Contents

1 Review of Propositional Logic

Task: Recall enough propositional logic to see how it matches up with set theory.

Definition: A <u>proposition</u> is any declarative sentence that is either true or false.

1.1 Connectives

	<u>Connectives</u>		<u>Notation in Maths</u>
and	$\wedge$		
or	$\vee$	"Inclusive or"	
not	$\neg$	Sometimes denoted $\sim$	
implies	$\rightarrow$	if/then; called implication	$\Rightarrow$
if and only if	$\leftrightarrow$	Called equivalence	$\Leftrightarrow$

1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	P$\wedge$Q
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	P$\vee$Q
F	F	F
F	T	T
T	F	T
T	T	T

P	$\neg$P
F	T
T	F

NB In some textbooks, T is denoted by 1, and F is denoted by 0.

P	Q	P$\rightarrow$Q
F	F	T
F	T	T
T	F	F
T	T	T

NB Note that the only instance when an implication (if/then statement) denoted by $P \rightarrow Q$ is false is when the hypothesis (P) is true, but the conclusion (Q) is false.

P	Q	P$\leftrightarrow$Q
F	F	T
F	T	F
T	F	F
T	T	T

NB The truth table for the equivalence says that both P and Q must have the same truth value, i.e. both be true or both be false for the equivalence to be true.

Priority of the Connectives

Highest to Lowest: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

1.2 Important Tautologies

$$\begin{array}{rcl} (P \rightarrow Q) & \leftrightarrow & (\neg P \vee Q) \\ (P \leftrightarrow Q) & \leftrightarrow & [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\ \neg(P \wedge Q) & \leftrightarrow & (\neg P \vee \neg Q) \\ \neg(P \vee Q) & \leftrightarrow & (\neg P \wedge \neg Q) \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{De Morgan Laws}$$

(these have parallels in in set theory)

As a result, $\neg$ and $\vee$ together can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Less obvious: One connective called the Sheffer stroke P|Q (which stands for "not both P and Q" or "P nand Q") can be used to represent all of $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ since $\neg$P $\leftrightarrow$ P|P and P $\vee$ Q $\leftrightarrow$ (P|P) | (Q|Q).

Recall that if P$\rightarrow$Q is a given implication, then Q$\rightarrow$P is called the converse of P$\rightarrow$Q, while $\neg$Q $\rightarrow$ $\neg$P is called the contrapositive of P $\rightarrow$ Q.

1.3 Indirect Arguments/Proofs by Contradiction/Reductio ad absurdum

Based on the tautology (P$\rightarrow$Q) $\leftrightarrow$ ($\neg$Q $\rightarrow$ $\neg$P)

Example: Famous argument that $\sqrt{2}$ is irrational.

Proof:

Suppose $\sqrt{2}$ is rational, then it can be expressed in fraction form as $\frac{a}{b}$ with a and b integers, $b \neq 0$. Let us **assume** that our fraction is reduced, **i.e.** the only common divisor of the numerator a and denominator b is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by b^2 yields

$$2b^2 = a^2$$

Therefore, 2 divides a^2, i.e. a^2 is even. If a^2 is even, then a is also even, namely $a = 2k$ for some integer k.

Substituting the value of $2k$ for a, we have $2b^2 = (2k)^2$ which means that $2b^2 = 4k^2$. Dividing both sides by 2, we have $b^2 = 2k^2$. That means 2 divides b^2, so b is even.

This implies that both a and b are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that the numerator a and the denominator b have no common divisor except 1. Since we found a contradiction, our assumption that $\sqrt{2}$ is rational must be false. Hence the theorem is true.

qed

2 Predicate logic and Quantifiers

Task: Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variables x, y, z, so their truth value may change depending on which values these variables assume: $P(x), Q(x, y), R(x, y, z)$

2.1 Introduce quantifiers

2.1.1 $\exists$ existential quantifier

Syntax: $\exists x P(x)$

Definition: $\exists x P(x)$ is true if $P(x)$ is true for some value of x. It is false otherwise.

2.1.2 $\forall$ universal quantifier

Syntax: $\forall x P(x)$

Definition: $\forall x P(x)$ is true if $P(x)$ is true for all allowable values of x. It is false otherwise.

2.1.3 $\exists!$ for one and only one (additional quantifier standard in maths)

Syntax: $\exists! x P(x)$

Definition: $\exists! x P(x)$ is true if $P(x)$ is true for exactly one value of x and false for all other values of x; otherwise, $\exists! x P(x)$ is false.

Example: $P(x) : x$ is/was the pope and x is Argentine.
(Compound statement; two sentences with connector $\wedge$ between them)
$\exists! x P(x)$ is true with x being Pope Francis.
Now, set $Q(x) : x$ is/was the pope and x is Brazilian.
$\exists! x Q(x)$ is false as there has not been a Brazilian pope so far.
In fact, $\exists x Q(x)$ is also false.

2.2 Alternation of Quantifiers

$\forall x \exists y \forall z \quad P(x,y,z)$

NB: The order cannot be exchanged as it might modify the truth value of the statement (think of examples with two quantifiers).

2.3 Negation of Quantifiers

$\neg(\exists x P(x)) \quad \leftrightarrow \quad \forall x \neg P(x)$
$\neg(\forall x P(x)) \quad \leftrightarrow \quad \exists x \neg P(x)$

3 Set Theory

Task: Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc.

Set theory started around 1870's $\rightarrow$ late development in mathematics but now taught early in one's maths education due to the Bourbaki school.

Definition: A set is a collection of objects. $x \in A$ means the element x is in the set A (**i.e.** belongs to A).

Examples:

1. All students in a class.
2. $\mathbb{N}$ the set of natural numbers starting at 0.

   $\mathbb{N}$ is defined via the following two axioms:

   (a) $0 \in \mathbb{N}$

   (b) if $x \in \mathbb{N}$, then $x+1 \in \mathbb{N}$ ($x \in \mathbb{N} \rightarrow x+1 \in \mathbb{N}$)
3. $\mathbb{R}$ set of real numbers also introduced axiomatically. The hardest axiom is the last one: completeness. $\mathbb{R}$ is constructed from $\mathbb{Q}$ in one of two ways: via Dedekind cuts or Cauchy sequences.

   $\mathbb{R}$ is the set of real numbers. The axioms governing $\mathbb{R}$ are:

   (a) Additive closure: $\forall x, y\, \exists z (x+y=z)$

   (b) Multiplicative closure: $\forall x, y, \exists z (x \times y = z)$

   (c) Additive associativity: $\forall x, y, z \quad x+(y+z)=(x+y)+z$

   (d) Multiplicative associativity: $\forall x, y, z \quad x \times (y \times z) = (x \times y) \times z$

   (e) Additive commutativity: $\forall x, y \quad x+y=y+x$

   (f) Multiplicative commutativity: $\forall x, y \quad x \times y = y \times x$

   (g) Distributivity: $\forall x, y, z \quad x \times (y+z) = (x \times y) + (x \times z)$ and $(y+z) \times x = (y \times x) + (z \times x)$

   (h) Additive identity: There is a number, denoted 0, such that for all $x, x+0=x$

   (i) Multiplicative identity: There is a number, denoted 1, such that for all $x, x \times 1 = 1 \times x = x$

(j) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$

(k) Multiplicative inverses: For every nonzero x there is a number, denoted x^{-1}, such that $x \times x^{-1} = x^{-1} \times x = 1$

(l) $0 \neq 1$

(m) Irreflexivity of $<$: $\sim (x < x)$

(n) Transitivity of $<$: If $x < y$ and $y < z$, then $x < z$

(o) Trichotomy: Either $x < y, y < x$, or $x = y$

(p) If $x < y$, then $x + z < y + z$

(q) If $x < y$ and $0 < z$, then $x \times z < y \times z$ and $z \times x < z \times y$

(r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.

4. $\emptyset$ is the empty set (The set with no elements).

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \rightarrow x \in B)] \wedge [\forall y(y \in B \rightarrow y \in A)]$

3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.
   NB: order is <u>irrelevant</u>.
   $A = \{0, 1, 2, 3, 4, 5\} = \{5, 0, 2, 3, 1, 4\}$

2. The formulaic/set builder method: give a formula that generates all elements of the set.
   $A = \{x \in \mathbb{N} \mid 0 \leq x \wedge x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$

Using $\mathbb{N}$ and the set-builder method, we can define:
$\mathbb{Z} = \{m - n \mid \forall m, n \in \mathbb{N}\}$
$n = 0$ and m any natural number $\Rightarrow$ we generate all of $\mathbb{N}$
$m = 0$ and n any natural number $\Rightarrow$ we generate all negative integers
$0 - 1 = -1$
$0 - 2 = -2$
etc.
$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$

Definition: A set A is called finite if it has a finite number of elements; otherwise, it is called infinite.

3.2 Set Operations

Task: Understand how to represent sets by Venn diagrams. Understand set union, intersection, complement, and difference.

Definition: Let A, B be sets. A is a <u>subset</u> of B if all elements of A are elements of B, **i.e.** $\forall x(x \in A \rightarrow x \in B)$. We denote that A is a subset of B by $A \subseteq B$

Example: $\mathbb{N} \subseteq \mathbb{Z}$

Definition: Let A, B be sets. A is a <u>proper</u> subset of B if $A \subseteq B \wedge A \neq B$, **i.e.** $A \subseteq B \wedge \exists x \in B \ s.t. \ x \notin A$.

Notation: $A \subset B$

Example: $\mathbb{N} \subset \mathbb{Z}$ since $\exists(-1) \in \mathbb{Z}$ such that $-1 \notin \mathbb{N}$.

NB: $\forall A$ a set, $\emptyset \subseteq A$

Recall: $B \subseteq C$ means $\forall x(x \in B \rightarrow x \in C)$, but $\emptyset$ has no elements, so in $\emptyset \subseteq A$ the quantifier $\forall$ operates on a domain with no elements. Clearly, we need to give meaning to $\exists$ and $\forall$ on empty sets.

<u>Boolean Convention</u>

$$\left.\begin{array}{l}\forall \text{ is true on the empty set}\\ \exists \text{ is false on the empty set}\end{array}\right\} \text{Consistent with common sense}$$

Definition: Let A, B be two sets. The <u>union</u> $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$

Definition: Let A, B be two sets. The <u>intersection</u> $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be sets. A and B are called <u>disjoint</u> if $A \cap B = \emptyset$

Definition Let A, B be two sets. $A - B = A \backslash B = \{x \mid x \in A \ \wedge \ x \notin B\}$

Examples:

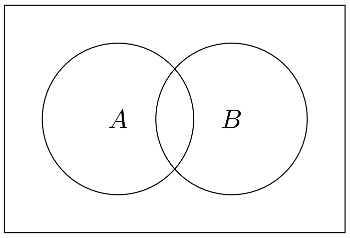
$$\begin{array}{ll} A = \{1,2,5\} & B = \{1,3,6\} \\ A \cup B = \{1,2,3,5,6\} & A \cap B = \{1\} \\ A \backslash B = \{2,5\} & B \backslash A = \{3,6\} \end{array}$$

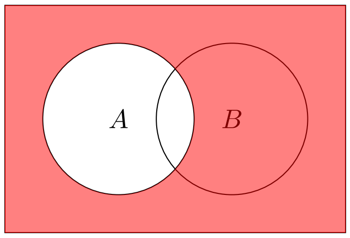
Definition: Let A, U be sets s.t. $A \subseteq U$. The <u>complement</u> of A in U $= U \backslash A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Remark: The notation A^C is unambiguous only if the universe U is clearly defined or understood.

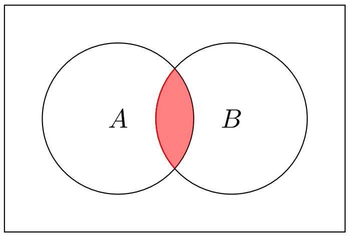
3.2.1 Venn Diagrams

Schematic representation of set operations.

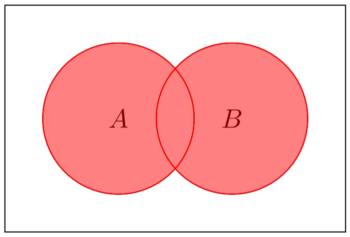




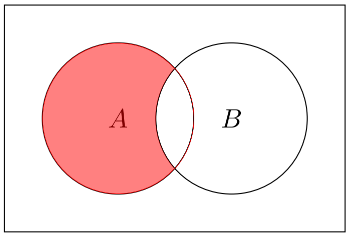
A^C



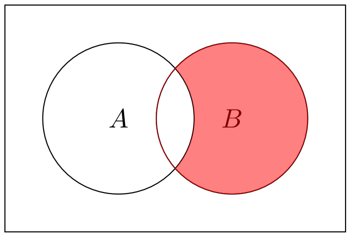
$A \cap B$



$A \cup B$



$A \backslash B$



$B \backslash A$

Pros of Venn diagrams:

Very easy to visualize

Cons of Venn diagrams:

1. Misleading if for example $A \subset B$ or sets are in some other non standard configuration;
2. Not helpful if a lot of sets are involved;
3. Not helpful if sets are infinite or have some peculiar structure.

Moral of the story: Venn diagrams will **NOT** be accepted as proof of any statement in set theory. Instead, we will introduce rigorous ways of proving assertions in set theory.

3.2.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation/property
$\wedge$	intersection $\cap$
$\vee$	union $\cup$
$\neg$	complement $(\ )^C$
$\rightarrow$	subset $\subseteq$
$\leftrightarrow$	equality of sets $=$

Recall:

Definition: Let A, B be two sets. The <u>intersection</u> $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be two sets. The <u>union</u> $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$

Definition: Let A, U be sets s.t. $A \subseteq U$. The <u>complement</u> of A in $U = U \backslash A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Definition: Let A, B be sets. A is a <u>subset</u> of B if all elements of A are elements of B, **i.e.** $\forall x(x \in A \rightarrow x \in B)$.

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \rightarrow x \in B)] \wedge [\forall y(y \in B \rightarrow y \in A)]$

As a result, various properties of set operations become obvious:

- Commutativity
  - $A \cap B = B \cap A$ comes from the tautology $(P \wedge Q) \leftrightarrow (Q \wedge P)$ (#31 on the list of tautologies posted in Course Documents)
  - $A \cup B = B \cup A$ comes from the tautology $(P \vee Q) \leftrightarrow (Q \vee P)$ (# 32 on the list of tautologies)
- Associativity
  - $(A \cup B) \cup C = A \cup (B \cup C)$ comes from the tautology $[(P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$ (# 33 on the list of tautologies)
  - $(A \cap B) \cap C = A \cap (B \cap C)$ comes from the tautology $[(P \wedge (Q \wedge R)] \leftrightarrow [(P \wedge Q) \wedge R]$ (# 34 on the list of tautologies)
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ comes from the tautology $[(P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$ (# 29 on the list of tautologies)

  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ comes from the tautology $[(P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$ (# 30 on the list of tautologies)

- De Morgan Laws in Set Theory
  - $(A \cap B)^C = A^C \cup B^C$ comes from the tautology $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ (# 18 on the list of tautologies)
  - $(A \cup B)^C = A^C \cap B^C$ comes from the tautology $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ (# 19 on the list of tautologies)

- Involutivity of the Complement
  - $(A^C)^C = A$ comes from the tautology $\neg(\neg P) \leftrightarrow P$ (# 3 on the list of tautologies)

NB: An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
  - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$ comes from the tautology
  $$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$$
  (# 14 on the list of tautologies)

- Criterion for proving equality of sets, which comes from the tautology $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$ (#22 on the list of tautologies)
  - $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

- Criterion for proving non-equality of sets
  - $A \neq B \leftrightarrow (A \backslash B) \cup (B \backslash A) \neq \emptyset$

3.3 Example Proof in Set Theory

Proposition: $\forall A, B$ sets. $(A \cap B) \cup (A \backslash B) = A$

Proof: Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

Show $(A \cap B) \cup (A \backslash B) \subseteq A$: $\forall x \in (A \cap B) \cup (A \backslash B), x \in (A \cap B)$ or $x \in A \backslash B$. If $x \in (A \cap B)$, then clearly $x \in A$ as $A \cap B \subseteq A$ by definition. If $x \in A \backslash B$, then by definition $x \in A$ and $x \notin B$, so definitely $x \in A$. In both cases, $x \in A$ as needed.

Show $A \subseteq (A \cap B) \cup (A \backslash B)$: $\forall x \in A$, we have two possibilities, namely $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B$. If $x \notin B$, then $x \in A$ and $x \notin B$, so $x \in A \backslash B$. In both cases, $x \in (A \cap B)$ or $x \in (A \backslash B)$, so $x \in (A \cap B) \cup (A \backslash B)$ as needed.

qed

3.4 The Power Set

Task: Understand what the power set of a set A is.

Definition: Let A be a set. The power set of A denoted $P(A)$ is the collection of all subsets of A.

Recall: $\emptyset \subseteq A$. It is also clear from the definition of a subset that $A \subseteq A$.

Examples:

1. $A = \{0, 1\}$
   $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
2. $A = \{a, b, c\}$
   $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
3. $A = \emptyset$
   $P(A) = \{\emptyset\}$
   $P(P(A)) = \{\emptyset, \{\emptyset\}\}$

NB: $\emptyset$ and $\{\emptyset\}$ are different objects. $\emptyset$ has no elements, whereas $\{\emptyset\}$ has one element.

Remark: $P(A)$ and A are viewed as living in separate worlds to avoid phenomena like Russell's paradox.

Q: If A has n elements, how many elements does $P(A)$ have?

A: 2^n

Theorem: Let A be a set with n elements, then $P(A)$ contains 2^n elements.

Proof: Based on the on/off switch idea.

$\forall x \in A$, we have two choices: either we include x in the subset or we don't (on vs off switch). A has n elements $\Rightarrow$ we have 2^n subsets of A.

qed

Alternate Proof: Using mathematical induction.

NB: It is an axiom of set theory (in the ZFC standard system) that every set has a power set, which implies no set consisting of all possible sets could exist, else what would its power set be?

3.5 Cartesian Products

Task: Understand sets like $\mathbb{R}^1$ in a more theoretical way.

Remark: Cartesian products allow us to pair up sets that have no relationship to each other, which is essential in computer science as it allows us to define objects like finite state acceptors, Turing machines, etc.

Recall from Calculus:

$\mathbb{R} = \mathbb{R}^1 \ni x$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ni (x_1, x_2)$

$\vdots$

$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{n times}} = \mathbb{R}^n \ni (x_1, x_2, ..., x_n)$

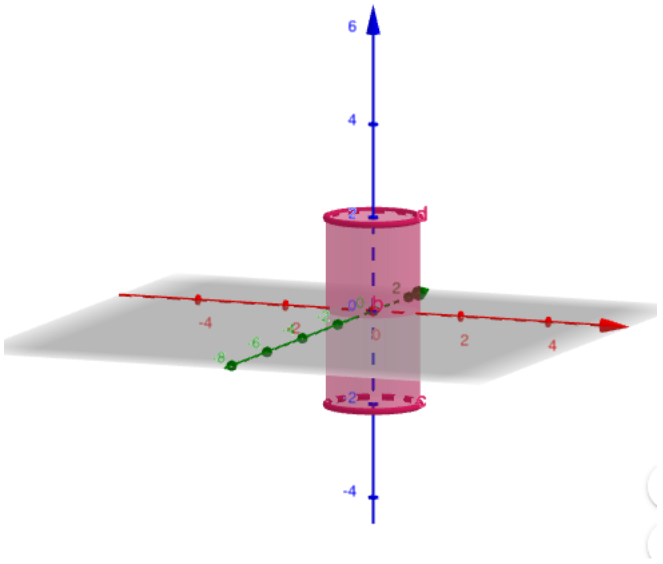
These are examples of Cartesian products.

Definition: Let A, B be sets. The Cartesian product denoted by $A \times B$ consists of all ordered pairs (x, y) $s.t.$ $x \in A \wedge y \in B$, **i.e.**

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

Further Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 5\}$
   $A \times B = \{(1, 1), (1, 5), (3, 1), (3, 5), (7, 1), (7, 5)\}$
   NB: The order in which elements in a pair matters: $(7, 1)$ is different from $(1, 7)$. This is why we call (x, y) an <u>ordered</u> pair.
2. $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \leftarrow$ circle of radius 1
   $B = \{z \in \mathbb{R} \mid -2 \leq z \leq 2\} = [-2, 2] \leftarrow$ closed interval
   $A \times B \leftarrow$ cylinder of radius 1 and height 4



3.5.1 Cardinality (number of elements) in a Cartesian product

If A has m elements and B has p elements, $A \times B$ has mp elements.

Examples:

1. $\#(A) = 3 \qquad A = \{1,3,7\}$
   $\#(B) = 2 \qquad B = \{1,5\}$
   $\#(A \times B) = 3 \times 2 = 6$
2. Both A and B are infinite sets, so $A \times B$ is infinite as well.

Remark: We can define Cartesian products of any length, **e.g.** $A \times A \times B \times A$, $B \times A \times B \times A \times B$, etc. If all sets are finite, the number of elements is the product of the numbers of elements of each factor. If $\#(A) = 3$ and $\#(B) = 2$ as above, $\#(A \times B \times A) = 3 \times 2 \times 3 = 18$ and $\#(B \times A \times B) = 2 \times 3 \times 2 = 12$.

4 Relations

Task: Define subsets of Cartesian products with certain properties. Understand the predicates " = " (equality) and other predicates in predicate logic in a more abstract light.

Start with $x = y$. The element x is some relation R to y (equality in this case). We can also denote it as xRy or $(x, y) \in E$

Let x, y in $\mathbb{R}$, then $E = \{(x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

The "diagonal" in $\mathbb{R} \times \mathbb{R}$ gives exactly the elements equal to each other.

More generally:

Definition: Let A, B be sets. A subset of the Cartesian product $A \times B$ is called a relation between A and B. A subset of the Cartesian product $A \times A$ is called a relation on A.

Remark: Note how general this definition is. To make it useful for understanding predicates, we will need to introduce key properties relations can satisfy.

Example: $A = \{1,3,7\} \qquad B = \{1,2,5\}$

We can define a relation S on $A \times B$ by $S = \{(1,1),(1,5),(3,2)\}$. This means $1S1$, $1S5$ and $3S2$ and no other ordered pairs in $A \times B$ satisfy S.

Remark: The relations we defined involve 2 elements, so they are often called binary relations in the literature.

4.1 Equivalence Relations

Task: Define the most useful kind of relation.

Definition: A relation R on a set A is called

1. reflexive iff (if and only if) $\forall x \in A, xRx$
2. symmetric iff $\forall x, y \in A, xRy \to yRx$
3. transitive iff $\forall x, y, z \in A, xRy \wedge yRz \to xRz$

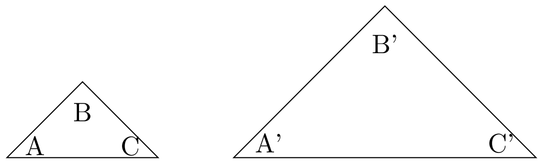
An equivalence relation on A is a relation that is reflexive, symmetric, and transitive.

Notation: Instead of xRy, an equivalence relation is often denoted by $x \equiv y$ or $x \sim y$.

Examples:

1. "$=$" equality is an equivalence relation.
   (a) $x = x$ reflexive
   (b) $x = y \Rightarrow y = x$ symmetric
   (c) $x = y \wedge y = z \Rightarrow x = z$ transitive
2. $A = \mathbb{N}$
   $x \equiv y \bmod 3$ is an equivalence relation. $x \equiv y \bmod 3$ means $x - y = 3m$ for some $m \in \mathbb{Z}$, **i.e.** x and y have the same remainder when divided by 3. The set of all possible remainders is $\{0, 1, 2\}$
   NB: In correct logic notation, $x \equiv y \bmod 3$ if $\exists m \in \mathbb{Z}\ s.t.\ x - y = 3m$
   (a) $x \equiv x \bmod 3$ since $x - x = 0 = 3 \times 0 \to$ reflexive
   (b) $x \equiv y \bmod 3 \Rightarrow y \equiv x \bmod 3$ because $x \equiv y \bmod 3$ means $x - y = 3m$ for some $m \in \mathbb{Z} \Rightarrow y - x = -3m = 3 \times (-m) \Rightarrow y \equiv x \bmod 3 \to$ symmetric
   (c) Assume $x \equiv y \bmod 3$ and $y \equiv z \bmod 3$
   $x \equiv y \bmod 3 \Rightarrow \exists m \in \mathbb{Z}$ s.t. $x - y = 3m \Rightarrow y = x - 3m$
   $y \equiv z \bmod 3 \Rightarrow \exists p \in \mathbb{Z}$ s.t. $y - z = 3p \Rightarrow y = z + 3p$
   Therefore, $x - 3m = z + 3p \Leftrightarrow x - z = 3p + 3m = 3(p + m)$
   Since $p, m \in \mathbb{Z}, p + m \in \mathbb{Z} \Rightarrow x \equiv z \bmod 3 \to$ transitive.
3. Let $f : A \to A$ be any function on a non-empty set A. We define the relation $R = \{(x, y) \mid f(x) = f(y)\}$
   (a) $\forall x \in A, f(x) = f(x) \Rightarrow (x, x) \in R \to$ reflexive
   (b) If $(x, y) \in R$, then $f(x) = f(y) \Rightarrow f(y) = f(x)$, **i.e.** $(y, x) \in R \to$ symmetric
   (c) If $(x, y) \in R$ and $(y, z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$, which by the transitivity of equality implies $f(x) = f(z)$, **i.e.** $(x, z) \in R$ as needed, so R is transitive as well.
   $f(x)$ can be $e^x, \sin x, |x|$, etc.

4. Let Γ be the set of all triangles in the plane. $ABC \sim A'B'C'$ if ABC and $A'B'C'$ are similar triangles, **i.e.** have equal angles.



   (a) $\forall ABC \in \Gamma, ABC \sim ABC$ so $\sim$ is reflexive
   (b) $ABC \sim A'B'C' \Rightarrow A'B'C' \sim ABC$ so $\sim$ is symmetric
   (c) $ABC \sim A'B'C'$ and $A'B'C' \sim A"B"C" \Rightarrow ABC \sim A"B"C"$, so $\sim$ is transitive

   Clearly (a), (b), (c) use the fact that equality of angles is an equivalence relation.

Exercise: For various predicates you've encountered, check whether reflexive, symmetric or transitive. Examples of predicates include $\neq$, $<$, $>$, $\leq$, $\geq$, $\subseteq$, $\rightarrow$, $\leftrightarrow$

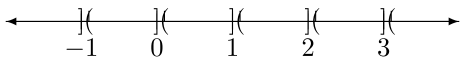
4.2 Equivalence Relations and Partitions

Task: Understand how equivalence relations divide sets.

Definition: Let A be a set. A partition of A is a collection of non-empty sets, any two of which are disjoint such that their union is A, **i.e.** $\lambda = \{A_\alpha \mid \alpha \in I\}$ s.t. $\forall \alpha, \alpha' \in I$ satisfying $\alpha \neq \alpha', A_\alpha \cap A_{\alpha'} = \emptyset$ and $\bigcup\limits_{\alpha \in I} A_\alpha = A$

Here I is an indexing act (may be infinite). $\bigcup\limits_{\alpha \in I} A_\alpha$ is the union of all the A_α's (possibly an infinite union)

Example $\{(n, n+1] \mid n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$



$\bigcup\limits_{n \in \mathbb{Z}} (n, n+1] = \mathbb{R}$

$(n, n+1] \cap (m, m+1] = \emptyset$ if $n \neq m$

Definition: If R is an equivalence relation on a set A and $x \in A$, the equivalence class of x denoted $[x]_R$ is the set $\{y \mid xRy\}$. The collection of all equivalence classes is called A modulo R and denoted A/R.

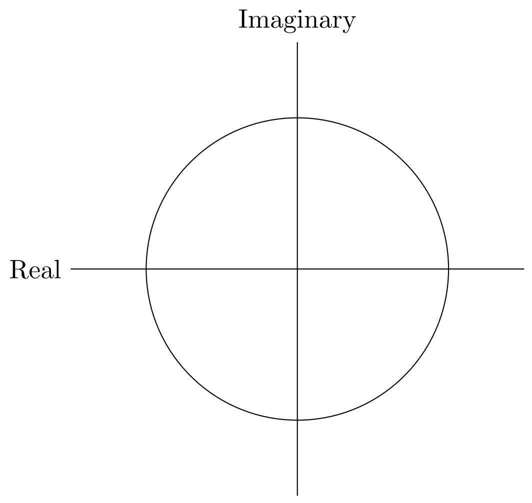
Examples:

1. $A = \mathbb{N}$ $\qquad x \equiv y \bmod 3$
   We have the equivalence classes $[0]_R, [1]_R$ and $[2]_R$ given by the three possible remainders under division by 3.
   $[0]_R = \{0, 3, 6, 9, ...\}$
   $[1]_R = \{1, 4, 7, 10, ...\}$
   $[2]_R = \{2, 5, 8, 11, ...\}$
   Clearly $[0]_R \cup [1]_R \cup [2]_R = \mathbb{N}$ and they are mutually disjoint $\Rightarrow R$ gives a partition of $\mathbb{N}$.

2. $ABC \sim A'B'C'$
   $[ABC] = \{$The set of all triangles with angles of magnitude $\measuredangle ABC, \measuredangle BAC, \measuredangle ACB\}$
   The union over the set of all $[ABC]$ is the set of all triangles and $[ABC] \cap [A'B'C'] = \emptyset$ if $ABC \nsim A'B'C'$ since it means these triangles have at least one angle that is different.

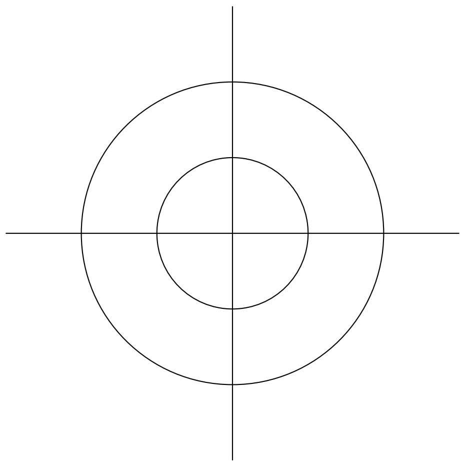
3. $A = \mathbb{C}$ $\qquad x \sim y$ if $|x| = |y|$ $\qquad$ equivalence relation
   $[x] = \{y \in \mathbb{C} \mid |x| = |y|\} = [r]$ for $r \in [0, +\infty)$ (meaning $r \geq 0$)

   circle of radius $|x|$



   $\bigcup\limits_{r \in [0,+\infty)} [r] = \mathbb{C}$
   $[r_1] \cap [r_2] \neq \emptyset$ if $r_1 \neq r_2$ since two distinct circles in $\mathbb{C} \simeq \mathbb{R}^2$ with empty intersection.

circles r_1 and r_2



Theorem: For any equivalence relation R on a set A, its equivalence classes form a partition of A, **i.e.**

1. $\forall x \in A, \exists y \in A$ s.t. $x \in [y]$ (every element of A sits somewhere)
2. $xRy \Leftrightarrow [x] = [y]$ (all elements related by R belong to the same equivalence class)
3. $\neg(xRy) \Leftrightarrow [x] \cap [y] = \emptyset$ (if two elements are not related by R, the they belong to disjoint equivalence classes)

Proof:

1. Trivial. Let $y = x$. $x \in [x]$ because R is an equivalence relation, hence reflexive, so xRx holds.
2. We will prove $xRy \Leftrightarrow [x] \subseteq [y]$ and $[y] \subseteq [x]$
   "$\Rightarrow$" Fix $x \in A, [x] = \{z \in A \mid xRz\} \Rightarrow \forall y \in A$ s.t. $xRy, y \in [x]$. Furthermore, $[y] = \{w \in A \mid yRw\}$
   $\Rightarrow \forall w \in [y], yRw$ but $xRy \Rightarrow xRw$ by transitivity. Therefore, $w \in [x]$. We have shown $[y] \subseteq [x]$.
   Since R is an equivalence relation, it is also symmetric. **i.e.** $xRy \Leftrightarrow yRx$. So by the same argument with x and y swapped $yRx \Rightarrow [x] \subseteq [y]$. Thus $xRy \Rightarrow [x] = [y]$.
   "$\Leftarrow$" $[x] = [y] \Rightarrow y \in [x]$ but $[x] = \{y \in A \mid xRy\}$
3. "$\Rightarrow$" We will prove the contrapositive. Assume $[x] \cap [y] \neq \emptyset \Rightarrow \exists z \in [x] \cap [y]$. $z \in [x]$ means xRz, whereas $z \in [y]$ means $yRz \Leftrightarrow zRy$ because R is symmetric. We thus have xRz and $zRy \Rightarrow xRy$ by the transitivity of R. xRy contradicts $\neg(xRy)$ so indeed $\neg(xRy) \Rightarrow [x] \cap [y] = \emptyset$
   "$\Leftarrow$" Once again we use the contrapositive:

Assume $\neg(\neg(xRy)) \Leftrightarrow xRy$. By part (2), $xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$ since $x \in [x]$ and $y \in [y]$, **i.e.** these equivalence classes are non-empty. We have obtained the needed contradiction.

qed

Q: What partition does "$=$" impose on $\mathbb{R}$?

A: $[x] = \{x\}$ since $E = \{(x, x) \mid x \in \mathbb{R}\}$ the diagonal.
The one-element equivalence class is the smallest equivalence class possible (by definition, an equivalence class cannot be empty as it contains x itself). We call such a partition the finest possible partition.

Remark: The theorem above shows how every equivalence relation partitions a set. It turns out every partition of a set can be used to define an equivalence relation: xRy if x and y belong to the same subset of the partition (check this is indeed an equivalence relation!). Therefore, there is a 1-1 correspondence between partitions and equivalence relations: to each equivalence relation there corresponds a partition and vice versa.

4.3 Partial Orders

Task: Understand another type of relation with special properties.

Definition: Let A be a set. A relation R on A is called anti-symmetric if $\forall x, y \in A$ s.t. $xRy \wedge yRx$, then $x = y$.

Definition: A partial order is a relation on a set A that is reflexive, anti-symmetric, and transitive.

Examples:

1. $A = \mathbb{R}$ $\quad\quad$ $\leq$ "less than or equal to" is a partial order
   (a) $\forall x \in \mathbb{R}$, $x \leq x \rightarrow$ reflexive
   (b) $\forall x, y \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq x \implies x = y \rightarrow$ anti-symmetric
   (c) $\forall x, y, z \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq z \implies x \leq z \rightarrow$ transitive
   Same conclusion if $A = \mathbb{Z}$ or $A = \mathbb{N}$
2. A is a set. Consider $P(A)$, the power set of A. The relation $\subseteq$ "being a subset of" is a partial order.
   (a) $\forall B \in P(A), B \subseteq B \rightarrow$ reflexive.
   (b) $\forall B, C \in P(A), B \subseteq C \wedge C \subseteq B \implies B = C$ (recall the criterion for proving equality of sets) $\rightarrow$ anti-symmetric
   (c) $\forall B, C, D \in P(A)$ s.t. $B \subseteq C \wedge C \subseteq D \implies B \subseteq D \rightarrow$ transitive

The most important example of a partial order is example (2) "being a subset of".

Q: Why is "being a subset of" a partial order as opposed to a total order?

A: There might exist subsets B, C of A s.t. neither $B \subseteq C$ nor $C \subseteq B$ holds, **i.e.** where B and C are not related via inclusion.

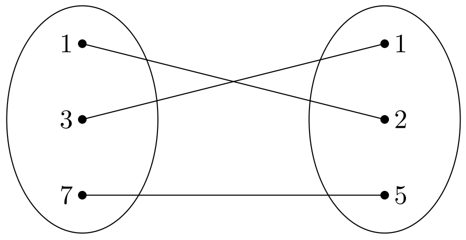
5 Functions

Task: Define a function rigorously and make sense of terminology associated to functions.

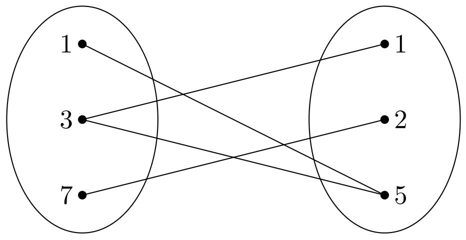
Definition: Let A, B be sets. A function $f : A \to B$ is a rule that assigns to every element of A one and only one element of B, **i.e.** $\forall x \in A\ \exists! y \in B$ s.t. $f(x) = y$. A is called the domain of f and B is called the codomain.

Examples:

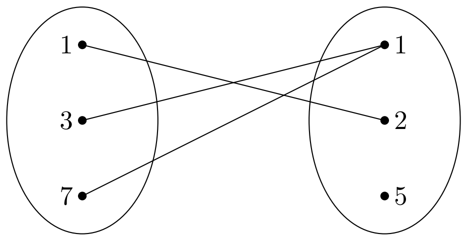
1. $A = \{1, 3, 7\}$
   $B = \{1, 2, 5\}$



Is a function.



Not a function; 3 sent to both 1 and 5



Is a function.

2. $A = B = \mathbb{R}$ $\quad F : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is called the identity function.

Definition: Let A, B be sets, and let $f : A \to B$ be a function. The range of f denoted by $f(A)$ is the subset of B defined by $f(A) = \{y \in B \mid \overline{\exists x \in A}$ s.t. $f(x) = y\}$.

Definition: Let A be a set. A Boolean function on A is a function $f : A \to \{T, F\}$, which has A as its domain and the set of truth values $\{T, F\}$ as is codomain. $f : A \to \{T, F\}$ thus assigns truth values to the elements of A.

Function are often represented by graphs. If $f : A \to B$ is a function, the graph of f denoted $\Gamma(f)$ is the subset of the Cartesian product of the domain with the codomain $A \times B$ given by $\{(x, f(x)) \mid x \in A\}$.

Q: Is it possible to obtain every subset of $A \times B$ as the graph of some function?

A: No! For $f : A \to B$ to be a function $\forall x \in A \quad \exists! y \in B$ s.t. $f(x) = y$, so for $\Gamma \subseteq A \times B$ to be the graph of some function, Γ must satisfy that $\forall x \in A \quad \exists! y \in B$ s.t. $(x, y) \in \Gamma$. Then we can define f by letting $y = f(x)$.

NB For the usual set-up of a function $f : \mathbb{R} \to \mathbb{R}$, this observation amounts to the "vertical line test," which you have seen before coming to university.

5.1 Composition of Functions

Task: Understand the natural operation that allows us to combine functions.

$$\begin{tikzcd} A \arrow[r, "f(x)"] \arrow[rr, bend right, "g \circ f : A \to C"'] & B \arrow[r, "g(x)"] & C \end{tikzcd}$$

Example:

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 2x$
$g : \mathbb{R} \to \mathbb{R} \qquad g(x) = \cos x$
$g \circ f(x) = g(f(x)) = g(2x) = \cos(2x)$
$f \circ g(x) = f(g(x)) = f(\cos x) = 2\,(\cos x) = 2\ \cos x$

5.2 Inverting Functions

Task: Figure out which properties a function has to satisfy so that its action can be undone, **i.e. when** we can define an inverse to the original function.

Given $f : A \to B$, want $f^{-1} : B \to A$ s.t. $f^{-1} \circ f : A \to A$ is the identity $f^{-1} \circ f(x) = f^{-1}(f(x)) = x$
$A \xrightarrow{f} B \xrightarrow{f^{-1}} A$
It turns out f has to satisfy two properties for f^{-1} to exist:

1. Injective
2. Surjective

Definition: A function $f : A \to B$ is called $\underline{\text{injective}}$ or an injection (sometimes called one-to-one) if $\underline{f(x) = f(y) \Rightarrow x = y}$

Examples:
$\sin x : [0, \frac{\pi}{2}] \to \mathbb{R}$ is injective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not injective because $\sin 0 = \sin \pi = 0$

Definition: A function $f : A \to B$ is called $\underline{\text{surjective}}$ or a surjection (sometimes called onto) if $\overline{\forall z \in B\ \exists x \in A \text{ s.t. } f(x) = z}$.

Remark: f assigns a value to each element of A by its definition as a function, but it is not required to cover all of B. f is surjective if its range is all of B.

Examples:
$\sin x : \mathbb{R} \to [-1, 1]$ is surjective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not surjective since $\nexists x \in \mathbb{R}$ s.t. $\sin x = 2$. We know $|\sin x| \leq 1\ \forall x \in \mathbb{R}$

Definition: A function $f : A \to B$ is called $\underline{\text{bijective}}$ or a bijection if f is $\underline{\text{both}}$ injective and surjective.

Example: $f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = 2x + 1$ is bijective.

- Check injectivity: $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2$ as needed.
- Check surjectivity: $\forall z \in \mathbb{R}\ f(x) = z$ means $2x + 1 = z$.
  Solve for x: $2x = z - 1 \Rightarrow x = \frac{z-1}{2} \in \mathbb{R} \Rightarrow f$ is surjective.

Remark: All bijective functions have inverses because we can define the inverse of a bijection and it will be a function:

- Surjectivity ensures f^{-1} assigns an element to every element of B (its domain).
- Injectivity ensures f^{-1} assigns to each element of B $\underline{\text{one and only one}}$ element of A.

Conclusion: $f : A \to B$ bijective $\Rightarrow f^{-1}$ exists, **i.e.** f^{-1} is a function. It turns out (reverse the arguments above) that f^{-1} exists $\Rightarrow f : A \to B$ is bijective.
Altogether we get the following theorem:

Theorem: Let $f : A \to B$ be a function. f^{-1} exists $\Leftrightarrow f : A \to B$ is bijective.

Q: How do we find the inverse function f^{-1} given $f : A \to B$?

A: If $f(x) = y$, solve for x as a function of y since $f^{-1}(f(x)) = f^{-1}(y) = x$ as $f^{-1} \circ f$ is the identity.

Example: $f(x) = 2x + 1 = y$. Solve for x in terms of y.
$f : \mathbb{R} \to \mathbb{R}$
$2x = y - 1 \qquad x = \frac{y-1}{2}$

5.3 Functions Defined on Finite Sets

Task: Derive conclusions about a function given the number of elements of the domain and codomain, if finite; understand the pigeonhole principle.

Proposition: Let A, B be sets and let $f : A \to B$ be a function. Assume A is finite. Then f is injective $\Leftrightarrow$ $f(A)$ has the same number of elements as A.

Proof:

A is finite so we can write it as $A = \{a_1, a_2, ..., a_p\}$ for some p. Then $f(A) = \{f(a_1), f(a_2), ..., f(a_p)\} \subseteq B$. A priori, some $f(a_i)$ might be the same as some $f(a_j)$. However, f injective $\Leftrightarrow f(a_i) \neq f(a_j)$ whenever $i \neq j \Leftrightarrow f(A)$ has exactly p elements just like A.

qed

Corollary 1 Let A, B be finite sets such that $\#(A) = \#(B)$. Let $f : A \to B$ be a function. f is injective $\Leftrightarrow$ f is bijective.

Proof:

"$\Rightarrow$" Suppose $f : A \to B$ is injective. Since A is finite, by the previous proposition, $f(A)$ has the same number of elements as A, but $f(A) \subseteq B$ and B has the same number of elements as $A \Rightarrow \#(A) = \#(f(A)) = \#(B)$, which means $f(A) = B$, **i.e.** f is also surjective $\Rightarrow f$ is bijective.

"$\Leftarrow$" f is bijective $\Rightarrow f$ is injective.

qed

Corollary 2 (The Pigeonhole Principle) Let A, B be finite sets, and let $f : A \to B$ be a function. If $\#(B) < \#(A)$, $\exists\, a, a' \in A$ with $a \neq a'$ such that $f(a) = f(a')$.

Remark: The name pigeonhole principle is due to Paul Erdös and Richard Rado. Before it was known as the principle of the drawers of Dirichlet. It has a simple statement, but it's a very powerful result in both mathematics and computer science.

Proof: Since $f(A) \subseteq B$ and $\#(B) < \#(A)$, $f(A)$ cannot have as many elements as A, so by the proposition, f cannot be injective, namely $\exists\, a, a' \in A$ with $a \neq a'$ (**i.e.** distinct elements) s.t. $f(a) = f(a')$.

qed

Examples:

1. You have 8 friends. At least two of them were born the same day of the week. #(days of the week) $= 7 < 8$.
2. A family of five gives each other presents for Christmas. There are 12 presents under the tree. We conclude at least one person got three presents or more.
3. In a list of 30 words in English, at least two will begin with the same letter. #(Letters in the English alphabet) $= 26 < 30$.

5.4 Behaviour of Functions on Infinite Sets

Let A be a set, and $f : A \to A$ be a function. If A is finite, then corollary 1 tells us f injective $\Leftrightarrow$ f bijective. What if A is not finite?

5.4.1 Hilbert's Hotel problem (jazzier name: Hilbert's paradox of the Grand Hotel)

A fully occupied hotel with infinitely many rooms can always accommodate an additional guest as follows: The person in Room 1 moves to Room 2. The person in Room 2 moves to Room 3 and so on, **i.e.** if the rooms are $x_1, x_2, x_3...$ define the function $f(x_1) = x_2, f(x_2) = x_3, ..., f(x_m) = x_{m+1}$.

Claim: As defined f is injective but not surjective (hence not bijective!). Let $H = \{x_1, x_2, ...\}$ be the hotel consisting of infinitely many rooms. $f : H \to H$ is given by $f(x_n) = x_{n+1}$. $f(H) = H\backslash\{x_1\}$. We can use this idea to prove:

Proposition: A set A is finite $\Leftrightarrow$ $\forall f : A \to A$ an injective function is also bijective.

Proof: "$\Rightarrow$" If the set A is finite, then it follows immediately from Corollary 1 that every injective function $f : A \to A$ is bijective.
"$\Leftarrow$" We prove the contrapositive. Suppose that the set A is infinite. We shall construct an injective function that is not bijective. Since A is infinite, there exists some infinite sequence $x_1, x_2, x_3, ...$ consisting of distinct elements of A, i.e. an element of A occurs at most once in this sequence. Then there exists a function $f : A \to A$ such that $f(x_n) = x_{n+1}$ for all integers $n \geq 1$ and $f(x) = x$ if x is an element of A that is not in the sequence $x_1, x_2, x_3, ...$ If x is not a member of the infinite sequence $x_1, x_2, x_3, ...,$ then the only element of A that gets mapped to x is the element x itself; if $x = x_n$, where $n > 1$, then the only element of A that gets mapped to x is x_{n-1}. It follows that the function f is injective. It is not surjective, however, since no element of A gets mapped to x_1. This function f is thus an example of a function from the set A to itself, which is injective but not bijective.

qed

6 Mathematical Induction

Task: Understand how to construct a proof using mathematical induction.

$\mathbb{N} = \{0, 1, 2, ...\}$ set of natural numbers.
Recall that $\mathbb{N}$ is constructed using 2 axioms:

1. $0 \in \mathbb{N}$
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

Remarks:

1. This is exactly the process of counting.
2. If we start at 1, then we construct $\mathbb{N}^* = \{1, 2, 3, 4, ...\} = \mathbb{N} \backslash \{0\}$

via the axioms

1. $1 \in \mathbb{N}^*$
2. if $n \in \mathbb{N}^*$, then $n + 1 \in \mathbb{N}^*$

$\mathbb{N}$ or $\mathbb{N}^*$ is used for mathematical induction.

6.1 Mathematical Induction Consists of Two Steps:

Step 1 Prove statement $P(1)$ called the base case.

Step 2 For any n, assume $P(n)$ and prove $P(n+1)$. This is called the inductive step. In other words, step 2 proves the statement $\forall n P(n) \to P(n + 1)$

Remark: Step 2 is not just an implication but infinitely many! In logic notation, we have:

Step 1 $P(1)$

Step 2 $\forall n(P(n) \to P(n + 1))$
Therefore, $\forall n P(n)$
Let's see how the argument proceeds:

1. $P(1)$ Step 1 (base case)
2. $P(1) \to P(2)$ by Step 2 with $n = 1$
3. $P(2)$ by Modus Ponens (tautology #10) applied to 1 & 2
4. $P(2) \to P(3)$ by Step 2 with $n = 2$
5. $P(3)$ by Modus Ponens (tautology #10) applied to 3 & 4
6. $P(3) \to P(4)$ by Step 2 with $n = 3$

7. $P(4)$ by Modus Ponens (tautology #10) applied to 5 & 6

   $\vdots$

8. $P(n)$ for any n.
   This is like a row of dominos: knocking over the first one in a row makes all the others fall. Another idea is climbing a ladder.

Examples:

1. Prove $1+3+5+...+(2n-1)=n^2$ by induction.

   Base Case: Verify statement for $n=1$
   When $n=1$, $2n-1=2\times 1-1=1^2$

   Inductive Step: Assume $P(n)$, **i.e.** $1+3+5+...+(2n-1)=n^2$ and seek to prove $P(n+1)$, **i.e.** the statement $1+3+5...+(2n-1)+[2(n+1)-1]=(n+1)^2$

   We start with LHS: $\underbrace{1+3+5+...+(2n-1)}_{n^2}+[2(n+1)-1]=$

   $n^2+2n+2-1=n^2+2n+1=(n+1)^2$

2. Prove $1+2+3+...+n=\frac{n(n+1)}{2}$ by induction.

   Base Case: Verify statement for $n=1$
   When $n=1, 1=\frac{1\times(1+1)}{2}=\frac{1\times 2}{2}=1$

   Inductive Step: Assume $P(n)$, **i.e.** $1+2+3+...+n=\frac{n\times(n+1)}{2}$ and seek to prove $1+2+3+...+n+(n+1)=\frac{(n+1)(n+2)}{2}$

   $\underbrace{1+2+3+...+n}_{\frac{n(n+1)}{2}}+n+1=\frac{n(n+1)}{2}+n+1=(n+1)(\frac{n}{2}+1)=$

   $(n+1)\frac{n+2}{2}=\frac{(n+1)(n+2)}{2}$ as needed.

Remarks:

1. For some arguments by induction, it might be necessary to assume not just $P(n)$ at the inductive step but also $P(1), P(2), ..., P(n-1)$. This is called strong induction.

   Base Case: Prove $P(1)$

   Inductive Step: Assume $P(1), P(2), ..., P(n)$ and prove $P(n+1)$.

   An example of result requiring the use of strong induction is the Fundamental Theorem of Arithmetic: $\forall n\in\mathbb{N}, n\geq 2, n$ can be expressed as a product of one or more prime numbers.

2. One has to be careful with arguments involving induction. Here is an illustration why:

   Polya's argument that all horses are the same colour:

   Base Case: $P(1)$ There is only one horse, so that has a colour.

   Inductive Step Assume any n horses are the same colour.

Consider a group of $n+1$ horses. Exclude the first horse and look at the other n. All of these are the same colour by our assumption. Now exclude the last horse. The remaining n horses are the same colour by our assumption. Therefore, the first horse, the horses in the middle, and the last horse are all of the same colour. We have established the inductive step.

Q: Where does the argument fail?

A: For $n = 2, P(2)$ is false because there are no middle horses to compare to.

3. <u>The Grand Hotel Cigar Mystery</u>

   Recall Hilbert's hotel - the grand Hotel. Suppose that the Grand Hotel does not allow smoking and no cigars may be taken into the hotel. In spite of the rules, the guest in Room 1 goes to Room 2 to get a cigar. The guest in Room 2 goes to Room 3 to get 2 cigars (one for him and one for the person in room 1), etc. In other words, guest in Room N goes to Room N+1 to get N cigars. They will each get back to their rooms, smoke one cigar, and give the rest to the person in Room N-1.

   Q: Where is the fallacy?

   A: This is an induction argument without a base case. No cigars are allowed in the hotel, so no guests have cigars. An induction cannot get off the ground without a base case.

7 Abstract Algebra

Task: Understand binary operations, semigroups, monoids, and groups as well as their properties.

7.1 Binary Operations

Definition: Let A be a set. A <u>binary operation</u> $*$ on A is an operation applied to any two elements $x, y \in A$ that yields an element $x * y$ in A. In other words, $*$ is a binary operation on A if $\forall x, y \in A, x * y \in A$.

Examples:

1. $\mathbb{R}, +$ addition on $\mathbb{R} : \forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$
2. $\mathbb{R}, -$ subtraction on $\mathbb{R} : \forall x, y \in \mathbb{R}, x - y \in \mathbb{R}$
3. $\mathbb{R}, \times$ multiplication on $\mathbb{R} : \forall x, y \in \mathbb{R}, x \times y \in \mathbb{R}$
4. $\mathbb{R}, /$, division on $\mathbb{R}$ is <u>NOT</u> a binary operation because $\forall x \in \mathbb{R}\ \exists\, 0 \in \mathbb{R}$ s.t. $\frac{x}{0}$ is undefined (not an element of $\mathbb{R}$)
5. Let A be the set of all lists or strings. Concatenation is a binary operation.

Definition: A binary operation $*$ on a set A is called <u>commutative</u> if $\forall x, y \in A, x * y = y * x$

Examples:

1. $\mathbb{R}, +$ is commutative since $\forall x, y \in \mathbb{R}, x + y = y + x$
2. $\mathbb{R}, \times$ is commutative since $\forall x, y \in \mathbb{R}, x \times y = y \times x$
3. $\mathbb{R}, -$ is not commutative since $\forall x, y \in \mathbb{R}, x - y \neq y - x$ in general. $x - y = y - x$ only if $x = y$
4. Let M_n be the set of n by n matrices with entries in $\mathbb{R}$, and let $*$ be matrix multiplication. $\forall A, B \in M_n, A * B \in M_n$, so $*$ is a binary operation, but $A * B \neq B * A$ in general. Therefore $*$ is not commutative.

Definition: A binary operation $*$ on a set A is called <u>associative</u> if $\forall x, y, z \in A$ $(x * y) * z = x * (y * z)$

Examples:

1. $\mathbb{R}, +$ is associative since $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$
2. $\mathbb{R}, \times$ is associative since $\forall x, y, z \in \mathbb{R}, (x \times y) \times z = x \times (y \times z)$
3. Intersection $\cap$ on sets is associative since $\forall A, B, C$ sets $(A \cap B) \cap C = A \cap (B \cap C)$.
4. Union $\cup$ on sets is associative since $\forall A, B, C$ sets $(A \cup B) \cup C = A \cup (B \cup C)$
5. $\mathbb{R}, -$ is not associative since $(1-3)-5 = -2-5 = -7$ but $1-(3-5) = 1-(-2) = 1+2 = 3$

Remark: When we are dealing with associative binary operations we can drop the parentheses, **i.e.** $(x * y) * z$ can be written $x * y * z$.

7.2 Semigroups

Definition: A <u>semigroup</u> is a set endowed with an associative binary operation. We denote the semigroup $(A, *)$

Examples:

1. $(\mathbb{R}, +)$ and $(\mathbb{R}, \times)$ are semigroups.
2. Let A be a set and let $P(A)$ be its power set. $(P(A), \cap)$ and $(P(A), \cup)$ are both semigroups.
3. $(M_n, *)$, the set of $n \times n$ matrices with entries in $\mathbb{R}$ with the operation of matrix multiplication (which is associative $\rightarrow$ a bit gory to prove) forms a semigroup.

Since $*$ is associative on a semigroup, we can define a^n :
$a^1 = a$
$a^2 = a * a$
$a^3 = a * a * a$
$\vdots$
Recursively, $a^1 = a$ and $a^n = a * a^{n-1}, \forall n > 1$

NB: In $(\mathbb{R}, \times), \forall a \in \mathbb{R}, a^n = \underbrace{a \times a \times ... \times a}_{n\ times}$, whereas in $(\mathbb{R}, +), \forall a \in \mathbb{R}, a^n = \underbrace{a + a + ... + a}_{n\ times} = na$. Be careful what $*$ stands for!

Theorem: Let $(A, *)$ be a semigroup. $\forall a \in A, a^m * a^n = a^{m+n}, \forall m, n \in \mathbb{N}^*$.

Proof: By induction on m.

Base Case: $m = 1$ $\quad a^1 * a^n = a * a^n = a^{1+n}$

Inductive Step: Assume the result is true for $m = p$, **i.e.** $a^p * a^n = a^{p+n}$ and seek to prove that $a^{p+1} * a^n = a^{p+1+n}$

$a^{p+1} * a^n = (a * a^p) * a^n = a * (a^p * a^n) = a * a^{p+n} = a^{p+1+n}$

Theorem: Let$(A, *)$ be a semigroup. $\forall a \in A, (a^m)^n = a^{mn}, \forall m, n \in \mathbb{N}^*$

Proof: By induction on n.

Base Case: $n = 1$ $\quad (a^m)^1 = a^m = a^{m \times 1}$

Inductive Step: Assume the result is true for $n = p$, **i.e.** $(a^m)^p = a^{mp}$ and seek to prove that $(a^m)^{p+1} = a^{m(p+1)}$

$(a^m)^{p+1} = (a^m)^p * a^m = a^{mp} * a^m = a^{mp+m} = a^{m(p+1)}$ by the previous theorem.

7.2.1 General Associative Law

Let $(A, *)$ be a semigroup. $\forall a_1, ..., a_s \in A, a_1 * a_2 * ... * a_s$ has the same value regardless of how the product is bracketed.

Proof Use associativity of $*$.

qed

NB: Unless $(A, *)$ has a commutative binary operation, $a_1 * a_2 * ... * a_s$ does depend on the <u>ORDER</u> in which the $a_j's$ appear in $a_1 * a_2 * ... * a_s$

7.3 Identity Elements

Definition: Let $(A, *)$ be a semigroup. An element $e \in A$ is called an identity element for the binary operation $*$ if $e * x = x * e = x, \forall x \in A$.

Examples:

1. $(\mathbb{R}, +)$ has 0 as the identity element.
2. $(\mathbb{R}, \times)$ has 1 as the identity element.
3. Given a set A, $(P(A), \cup)$ has $\emptyset$ (the empty set) as its identity element, whereas $(P(A), \cap)$ has A as its identity element.
4. $(M_n, *)$ has I_n, the identity matrix, as its identity element.

Theorem A binary operation on a set cannot have more than one identity element, **i.e.** if an identity element exists, then it is unique.

Proof: Assume not (proof by contradiction). Let e and e' both be identity elements for a binary operation on a set A. $e = e * e' = e'$

qed

7.4 Monoids

Definition: A <u>monoid</u> is a set A endowed with an associative binary operation $*$ that has an identity element e. In other words, a monoid is a semigroup $(A, *)$, where $*$ has an identity element e.

Definition: A monoid $(A, *)$ is called <u>commutative</u> (or <u>Abelian</u>) if the binary operation $*$ is commutative.

Examples:

1. $(\mathbb{R}, +)$ is a commutative monoid with $e = 0$.
2. $(\mathbb{R}, \times)$ is a commutative monoid with $e = 1$.
3. Given a set A, $(P(A), \cup)$ is a commutative monoid with $e = \emptyset$.
4. $(M_n, *)$ is a monoid since $e = I_n$, but it is not commutative since matrix multiplication is not commutative.
5. $(\mathbb{N}, +)$ is a commutative monoid with $e = 0$, whereas $(\mathbb{N}^*, +)$ is merely a semigroup (recall $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$)

Let $(A, *)$ be a monoid and let $a \in A$. We define $a^0 = e$, the identity element.

Theorem: Let $(A, *)$ be a monoid and let $a \in A$. Then $a^m * a^n = a^{m+n}, \forall m, n \in \mathbb{N}$.

Remark: Recall that we proved this theorem for semigroups if $m, n \in \mathbb{N}^*$. We now need to extend that result.

Proof: A monoid is a semigroup $\implies \forall a \in A, a^m * a^n = a^{m+n}$ whenever $m, n \in \mathbb{N}^*$, **i.e.** $m > 0$ and $n > 0$. Now let $m = 0$. $a^m * a^n = a^0 * a^n = e * a^n = a^n = a^{0+n}$
If $n = 0, a^m * a^n = a^m * a^0 = a^m * e = a^m = a^{m+0}$.

qed

Theorem: Let $(A, *)$ be a monoid, $\forall a \in A\ \forall m, n \in \mathbb{N}$, $(a^m)^n = a^{mn}$.

Remark: Once again, we had this result for semigroups when $m > 0$ and $n > 0$.

Proof: By the remark, we only need to prove the result when $m = 0$ or $n = 0$. If $m = 0, (a^0)^n = (e)^n = e = a^0 = a^{0\times n}$. If $n = 0$, then $(a^m)^0 = e = a^0 = a^{0\times m}$.

qed

7.5 Inverses

Task: Understand what an inverse is and what formal properties it satisfies.

Definition: Let $(A, *)$ be a monoid with identity element e and let $x \in A$. An element y of A is called the <u>inverse</u> of x if $x * y = y * x = e$. If an element $x \in A$ has an inverse, then x is called <u>invertible</u>.

Examples:

1. $(\mathbb{R}, +)$ has identity element 0. $\forall x \in \mathbb{R}, (-x)$ is the inverse of x since $x + (-x) = (-x) + x = 0$.
2. $(\mathbb{R}, \times)$ has identity element 1. $x \in \mathbb{R}$ is invertible only if $x \neq 0$. If $x \neq 0$, the inverse of x is $\frac{1}{x}$ since $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$.
3. $(M_n, *)$ the identity element is I_n. $A \in M_n$ is invertible if $\det(A) \neq 0$. A^{-1} the inverse is exactly the one you computed in linear algebra. If $\det(A) = 0, A$ is <u>NOT</u> invertible.
4. Given a set $A, (P(A), \cup)$ has $\emptyset$ as its identity element. Of all the elements of $P(A)$, only $\emptyset$ is invertible and has itself as its inverse: $\emptyset \cup \emptyset = \emptyset \cup \emptyset = \emptyset$.

Theorem: Let $(A, *)$ be a monoid. If $a \in A$ has an inverse, then that inverse is unique.

Proof: By contradiction: Assume not, then $\exists a \in A$ s.t. both b and c in A are its inverses, **i.e.** $a * b = b * a = e$, the identity element of $(A, *)$, and $a * c = c * a = e$, where $b \neq c$. Then $b = b * e = b * (a * c) = (b * a) * c = e * c = c$. $\Rightarrow\Leftarrow$

qed

Since every invertible element a of a monoid $(A, *)$ has a unique inverse, we can denote the inverse by the more standard notation a^{-1}.

Next, we need to understand inverses of elements obtained via the binary operation:

Theorem: Let $(A, *)$ be a monoid, and let a, b be invertible elements of A. Then $a * b$ is also invertible, and $(a * b)^{-1} = b^{-1} * a^{-1}$.

Remark: You might remember this formula from linear algebra when you looked at the inverse of a product of matrices AB.

Proof: Let e be the identity element of $(A, *)$. $a * a^{-1} = a^{-1} * a = e$, and $b * b^{-1} = b^{-1} * b = e$. We would like to show $b^{-1} * a^{-1}$ is the inverse of $a * b$ by computing $(a * b) * (b^{-1} * a^{-1})$ and $(b^{-1} * a^{-1}) * (a * b)$ and showing both are e.

$(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1} = a * e * a^{-1} = a * a^{-1} = e$

$(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b = b^{-1} * e * b = (b^{-1} * e) * b = b^{-1} * b = e$

Thus $b^{-1} * a^{-1}$ satisfies the conditions needed for it to be the inverse of $a * b$. Since an inverse is unique, $a * b$ is invertible and $b^{-1} * a^{-1}$ is its inverse.

qed

Theorem: Let $(A, *)$ be a monoid, and let $a, b \in A$. Let $x \in A$ be invertible. $a = b * x \Leftrightarrow b = a * x^{-1}$. Similarly, $a = x * b \Leftrightarrow b = x^{-1} * a$

Proof: Let e be the identity element of $(A, *)$.

First $a = b * x \Leftrightarrow b = a * x^{-1}$:

"$\Rightarrow$" Assume $a = b * x$. Then $a * x^{-1} = (b * x) * x^{-1} = b * x * x^{-1} = b * e = b$ as needed.

"$\Leftarrow$" Assume $b = a * x^{-1}$. Then $b * x = (a * x^{-1}) * x = a * (x^{-1} * x) = a * e = a$ as needed.

Apply the same type of argument to show $a = x * b \Leftrightarrow b = x^{-1} * a$.

qed

Let $(A, *)$ be a monoid. We can now make sense of a^n for $n \in \mathbb{Z}, n < 0$ for every $a \in A$ invertible. Since n is a negative integer, $\exists p \in \mathbb{N}$ s.t. $n = -p$. Set $a^n = a^{-p} = (a^p)^{-1}$.

Theorem: Let $(A, *)$ be a monoid, and let $a \in A$ be invertible. Then $a^m * a^n = a^{m+n}$ $\forall m, n \in \mathbb{Z}$.

Proof: When $m \geq 0$ and $n \geq 0$, we have already proven this result. The rest of the proof splits into cases.

Case 1: $m = 0$ or $n = 0$

If $m = 0$, $n \in \mathbb{Z}$, $a^m * a^n = a^0 * a^n = e * a^n = a^n = a^{0+n}$ as needed.

If $m \in \mathbb{Z}$, $n = 0$, $a^m * a^n = a^m * a^0 = a^m * e = a^m = a^{m+0}$ as needed.

Case 2: $m < 0$ and $n < 0$

$m < 0 \Rightarrow \exists p \in \mathbb{N}\ s.t.\ p = -m.\ n < 0 \Rightarrow \exists q \in \mathbb{N}\ s.t.\ q = -n.$

$a^m = a^{-p} = (a^p)^{-1}$ and $a^n = a^{-q} = (a^q)^{-1}$

$a^m * a^n = (a^p)^{-1} * (a^q)^{-1} = (a^q * a^p)^{-1} = (a^{p+q})^{-1} = a^{-(p+q)} = a^{-q-p} = a^{m+n}$

Case 3: m and n have opposite signs.
Without loss of generality, assume $m < 0$ and $n > 0$ (the case $m > 0$ and $n < 0$ is handled by the same argument). Since $m < 0, \exists p \in \mathbb{N}\, s.t.\, p = -m$. This case splits into two subcases:

Case 3.1: $m + n \geq 0$

Set $q = m + n$. Then $a^{m+n} = a^q = e * a^q = (a^p)^{-1} * a^p * a^q = (a^p)^{-1} * a^{p+q} = a^{-p} * a^{p+q} = a^m * a^{-m+m+n} = a^m * a^n$

Case 3.2: $m + n < 0$

Set $q = -(m+n) = -m-n \in \mathbb{N}^*$. Then $a^{m+n} = a^{-q} = (a^q)^{-1} * e = (a^q)^{-1} * (a^{-n} * a^n) = (a^q)^{-1} * (a^n)^{-1} * a^n = (a^n * a^q)^{-1} * a^n = (a^{n+q})^{-1} * a^n = (a^{n-m-n})^{-1} * a^n = (a^{-m})^{-1} * a^n = (a^p)^{-1} * a^n = a^m * a^n$

Theorem: Let $(A, *)$ be a monoid, and let a be an invertible element of A. $\forall m, n \in \mathbb{Z}, (a^m)^n = a^{mn}$.

Proof: We consider 3 cases:

Case 1: $n > 0$, **i.e.** $n \in \mathbb{N}^*$. $m \in \mathbb{Z}$ with no additional restrictions. We proceed by induction on n.

Base Case: $n = 1$ $\quad (a^m)^1 = a^m = a^{m \times 1}$

Inductive Step: We assume $(a^m)^n = a^{mn}$ and seek to prove $(a^m)^{n+1} = a^{m(n+1)}$. Start with $(a^m)^{n+1} = (a^m)^n * (a^m)^1 = a^{mn} * a^m = a^{mn+m} = a^{m(n+1)}$

Case 2: $n = 0$; no restriction on $m \in \mathbb{Z}$

$(a^m)^n = (a^m)^0 = e = a^0 = a^{m \times 0} = a^{mn}$

Case 3: $n < 0$; no restriction on $m \in \mathbb{Z}$.

Since $n < 0, \exists p \in \mathbb{N}\, s.t.\, p = -n$. By case 1, $(a^m)^p = a^{mp}$

$(a^m)^n = (a^m)^{-p} = ((a^m)^p)^{-1} = (a^{mp})^{-1} = a^{-mp} = a^{m(-p)} = a^{mn}$

7.6 Groups

A notion formally defined in the 1870's even though theorems about groups were proven as early as a century before that.

Definition: A group is a monoid in which every element is invertible. In other words, a group is a set A endowed with a binary operation $*$ satisfying the following properties:

1. $*$ is associative, **i.e.** $\forall x, y, z \in A, (x * y) * z = x * (y * z)$
2. There exists an identity element $e \in A$, **i.e.** $\exists e \in A\, s.t. \forall a \in A, a * e = e * a = a$
3. Every element of A is invertible, **i.e.** $\forall a \in A\ \exists a^{-1} \in A\, s.t.\, a * a^{-1} = a^{-1} * a = e$

Notation for Groups: $(A, *)$ or $(\underbrace{A}_{set}, \underbrace{*}_{operation}, \underbrace{e}_{identity})$

Remark: Closure under the operation $*$ is implicit in the definition **i.e.** $\forall a, b \in A, a * b \in A$

Definition: A group $(A, *, e)$ is called commutative or Abelian if its operation $*$ is commutative.

Examples:

1. $(\mathbb{R}, +, 0)$ is an Abelian group.

   $-x$ is the inverse of $x, \forall x \in \mathbb{R}$

2. $(\mathbb{Q}^*, \times, 1)$ $\quad \mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$ $\quad (\mathbb{Q}^*, \times, 1)$ is Abelian

   $\forall q \in \mathbb{Q}^*, q^{-1} = \frac{1}{q}$ is the inverse.

3. $(\mathbb{R}^3, +, 0)$ vectors in $\mathbb{R}^3$ with vector addition forms an Abelian group.

   $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$ vector addition.

   $0 = (0, 0, 0)$ is the identity. $(-x, -y, -z) = -(x, y, z)$ is the inverse of (x, y, z).

4. $(\widetilde{M}_n, *, I_n)$ $n \times n$ invertible matrices with real coefficients under matrix multiplication with I_n as the identity element forms a group, which is NOT Abelian.

5. Set $A = \mathbb{Z}$ and recall the equivalence relation $x \equiv y \bmod 3$ **i.e.** x and y have the same remainder under the division by 3. Recall that $\mathbb{Z}/\sim = \{0, 1, 2\}$, **i.e.** the set of equivalence classes under the partition determined by this equivalence relation. We denote $\mathbb{Z}/\sim = \{0, 1, 2\} = \mathbb{Z}_3$

   Consider $(\mathbb{Z}_3, \oplus_3, 0)$ where $\oplus_3$ is the operation of addition modulo 3, **i.e.** $1 + 0 = 1, 1 + 1 = 2, 1 + 2 = 3 \equiv 0 \bmod 3$.

Claim: $(\mathbb{Z}_3, \oplus_3, 0)$ is an Abelian group.

Proof of Claim: Associativity of $\oplus_3$ follows from the associativity of $+$, addition on $\mathbb{Z}$. Clearly, 0 is the identity (don't forget 0 stands for all elements with remainder 0 under division by 3, **i.e.** $\{0, 3, -3, 6, -6, ...\}$). To compute inverses recall that $a \oplus_3 a^{-1} = 0$, 0 is the inverse of 0 because $0 + 0 = 0$. 2 is the inverse of 1 because $1 + 2 = 3 \equiv 0 \bmod 3$, and 1 is the inverse of 2 because $2 + 1 = 3 \equiv 0 \bmod 3$.
More generally, consider the equivalence relation on $\mathbb{Z}$ given by $x \equiv y \bmod n$ for $n \geq 1$. $\mathbb{Z}/\sim = \{0, 1, ..., n-1\} = \mathbb{Z}_n$. All possible remainders under division by n are the equivalence classes. Let $\oplus_n$ be addition mod n. By the same argument as above, $(\mathbb{Z}_n, \oplus_n, 0)$ is an Abelian group.

Q: What if we consider multiplication mod n, **i.e.** $\otimes_n$. Is $(\mathbb{Z}_n, \otimes_n, 1)$ a group?

A: No! $(\mathbb{Z}_n, \otimes_n, 1)$ is not a group because 0 is not invertible: for any $a \in \mathbb{Z}_n$, $0 \otimes_n a = a \otimes_n 0 = 0 \neq 1$.

Q: Can this be fixed?

A: Troubleshoot how to get rid of 0.
Consider $\mathbb{Z}_n^* = \mathbb{Z}_n \backslash \{0\} = \{1, 2, ..., n-1\}$ all non-zero elements in $\mathbb{Z}_n^*$. This eliminates 0 as an element, but can 0 arise any other way from the binary operation? It turns out the answer depends on n. If n is not prime, say $n = 6$, we get **zero divisors**, i.e. elements that yield 0 when multiplied. These are $\underline{\text{precisely}}$ the factors of n. For $n = 6$, $\mathbb{Z}_6^* = \{1, 2, 3, 4, 5\}$ $\underline{\text{but}}$ $2 \otimes_6 3 = 6 \equiv 0 \bmod 6$, so 2 and 3 are zero divisors.

Claim: If n is prime, then $(\mathbb{Z}_n^*, \otimes_n, 1)$ is an Abelian group.
Used in cryptography $\rightarrow$ n is taken to be a very large prime number. As an example, let us look at the multiplication table for $\mathbb{Z}_5^*$ to see the inverse of various elements: $\mathbb{Z}_5^* = \mathbb{Z}_5 \backslash \{0\} = \{0, 1, 2, 3, 4\} \backslash \{0\} = \{1, 2, 3, 4\}$

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$1^{-1} = 1$	$1 \otimes_5 1 = 1$
$2^{-1} = 3$	$2 \otimes_5 3 = 6 \equiv 1 \bmod 5$
$3^{-1} = 2$	$3 \otimes_5 2 = 6 \equiv 1 \bmod 5$
$4^{-1} = 4$	$4 \otimes_5 4 = 16 \equiv 1 \bmod 5$

The fact that $(\mathbb{Z}_n^*, \otimes_n, 1)$ is Abelian follows from the commutativity of multiplication on $\mathbb{Z}$.

6. Let $(A, *, e)$ be any group, and let $a \in A$.
Consider $A' = \{a^m \mid m \in \mathbb{Z}\}$ all powers of a. It turns out $(A', *, e)$ is a group called the $\underline{\text{cyclic group}}$ determined by a. $(A', *, e)$ is Abelian $\underline{\text{regardless}}$ of whether the original group was Abelian or not because of the theorem we proved on powers of a: $\forall m, n \in \mathbb{Z}$ $a^m * a^n = a^{m+n} = a^{n+m} = a^n * a^m$.
Cyclic groups come in two flavours: finite (A' is a finite set) and infinite (A' is an infinite set).
For example, let $(A, *, e) = (\mathbb{Q}^*, \times, 1)$
If $a = -1$ $\quad A' = \{(-1)^m \mid m \in \mathbb{Z}\} = \{-1, 1\}$ is finite.
If $a = 2$ $\quad A' = \{2^m \mid m \in \mathbb{Z}\} = \{1, 2, \frac{1}{2}, 4, \frac{1}{4}, ...\}$ is infinite.

7.7 Homomorphisms and Isomorphisms

Task: Understand the most natural functions between objects in abstract algebra such as semigroups, monoids or groups.

Definition: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. A function $f : A \rightarrow B$ is called a $\underline{\text{homomorphism}}$ if

$$f(x * y) = f(x) * f(y) \; \forall x, y \in A.$$

In other words, if f is a function that respects (behaves well with respect to) the binary operation.

Examples:

1. Consider $(\mathbb{Z}, +, 0)$ and $(\mathbb{R}^*, \times, 1)$.
   Pick $a \in \mathbb{R}^*$, then $f(n) = a^n$ is a homomorphism between $(\mathbb{Z}, +, 0)$ and $(\mathbb{R}^*, \times, 1)$ because $(\mathbb{R}^*, \times, 1)$ is a group, and we proved for groups that $a^{m+n} = f(m+n) = a^m * a^n = f(m) * f(n)\ \forall m, n \in \mathbb{Z}$.
2. More generally, $\forall a \in A$ invertible, where $(A, *)$ is a monoid with identity element e, $f(m) = a^m$ gives a homomorphism between $(\mathbb{Z}, +, 0)$ and $(A', *, e)$, where as before $A' = \{a^m \mid m \in \mathbb{Z}\} \subset A$.
   We get even better behaviour if we require $f : A \to B$ to be bijective.

Definition: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. A function $f : A \to B$ is called an isomorphism if $f : A \to B$ is both bijective AND a homomorphism.

Examples:

1. Let $A' = \{2^m \mid m \in \mathbb{Z}\} = \{1, 2, \frac{1}{2}, 4, \frac{1}{4}, ...\}$
   $f(m) = 2^m$ from $(\mathbb{Z}, +, 0)$ to $(A', \times, 1)$ is an isomorphism since $2^m \neq 2^n$ if $m \neq n$.
2. Let $A' = \{(-1)^m \mid m \in \mathbb{Z}\} = \{-1, 1\}$
   $f(m) = (-1)^m$ from $(\mathbb{Z}, +, 0)$ to $(A', \times, 1)$ is NOT an isomorphism since it's not injective $(-1)^2 = (-1)^4 = 1$.

Theorem: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. The inverse $f^{-1} : B \to A$ of any isomorphism $f : A \to B$ from A to B is itself an isomorphism.

Proof: If $f : A \to B$ is an isomorphism $\Rightarrow f : A \to B$ is bijective $\Rightarrow f^{-1} : B \to A$ is bijective (proven when we discussed functions).

To show $f^{-1} : B \to A$ is a homomorphism, let $u, v \in B$. $\exists x, y \in A$ s.t. $x = f^{-1}(u)$ and $y = f^{-1}(v)$, but then $u = f(x)$ and $v = f(y)$.

Since $f : A \to B$ is a homomorphism, $f(x * y) = f(x) * f(y) = u * v$. Then $f^{-1}(u * v) = f^{-1}(f(x * y)) = x * y = f^{-1}(u) * f^{-1}(v)$ as needed.

qed

Definition: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. If $\exists f : A \to B$ an isomorphism betwen A and B, then $(A, *)$ and $(B, *)$ are said to be isomorphic.

Remark: "Isomorphic" comes from "iso" same and "morphē " form: the same abstract algebra structure on both $(A, *)$ and $(B, *)$ given to you in two different guises. As the French would say: "Même Marie, autre chapeau" same Mary, different hat.

8 Formal Languages

Task: Use what we learned about structures in abstract algebra in order to make sense of formal languages and grammars.
Let A be a finite set. When studying formal languages, we call A an alphabet and the elements of A letters.

Examples:

1. $A = \{0, 1\}$ binary digits
2. $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ decimal digits
3. $A =$ letters of the English alphabet

Definition: $\forall n \in \mathbb{N}^*$, we define a word of length n in the alphabet A as being any string of the form $a_1 a_2 \cdots a_n$ *s.t.* $a_i \in A \quad \forall i, 1 \leq i \leq n$. Let A^n be the set of all words of length n over the alphabet A.

Remark: There is a one-to-one correspondence between the string $a_1 a_2 \cdots a_n$ and the ordered n-tuple $(a_1, a_2, ..., a_n) \in A^n = \underbrace{A \times ... \times A}_{n\ times}$, the Cartesian product of n copies of A.

Definition: Let $A^+ = \bigcup\limits_{n=1}^{\infty} A^n = A^1 \cup A^2 \cup A^3 \cup ....$ A^+ is the set of all words of positive length over the alphabet A.

Examples:

1. $A = \{0, 1\}, A^+$ is the set of all binary strings of finite length that is at least one, **i.e.** $0, 1, 01, 10, 00, 11$, etc.
2. If $A =$ letters of the English alphabet, then A^+ consists of all non-empty strings of finite length of letters from the English alphabet.

It is useful to also have the empty word ε in our set of strings. ε has length 0. Define $A^0 = \{\varepsilon\}$ and then adjoin the empty word ε to A^+. We get $A^* = \{\varepsilon\} \cup A^+ = A^0 \cup \bigcup\limits_{n=1}^{\infty} A^n = \bigcup\limits_{n=0}^{\infty} A^n$.

Notation: We denote the length of a word w by $\mid w \mid$.

Next introduce an operation on A^*.

Definition: Let A be a finite set, and let w_1 and w_2 be words in A^*. $w_1 = a_1 a_2 ... a_m$ and $w_2 = b_1 b_2 ... b_n$. The concatenation of w_1 and w_2 is the word $w_1 \circ w_2$, where $w_1 \circ w_2 = a_1 a_2 ... a_m b_1 b_2 ... b_n$. Sometimes $w_1 \circ w_2$ is denoted as just $w_1 w_2$. Note that $\mid w_1 \circ w_2 \mid = \mid w_1 \mid + \mid w_2 \mid$.
Concatenation of words is:

1. associative
2. NOT commutative if A has more than one element.

Proof of (1): Let $w_1, w_2, w_3 \in A^*$. $w_1 = a_1a_2...a_m$ for some $m \in \mathbb{N}$, $w_2 = b_1b_2...b_n$ for some $n \in \mathbb{N}$, and $w_3 = c_1c_2...c_p$ for some $p \in \mathbb{N}$. $(w_1 \circ w_2) \circ w_3 = w_1 \circ (w_2 \circ w_3) = a_1a_2...a_mb_1b_2...b_nc_1c_2...c_p$.

qed

Proof of (2): Since A has at least two elements, $\exists a, b \in A$ s.t. $a \neq b$.

$a \circ b = ab \neq ba = b \circ a$.

qed

A^* is closed under the operation of concatenation $\Rightarrow$ concatenation is a binary operation on A^* as $\forall w_1, w_2 \in A^*$, $w_1 \circ w_2 \in A^*$.

Theorem Let A be a finite set. $(A^*, \circ)$ is a monoid with identity element ε.

Proof: Concatenation $\circ$ is an associative binary operation on A^* as we showed above. Moreover, $\forall w \in A^*, \varepsilon \circ w = w \circ \varepsilon = w$, so ε is the identity element of A^*.

qed

Definition: Let A be a finite set. A language over A is a subset of A^*. A language L over A is called a formal language is $\exists$ a finite set of rules or algorithm that generates exactly L, **i.e.** all words that belong to L and no other words.

Theorem: Let A be a finite set.

1. If L_1 and L_2 are languages over $A, L_1 \cup L_2$ is a language over A.
2. If L_1 and L_2 are languages over $A, L_1 \cap L_2$ is a language over A.
3. If L_1 and L_2 are languages over A, the concatenation of L_1 and L_2 given by $L_1 \circ L_2 = \{w_1 \circ w_2 \in A^* \mid w_1 \in L_1 \wedge w_2 \in L_2\}$ is a language over A.
4. Let L be a language over A. Define $L^1 = L$ and inductively for any $n \geq 1$, $L^n = L \circ L^{n-1}$. L^n is a language over A. Furthermore, $L^* = \{\varepsilon\} \cup L^1 \cup L^2 \cup L^3 \cup ... = \bigcup_{n=0}^{\infty} L^n$ is a language over A.

Proof: By definition, a language over A is a subset of A^*. Therefore, if $L_1 \subseteq A^*$ and $L_2 \subseteq A^*$, then $L_1 \cup L_2 \subseteq A^*$ and $L_1 \cap L_2 \subseteq A^*$. $\forall w_1 \circ w_2 \in L_1 \circ L_2$, $w_1 \circ w_2 \in A^*$ because $w_1 \in A^n$ for some n and $w_2 \in A^m$ for some m, so $w_1 \circ w_2 \in A^{m+n} \subseteq A^* = \bigcup_{n=0}^{\infty} A^n$.

Applying the same reasoning inductively, we see that $L \subset A^* \Rightarrow L^* \subseteq A^*$ as $L^n \subseteq A^* \; \forall n \geq 0$.

qed

Remark: This theorem gives us a theoretic way of building languages, but we need a practical way. The practical way of building a language is through the notion of a grammar.

Definition: A (formal) grammar is a set of production rules for strings in a language.
To generate a language we use:

1. the set A, which is the alphabet of the language;
2. a start symbol <s>;
3. a set of production rules.

Example: $A = \{0, 1\}$; start symbol <s>; 2 production rules given by:

1. <s> $\to$ 0<s>1
2. <s> $\to$ 01

Let's see what we generate: via rule 2, <s> $\to$ 01, so we get <s> $\Rightarrow$ 01
Via rule 1, <s> $\to$ 0<s>1, then via rule 2, 0<s>1 $\to$ 0011. We write the process as <s> $\Rightarrow$ 0<s>1 $\Rightarrow$ 0011.
Via rule 1, <s> $\to$ 0<s>1, then via rule 1 again 0<s>1 $\to$ 00<s>11, then via rule 2, 00<s>11 $\to$ 000111.
We got <s> $\Rightarrow$ 0<s>1 $\Rightarrow$ 00<s>11 $\Rightarrow$ 000111.
The language L we generated thus consists of all strings of the form 0^m1^m (m 0's followed by m 1's) for all $m \geq 1, m \in \mathbb{N}$
We saw 2 types of strings that appeared in this process of generating L:

1. terminals, **i.e.** the elements of A
2. nonterminals, **i.e.** strings that don't consist solely of 0's and 1's such as <s>, 0<s>1, 00<s>11, etc.

The production rules then have the form:

nonterminal $\to$ word over the alphabet V = {terminals, non-terminals}

<T> $\to$ w

In our notation, the set of nonterminals is $V \backslash A$, so <T>$\in V \backslash A$ and $w \in V^* = \bigcup\limits_{n=0}^{\infty} V^n$. To the production rule <T>$\to w$, we can associate the ordered pair $(\text{<T>}, w) \in (V \backslash A) \times V^*$, so the set of production rules, which we will denote by P, is a subset of the Cartesian product $(V \backslash A) \times V^*$.
Grammars come in two flavours:

1. Context-free grammars where we can replace any occurrence of <T> by w if <T>$\to w$ is one of our production rules.
2. Context-sensitive grammars only certain replacements of <T> by w are allowed, which are governed by the syntax of our language L.

The example we had was of a context-free grammar. We can now finally define context free-grammars.

Definition: A <u>context-free grammar</u> $(V, A, \text{<s>}, P)$ consists of a finite set V, a subset A of V, an element <s> of $V\backslash A$, and a finite subset P of the Cartesian product $V\backslash A \times V^*$.

Notation: $(\underbrace{V}_{set\ of\ terminals\ and\ non\ terminals}, \underbrace{A}_{set\ of\ terminals}, \underbrace{<s>}_{start\ symbol}, \underbrace{P}_{set\ of\ production\ rules})$

Example: $A = \{0, 1\}$; start symbol <s>; 3 production rules given by:

1. <s>$\rightarrow$ 0<s>1
2. <s>$\rightarrow$ 01
3. <s>$\rightarrow$ 0011

We notice here that the word 0011 can be generated in 2 ways in this context free grammar:

By rule 3, <s>$\rightarrow$ 0011 so <s>$\Rightarrow$ 0011

$\vee$

By rule 1, <s>$\rightarrow$ 0<s>1 and by rule 2, 0<s>1 $\rightarrow$ 0011. Therefore, <s>$\Rightarrow$ 0<s>1 $\Rightarrow$ 0011.

Definition: A grammar is called <u>ambiguous</u> if it generates the same string in more than one way.
Obviously, we prefer to have unambiguous grammars, else we waste computer operations.
Next, we need to spell out how words <u>relate</u> to each other in the production of our language via the grammar:

Definition: Let w' and w'' be words over the alphabet V = {terminals, non-terminals}. We say that <u>w' directly yields w''</u> if $\exists$ words u and v over the alphabet V and a production rule <T>$\rightarrow$ w of the grammar s.t. $w' = u$ <T> v and $w'' = uwv$, where either or both of the words u and v may be the empty word.
In other words, w' directly yields $w'' \Leftrightarrow \exists$ production rule <T>$\rightarrow$ w in the grammar s.t. w" may be obtained from w' by replacing a single occurrence of the nonterminal <T> within the word w' by the word w.

Notation: w' directly yields w'' is denoted by $w' \Rightarrow w''$

Definition: Let w' and w'' be words over the alphabet V. We say that w' yields w'' if either $w' = w''$ or else $\exists$ words $w_0, w_1, ...w_n$ over the alphabet V s.t. $w_0 = w', w_n = w'', w_{i-1} \Rightarrow w_i$ for all $i, 1 \leq i \leq n$. In other words, $w_0 \Rightarrow w_1 \Rightarrow w_2 \Rightarrow ... \Rightarrow w_{n-1} \Rightarrow w_n$

Notation: w' yields w'' is denotes by $w' \overset{*}{\Rightarrow} w''$.

Definition: Let $(V, A, \text{<s>}, P)$ be a context-free grammar. The language generated by this grammar is the subset L or A^* defined by $L = \{w \in A^* \mid \text{<s>} \overset{*}{\Rightarrow} w\}$

In other words, the language L generated by a context-free grammar $(V, A, \text{<s>}, P)$ consists of the set of all finite strings consisting entirely of terminals that may be obtained from the start symbol <s> by applying a finite sequence of production rules of the grammar, where the application of one production rule causes one and only one nonterminal to be replaced by the string in V^* corresponding to the right-hand side of the production rule.

8.1 Phrase Structure Grammars

Definition: A phrase structure grammar $(V, A\ \text{<s>}, P)$ consists of a finite set V, a subset A of V, an element <s> of $V \backslash A$, and a finite subset P of $(V^* \backslash A^*) \times V^*$

In a context-free grammar, the set of production rules $P \subset (V \backslash A) \times V^*$. In a phrase structure grammar, $P \subset (V^* \backslash A^*) \times V^*$. In other words, a production rule in a phrase structure grammar $r \to w$ has a left-hand side r that may contain more than one nonterminal. It is required to contain at least one nonterminal.

For example, if $A = \{0, 1\}$ and <s> is the start symbol in a phrase structure grammar grammar, $0\text{<s>}0\text{<s>}0 \to 00010$ would be an acceptable production rule in a phrase structure grammar but not in a context-free grammar.

The notions $w' \Rightarrow w''$ (w' directly yields w'') and $w' \overset{*}{\Rightarrow} w''$ (w' yields w'') are defined the same way as for context-free grammars except that our production rules may, of course, be more general as we saw in the example above.

Definition: Let $(V, A\ \text{<s>}, P)$ be a phrase structure grammar. The language generated by this grammar is the subset L or A^* defined by $L = \{w \in A^* \mid \text{<s >} \overset{*}{\Rightarrow} w\}$

Remark: The term phrase structure grammars was introduced by Noam Chomsky.

Definition: A language L generated by a context-free grammar is called a context-free language.

We now want to understand a particularly important subclass of context-free languages called regular languages.

8.2 Regular Languages

Task: Understand when a language is regular and how regular languages are produced. Understand basics of automata theory.

History: The term regular language was introduced by Stephen Kleene in 1951. A more descriptive name is finite-state language as we will see that a language is regular $\Leftrightarrow$ it can be recognised by a finite state acceptor, which is a type of finite state machine.
The definition of a regular language is very abstract, though. First, describe what operations the collection of regular languages is closed under:

Let A be a finite set, and let A^* be the set of all words over the alphabet A. The regular languages over the alphabet A constitute the smallest collection C of subsets of A^* satisfying that:

1. All finite subsets of A^* belong to C.
2. C is closed under the Kleene star operation (if $M \subseteq A^*$ is inside C, **i.e.** $M \in C$, then $M^* \in C$)
3. C is closed under concatenation (if $M \subseteq A^*, N \subseteq A^*$ satisfy that $M \in C$ and $N \in C$, then $M \circ N \in C$)
4. C is closed under union (if $M \subseteq A^*$ and $N \subseteq A^*$ satisfy that $M \in C$ and $N \in C$, then $M \cup N \in C$)

Definition: Let A be a finite set, and let A^* be the set of words over the alphabet A. A subset L of A^* is called a regular language over the alphabet A if $L = L_m$ for some finite sequence $L_1, L_2, ..., L_m$ of subsets of A^* with the property that $\forall i, 1 \leq i \leq m, L_i$ satisfies one of the following:

1. L_i is a finite set
2. $L_i = L_j^*$ for some $j, 1 \leq j < i$ (the Kleene star operation applied to one of the previous $L_j's$)
3. $L_i = L_j \circ L_k$ for some j, k such that $1 \leq j, k < i$ (L_i is a concatenation of previous $L_j's$)
4. $L_i = L_j \cup L_k$ for some j, k such that $1 \leq k, j < i$ (L_i is a union of previous $L_j's$)

Example 1: Let $A = \{0, 1\}$. Let $L = \{0^m 1^n \mid m, n \in \mathbb{N} \quad m \geq 0, n \geq 0\}$
L is a regular language. Note that L consists of all strings of first 0's, then 1's or the empty string ε. $0^m 1^n$ stands for m 0's followed by n 1's, **i.e.** $0^m \circ 1^n$. Let us examine $L' = \{0^m \mid m \in \mathbb{N}, m \geq 0\}$ and $L'' = \{1^n \mid n \in \mathbb{N}, n \geq 0\}$

Q: Can we obtain them via operatons listed among 1-4?

A: Yes! Let $M = \{0\} \qquad M \subseteq A \subseteq A^*$ and $M^* = L' = \{0^m \mid m \in \mathbb{N} \quad m \geq 0\}$. Let $N = \{1\} \quad N \subseteq A \subseteq A^*$ and $N^* = L'' = \{1^n \mid n \in \mathbb{N}, n \geq 0\}$. In other words, we can do $L_1 = \{0\}, L_2 = \{1\}, L_3 = L_1^*, L_4 = L_2^*, L_5 = L_3 \circ L_4 = L$. Therefore, L is a regular language.

Example 2 Let $A = \{0, 1\}$. Let $L = \{0^m 1^m \mid m \in \mathbb{N}, m \geq 1\}$. L is the language we used as an example earlier. It turns out L is NOT regular. This language consists of strings of 0's followed by an equal number of strings of 1's. For a machine to decide that the string $0^m 1^m$ is inside the language, it must store the number of 1's, as it examines the number of 0's or vice versa. The number of strings of the type $0^m 1^m$ is not finite, however, so a finite-state machine cannot recognise this language. Heuristically, regular languages correspond to problems that can be solved with finite memory, **i.e.** we only need to remember one of finitely many things. By contrast, nonregular languages correspond to problems that cannot be solved with finite memory.

Theorem: The collection of regular languages L is also closed under the following two operations:

1. Intersection, **i.e.** if L', L'' are regular languages (**i.e.** $L' \in C$ and $L'' \in C$), then their intersection $L' \cap L''$ is a regular language.
2. Complement, **i.e.** if L is a regular language (**i.e.** $L \in C$), then $A^* \backslash L$ is a regular language ($A^* \backslash L \in C$).

Remark: These two properties did not come into the definition of a regular language, but they are true and often quite useful.

8.3 Finite State Acceptors and Automata Theory

Definition: An automaton is a mathematical model of a computing device. Plural of automaton is automata.

Basic idea: Reason about computability without having to worry about the complexity of actual implementation.
It is most reasonable to consider at the beginning just finite states automata, **i.e.** machines with a finite number of internal states. The data is entered discretely, and each datum causes the machine to either remain in the same internal state or else make the transition to some other state determined solely by 2 pieces of information:

1. The current state
2. The input datum

In other words, if S is the finite set of all possible states of our finite state machine, then the transition mapping t that tells us how the internal state of the machine changes on inputting a datum will depend on the current state $s \in S$ and the input datum a, **i.e.** the machine will enter a (potentially) new state $s' = t(s, a)$.

Want to use finite state machines to recognise languages over some alphabet A. Let L be our language.

$\underline{\text{Input}}$	$\underline{\text{Output}}$
Word $w = a_1...a_n, a_i \in A\, \forall i$	Yes if $w \in L$
	No if $w \notin L$

Since our finite state machine accepts (**i.e.** returns yes to) w if $w \in L$, we call our machine a finite state acceptor. We want to give a rigorous definition of a finite state acceptor. To check $w = a_1...a_n$, we input each a_i starting with a_1 and trace how the internal state of the machine changes. S is our set of states of the machine (a finite set). The transition mapping t takes the pair(s, a) and returns the new state $s' = t(s, a)$ (where $s \in S$ and $a \in A$) that the machine has reached so $t : S \times A \to S$.
Some elements and subsets of S are important to understand:

1. The initial state $i \in S$ where the machine starts
2. The subset $F \subseteq S$ of finishing states

It turns out that knowing S, F, i, t, A specifies a finite state acceptor completely.

Definition: A finite state acceptor (S, A, i, t, F) consists of a finite set S of states, a finite set A that is the input alphabet, a starting state $i \in S$, a transition mapping $t : S \times A \to S$, and a set F of finishing states, where $F \subseteq S$.

Definition: Let (S, A, i, t, F) be a finite state acceptor, and let A^* denote the set of words over the input alphabet A. A word $a_1...a_n$ of length n over the alphabet A is said to be recognised or accepted by the finite state acceptor if $\exists s_0, s_1, ..., s_n \in S$ states s.t. $s_0 = i$ (the initial state), $s_n \in F$, and $s_i = t(s_{i-1}, a_i)\, \forall i \quad 1 \leq i \leq n$.

Definition: Let (S, A, i, t, F) be a finite state acceptor. A language L over the alphabet A is said to be recognised or accepted by the finite state acceptor if L is the set consisting of all words recognized by the finite state acceptor.

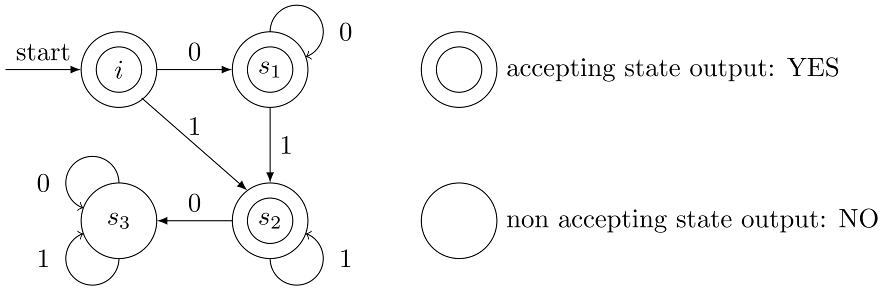
In the definition of a finite state acceptor, t is the transition mapping, which may or may not be a function (hence the careful terminology). This is because finite state acceptors come in 2 flavours:

1. Deterministic: every state has exactly one transition for each possible input, **i.e.** $\forall (s, a) \in S \times A\ \exists!\ t(s, a) \in S$. In other words, the transition mapping is a function.
2. Non-deterministic: an input can lead to one, more than one or no transition for a given state. Some $(s, a) \in S \times A$ might be assigned to more than one element of S, **i.e.** the transition mapping is not a function.

Surprisingly $\exists$ algorithm that transforms a non-deterministic (though more complex one) into a deterministic one using the power set construction. As a result, we have the following theorem:

Theorem: A language L over some alphabet A is a regular language $\Leftrightarrow$ L is recognised by a deterministic finite state acceptor with input alphabet A $\Leftrightarrow$ L is recognised by a non-deterministic finite state acceptor with input alphabet A.

Example: Build a deterministic finite state acceptor for the regular language $L = \{0^m 1^n \mid m, n \in \mathbb{N}, m \geq 0, n \geq 0\}$



Accepting states in this examples: i, s_1, s_2
Non accepting states: s_3
Start state: i
Here $S = \{i, s_1, s_2, s_3\}$ $\quad F = \{i, s_1, s_2\}$ $\quad A = \{0, 1\}$ $\quad t : S \times A \to S$ $\quad t(i,0) = s_1$ $\quad t(i,1) = s_2$ $\quad t(s_1,0) = s_1$ $\quad t(s_1,1) = s_2$
$t(s_2,0) = s_3$ $\quad t(s_2,1) = s_2$ $\quad t(s_3,0) = s_3$ $\quad t(s_3,1) = s_3$
Let's process some strings:

String	ε (empty string)
State (i)	i
Output	YES

String	0	0	1	1	1
State i	s_1	s_1	s_2	s_2	s_2
Output	YES				

String	1	1
State i	s_2	s_2
Output	YES	

String	1
State i	s_2
Output	YES

String	0	1	0	1
State i	s_1	s_2	s_3	s_3
Output	NO			

Now that we really understand what a finite state acceptor is, we can develop a criterion for recognising regular languages called the Myhill-Nerode theorem based on an equivalence relation we can set up on words in our language over the alphabet A.

Definition: Let $x, y \in L$, a language over the alphabet A. We call x and y equivalent over L denoted by $x \equiv_L y$ if $\forall w \in A^*, xw \in L \Leftrightarrow yw \in L$.

Note: xw means the concatenation $x \circ w$, and yw is the concatenation $y \circ w$.

Idea: If $x \equiv_L y$, then x and y place our finite state acceptor into the $\underline{\text{same state}}$ s.

Notation: Let $L/\equiv$ be the set of equivalence classes determined by the equivalence relation $\equiv_L$.

The Myhill-Nerode Theorem: Let L be a language over the alphabet A. If the set $L/\equiv$ of equivalence classes in L is infinite, then L is not a regular language.

Stretch of Proof: All elements of one equivalence class in $L/\equiv$ place our automaton into the same state s. Elements of distinct equivalence classes place the automaton into distinct states, **i.e.** if $[x], [y] \in L/\equiv$ and $[x] \neq [y]$, then all elements of $[x]$ place the automaton into some state s, while all elements of $[y]$ place the automaton into some state s', with $s \neq s' \Rightarrow$ an automaton that can recognise L has $\underline{\text{as many}}$ states at the number of equivalence classes in $L/\equiv$, but $L/\equiv$ is $\underline{\text{NOT}}$ finite $\Rightarrow L$ cannot be recognised by a finite state automaton $\Rightarrow L$ is not regular by the theorem above.

qed

8.4 Regular Grammars

Task: Understand what is the form of the production rules of a grammar that generates a regular language.

Recall: that a context-free grammar is given by $(V, A, <s>, P)$ where every production rule <T>$\rightarrow w$ in P causes $\underline{\text{one and only one}}$ nonterminal to be replaced by a string in V^*.

Definition: A context-free grammar $(V, A, <s>, P)$ is called a $\underline{\text{regular grammar}}$ if every production rule in P is of one of the three forms:

(i) <A>$\rightarrow$ b<B>

(ii) <A>$\rightarrow$ b

(iii) <A>$\rightarrow \varepsilon$

where <A> and <B> are nonterminals, b is a terminal, and ε is the empty word. A regular grammar is said to be in normal form if all its production rules are of types (i) and (iii).

Remark: In the literature, you often see this definition labelled left-regular grammar as opposed to right-regular grammar, where the production rules of type (i) have the form <A>→<B>b, (**i.e.** the terminal is one the right of the nonterminal). This distinction is not really important as long as we stick to one type throughout since both left-regular grammars and right-regular grammars generate regular languages.

Lemma: Any language generated by a regular grammar may be generated by a regular grammar in normal form.

Proof: Let <A>→b be a rule of type (ii). Replace it by two rules: <A>→b<F> and <F>→ ε, where <F> is a new nonterminal. Add <F> to the set V. We do the same for every rule of type (ii) obtaining a bigger set V, but now our production rules are only of type (i) and (iii) and we are generating the same language.

qed

Example: Recall the regular language $L = \{0^m 1^n \mid m, n \in \mathbb{N}, m \geq 0, n \geq 0\}$. We can generate it from the regular grammar in normal form given by production rules:

1. <s>→ 0<A>
2. <A>→ 0<A>
3. <A>→ ε
4. <s>→ ε
5. <A>→ 1<B>
6. <B>→ 1<B>
7. <s>→ 1 <B>
8. <B>→ ε

Rules (1), (2), (5), (6), (7) are of type (i), while rules (3), (4) and (8) are of type (iii).
(1) and (3) give 0. (1), (2) applied $m-1$ times and (3) give 0^m for $m \geq 2$.
(7) and (8) give 1. (7), (6) applied $n-1$ times and (8) give 1^n for $n \geq 2$.
(1), (5) and (8) give 01. (1), (5), (6) applied $n-1$ times and (8) give 01^n for $n \geq 2$.
(1), (2) applied $m-1$ times, (5) and (8) give $0^m 1$ for $m \geq 2$.
(1), (2) applied $m-1$ times, (5), (6) applied $n-1$ times, and (8) give $0^m 1^n$ for $m \geq 2, n \geq 2$.
Rule (4) gives the empty word $\varepsilon = 0^0 1^0$.

Q: Why does a regular grammar yield a regular language, **i.e.** one recognised by a finite state acceptor?

A: Not obvious from the definition, but we can construct the finite state acceptor from the regular grammar as follows: our regular grammar is given by $(V, A, <s>, P)$. Want a finite state acceptor (S, A, i, t, F). Immediately, we see the alphabet A is the same and $i =$<s>. This gives us the idea of associating to every nonterminal symbol in $V\backslash A$ a state. <s>$\in V\backslash A$, so that's good. Next we ask:

Q: Is it sufficient for $S = V\backslash A$?

A: No! Our set F of finishing/accepting states should be nonempty. So we add an element $\{f\}$ to $V\backslash A$, where our acceptor will end up when a word in our language. Thus, $S = (V\backslash A) \cup \{f\}$ and $F = \{f\}$. $F \subseteq S$ as needed.

Q: How do we define t?

A: Use the production rules in P! For every rule of type (i), which is of the form <A>$\to$ b<B> set t(<A>,b) =<B>. This works out well because our nonterminals <A> and <B> are states of the acceptor and the terminal $b \in A$ so t takes an element of $S \times A$ to an element of S as needed. Now look at production rules of type (ii), <A>$\to b$ and of type (iii), <A>$\to \varepsilon$. Those are applied when we finish constructing a word w in our language L, **i.e.** at the very last step, so our acceptor should end up in the finishing state f whenever a production rule of type (ii) or (iii) is applied. Write a production rule of type (ii) or (iii) as <A>$\to w$, then we can set $t(\text{<A>}, w) = f$. We have finished constructing t as well. Technically, $t : S \times (A \cup \{\varepsilon\}) \to S$ instead of $t : S \times A \to S$, but we can easily fix the transition function t by combining the last two transitions for each accepted word.

Remark: The same general principles as we used above allow us to go from a finite state acceptor to a regular grammar. This gives us the following theorem:

Theorem: A language L is regular $\Leftrightarrow L$ is recognised by a finite state acceptor $\Leftrightarrow L$ is generated by a regular grammar.

8.5 Regular expressions

Task: Understand another equivalent way of characterizing regular languages due to Kleene in the 1950's.

Definition: Let A be an alphabet.

1. $\emptyset$, ϵ, and all elements of A are regular expressions;
2. If w and w' are regular expressions, then $w \circ w'$, $w \cup w'$, and w^* are regular expressions.

Remark: This definition is an inductive one.

NB It is important not to confuse the regular expressions $\emptyset$ and ϵ. The expression ϵ represents the language consisting of a single string, namely ϵ, the empty string, whereas $\emptyset$ represents the language that does not contain any strings. Recall that a language L is any subset of

$$A^* = \bigcup_{n=0}^{\infty} A^n = A^0 \cup A^1 \cup A^2 \cup \cdots ,$$

where $A^0 = \{\epsilon\}$, the set of words of length 0, $A^1 =$ the set of words of length 1, and $A^2 =$ the set of words of length 2.

Precedence order of operations if parentheses are not present:

First $*$, then $\circ$ (concatenation), then $\cup$ (union).

Examples: (1) $A = \{0,1\}$

$$\begin{aligned} 1^* \circ 0 &= \{w \in A^* \mid w = 1^m 0 \text{ for } m \in \mathbb{N}, m \geq 0\} = \{0, 10, 110, 1110, \dots\} \\ &= 1^*0. \end{aligned}$$

We can omit the concatenation symbol.

(2) $A = \{0,1\}$

$$\begin{aligned} A^* \circ 1 \circ A^* &= \{w \in A^* \mid w \text{ contains at least one } 1\} \\ &= \{u \circ 1 \circ v \mid u, v \in A^*\} = A^*1A^* \end{aligned}$$

(3) $A = \{0,1\}$

$$(A \circ A)^* = \{w \in A^* \mid w \text{ is a word of even length}\}.$$

Recall that $L^* = \bigcup_{n=0}^{\infty} L^n$, where $L^0 = \{\epsilon\}$, $L^1 = L$, and inductively $L^n = L \circ L^{n-1}$. Here $L = \{00, 01, 10, 11\}$.

(3') $(A^* \circ A^*)^* = A^*$.

(4) $A = \{0,1\}$ $\quad (0 \cup \epsilon) \circ (1 \cup \epsilon) = \{\epsilon, 0, 1, 01\}$.

(5) $\epsilon^* = \{\epsilon\}$.

(6) $\emptyset^* = \{\epsilon\}$. The star operation concatenates any number of words from the language. If the language is empty, then the star operation can only put together 0 words, which yields only the empty word.

Use of regular expressions in programming:

$\rightarrow$ design of compilers for programming languages

Elemental objects in a programming language, which are called tokens (for example variables names and constants) can be described with regular expressions. We get the syntax of a programming language this way. There exists an algorithm for recognizing regular expressions that has been implemented $\Longrightarrow$ an automatic system generates the lexical analyzer that checks the input in a compiler.

$\rightarrow$ eliminate redundancy in programming

The same regular expression can be generated in more than one way (obvious from the definition of a regular expression) $\Longrightarrow$ there exists an equivalence relation on regular expressions and algorithms that check when two regular expressions are equivalent.

Theoretical importance of regular expressions

For the study of formal languages and grammars, the importance of regular expressions comes from the following theorem:

Theorem: A language is regular $\Longleftrightarrow$ some regular expression describes it.

Sketch of proof: Recall the definition of a regular language as the language obtained in finitely many steps from finite subsets of words via union, concatenation or the Kleene star. We can construct a regular expression from the definition of the regular language in question, and vice versa starting with a regular expression, we can define a finite sequence of L_i's such that each L_i is a finite set of words or is obtained from previous L_i's via union, concatenation or the Kleene star.

qed

Finally, we can state the complete characterization of regular languages:

Theorem: The following are equivalent:

(i) L is a regular language.

(ii) L is recognized by a (deterministic or non-deterministic) finite state acceptor.

(iii) L is produced by a regular grammar.

(iv) L is given by a regular expression.

Remark: It is possible to prove directly that (iv) $\Longleftrightarrow$ (ii), but the construction is rather complicated. Instead, we sketched above the proof that (i) $\Longleftrightarrow$ (iv), and we had previously stated that (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii), so we now have that (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

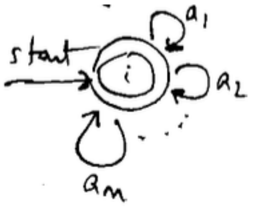
Example: Let $L = \{0^m 1^n \mid m, n \in \mathbb{N}, m \geq 0, n \geq 0\}$ be the regular language we considered before. We now give a regular expression for L: $L = 0^* \circ 1^*$. Recall we previously show this language is regular from the definition of a regular language, so solving this problem is a direct illustration of the implication (i) $\Longleftrightarrow$ (iv).

8.6 The Pumping Lemma

Task: Understand another criterion for figuring out when a language is regular.

Let a finite set A be the alphabet, and let L be a language over A. Then $L \subset A^*$. We make the following two crucial observations:

1. If L is finite, then clearly there exists a finite state acceptor that recognizes $L \Rightarrow L$ is regular.
2. If $L = A^*$, then L is likewise regular. Here is why: Let $A = \{a_1, \ldots, a_n\}$. The acceptor



with just one state i recognizes A^*.

Question: If L is infinite, but $L \subsetneqq A^*$, how can we tell whether L is regular?

Answer: The Myhill-Nerode Theorem would have us look at equivalence classes of words, but that analysis can be complicated at times. The Pumping Lemma provides another way of checking whether L is regular.

The Pumping Lemma: If L is a regular language, then there is a number p (the pumping length) where if w is any word in L of length at least p, then $w = xuy$ for words x, y, and u satisfying:

1. $u \neq \epsilon$ (i.e., $|u| > 0$, the length of u is positive);
2. $|xu| \leq p$;
3. $xu^n y \in L \ \forall\, n \geq 0$.

Remark: p can be taken to equal the number of states of a deterministic finite state acceptor that recognizes L (we know such a finite state acceptor exists because L is regular).

Sketch of proof: The name of the lemma comes from the fact that if L is regular, then all of its words can be pumped through a finite state acceptor that recognizes L. We assume this acceptor is deterministic and has p states. We will show the Pumping Lemma is a consequence of the Pigeonhole Principle we studied in the unit on functions. If a word w has length l, then the finite state acceptor must process l pieces of information ($w = a_1 a_2 \cdots a_l$, where $a_k \in A \ \forall\, k,\ 1 \leq k \leq l$) $\implies$ it passes through $l+1$ states starting with the initial state. In the hypotheses of the lemma, we assume $|w| = l \geq p$, but $p = \#(\text{states of the acceptor}) \implies$ the acceptor passes through $l + 1 \geq p + 1$ states to process w and therefore at least one state is repeated among the first $p+1$. Let s_1, s_2, ..., s_{l+1} be the sequence of states. $|w| = l \geq p \implies s_i = s_j$ with $i < j \leq p+1$. Now we set x to be the part of w that makes the acceptor pass through states s_1, s_2, ..., s_i, i.e., $x = a_1 a_2 \cdots a_{i-1}$ (the first $i-1$ letters in w). We set u to be the part of w that makes the acceptor pass through states s_i, s_{i+1}, s_{i+2}, ..., s_j. In other words, $u = a_i a_{i+1} \cdots a_{j-1}$. Since $i < j$, $|u| \geq 1 \implies u \neq \epsilon$. Finally, set y to be the part of w (the tail end) that makes the acceptor pass through states s_j, s_{j+1}, ..., s_{l+1}, i.e., $y = a_j a_{j+1} \cdots a_l$. Since $j \leq p+1$, $j-1 \leq p$, so $|xu| = |a_1 a_2 \cdots a_{j-1}| = j-1 \leq p$ as needed. Furthermore, $s_i = s_j$, so at the beginning of u and at its end the acceptor is in the same state $s_i = s_j \implies xu^n y$ is accepted for every $n \geq 0 \implies xu^n y \in L$ as needed. We have obtained conditions (1)-(3).

qed

Applications of the Pumping Lemma

As a statement, the Pumping Lemma is the implication $P \to Q$ with P being the sentence "L is a regular language" and Q being the decomposition of every w, $|w| \geq p$ as $w = xuy$. We use the contrapositive $\neg Q \to \neg P$ (tautologically equivalent to $P \to Q$) as our criterion for detecting non-regular languages.

Examples:

1. $L = \{0^m 1^m \mid m \in \mathbb{N}, m \geq 0\}$ is not regular. Let $w = 0^m 1^m$. We cannot decompose w as $w = xuy$ because whatever we let u be, we get a contradiction to $xu^n y \in L \; \forall n \geq 0$. If $u \in 0^*$ (string of 0's), $x \in 0^*$ and $y = 0^s 1^m$ (string of s 0's with $s \geq 0$ and m 1's). If $n \geq 2$, $xu^n y \notin L$ because $xu^n y$ has more 0's than 1's.
   If $u \in 1^*$, we get a contradiction the same way (more 1's than 0's in this case).
   If $u \in 0^* 1^*$, $xu^2 y \notin L$ for any x, y words!
2. $L = \{0^m \mid m \text{ is prime}\}$ is not regular.
   Since $w = 0^m$, x, u, y can consist only of 0's, so then $x = 0^i$, $u = 0^j$, $y = 0^k$. If $xu^n y \in L \; \forall n \geq 0$, then $i + nj + k$ is prime $\forall n \geq 0$, which is impossible.
   Set $n = i + 2j + k + 2$, then

$$\begin{aligned} i + nj + k &= i + (i + 2j + k + 2)j + k = i + ij + 2j^2 + jk + 2j + k \\ &= i(j+1) + 2j(j+1) + k(j+1) = (j+1)(i + 2j + k), \end{aligned}$$

   where $|u| > 0$, so $j \geq 1$. Therefore, $n = (j+1)(i+2j+k)$ is not prime!

Practice at home: `weitz.de/pump` (on Edi Weitz's website)

The pumping game, an online game to help you understand the Pumping Lemma.

8.7 Applications of Formal Languages and Grammars as well as Automata Theory

1. Compiler architecture uses context-free grammars
2. Parsers - recognise if commands comply with the syntax of a language
3. Pattern matching and data mining - guess the language from a given set of words (applied in CS, genetics, etc.)
4. Natural language processing - example in David Wilkins' notes pp.40-44
5. Checking proofs by computers/automatic theorem proving - simpler example of this kind in David Wilkins' notes pp.45-57 that pertains to propositional logic
6. The theory of regular expressions enables
   (a) grep/awk/sed in Unix
   (b) More efficient coding (avoiding unnecessary detours in your code)
7. Biology - John Conway's game of life is a cellular automaton
8. Modelling of AI characters in games uses the finite state automaton idea. Our character can choose among different behaviours based on stimuli - like a finite state automaton reacting to input
9. Strategy and tactics in games - teach the opposition to recognise certain patterns, then suddenly change them to gain an advantage and score - used in football, fencing, etc.
10. Learning a sport/a numerical instrument/a new field or subject - split the information into blocks and learn how to combine them into meaningful patterns - uses notions from context-sensitive grammars.
11. Finite state automata and probability $\rightsquigarrow$ Markov chains - chaos theory, financial mathematics.

etc...

9 Graph Theory

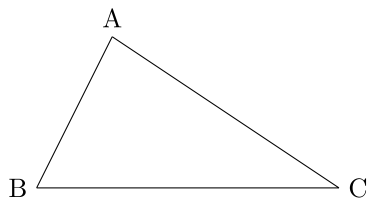
Task: Introduce terminology related to graphs; understand different types of graphs; learn how to put together arguments involving graphs.
An undirected graph consists of:

1. A finite set of points V called vertices
2. A finite set E of edges joining two distinct vertices of the graph.

Understand the meaning of an edge better: Let V be the set of vertices. Consider $P(V)$, the power set of V. Let $V_2 \subseteq P(V)$ consist of all subsets of V containing exactly 2 points, **i.e.** $V_2 = \{A \in P(V) \mid \#(A) = 2\}$ Identify each element in V_2 with the edge joining the two points. In other words, if $\{a, b\} \in V_2$, then we can let ab be the edge corresponding to $\{a, b\}$.

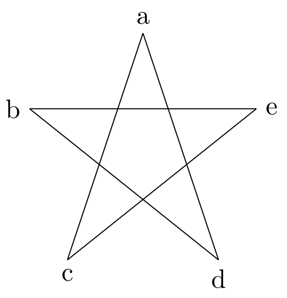
Examples:

1. A triangle is an undirected graph.
   $V = \{A, B, C\}$



   3 possible 2 element subsets of V: $\{A, B\} \to AB$
   $\{A, C\} \to AC$
   $\{B, C\} \to BC$
   $E = \{AB, AC, BC\}$

2. A pentagram is an example of an undirected graph.
   $V = \{a, b, c, d, e\}$



   $E = \{ac, ad, be, ce, bd\}$

Convention: The set of vertices cannot be empty, **i.e.** $V \neq \emptyset$.

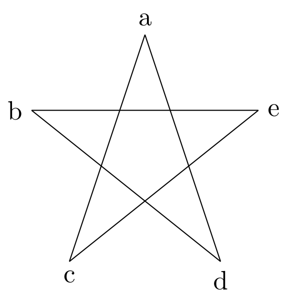
Q: If $V \neq \emptyset$, what is the simplest possible undirected graph?

A: A graph consisting of a single point, **i.e.** with one vertex and zero edges.

Definition: A graph is called trivial if it consists of one vertex and zero edges. Next, study how vertices and edges relate to each other.

Definition: If v is a vertex of some graph, if e is an edge of that graph, and it $e = vv'$ for v' another vertex, then the vertex v is called <u>incident</u> to the edge e and the edge e is called <u>incident</u> to the vertex v.

Example:
b is incident to edges be and bd
be is incident to vertices b and e

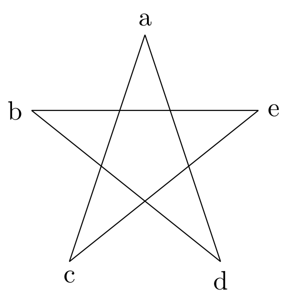


Definition: Let (V, E) be an undirected graph. Two vertices $A, B \in V$ $A \neq B$ are called <u>adjacent</u> if $\exists$ edge $AB \in E$.
We represent the incidence relations among the vertices V and edges E of an undirected graph via:

1. An incidence table
2. An incidence matrix

Legend:
1 an incidence relation holds
0 an incidence relation does not hold

From the pentagram:
$V = \{a, b, c, d, e\}$
$E = \{ac, ad, be, bd, ce\}$



The incidence table is:

	ac	ad	be	bd	ce
a	1	1	0	0	0
b	0	0	1	1	0
c	1	0	0	0	1
d	0	1	0	1	0
e	0	0	1	0	1

Correspondingly, the incidence matrix is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Note that for the incidence matrix to make sense, we need to know that vertices were considered in the order $\{a, b, c, d, e\}$ and edges in the order $\{ac, ad, be, bd, ce\}$. If we shuffle either set, the incidence matrix changes. With this in mind, we can now rigorously define the incidence matrix:

Definition: Let (V, E) be an undirected graph with m vertices and n edges. Let vertices be ordered as $v_1, v_2, ..., v_m$, and let the edges be ordered $e_1, e_2, ..., e_n$. The incidence matrix for such a graph is given by $\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$, where the entry a_{ij} in row i and column j has the value 1 if the i^{th} vertex is incident to the j^{th} edge and has value 0 otherwise.
Similarly, we can define the adjacency table and the adjacency matrix of a graph:

Definition: Let (V, E) be an undirected graph with m vertices, and let these vertices be ordered as $v_1, v_2, ..., v_m$. The adjacency matrix for this graph is given by $\begin{pmatrix} b_{11} & b_{12} & ... & b_{1m} \\ b_{21} & b_{22} & ... & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & ... & b_{mm} \end{pmatrix}$ where $b_{ij} = 1$ if v_i and v_j are adjacent to each other and $b_{ij} = 0$ if v_i and v_j are not adjacent to each other.

Remark: "Being adjacent to" is a symmetric relation on the set of vertices V, so the adjacency matrix is symmetric, **i.e.** $b_{ij} = b_{ji} \quad \forall i, j \quad 1 \leq i, j \leq m$. It is not reflexive so all the entries on the diagonal are zero.

9.1 Complete graphs

Definition: A graph (V, E) is called complete if $\forall v, v' \in V$ s.t. $v \neq v'$, the edge $vv' \in E$. In other words, a complete graph has the highest number of edges possible given its number of vertices.

Examples:

1. The triangle is a complete graph.
2. The pentagram is not a complete graph.

Notation: A complete graph with n vertices is denoted by K_n.

Q: How does the adjacency matrix of a complete graph look like?

A: All entries are 1 except on the diagonal, where they are all zero.

9.2 Bipartite graphs

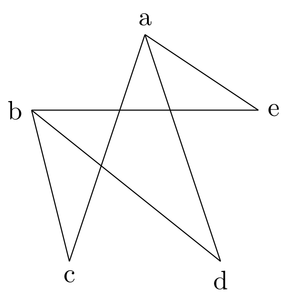
Definition: A graph (V, E) is called bipartite is $\exists$ subsets V_1 and V_2 s.t.

1. $V_1 \cup V_2 = V$
2. $V_1 \cap V_2 = \emptyset$
3. Every edge in E is of the form vw with $v \in V_1$ and $w \in V_2$.

A bipartite graph is called a complete bipartite graph if $\forall v \in V_1$ $\forall w \in V_2 \quad \exists vw \in E$.

Notation: A complete bipartite graph where the set V_1 has p elements and the set V_2 has q elements is denoted by $K_{p,q}$.

Example:
$V_1 = \{a, b\}$
$V_2 = \{c, d, e\}$
$V = \{a, b, c, d, e\}$
$E = \{ac, ad, ae, bc, bd, be\}$
is a complete bipartite graph.



Next, relate graphs to each other via functions with special properties.

9.3 Isomorphisms of Graphs

Definition: An isomorphism between two graphs (V, E) and (V', E') is a bijective function $\varphi : V \to V'$ satisfying that $\forall a, b \in V$ with $a \neq b$ the edge $ab \in E \Leftrightarrow$ the edge $\varphi(a)\varphi(b) \in E'$.

Recall: A function $\varphi : V \to V'$ is bijective $\Leftrightarrow$ it has an inverse $\varphi^{-1} : V' \to V$. The bijection $\varphi : V \to V'$ that gives the isomorphism between (V, E) and (V', E') thus sets up the following:

1. A 1-1 correspondence of the vertices V of (V, E) with the vertices V' of $(V', E') \rightsquigarrow$ comes from $\varphi : V \to V'$ being bijective.
2. A 1-1 correspondence of the edges E of (V, E) with the edges E' of $(V', E') \rightsquigarrow$ comes from the additional property in the definition of an isomorphism that $\forall a, b \in V$ with $a \neq b, ab \in E \Leftrightarrow \varphi(a)\varphi(b) \in E'$.

Definition: If there exists an isomorphism $\varphi : V \to V'$ between two graphs (V, E) and (V', E'), then (V, E) and (V', E') are called isomorphic.

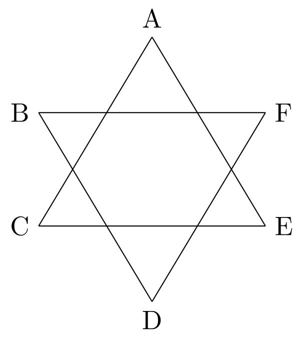
Remark: Just like an isomorphism of groups discussed earlier in the course, an isomorphism of graphs means (V, E) and (V', E') have the same "iso" form "morph$\bar{e}$". "Being isomorphic" is an equivalence relation, so we get classes of graphs that have the same form as our equivalence classes.

9.4 Subgraphs

Task: Understand sub-objects of a graph.

Definition: Let (V, E) and (V', E') be graphs. The graph (V', E') is called a subgraph of (V, E) if $V' \subseteq V$ and $E' \subseteq E$, **i.e.** if (V', E') consists of a subset V' of the vertices of (V, E) and a subset E' of edges (V, E) between vertices in V'.

Example: Star of David on the flag of Israel
$V = \{a, b, c, d, e, f\}$
$E = \{ac, ce, ae, bf, fd, bd\}$



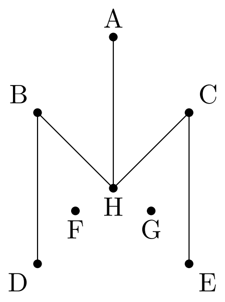
2 triangle subgraphs of the star of David:
$V' = \{a, c, e\}$ $E' = \{ac, ce, ae\}$
$V" = \{b, f, d\}$ $E" = \{bf, fd, bd\}$

9.5 Vertex Degrees

Task: Use numbers to understand incidence relationships.

Definition: Let (V, E) be a graph. The degree deg v of a vertex $v \in V$ is defined as the number of edges of the graph that are incident to v, **i.e.** the number of edges with v as one of their endpoints.

Example:



def f = deg $g = 0$
deg d = deg e = deg $a = 1$
deg b = deg $c = 2$
deg $h = 3$

Definition: A vertex of degree 0 is called an isolated vertex.

Definition: A vertex of degree 1 is called a pendant vertex.

Theorem: Let (V, E) be a graph. Then $\sum_{v \in V}$deg $v = 2\#(E)$, where $\sum_{v \in V}$deg v is the sum of the degrees of all the vertices of the graph, and $\#(E)$ is the number of edges of the graph.

Proof: $\sum_{v \in V}$deg v is the sum of all the entries in the adjacency matrix. Every edge $vv' \in E$ contributes 2 to the sum $\sum_{v \in V}$deg v, 1 for the vertex v and 1 for the vertex $v' \Rightarrow$ each edge must be counted twice, so $\sum_{v \in V}$deg $v = 2\#(E)$.

qed

Corollary: $\sum_{v \in V}$deg v is an even integer.

Proof: Since $\sum_{v\in V}$deg $v = 2\#(E)$, and $\#(E) \in \mathbb{N}$, the result follows.

qed

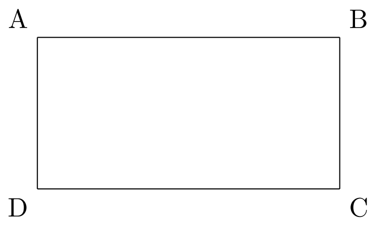
Corollary: In any graph, the number of vertices of odd degrees must be even.

Proof: Assume not, then $\sum_{v\in V}$deg v is an odd integer as $odd + even = odd \Rightarrow\Leftarrow$ to the previous corollary.

qed

Definition: A graph is called k-regular for some non-negative integer k if every vertex of the graph has degree equal to k.

Example: A rectangle is 2-regular.
deg $a =$ deg $b =$ deg $c =$ deg $d = 2$.



Definition: A graph (V, E) is called <u>regular</u> is $\exists k \in \mathbb{N}$ s.t. (V, E) is k-regular.

Corollary: Let (V, E) be a k-regular graph. Then $k\#(V) = 2\#(E)$ where $\#(V)$ is the number of vertices and $\#(E)$ is the number of edges.

Proof: By the theorem, $\sum_{v\in V}$deg $v = 2\#(E)$, but (V, E) is k-regular $\Rightarrow$ deg $v = k$ $\forall v \in V$. Therefore $\sum_{v\in V}$deg $v = \#(V) \times k = 2\#(E)$.

qed

Example: Consider a complete graph (V, E) with n vertices. (V, E) is $(n-1)$-regular because every vertex is adjacent to all the remaining $(n-1)$ vertices.

Corollary: A complete bipartite graph $K_{p,q}$ is regular $\Leftrightarrow p = q$

Proof: Recall that $V = V_1 \cup V_2$ $V_1 \cap V_2 = \emptyset$ for a bipartite graph, where $\#(V_1) = p$ and $\#(V_2) = q$.
"$\Leftarrow$" If $p = q, \forall v \in V_1$ satisfies that deg $v = p = q$ and $\forall v \in V_2$ satisfies that deg $v = p = q$ since the graph is complete $\Rightarrow K_{p,q}$ is p-regular.
"$\Rightarrow$" $K_{p,q}$ is regular $\Rightarrow \forall v \in V_1$ and $\forall v' \in V_2$, deg $v =$ deg v', but $K_{p,q}$ is complete $\Rightarrow$ v is adjacent to all vertices in V_2, **i.e.** deg $v = \#(V_2)$ and v' is adjacent to all vertices in V_1, **i.e.** deg $v' = \#(V_1)$. Therefore, $\#(V_1) = \#(V_2)$.

qed

9.6 Walks, trails and paths

Task: Make rigorous the notion of traversing parts of a graph in order to understand its structure better.

Definition: Let (V, E) be a graph. A walk $v_0v_1v_2...v_n$ of length n in the graph from vertex a to vertex b is determined by a finite sequence $v_0, v_1, v_2, ..., v_n$ of vertices of the graph s.t. $v_0 = a, v_n = b$ and $v_{i-1}v_i$ is an edge of the graph for $i = 1, 2, ..., n$.

Definition: A walk $v_0v_1v_2...v_n$ in a graph is said to traverse the edges $v_{i-1}v_i$ and to pass through the vertices $v_0, v_1, ..., v_n$. Length of walk = # of edges traversed $\Rightarrow$ the smallest possible number is zero edges. As a result, we have the following definition:

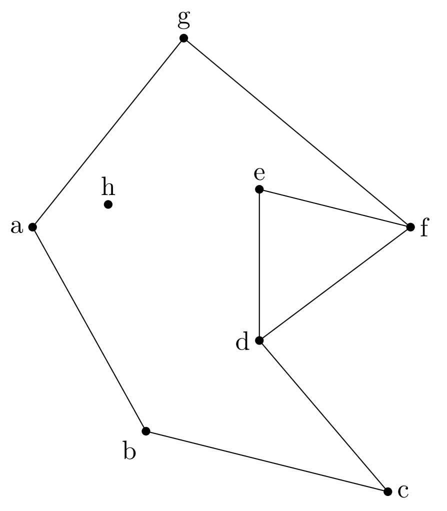
Definition: A walk that consists of a single vertex $v \in V$ and has length zero is called trivial.

Definition: Let (V, E) be a graph. A trail $v_0v_1v_2...v_n$ of length n in the graph from some vertex a to some vertex b is a walk of length n from a to b with the property that edges $v_{i-1}v_i$ are distinct for $i = 1, 2, ..., n$. In other words, a trail is a walk in the graph, which traverses edges of the graph at most once.

Definition: Let (V, E) be a graph. A path $v_0v_1v_2...v_n$ of length n in the graph from some vertex a to some vertex b is a walk of length n from a to b with the property that vertices v_0, v_1, ..., v_n are distinct. In other words, a path is a walk in the graph, which passes through the vertices of the graph at most once.

Definition: A walk, trail or path in a graph is called trivial if it is a walk of length zero consisting of a single vertex $v \in V$; otherwise, the walk, trail, or path is called non-trivial.

Example:



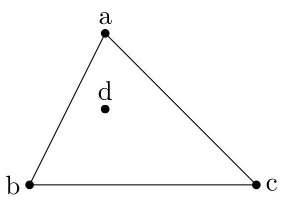
1. h is a trivial walk/trail/path
2. $defd$ is a trail, but not a path because we pass through the vertex d twice.
3. def is a path
4. $gfdefdc$ is a walk but not a trail or a path

9.7 Connected Graphs

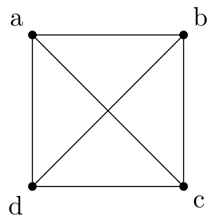
Task: Use the ideas above related to traversing parts of a graph in order to define a particularly important category of graphs.

Definition: An undirected graph (V, E) is called connected if $\forall u, v \in V$ vertices, $\exists$ path in the graph from u to v.

Examples: 1. Is not connected as d is not connected to any other vertex.



2. Is connected. $\exists$ path between any two of the vertices.



Theorem: Let (V, E) be a undirected graph, and let $u, v \in V$. $\exists$ path between u and v in the graph $\Leftrightarrow \exists$ walk in the graph between u and v.

Proof: "$\Rightarrow$" trivial: A path is a walk.
"$\Leftarrow$" $\exists$ walk between u and v. Choose the walk of least length between u and v, (**i.e.** $\nexists$ a walk of lower length than this one) and prove it is a path. Let this walk be $a_0a_1...a_n$ with $a_0 = u$ and $a_n = v$. Assume $\exists j, k$ with $0 \leq j, k \leq n$ s.t. $j < k$ and $a_j = a_k$, but then $a_0a_1...a_ja_{k+1}...a_n$ would be a walk from u to v of $\underline{\text{strictly smaller}}$ length than $a_0a_1...a_n$. $\Rightarrow\Leftarrow$ as we chose $a_0a_1...a_n$ to be of minimal length $\Rightarrow a_j \neq a_k \forall j, k$ s.t. $0 \leq j, k \leq n \Rightarrow a_0a_1...a_n$ is a path between u and v.

qed

Corollary: An undirected graph (V, E) is connected $\Leftrightarrow \forall u, v \in V$ $\exists$ walk in the graph between u and v.

9.8 Components of a graph

Task: Divide a graph into subgraphs that are isolated from each other.

Let (V, E) be an undirected graph. We define a relation $\sim$ on the set of vertices V, where $a, b \in V$ satisfy $a \sim b$ iff $\exists$ walk in the graph from a to b.

Lemma: Let (V, E) be an undirected graph. The relation $a \sim b$ or $a, b \in V$, which holds iff $\exists$ walk in the graph between a and b is an equivalence relation.

Proof: We must show $\sim$ is reflexive, symmetric, and transitive.

Reflexive: $\forall v \in V, v \sim v$ since the trivial walk is a walk from v to itself.

Symmetric: If $a \sim b$ for $a, b \in V$, then $\exists$ walk $v_0v_1...v_n$ where $v_0 = a$ and $v_n = b$. This walk can be reversed to $v_nv_{n-1}...v_1v_0$, which now goes from $v_n = b$ to $v_0 = a$. Therefore, $b \sim a$ as needed.

Transitive: If $a \sim b$ and $b \sim c$, for $a, b, c \in V$, there $\exists$ walk $av_1v_2...v_{n-1}b$ from a to b and $\exists$ walk $bw_1w_2...w_{m-1}c$ from b to c. We put these two walks together (concatenate them) to yield the walk $av_1v_2...v_{n-1}bw_1w_2...w_{m-1}c$ from a to c. Therefore $a \sim c$.

qed

The equivalence relation $\sim$ on V partitions it into disjoint subsets $V_1, V_2, ...V_p$, where

1. $V_1 \cup V_2 \cup ... \cup V_p = V$
2. $V_i \cap V_j = \emptyset$ if $i \neq j$
3. Two vertices $a, b \in V_i \Leftrightarrow a \sim b$, **i.e.** $\exists$ walk in (V, E) from a to b

Note that an edge is a walk of length 1, so if $a, b \in V$ satisfy that $\exists\, ab \in E$, then a and b belong to the same V_i. As a result, we can partition the set of edges as follows:

$$E_i = \{ab \in E \mid a, b \in V_i\}$$

Clearly, $E_1 \cup E_2 \cup ... \cup E_p = E$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. Furthermore, $(V_1, E_1), (V_2, E_2), ..., (V_p, E_p)$ are subgraphs of (V, E), and these subgraphs are disjoint since $V_i \cap V_j = \emptyset$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. The subgraphs (V_i, E_i) are called the components (or connected components) of the graph (V, E).

Lemma: The vertices and edges of any walk in an undirected graph are all contained in a single component of that graph.

Proof: Let $v_0 v_1 ... v_n$ be a walk in a graph (V, E), then $v_0 v_1 ... v_r$ is a walk in (V, E) $\forall r$ $1 \leq r \leq n \Rightarrow v_0 \sim v_r$ $\forall r$ $1 \leq r \leq n \Rightarrow v_r$ belongs to the same component of the graph as v_0. The same is true for all the edges $v_{i-1} v_i$ for $1 \leq i \leq n$.

qed

Lemma: Each component of an undirected graph is connected.

Proof: Let (V, E) be a graph and let (V_i, E_i) be any component of (V, E). $\forall u, v \in V_i$, by definition $\exists$ walk in (V, E) between u and v. By previous lemma, however, all vertices and edges of this walk are in $(V_i, E_i) \Rightarrow$ the walk between u and v is a walk in (V_i, E_i), but this assertion is true $\forall u, v \in V_i \Rightarrow (V_i, E_i)$ is connected.

qed

Moral of the story Any undirected graph can be represented as a disjoint union of connected subgraphs, namely its components $\Rightarrow$ the study of undirected graphs reduces to the study of connected graphs, as components don't share either vertices or edges.

9.9 Circuits

Task: Use closed walks to understand the structure of graphs better.

Definition: Let (V, E) be a graph. A walk $v_0v_1...v_n$ in (V, E) is called <u>closed</u> if $v_0 = v_n$, **i.e.** if it starts and ends at the same vertex.

Definition: Let (V, E) be a graph. A <u>circuit</u> is a nontrivial closed trail in (V, E), **i.e.** a closed walk with no repeated edges passing through at least two vertices.

Definition: A circuit is called <u>simple</u> if the vertices $v_0, v_1, v_2, ...v_{n-1}$ are distinct.

NB: This is the strongest condition regarding vertices that we can impose since $v_0 = v_n$.

Alternative terminology: Some authors use <u>cycle</u> to denote a simple circuit, while for others <u>cycle</u> denotes a circuit regardless of whether it is simple or not.

Q: When does a graph have simple circuits?

A: We can give 2 criteria for the existence of simple circuits:

1. Every vertex has degree ≥ 2.
2. $\exists u, v \in V$ s.t. $\exists$ 2 distinct paths from u to v.

Theorem: If (V, E) has no isolated or pendant vertices, **i.e.** $\forall v \in V$ deg $v \geq 2$, then (V, E) contains at least one simple circuit.

Proof: Consider all paths in (V, E). The maximum length of a path is $\#(V)-1$ since a path of length p passes through $p+1$ vertices. Take a path $v_0v_1...v_m$ in (V, E) of maximum length, **i.e.** any other path in (V, E) has length $\leq m =$ length of $v_0v_1...v_m$. Now consider the vertex v_m. deg $v_m \geq 2$ by assumption. We know v_{m-1} is adjacent to v_m since the edge $v_{m-1}v_m$ is part of the path $v_0v_1...v_m$, but deg $v_m \geq 2$ means $\exists w \in V$ s.t. $wv_m \in E$. If $w \neq v_i$ for $0 \leq i \leq m-2$, then $v_0v_1...v_mw$ is a path in (V, E) longer than $v_0v_1...v_m \Rightarrow\Leftarrow$ to the fact that $v_0v_1...v_m$ was chosen of maximal length. Therefore, $w = v_i$ for some $0 \leq i \leq m-2$, but then $v_iv_{i+1}...v_mv_i$ is a simple circuit in the graph.

qed

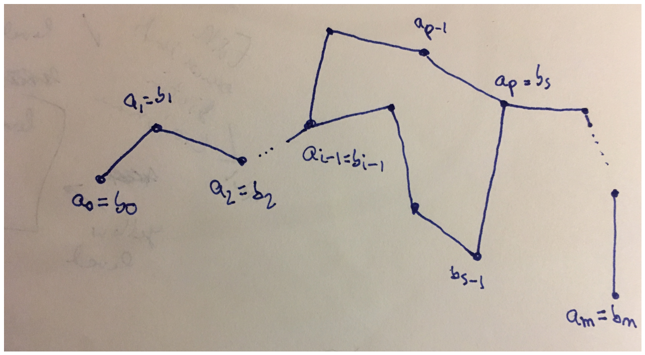
Theorem: Let (V, E) be an undirected graph and let $u, v \in V$ be vertices s.t. $u \neq v$ and $\exists$ at least two distinct paths in (V, E) from u to v. Then the graph contains at least one simple circuit.

Proof: Let $a_0a_1a_2...a_m$ and $b_0b_1...b_n$ be the two distinct paths in the graph between u and v, **i.e.** $a_0 = b_0 = u$ and $a_m = b_n = v$. WLOG let $m \leq n$. Since the paths are distinct $\exists i$ with $0 \leq i \leq m$ s.t. $a_i \neq b_i$. Choose the smallest i for which $a_i \neq b_i$, **i.e.** $a_0 = b_0, a_1 = b_1, ..., a_{i-1} = b_{i-1}$, but $a_i \neq b_i$. We have thus eliminated the redundancies at the beginning of the paths. We now need to eliminate redundancies at the other end of the paths. We know $a_m = b_n$ so $a_j \in \{b_k \mid i-1 < k \leq n\}$ is certainly satisfied for $j = m$, but we want to choose the <u>smallest</u> index for which this condition is satisfied. Let this index be $p \Rightarrow a_p \in \{b_k \mid i-1 < k \leq n\}$, **i.e.** $a_p = b_s$ for some s s.t. $i-1 < s \leq n$. Since p is the smallest index satisfying $a_p \in \{b_k \mid i-1 < k \leq n\}$,

$$a_i, a_{i+1}, ..., a_{p-1} \notin \{b_k \mid i-1 < k \leq n\}$$

$\Rightarrow \underbrace{a_{i-1}a_i...a_p}_{\textit{indices running in increasing order}} \underbrace{b_{s-1}...b_i}_{\textit{indices running in decreasing order}} a_{i-1}$ is a simple circuit in (V, E) (recall $a_p = b_s$ and $a_{i-1} = b_{i-1}$) $\Rightarrow (V, E)$ has at least one simple circuit.

qed


a_{p-1}
$a_p = b_s$
$a_1 = b_1$
$a_{i-1} = b_{i-1}$
$a_0 = b_0$
$a_2 = b_2$
b_{s-1}
$a_m = b_m$