Algorithms and Computability

Lecture 6:
Amortized Analysis

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Amortized analysis

- Main goals of the lecture:
  - *to understand what is **amortized analysis**, when it is used, and how it differs from the average-case analysis;*
  - *to be able to apply the techniques of the **aggregate analysis**, the **accounting method**, and the **potential method** to analyze operations on simple data structures.*

Sequence of operations

- *The problem*:
  - We have a data structure
  - We perform a sequence of operations
    - Operations may be of different types (e.g., *insert, delete*)
    - Depending on the state of the structure the actual cost of an operation may differ (e.g., *inserting into a sorted array*)
  - Just analyzing the worst-case time of a single operation may not say too much
  - We want the average running time of an operation (*but from the worst-case sequence of operations*!).

Case study: Dijkstra’s and Prim’s

```
Dijkstra(G,s)
...                  // initialize
Q.init(G.V())        // Q is a priority queue on v.d
while not Q.isEmpty()
   u ← Q.extractMin()  // with min u.d: called V times
   for each v ∈ u.adjacent() do
      Relax(u, v, G)   // updates v.d
      Q.modifyKey(v)   // called E times
```

- *What is the running time*?
  - Depends on the data structure:
  - V*$cost$(extractMin) + E*$cost$(modifyKey)
    - Simple array: $V * V + E * 1 = \Theta(V^2)$
    - Binary heap: $V * \Theta(\lg V) + E * \Theta(\lg V) = \Theta(E \lg V)$
    - *Fibonacci heap*: $V * \Theta(\lg V) + E * \Theta(1) = \Theta(V \lg V + E)$

Binary counter example

- *Example data structure: a binary counter*
  - Operation: *Increment*
  - Implementation: An array of bits $A[0..k–1]$

```
Increment(A)
i ←  0
while i < k and A[i] = 1 do
    A[i] ← 0
    i ← i + 1
if i < k then A[i] ← 1
```

k-1					…							3	2	1	0
0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1
0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0

- *How many bit assignments do we have to do in the* ***worst-case*** *to perform Increment(A)*?
  - But usually we do much less bit assignments!

Analysis of the binary counter

- *How many bit-assignments do we do on average*?
  - Let's consider a sequence of *n Increments*
  - Let's compute the sum of bit assignments:
    - *A*[0] assigned on each operation: *n* assignments
    - *A*[1] assigned every two operations: *n*/2 assignments
    - *A*[2] assigned every four ops: n/4 assignments
    - *A*[*i*] assigned every 2^i ops: n/2^i assignments

$$\sum_{i=0}^{\lfloor lg\, n \rfloor} \left\lfloor \frac{n}{2^i} \right\rfloor = n \sum_{i=0}^{\lfloor lg\, n \rfloor} \left\lfloor \frac{1}{2^i} \right\rfloor < 2n$$

- Thus, a single operation takes 2n/n = 2 = *O*(1) **amortized** time

Aggregate analysis

- ***Aggregate** analysis – a simple way to do amortized analysis*
  - Treat all operations equally
  - Compute the *worst-case* running time of a sequence of n operations.
  - Divide by n to get an amortized running time

Another look at the binary counter

- *Another way of looking at it (proving the amortized time)*:
  - To assign a bit, I have to pay \$1
  - When I assign "1", I pay \$1, plus I put \$1 in my "savings account" associated with that bit.
  - When I assign "0", I can do it using a dollar from the savings account on that bit
  - *How much do I have to pay for the* Increment(*A*) *for this scheme to work*?
    - Only one assignment of "1" in the algorithm. Obviously, \$2 will always pay for the entire operation

```
Increment(A)
i ← 0
while i < k and A[i] = 1 do
    A[i] ← 0
    i ← i + 1
if i < k then A[i] ← 1
```

```
k-1        …        3210
0000000000100111
           …    $   $$$

0000000000101000
           …    $ $
```

Accounting method

- Principles of the **accounting method**
  - 1. Associate credit accounts with different parts of the structure
  - 2. Associate amortized costs with operations and show how they credit or debit accounts
    - Different costs may be assigned to different operations
  - Requirement (c – real cost, $\hat{c}$ – amortized cost):

$$\sum_{i=1}^{n} \hat{c}_i \geqslant \sum_{i=1}^{n} c_i$$

  - This is equivalent to requiring that the sum of all credits in the data structure is *non-negative* after any sequence of operations
    - What would it mean not to satisfy this requirement?
  - 3. Show that this requirement is satisfied

Stack example

- Start with an empty stack and consider a sequence of n operations: *Push, Pop,* and *Multipop*(k).
  - *What is the worst-case running time of an operation from this sequence?*
  - 1. Let's associate an account with each element in the stack
  - 2. After pushing an element, put a dollar into the account associated with it,
    - then *Pop* and *Multipop* can work only using money in the accounts (amortized cost 0)
    - *Push* has amortized cost 2
  - 3. The total credit in the structure is always ≥ 0
  - Thus, the amortized cost of an operation is $O(1)$

Potential method

- *We can have one account associated with the whole structure*:
  - We call it a **potential**
  - It's a function that maps a state of the data structure after operation i to a number: $\Phi(D_i)$
    - $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

- The main step of this method is defining the potential function
  - Requirement: $\Phi(D_n) - \Phi(D_0) \geq 0$
- Once we have Φ, we can compute the amortized costs of operations

Binary counter example

- *How do we define the potential function for the binary counter?*
  - Potential of A: b_i – a number of “1”s
  - *What is* $\Phi(D_i) - \Phi(D_{i-1})$*, if the number of bits set to 0 in operation i is* t_i*?*
  - *What is the amortized cost of* Increment(A)?
    - We showed that: $\Phi(D_i) - \Phi(D_{i-1}) \le 1 - t_i$
    - Real cost: $c_i \le t_i + 1$
    - Thus, $\hat{c}_i \le c_i + \Phi(D_i) - \Phi(D_{i-1}) \le (t_i + 1) + (1 - t_i) = 2$

```
Increment(A)
i ← 0
while i < k and A[i] = 1 do
   A[i] ← 0
   i ← i + 1
if i < k then A[i] ← 1
```

Potential method

- *We can analyze the counter even if it does not start at 0 using the potential method*:
  - Let's say we start with b_0 and end with b_n "1"s
  - Observe that: $$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \hat{c}_i - \Phi(D_n) + \Phi(D_0)$$
  - We have that: $\hat{c}_i \leq 2$
  - This means that: $$\sum_{i=1}^{n} c_i \leqslant 2n - b_n + b_0$$
  - Note that $b_0 \leq k$. This means that the total cost $\leq 2n+k$. If $k = O(n)$, then the total cost is $O(n)$. In other words: if $n = \Omega(k)$, the amortized cost per increment is $O(1)$.

Dynamic table

- *It is often useful to have a dynamic table:*
  - The table that expands and contracts as necessary when new elements are added or deleted.
    - **Expands** when insertion is done and the table is already full
    - **Contracts** when deletion is done and there is “too much” free space
  - Contracting or expanding involves **relocating**
    - Allocate new memory space of the new size
    - Copy all elements from the table into the new space
    - Free the old space
  - Worst-case time for insertions and deletions:
    - Without relocation: $O(1)$
    - With relocation: $O(m)$, where m – the number of elements in the table

Requirements

- Load factor
  - *num* – current number of elements in the table
  - *size* – the total number of elements that can be stored in the allocated memory
  - *Load factor* α *= num/size*

- It would be nice to have these two properties:
  1) Amortized cost of insert and delete is constant
  2) The load factor is always above so

Naïve insertions

- *Let's look only at insertions*: *Why not expand the table by some constant when it overflows?*
  - *What is the amortized cost of an insertion in a sequence of n insertions?*
    - Let's start with 100 and expand with 100 when full (and count element insertions and copying)
    - *Go to Socrative and vote:*
    - A: ≈2 B: ≈ 1 + n/200 C: ≈ 1 + n/100 D: other
  - *Does it satisfy the two requirements?*

Aggregate analysis / accounting

- *The "right" way to expand – double the size of the table*
  - Let's do an aggregate analysis
  - The cost of the *i*-th insertion is:
    - *i*, if *i* –1 is an exact power of 2
    - 1, otherwise
  - Let's sum up…

$$\sum_{i=1}^{n} c_i = n + \sum_{j=1}^{\lfloor \lg n \rfloor} 2^j \leqslant n + \frac{2^{\lg n+1}-1}{2-1} = 3n-1$$

  - The total cost of *n* insertions is then < 3*n*
  - Accounting method gives the intuition:
    - Pay $1 for inserting the element
    - Put $1 into element's account for reallocating it later
    - Put $1 into the account of another element to pay for a later relocation of that element

x	x	x	x	x	x	x	x	x							
$								$							

x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$	$	$	$	$	$	$	$	$	$	$	$	$	$	$	$

Potential function

- *What potential function do we want to have?*
  - It is zero right after expansion ($num = size/2$) and grows…
  - ...to $size$ right before the next expansion ($num = size$)
  - Thus, it has to grow by 2 on each insertion.
  - $\Phi_i = 2(num_i - size_i/2) = 2num_i - size_i$
  - It is always non-negative
  - Amortized cost of insertion:
    - Insertion does not trigger an expansion ($size_{i-1}=size_i$):
      - $\Delta\Phi_i = \Phi_i - \Phi_{i-1} = 2(num_{i-1}+1) - size_i - 2num_{i-1} + size_i = 2$
      - $\hat{c}_i = c_i + \Delta\Phi_i = 1+2 = 3$
    - Insertion triggers an expansion ($size_{i-1}=num_{i-1}$, $size_i = 2num_{i-1}$):
      - $\Delta\Phi_i = \Phi_i - \Phi_{i-1} = 2(num_{i-1}+1) - size_i - 2num_{i-1} + size_{i-1} = 2(num_{i-1}+1) - 2num_{i-1} - 2num_{i-1} + num_{i-1} = 2 - num_{i-1}$
      - $\hat{c}_i = c_i + \Delta\Phi_i = num_{i-1}+1 + 2 - num_{i-1} = 3$
    - Both cases: 3

Deletions

- *Deletions*: *What if we contract whenever the table is about to get less than half full*?
  - *Would the amortized running times of a sequence of insertions and deletions be constant*?
  - Problem: we want to avoid doing re-allocations often without having accumulated “the money” to pay for that!

Deletions

- *Idea: delay contraction!*
  - Contract only when *num* = *size*/4
  - Second requirement still satisfied: $\alpha \geq \frac{1}{4}$

- *Consider the following sequence of operations (starting with an empty table of size 1):*
  - 6 ins, 3 dels, 5 ins, 7 dels, 7 ins
  - *How many contractions and expansions are performed?*
  - *What is the final size of the table?*

Deletions

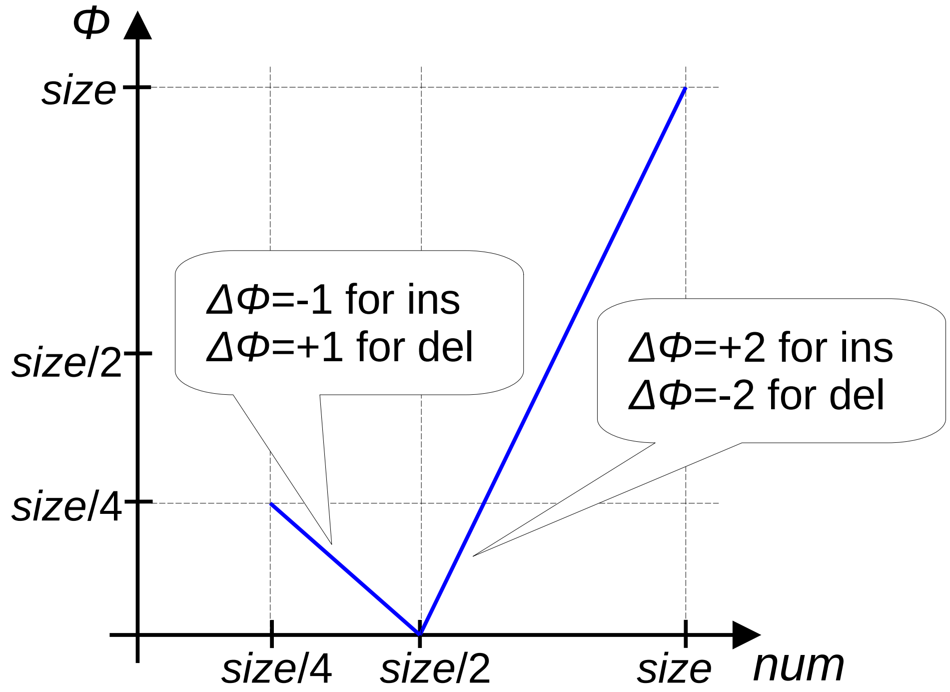
- *Idea: delay contraction!*
  - Contract only when *num* = *size*/4
  - Second requirement still satisfied: $\alpha \geq 1/4$

- *Consider the following sequence of operations (starting with an empty table of size* 1*):*
  - 6 ins, 3 dels, 5 ins, 7 dels, 7 ins
  - *How many contractions and expansions are performed?*
  - *What is the final size of the table?*

Deletions

- Contraction: *num = size/4*
- *How do we define the potential function?*

$$\Phi_i = \begin{cases} 2 \cdot num_i - size_i & \text{if } \alpha \geqslant 1/2 \\ size_i/2 - num_i & \text{if } \alpha < 1/2 \end{cases}$$



- It is always non-negative
- Let's compute the amortized running time of deletions:
  - α < ½ (with contraction, without contraction)