \left[\text{Reminder: } \mathrm{P} = \bigcup_{k \geq 0} \mathrm{T\scriptstyle{IME}}(n^k)\right]$$

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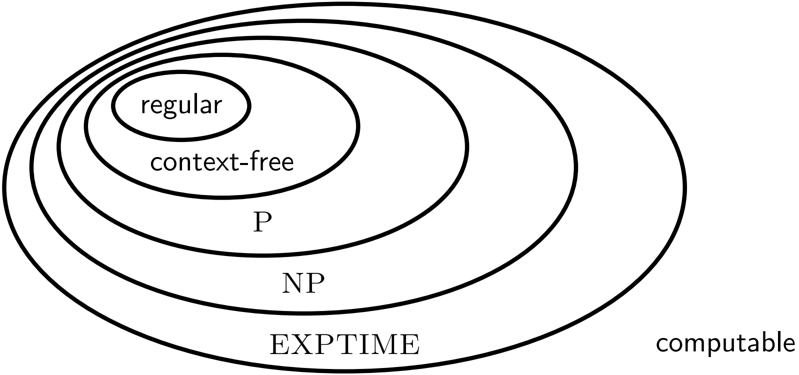
Remark

$\mathrm{P} \subseteq \mathrm{NP}$

- Because every halting DTM with polynomial time complexity is a halting NTM with polynomial time complexity

Complexity Classes



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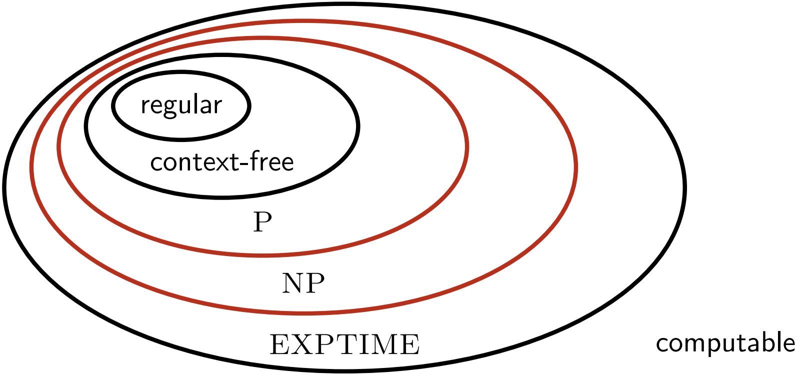

- EXPTIME: problems solved by DTMs in exponential time
- It is unknown whether the marked inclusions are strict
- We only know that $\mathrm{P} \subsetneq \mathrm{EXPTIME}$

Another Example

- A nondeterministic polynomial-time algorithm for Clique
- Recall that we encode a graph G by (a word over $\mathbb{B}$ representing) its adjacency matrix, i.e., entry (i, j) is 1 if and only if there is an edge from the i-th to the j-th vertex

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Problems in NP

The halting NTMs we have seen show that

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In fact, the other problems we have seen are also in NP:

- Nonprimality
- Hamiltonian path
- Linear programming
- 3-coloring
- And many more

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- Hence, a nondeterministic algorithm can solve the complement problem efficiently
- Open question whether this problem and SAT have different complexity

Agenda

1. Motivation
2. Nondeterministic Time Complexity
3. **Polynomial-Time Reductions**

Polynomial-Time Computable Functions

Definition

A function $f\colon \Sigma_1^* \to \Sigma_2^*$ is a computable function
if and only if there exists a DTM M_f
that on every input $w \in \Sigma_1^*$

- always halts and accepts
- with just $f(w)$ on its tape and
- the head at the first letter of $f(w)$

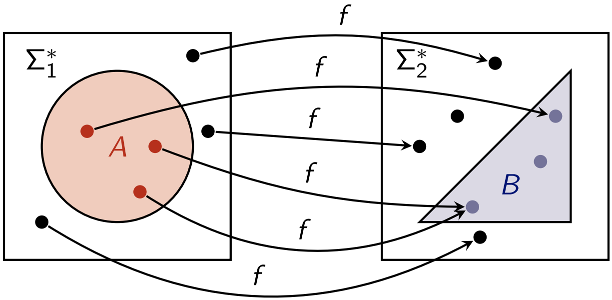
Definition

A function $f\colon \Sigma_1^* \to \Sigma_2^*$ is a polynomial-time computable function if and only if there exists a DTM M_f with polynomial time complexity that on every input $w \in \Sigma_1^*$

- always halts and accepts
- with just $f(w)$ on its tape and
- the head at the first letter of $f(w)$

Examples

- $f(w) = ww$ is polynomial-time computable
- $f(\ulcorner m \urcorner \# \ulcorner n \urcorner) = \ulcorner m \cdot n \urcorner$ is polynomial-time computable (where $\ulcorner n \urcorner$ denotes the binary encoding of $n \in \mathbb{N}$)



Definition

Let $A \subseteq \Sigma_1^*$ and $B \subseteq \Sigma_2^*$ be languages. We say that A is mapping reducible to B, written $A \leq_m B$, if and only if

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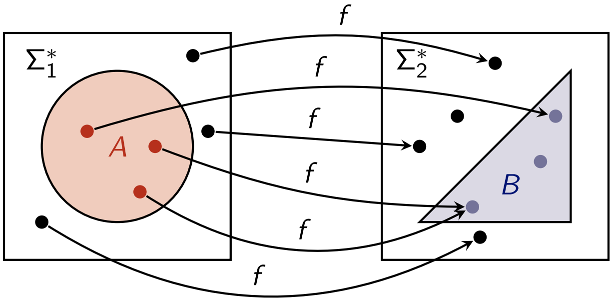

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Applications

Recall: Let $A \leq_m B$

- If B is computable, then A is computable
- If B is computably-enumerable, then A is computably-enumerable

A similar result holds for polynomial-time reductions and the complexity classes P and NP

Theorem

Let $A \leq_p B$

1. *If $B \in \mathrm{P}$, then $A \in \mathrm{P}$*
2. *If $B \in \mathrm{NP}$, then $A \in \mathrm{NP}$*

Proof

We prove "If $B \in \mathrm{P}$, then $A \in \mathrm{P}$". The proof for NP is analogous

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- Consider the following DTM M'': On input w:
  1. Simulate M' (yielding $f(w)$ on the tape)
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- M'' is halting, as M' and M both always halt

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- As $A \leq_p B$, there is a polynomial-time computable function f such that $w \in A \Leftrightarrow f(w) \in B$. Then, there is a DTM M' computing f; say it runs within time $\mathcal{O}(n^{k'})$
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  1. Simulate M' (yielding $f(w)$ on the tape)
  2. Simulate M on $f(w)$
  3. Accept if M accepts $f(w)$, reject if M rejects $f(w)$
- M'' is halting, as M' and M both always halt
- M'' computes A because $w \in A \Leftrightarrow f(w) \in B$

Proof

We prove "If $B \in \mathrm{P}$, then $A \in \mathrm{P}$". The proof for NP is analogous

- Let M be a halting DTM for B that runs within time $\mathcal{O}(n^k)$
- As $A \leq_p B$, there is a polynomial-time computable function f such that $w \in A \Leftrightarrow f(w) \in B$. Then, there is a DTM M' computing f; say it runs within time $\mathcal{O}(n^{k'})$
- Consider the following DTM M'': On input w:
  1. Simulate M' (yielding $f(w)$ on the tape)
  2. Simulate M on $f(w)$
  3. Accept if M accepts $f(w)$, reject if M rejects $f(w)$
- M'' is halting, as M' and M both always halt
- M'' computes A because $w \in A \Leftrightarrow f(w) \in B$
- M'' runs within time $(n^{k'})^k = n^{k' \cdot k}$
- Hence, $A \in \mathrm{P}$

Conclusion

We have seen:

- The complexity class NP: The class of languages accepted by polynomial-time NTMs
- Polynomial-time reductions

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Recall: $\mathrm{P} \subseteq \mathrm{NP}$

The (literally) Million Dollar Question

Is verifying certificates easier than finding certificates or not:

$$\mathrm{P} = \mathrm{NP} \text{ or } \mathrm{P} \subsetneq \mathrm{NP}?$$

- One of the most challenging and most important questions of (theoretical) computer science
- More on that next time

Reading

Sections 3.1 to 3.4 of "Computability and Complexity"
(pages 141 to 178):

Note

- The book uses slightly different notation and definitions
- These sections also cover the complement class of NP (called coNP), which is not covered in this course