Algorithms and Computability

Lecture 7: Introduction and Turing Machines

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slides courtesy of Martin Zimmermann

An Algorithm

What does this algorithm do?

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def mystery(n):
  for i in range(2,int(sqrt(n))+1):
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  Example

  Testing a 256 bit number for primality this way takes roughly 3.4×10^{38} divisions. At 1 billion divisions per second, that takes approximately 1.5×10^{22} years

What about this algorithm?

```
def mystery2(n):
  if n = a^b for some b > 1:
    return False
  r = min{ r | ord(n,r) > log(n)^2 }
  if 1 < gcd(a,n) < n for some a ≤ r:
    return False;
  if n ≤ r:
    return True
  for a in range(1,sqrt(φ(r))*log(n):
    if (X+a)^n ≠ X^n+a (mod (X^r-1), n):
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- Requires some number theory to prove correct and efficient!
- Agrawal, Kayal, and Saxena received the Gödel and Fulkerson prizes for this work

A Thief

A thief has a knapsack holding at most W kg of loot. The thief robs a store that has items $1, \ldots, n$ of weight w_j and value c_j (each item only once). What is the maximal value the thief can put in the knapsack?

Example

$W = 50$ and the following items:

item	weight	value	value per kg
1	10 kg	\$60	\$6
2	20 kg	\$100	\$5
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- "Greedy" solution (item 1 and item 2) has value \$160
- Optimal solution (item 2 and item 3) has value \$220

Collatz

Does the following algorithm return `True` for every possible input $n \geq 1$?

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def collatz(n):
  while(n > 1):
    if n%2 == 0:
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- Can you compute whether a given program returns `True` for every input?
- All $n \leq 2^{68}$ have been checked, and yield `True`. But that does not mean much!
- Nobody knows whether the above algorithm always returns `True`! Erdős: "Mathematics may not be ready for such problems"

Purpose

In previous courses, you have

- seen algorithms that solve specific tasks, e.g., sorting an array, searching a path through a graph, and
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- Actually, what is an algorithm? And what is a problem? And what does “efficiently” mean?
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- Actually, what is an algorithm? And what is a problem? And what does "efficiently" mean?
- How can we still solve *hard* problems?

And we will see how all the examples we have seen so far relate to these questions

Agenda

1. **Setting the Stage**

2. Turning Machines

Motivating Question

- Actually, what is an algorithm? And what is a problem?

What is a Problem?

In the theoretical parts of this course, we are mostly concerned with so-called decision problems, e.g.,

- Is a given number prime?
- Does a given graph have a path from a given source vertex to a given destination vertex
- Can the thief put $250 worth of loot in the knapsack?
- Is a given formula of propositional logic satisfiable?
- Can the vertices of a given graph be colored with three colors such that no two neighbors have the same color?
- Does a given program ever output `False`?
- ...

General format: Yes/No question over a (typically infinite) set of inputs

Abstraction

There are just too many different types of inputs, e.g., numbers, graphs, formulas, programs, sets of linear inequalities, polynomials, etc.

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To simplify our setting, we only consider sequences of symbols as inputs (i.e., words over an alphabet), e.g.,

- a number is encoded in binary or decimal,
- a graph is encoded by (a linearization) of its adjacency matrix,
- a program is given by its source code

Reminder: Formal Languages

- An alphabet is a finite, nonempty set of letters

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  Examples:

  - The set $\mathbb{B} = \{0, 1\}$ of binary digits
  - The set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of decimal digits
  - The Latin alphabet $\{a, b, c, \ldots, z\}$
  - $\Sigma_L = \{\neg, \wedge, \vee, (,), p, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

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Examples:

- 0, 1, 110, 11111100111 over $\mathbb{B}$
- *alan* and *mathison* over the Latin alphabet, but also *crwth* and *ghfbfdtnjs*
- $(p0 \wedge p1) \vee p23$ over Σ_L, but also $\wedge\wedge)(23p$

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Example:

- If $w_1 = \mathit{algorithms}$ and $w_2 = \mathit{computability}$, then $w_1 w_2 = \mathit{algorithmscomputability}$

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Examples:

- $|0| = |1| = 1$, $|110| = 3$, and $|11111100111| = 11$
- $|alan| = 4$ and $|mathison| = 8$
- $|\varepsilon| = 0$

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Example:

- $\mathbb{B}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$

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Examples:

- $\{0^n 1 \mid n \geq 0\}$ and $\{0^n 1^n \mid n \geq 0\}$ over $\mathbb{B}$
- the set $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$ of prime numbers
- the set of words in today's newspaper

Reminder: Operations on Languages

Let L_1, L_2 be two languages over Σ

- union:

$$L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ or } w \in L_2\}$$

- intersection:

$$L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ and } w \in L_2\}$$

- complement (w.r.t. Σ^*):

$$\overline{L_1} = \{w \in \Sigma^* \mid w \notin L_1\}$$

- concatenation:

$$L_1 \cdot L_2 = \{w \in \Sigma^* \mid w = w_1 w_2 \text{ with } w_1 \in L_1 \text{ and } w_2 \in L_2\}$$

- Kleene star (iteration):

$$(L_1)^* = \{w \in \Sigma^* \mid w = w_1 w_2 \cdots w_k \text{ for some } k \geq 0 \text{ and } w_i \in L_1 \text{ for all } i \in \{1, 2, \ldots, k\}\}$$

From Problems to Languages

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- Does a given program ever output `False`?

$$\{w \in \{a, b, \ldots\}^* \mid w \text{ is Python source code of a function that outputs } \texttt{False} \text{ for some input}\}$$

Solving Problems = Language Membership

From now on: Decision problems = formal languages

- So, to solve a decision problem $L \subseteq \Sigma^*$, we "just" need an algorithm that, given an input $w \in \Sigma^*$, returns `True` if $w \in L$ and `False` if $w \notin L$
- This is easy enough for some problems, but seems much harder for others (e.g., Python programs outputting `False`)

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- This is easy enough for some problems, but seems much harder for others (e.g., Python programs outputting `False`)
- So, can every decision problem be algorithmically solved?
- To answer this question, we need a formal definition of "algorithm" to be able to argue that there is a problem that is not solved by any algorithm

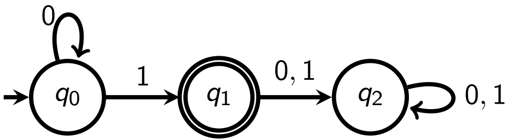
In the remainder of this lecture, we present one such definition (we will later in the course discuss to which extent such a definition is actually possible)

A deterministic finite automaton (DFA) has the form $(Q, \Sigma, q_I, \delta, F)$ where

- Q is a finite set of states,
- Σ is an alphabet,
- $q_I \in Q$ is the initial state,
- $\delta\colon Q \times \Sigma \to Q$ is the transition function, and
- $F \subseteq Q$ is a set of accepting states

Example

A DFA



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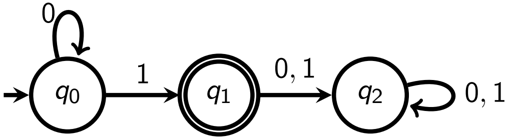
A DFA for the language $\{0^n 1 \mid n \geq 0\}$:



Reminder: Finite Automata

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- So, DFAs can be seen as a (very weak) formalization of algorithms for decision problems
- In the remainder of this course, we study a much stronger formalization

Agenda

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2. Turing Machines

Alan Turing: On Computable Numbers (1936)

Computing is normally done by writing certain symbols on paper. "We may suppose this paper is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is no essential of computation. I assume then that the computation is carried out on one-dimensional paper, i.e. on a tape divided into squares [...]

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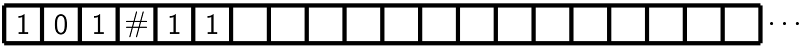
We will also suppose that the number of states of mind which need be taken into account is finite [...]
Let us imagine the operations performed by the computer to be split up into "simple operations" which are so elementary that it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape. We know the state of the system if we know the sequence of symbols on the tape, which of these are observed by the computer [...], and the state of mind of the computer. We may suppose that in a simple operation not more than one symbol is altered"

- A one-sided infinite tape of paper, divided into squares (often called cells)



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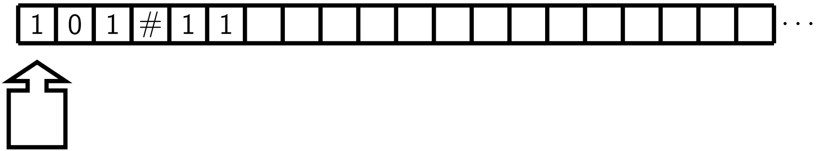

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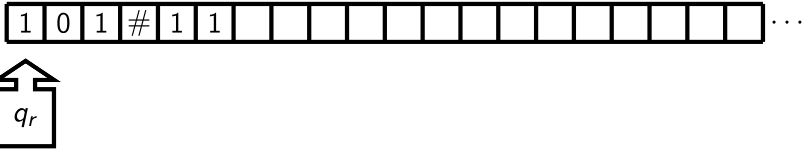
Turing Machine – Conceptual View



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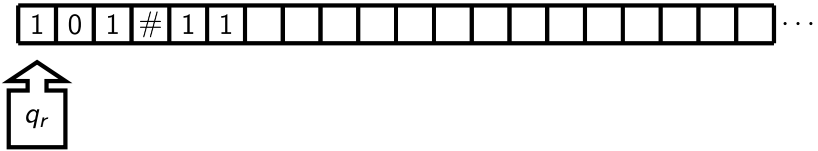

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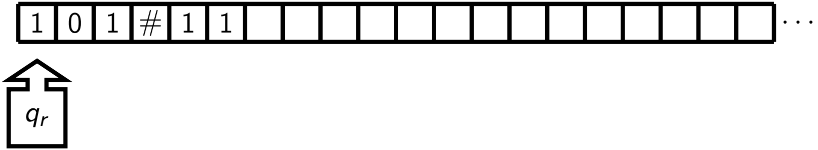
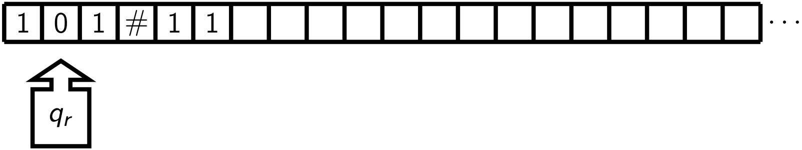
- A single square that is currently observed (with a reading/writing head)
- A "state of mind" (one of finitely many)
- Rules updating the state and currently observed square



- If state is q_r and symbol is 0 then change to state q_r, change symbol to 0, and move in direction 'right'
- If state is q_r and symbol is 1 then change to state q_r, change symbol to 1, and move in direction 'right'
- If state is q_r and symbol is # then change to state q_r, change symbol to #, and move in direction 'right'
- If state is q_r and symbol is 'empty' then change to state q_s, change symbol to 'empty', and move in direction 'left'



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- If state is q_r and symbol is 1 then change to state q_r, change symbol to 1, and move in direction 'right'
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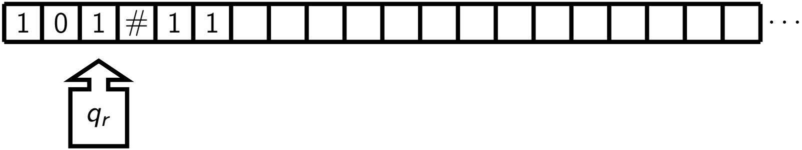
- If state is q_r and symbol is 'empty' then change to state q_s, change symbol to 'empty', and move in direction 'left'

Turing Machine – Conceptual View



- If state is q_r and symbol is 0 then change to state q_r, change symbol to 0, and move in direction 'right'
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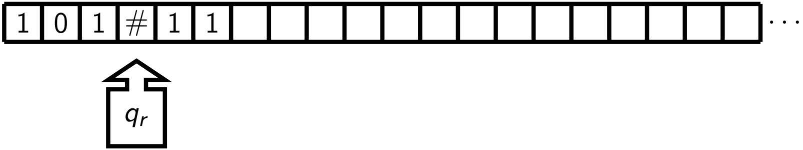
Turing Machine – Conceptual View



- If state is q_r and symbol is 0 then change to state q_r, change symbol to 0, and move in direction 'right'
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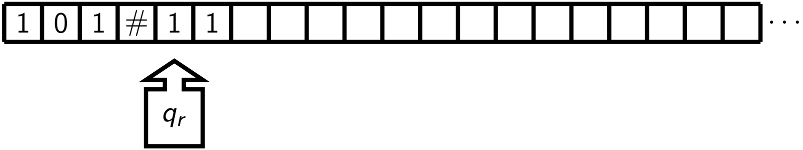
- If state is q_r and symbol is 'empty' then change to state q_s, change symbol to 'empty', and move in direction 'left'

Turing Machine – Conceptual View



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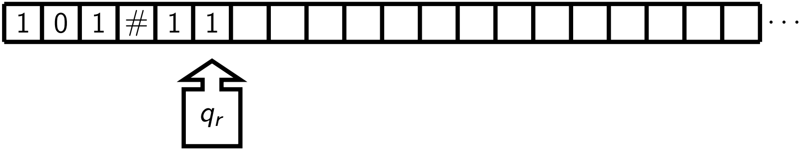
Turing Machine – Conceptual View



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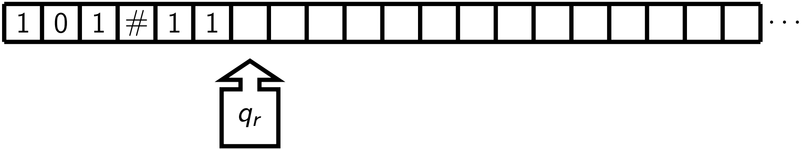
Turing Machine – Conceptual View



- If state is q_r and symbol is 0 then change to state q_r, change symbol to 0, and move in direction 'right'
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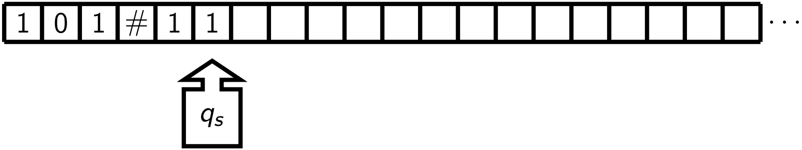
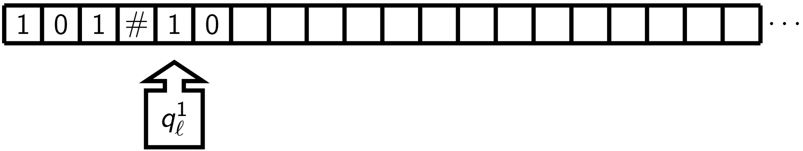
Turing Machine – Conceptual View

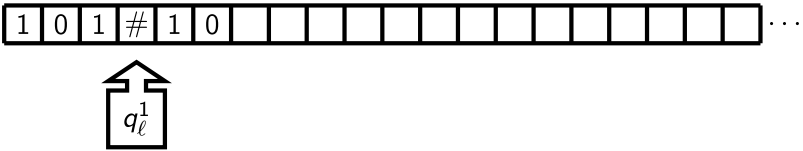


- If state is q_s and symbol is 1 then change to state q_ℓ^1, change symbol to 0, and move in direction 'left'
- If state is q_ℓ^1 and symbol is 1 then change to state q_ℓ^1, change symbol to 1, and move in direction 'left'
- If state is q_ℓ^1 and symbol is # then change to state q_a^1, change symbol to #, and move in direction 'left'
- If state is q_a^1 and symbol is 1 then change to state q_a^1, change symbol to 0, and move in direction 'left'



- `If state is` q_s `and symbol is` 1 `then change to state` q_ℓ^1, `change symbol to` 0, `and move in direction 'left'`
- `If state is` q_ℓ^1 `and symbol is` 1 `then change to state` q_ℓ^1, `change symbol to` 1, `and move in direction 'left'`
- `If state is` q_ℓ^1 `and symbol is` $\#$ `then change to state` q_a^1, `change symbol to` $\#$, `and move in direction 'left'`
- `If state is` q_a^1 `and symbol is` 1 `then change to state` q_a^1, `change symbol to` 0, `and move in direction 'left'`

Turing Machine – Conceptual View



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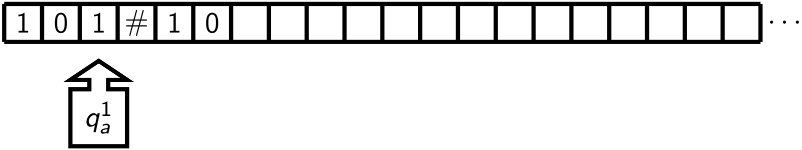
Turing Machine – Conceptual View



- If state is q_s and symbol is 1 then change to state q_ℓ^1, change symbol to 0, and move in direction 'left'
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- If state is q_ℓ^1 and symbol is # then change to state q_a^1, change symbol to #, and move in direction 'left'
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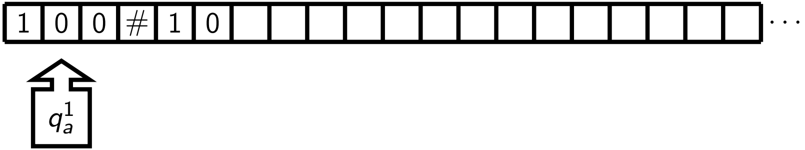
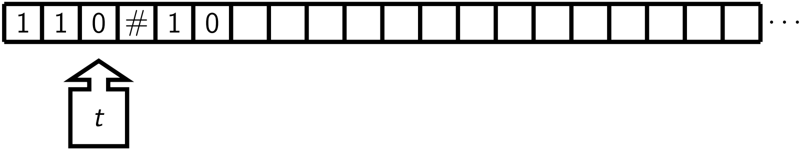


- `If state is` q_a^1 `and symbol is` 0 `then change to state` t, `change symbol to` 1, `and move in direction 'right'`

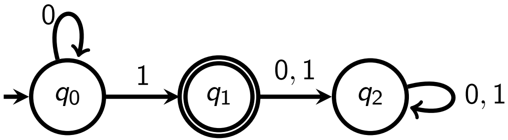
Turing Machine – Conceptual View



- If state is q_a^1 and symbol is 0 then change to state t, change symbol to 1, and move in direction 'right'

DFA as Turing Machines

Meaning of a transition $\delta(q, a) = q'$: `if in state` q `and next letter is` a, `change to state` q'



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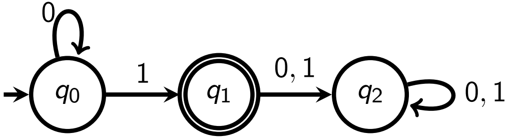
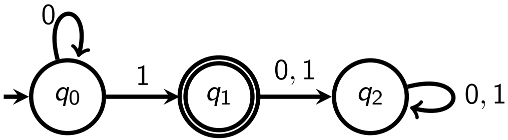


We can consider a DFA as a Turing machine that

- moves its head to the right until the first blank cell is reached and
- only reads the tape, but never writes to it

Meaning of a transition $\delta(q, a) = q'$: `if in state` q `and next letter is` a, `change to state` q'



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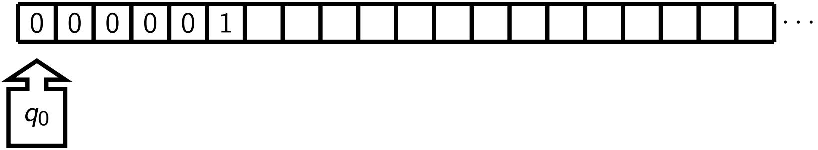

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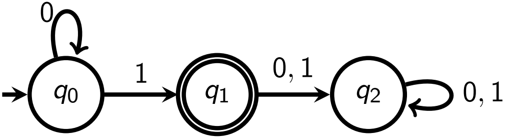

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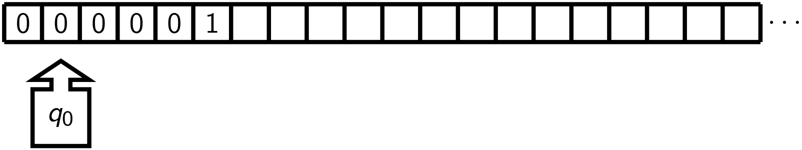

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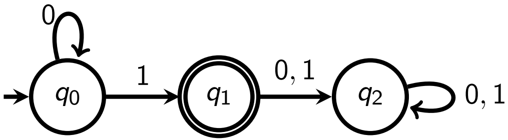

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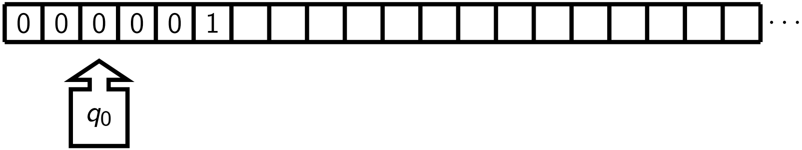

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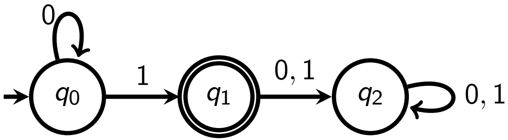

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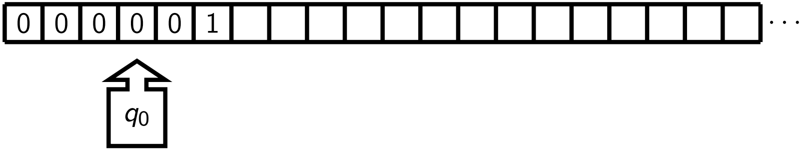

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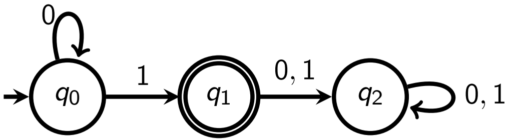

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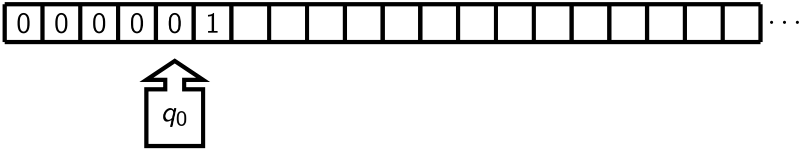

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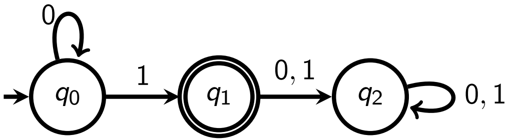

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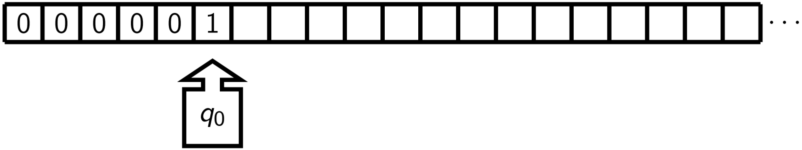

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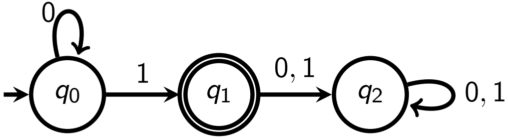

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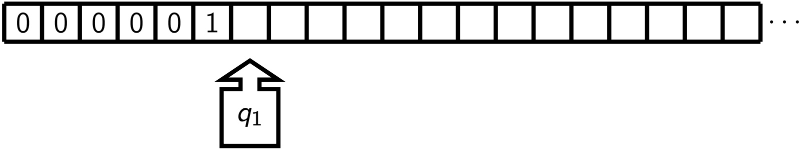

Turing Machine – Formal Definition

A deterministic Turing machine (DTM) is a 7-tuple $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$

- Q is a finite set of states,
- Σ is the input alphabet,
- $\Gamma \supseteq \Sigma$ is the tape alphabet s.t. $\sqcup \in \Gamma \setminus \Sigma$ (the blank symbol),
- $s \in Q$ is the starting (or initial) state,
- $t \in Q$ is the accepting state,
- $r \in Q$ is the rejecting state (we require $t \neq r$), and
- $\delta\colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$ is the transition function

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Notation

We write $\delta(q, a) = (q', a', d)$ (using -1 for `left` and $+1$ for `right`) instead of

“`If state is` q `and symbol is` a `then change to state` q', `change symbol to` a’, `and move in direction` d”

Running a Turing Machine: Intuition

Let $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$ be a Turing machine. Given an input $w \in \Sigma^*$:

- Initialization: w is on the tape (all other cells are blank), reading head is on first letter of w (if w is nonempty), in state s
- Execution: Apply transition function repeatedly until termination, thereby updating the tape contents, the state, and the position of the reading head
- Termination: Stop if either state t or r is reached

Question

Is there another option than reaching t or r?

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Question

Is there another option than reaching t or r? Yes, it can loop forever

Configurations

Definition

Let $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$ be a DTM. A configuration of M is a triple $[q, \tau, \ell]$ where

- $q \in Q$ is the current state,
- $\tau \colon \mathbb{N}^+ \to \Gamma$ represents the current tape content, and
- $\ell \in \mathbb{N}^+$ denotes the current location of the reading head

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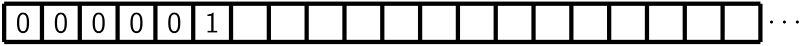

is encoded by $[q_0, \tau, 3]$ with $\tau(n) = \begin{cases} 0 & \text{if } n \leq 5, \\ 1 & \text{if } n = 6, \\ \text{␣} & \text{if } n \geq 7 \end{cases}$

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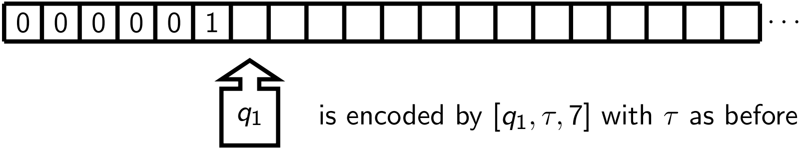

is encoded by $[q_1, \tau, 7]$ with τ as before

A Remark on Notation

- Although the tape is infinite, only finitely many cells are non-blank at any time
- Accordingly, for each τ function there is an n_0 such that $\tau(n) = \textvisiblespace$ for all $n > n_0$. We then often write $\tau(1)\tau(2)\cdots\tau(n_0)\textvisiblespace\ldots$ for τ

A Remark on Notation

- Although the tape is infinite, only finitely many cells are non-blank at any time
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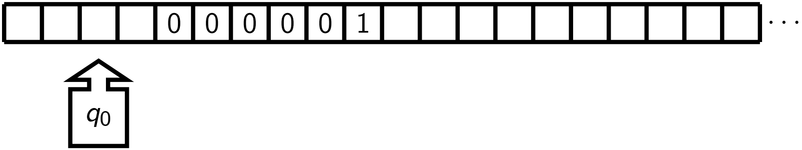
Example

We can represent $\tau(n) = \begin{cases} 0 & \text{if } n \leq 5, \\ 1 & \text{if } n = 6, \\ \sqcup & \text{if } n \geq 7 \end{cases}$ by $000001\sqcup\ldots$

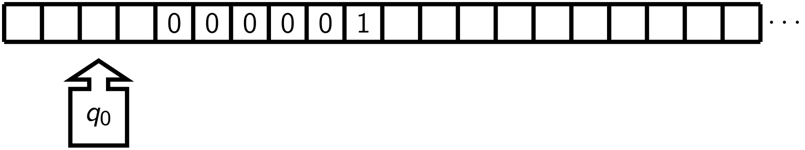
Quiz 1



What is the encoding of the above configuration?

- $[q_0, 000001\textvisiblespace \ldots, 3]$,
- $[q_0, \textvisiblespace\textvisiblespace 000001\textvisiblespace \ldots, 3]$,
- $[q_0, \textvisiblespace\textvisiblespace\textvisiblespace\textvisiblespace 000001\textvisiblespace \ldots, 3]$,
- something else?

Quiz 1



What is the encoding of the above configuration?

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- $[q_0, \sqcup\sqcup\sqcup\sqcup000001\sqcup\ldots, 3]$,
- something else?

Item 3 is correct

More on Configurations

Definition

Let $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$ be a DTM

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$$\tau'(n) = \begin{cases} a & \text{if } n = \ell, \\ \tau(n) & \text{otherwise} \end{cases}$$

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$$\tau'(n) = \begin{cases} a & \text{if } n = \ell, \\ \tau(n) & \text{otherwise} \end{cases}$$
- A configuration is accepting if its state is t
- A configuration is rejecting if its state is r
- A configuration is halting if it is accepting or rejecting

Definition

Let α and β be configurations of a DTM M

- We write $\alpha \vdash_M \beta$ if β is the successor configuration of α

Runs

Definition

Let α and β be configurations of a DTM M

- We write $\alpha \vdash_M \beta$ if β is the successor configuration of α
- For $n \geq 0$, we write $\alpha \vdash_M^n \beta$ if there exist configurations $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that
  - $\gamma_0 = \alpha$,
  - $\gamma_n = \beta$, and
  - $\gamma_0 \vdash_M \gamma_1 \vdash_M \cdots \vdash_M \gamma_{n-1} \vdash_M \gamma_n$

Runs

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  - $\gamma_n = \beta$, and
  - $\gamma_0 \vdash_M \gamma_1 \vdash_M \cdots \vdash_M \gamma_{n-1} \vdash_M \gamma_n$
- We write $\alpha \vdash_M^* \beta$ if there exists an $n \geq 0$ such that $\alpha \vdash_M^n \beta$

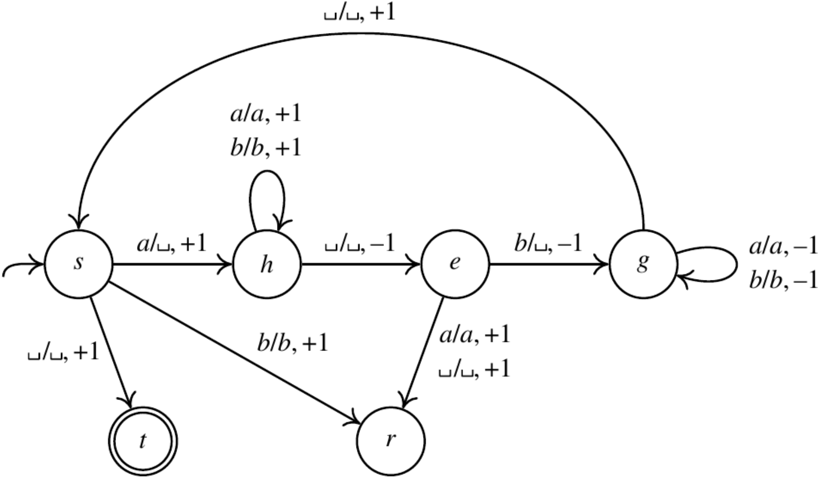
Example runs

- Consider the following DTM $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$ from Example 2.1.3 in "Computability and Complexity" with
  - $Q = \{s, t, r, h, e, g\}$,
  - $\Sigma = \{a, b\}$,
  - $\Gamma = \{a, b, \sqcup\}$, and
  - δ given by the following table:

δ	a	b	$\sqcup$
s	$(h, \sqcup, +1)$	$(r, b, +1)$	$(t, \sqcup, +1)$
h	$(h, a, +1)$	$(h, b, +1)$	$(e, \sqcup, -1)$
e	$(r, a, +1)$	$(g, \sqcup, -1)$	$(r, \sqcup, +1)$
g	$(g, a, -1)$	$(g, b, -1)$	$(s, \sqcup, +1)$

- Give the runs on the inputs ab and abb

Alternative graphical representation



Accepting, Rejecting, and Looping

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We say M halts on input w if it accepts or rejects w (i.e., it does not loop)

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  Note: M may not terminate on all inputs $w \notin L(M)$!

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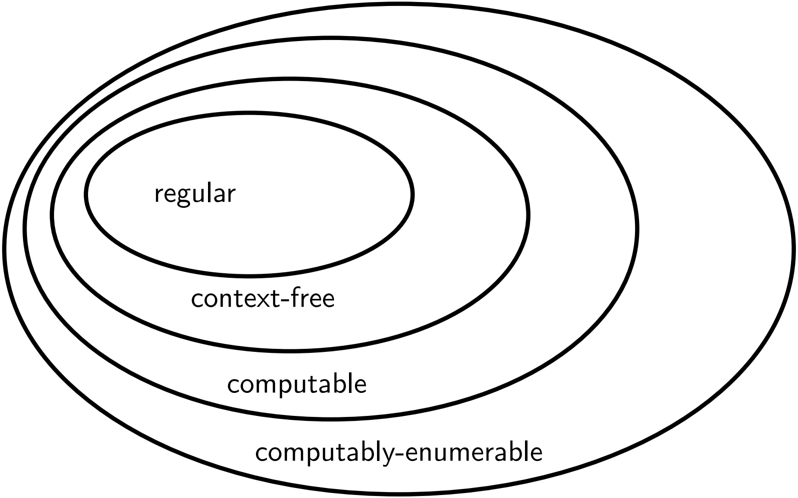
Language Classes



Are all inclusions strict?

Is there a language that is not computably-enumerable?

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- It is finite (either $\{0\}$ or $\{1\}$) and hence trivial
  - Note that currently nobody knows which one it is

    But that is a different question!

Conclusion

We have seen

- Problem = formal language
- DTMs are an abstract model of computation
- The difference between computable and computably-enumerable languages:
  - L is computably-enumerable $\Leftrightarrow$ there exists a DTM M such that $L(M) = L$, i.e.,
    - $w \in L \Rightarrow M$ accepts w
    - but $w \notin L \Rightarrow M$ rejects w or loops
  - L is computable $\Leftrightarrow$ there exists a halting DTM M such that $L(M) = L$, i.e.,
    - $w \in L \Rightarrow M$ accepts w
    - $w \notin L \Rightarrow M$ rejects w

Reading

We follow Hubie Chen: Computability and Complexity

- MIT Press
- ISBN: 9780262048620

This lecture:

- The Introduction and Agreements (pages xiii to xvii)
- Section 2.1 (pages 71 to 85)
- Also skim Section 1 (pages 1 to 21 suffice) to get used to the notation in the book (and slides) and to recall what you have learned about finite automata on the 4th semester

Finally, you may want to take a look at Turing's paper introducing what we today call Turing machines, keeping in mind that it was published 1936 before the advent of *nonhuman* computers:

`https://www.cs.virginia.edu/~robins/Turing_Paper_1936.pdf`

Reading beyond this course

- Douglas R. Hofstadter: Gödel, Escher, Bach (1979)
- Won Pulitzer Prize for General Nonfiction