Algorithms and Computability

Lecture 9: Computability

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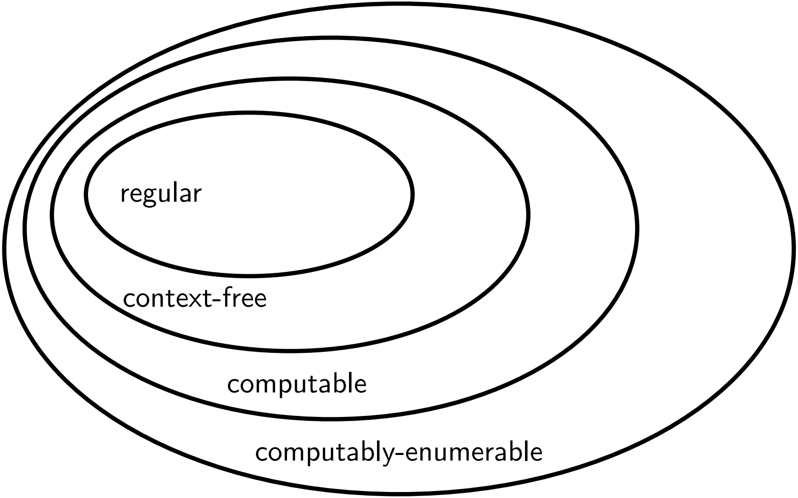
slides courtesy of Martin Zimmermann

We have seen

- the Church-Turing Thesis ("Everything that can be computed can be computed by a Turing machine"),
- multi-tape DTMs and
- nondeterministic TMs, and
- their equivalence: The same classes of languages are computably-enumerable (computable) by
  1. Deterministic one-tape TMs,
  2. Deterministic multi-tape TMs,
  3. Nondeterministic one-tape TMs, and
  4. Nondeterministic multi-tape TMs (not shown)

Today

Understand the structure of computable and computably-enumerable languages:



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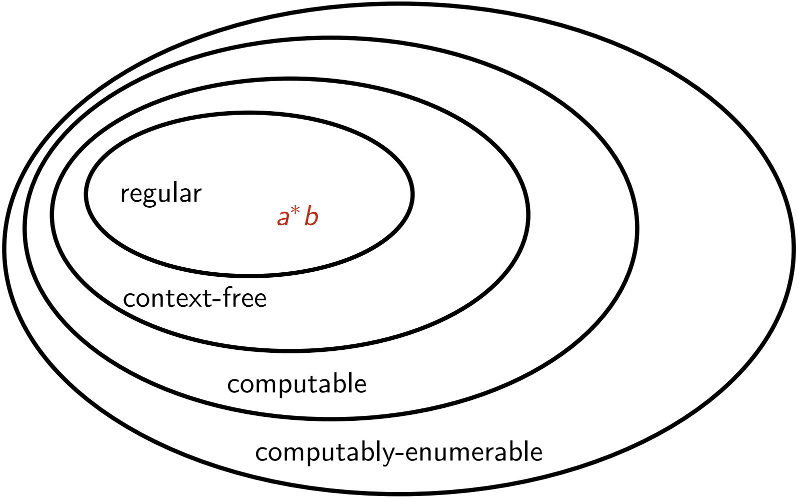

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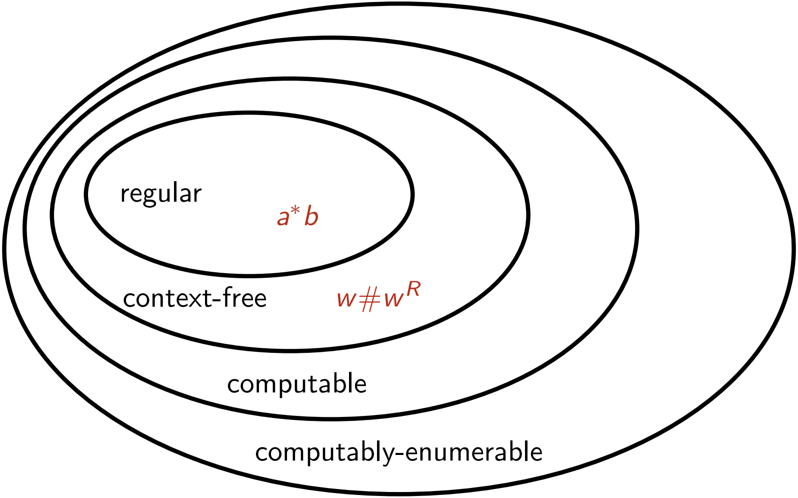

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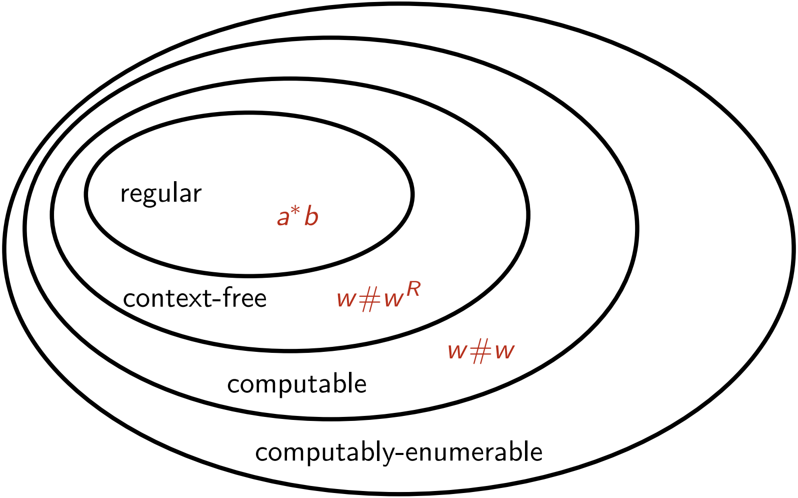

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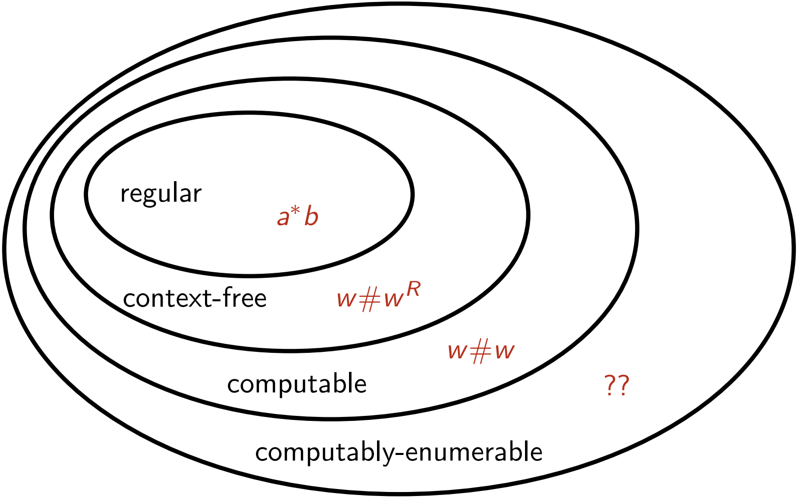

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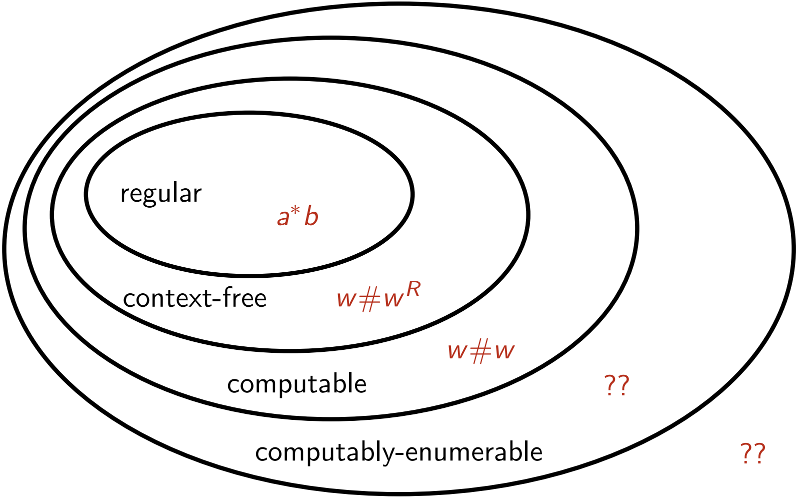

Agenda

1. Warm-up: Hotels and Barbers
2. The Halting Problem
3. Closure Properties

Hilbert's Grand Hotel

You have a hotel with ten rooms, all occupied. Can you accommodate a newly arriving guest?

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Infinite sets (and infinite hotels) behave quite differently from finite ones!

Are there Different Types of Infinities?

Question

Are there more subsets of natural numbers than natural numbers?

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$$\left\{ \begin{array}{c} 3, \\ 7, \\ 13 \end{array} \right\} \qquad\qquad \left\{ \begin{array}{c} a, \\ p, \\ u \end{array} \right\}$$

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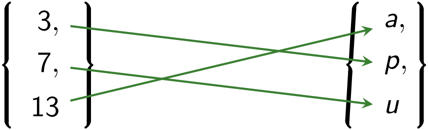

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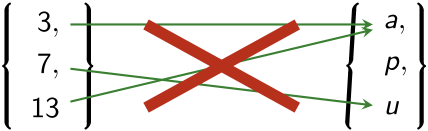

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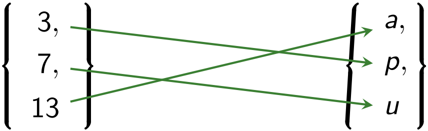

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We use the same approach for infinite sets

Unintuitive Consequences

“Two sets have the same cardinality if there is a bijection between them”

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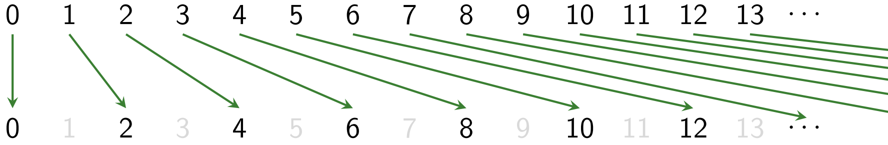
"Two sets have the same cardinality if there is a bijection between them"

- So, the natural numbers and the even natural numbers have the same cardinality



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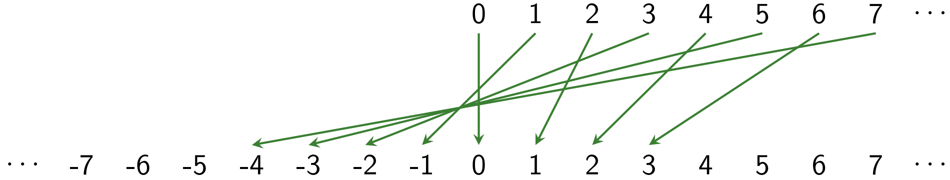

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- So, the natural numbers and the even natural numbers have the same cardinality
- And the natural numbers and the integers have the same cardinality
- The natural numbers and Σ^* also have the same cardinality
- The latter result also implies that even the natural numbers and the rational numbers have the same cardinality
- But Cantor's diagonalization argument shows that the cardinality of the reals is strictly larger than that of the natural numbers

Back to our Question

- Assume the natural numbers and the subsets of the natural numbers have the same cardinality
- Then, there is a bijection f between the natural numbers and the subsets of the natural numbers
- So, the list $f(0), f(1), f(2), \ldots$ contains all subsets of the natural numbers

Back to our Question

- We can arrange the list $f(0), f(1), f(2), \ldots$ in an infinite table such that row i represents the subset $f(i)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$f(0)$	✓	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓	✗	$\cdots$
$f(1)$	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓	$\cdots$
$f(2)$	✗	✗	✓	✓	✗	✓	✗	✓	✗	✗	✗	✓	✗	✓	$\cdots$
$f(3)$	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	$\cdots$
$f(4)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	$\cdots$
$f(5)$	✗	✓	✓	✗	✓	✗	✗	✗	✓	✗	✗	✗	✗	✗	$\cdots$
$f(6)$	✓	✗	✗	✓	✗	✗	✓	✗	✗	✓	✗	✗	✓	✗	$\cdots$
$f(7)$	✓	✗	✗	✓	✗	✗	✗	✗	✗	✓	✗	✗	✓	✗	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Back to our Question

- We can arrange the list $f(0), f(1), f(2), \ldots$ in an infinite table such that row i represents the subset $f(i)$
- Consider the set $\overline{D} = \{3, 5, 7, \ldots\}$ obtained by flipping the diagonal

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- Consider the set $\overline{D} = \{3, 5, 7, \ldots\}$ obtained by flipping the diagonal
- It differs from every set $f(i)$ in the membership of i

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This process, due to Georg Cantor, is called **diagonalization** and has many applications in set theory, logic, and computability

Barber Paradox

Suppose a town's male barber shaves precisely every man in town who does not shave himself.
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The paradox relies on **self-reference**, i.e., "does the barber shave *himself*?"

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Can we apply self-reference to Turing machines?

Quiz 1

The number 0 is interesting, as $n + 0 = n$ for all n. The number 1 is interesting, as $n \cdot 1 = n$ for all n. The number 2 is interesting, as it is the only even prime number. The number 3 is interesting, as it is the only number n that satisfies $n = \sum_{0 \leq j < n} j$

Is there an uninteresting number?

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Is there an uninteresting number?

This is another paradox
If there was an uninteresting number, that fact itself would make it interesting

More paradoxes:
`https://en.wikipedia.org/wiki/List_of_paradoxes`

Agenda

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2. **The Halting Problem**
3. Closure Properties

Encoding of Turing Machines

Let $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$ be a DTM. We assume without loss of generality that $Q = \{1, 11, \ldots, 1^{|Q|}\}$ and $\Sigma = \{0, 1\}$.
Also, let $rep_\Gamma \colon \Gamma \to \{1, 11, \ldots, 1^{|\Gamma|}\}$ be an encoding of Γ such that $rep_\Gamma(0) = 1$, $rep_\Gamma(1) = 11$, and $rep_\Gamma(\text{␣}) = 111$

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Then, M is encoded by the word $\ulcorner M \urcorner$ over $\{0, 1\}$ defined as follows:

$$1^{|Q|}\,0\,1^{|\Gamma|}\,0\,s\,0\,t\,0\,r\,0\,w_\delta$$

where w_δ is the list of encodings of transitions.
Each $\delta(q, b) = (p, a, d)$ is encoded by

$$q\,0\,rep_\Gamma(b)\,0\,p\,0\,rep_\Gamma(a)\,0\,dir(d)\,0$$

where $dir(-1) = 1$ and $dir(+1) = 11$

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Note

Basis of modern computer architecture: program and input are both data stored in memory (von Neumann architecture)

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Theorem (Turing 1936)

The halting problem is not computable

Intuition

- List behavior of all halting DTMs M_i on all inputs w_j in matrix

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	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\cdots$
M_0	acc	rej	acc	rej	acc	rej	acc	rej	acc	rej	$\cdots$
M_1	rej	acc	rej	acc	rej	acc	rej	acc	rej	acc	$\cdots$
M_2	rej	rej	acc	acc	rej	acc	rej	acc	rej	rej	$\cdots$
M_3	rej	rej	rej	rej	rej	rej	rej	rej	rej	rej	$\cdots$
M_4	acc	acc	acc	acc	acc	acc	acc	acc	rej	acc	$\cdots$
M_5	rej	acc	acc	rej	acc	rej	rej	rej	acc	rej	$\cdots$
M_6	acc	rej	rej	acc	rej	rej	acc	rej	rej	acc	$\cdots$
M_7	acc	rej	rej	acc	rej	rej	rej	rej	rej	acc	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Intuition

- List behavior of all halting DTMs M_i on all inputs w_j in matrix
- Flipping the diagonal gives language $\{w_3, w_5, w_7, \ldots\}$ that is different from every computable language

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M_1	rej	acc	rej	acc	rej	acc	rej	acc	rej	acc	$\cdots$
M_2	rej	rej	acc	acc	rej	acc	rej	acc	rej	rej	$\cdots$
M_3	rej	rej	rej	rej	rej	rej	rej	rej	rej	rej	$\cdots$
M_4	acc	acc	acc	acc	acc	acc	acc	acc	rej	acc	$\cdots$
M_5	rej	acc	acc	rej	acc	rej	rej	rej	acc	rej	$\cdots$
M_6	acc	rej	rej	acc	rej	rej	acc	rej	rej	acc	$\cdots$
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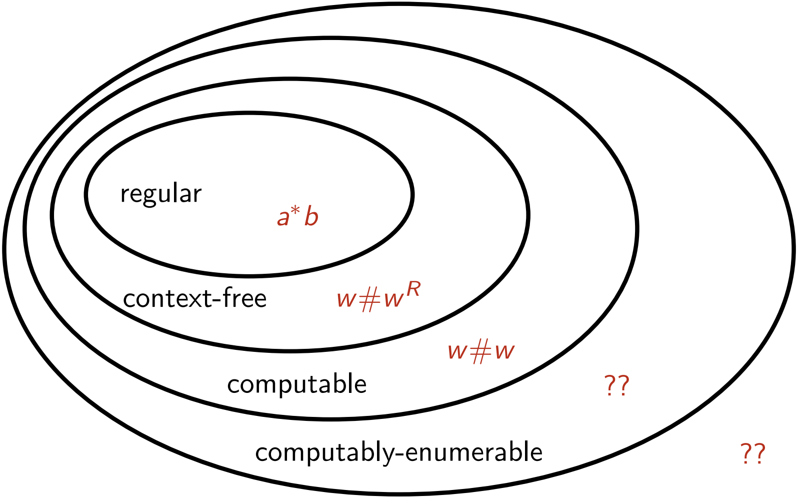
Both cases lead to a contradiction, so H cannot exist
Hence, the halting problem HP is not computable

What about Computably-enumerability?

Is the halting problem at least computably-enumerable?

Exercise 5: Yes, it is!

What Have We Achieved?



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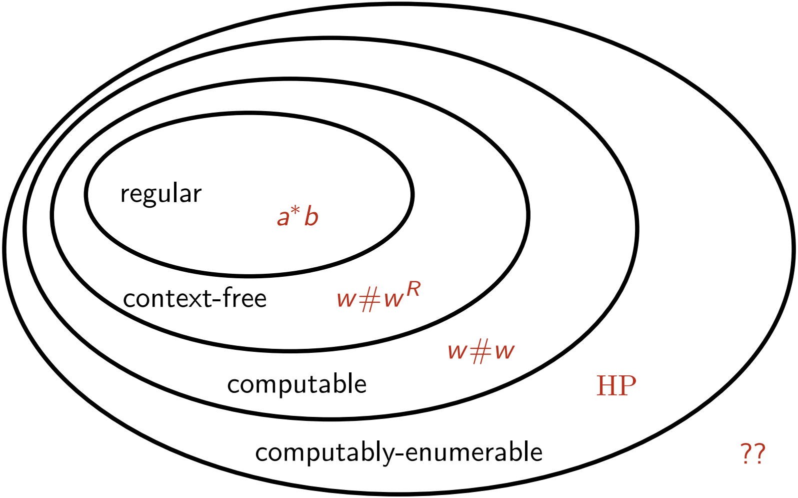

Agenda

1. Warm-up: Hotels and Barbers
2. The Halting Problem
3. **Closure Properties**

Reminder: Operations on Languages

Let L_1, L_2 be two languages over Σ

- union:

$$L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ or } w \in L_2\}$$

- intersection:

$$L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ and } w \in L_2\}$$

- complement (w.r.t. Σ^*):

$$\overline{L_1} = \{w \in \Sigma^* \mid w \notin L_1\}$$

- concatenation:

$$L_1 \cdot L_2 = \{w \in \Sigma^* \mid w = w_1 w_2 \text{ with } w_1 \in L_1 \text{ and } w_2 \in L_2\}$$

- Kleene star (iteration):

$$(L_1)^* = \{w \in \Sigma^* \mid w = w_1 w_2 \cdots w_k \text{ for some } k \geq 0 \text{ and } w_i \in L_1 \text{ for all } i \in \{1, 2, \dots, k\}\}$$

Why Care about Closure Properties?

There are two main applications of closure properties:

1. Show that a language is in some class of languages

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Example

- Regular languages are closed under intersection
- $L_s = \{w \in \mathbb{B}^* \mid w \text{ starts with a } 1\}$ is regular
- $L_e = \{w \in \mathbb{B}^* \mid w \text{ ends with a } 1\}$ is regular
- So, $\{w \in \mathbb{B}^* \mid w \text{ starts and ends with a } 1\} = L_s \cap L_e$ is regular as well

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- $L_{prs} = \{x\#y \mid x, y \in \mathbb{B}^*\}$ is regular
- Assume $L_{ne} = \{x\#y \mid x, y \in \mathbb{B}^*$ such that $x \neq y\}$ is regular

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- Then, $L = \overline{L_{ne}} \cap L_{prs}$ is regular as well

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- Then, $L = \overline{L_{ne}} \cap L_{prs}$ is regular as well
- But $L = \{w\#w \mid w \in \mathbb{B}^*\}$, which we know is not regular
- Contradiction. So, our assumption is wrong and L_{ne} is not regular

Intersections

Given one-tape DTMs M_1 for $L_1 \subseteq \Sigma^*$ and M_2 for $L_2 \subseteq \Sigma^*$, we construct a two-tape DTM for $L_1 \cap L_2$:

On input w:

1. Copy w on second tape, place both heads at first letter of w
2. Do one step of M_1 on first tape and one step of M_2 on second tape (if one halts, only simulate the other)
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2. Run M_i on w
3. If M_i accepts, accept as well
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Complements of Computable Languages

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Does the same construction work for computably-enumerable languages?

Closure Properties

Theorem

1. *The computable languages are closed under union, intersection, complement, concatenation, and iteration*
2. *The computably-enumerable languages are closed under union, intersection, concatenation, and iteration, but* **not** *under complement*

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- We sketched some of the proofs here, but not all
- Some more are on the exercise sheet
- The others are similar (and can be found in the literature)

Conclusion

We have seen

- Diagonalization
- Non-computability of the halting problem
- Closure properties of computable and computably-enumerable languages

Reading

In "Computability and Complexity":

- Section 2.3 on universal Turing machines
- Section 2.4 on non-computable problems
- Section 2.5 on closure properties