Bayesian Statistics I

Lecture 11 - Bayesian Model Comparison

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Overview

- Bayesian model comparison
- Marginal likelihood
- Log Predictive Score

Using likelihood for model comparison

- Consider two models for the data $\mathrm{y} = (y_1, ..., y_n)$: M_1 and M_2.
- Let $p(\mathrm{y}|\theta_k, M_k)$ denote the data density under model M_k.
- If we know θ_1 and θ_2, the **likelihood ratio** is useful

$$\frac{p(\mathrm{y}|\theta_1, M_1)}{p(\mathrm{y}|\theta_2, M_2)}.$$

- The **likelihood ratio** with **ML estimates** plugged in:

$$\frac{p(\mathrm{y}|\hat{\theta}_1, M_1)}{p(\mathrm{y}|\hat{\theta}_2, M_2)}.$$

- Bigger models always win in estimated likelihood ratio.
- **Hypothesis tests** are problematic for non-nested models. End results are not very useful for analysis.

Bayesian model comparison

- **Posterior model probabilities**

$$\underbrace{\Pr(M_k|\mathrm{y})}_{\text{posterior model prob.}} \propto \underbrace{p(\mathrm{y}|M_k)}_{\text{marginal likelihood}} \cdot \underbrace{\Pr(M_k)}_{\text{prior model prob.}}$$

- The **marginal likelihood** for model M_k with parameters θ_k

$$\underbrace{p(\mathrm{y}|M_k)} = \int p(\mathrm{y}|\theta_k, M_k)p(\theta_k|M_k)d\theta_k.$$

- θ_k is 'removed' by the averaging wrt prior. **Priors matter!**
- The **Bayes factor**

$$B_{12}(\mathrm{y}) = \frac{p(\mathrm{y}|M_1)}{p(\mathrm{y}|M_2)}.$$

Jeffreys scale of evidence for the Bayes factor

- Barely worth mentioning: $1 < \mathrm{BF} \leq 3$
- Positive: $3 < \mathrm{BF} \leq 20$
- Strong: $20 < \mathrm{BF} \leq 150$
- Very strong: > 150

Bayesian hypothesis testing - Bernoulli

- **Hypothesis testing** is just a special case of model selection:

$$M_0 : x_1, ..., x_n \overset{iid}{\sim} \text{Bernoulli}(\theta_0)$$

$$M_1 : x_1, ..., x_n \overset{iid}{\sim} \text{Bernoulli}(\theta), \theta \sim \text{Beta}(\alpha, \beta)$$

$$p(x_1, ..., x_n | M_0) = \theta_0^s (1 - \theta_0)^f,$$

$$\begin{aligned} p(x_1, ..., x_n | M_1) &= \int_0^1 \theta^s (1-\theta)^f B(\alpha, \beta)^{-1} \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta \\ &= B(\alpha + s, \beta + f) / B(\alpha, \beta), \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function.

- **Posterior model probabilities**

$$\Pr(M_k | x_1, ..., x_n) \propto p(x_1, ..., x_n | M_k) \Pr(M_k), \text{ for } k = 0, 1.$$

- The **Bayes factor**

$$\text{BF}(M_0; M_1) = \frac{p(x_1, ..., x_n | M_0)}{p(x_1, ..., x_n | M_1)} = \frac{\theta_0^s (1 - \theta_0)^f B(\alpha, \beta)}{B(\alpha + s, \beta + f)}.$$

Normal example

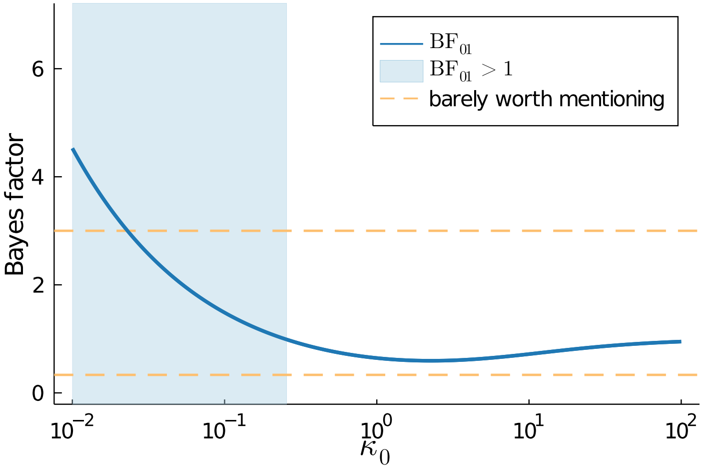
- **Model**: $x_1, \ldots, x_n \overset{\text{iid}}{\sim} N(\theta, \sigma^2)$, σ^2 known.
- **Prior**: $\theta \sim N(\mu_0, \sigma^2/\kappa_0)$.
- **Likelihood**: $\bar{x}$ is **sufficient** for θ and $\bar{x}|\theta \sim N(\theta, \sigma^2/n)$.
- **Marginal likelihood**: $p(\bar{x}|M_1) = N\left(\mu_0, \sigma^2(1/n + 1/\kappa_0)\right)$.
- Testing a **sharp null**: $M_0 : \theta = \mu_0$ vs $M_1 : \theta \neq \mu_0$.

$$B_{01} = \frac{p(\bar{x}|M_0)}{p(\bar{x}|M_1)} = \frac{N\left(\bar{x}|\mu_0, \sigma^2/n\right)}{N\left(\bar{x}|\mu_0, \sigma^2(1/n + 1/\kappa_0)\right)}$$

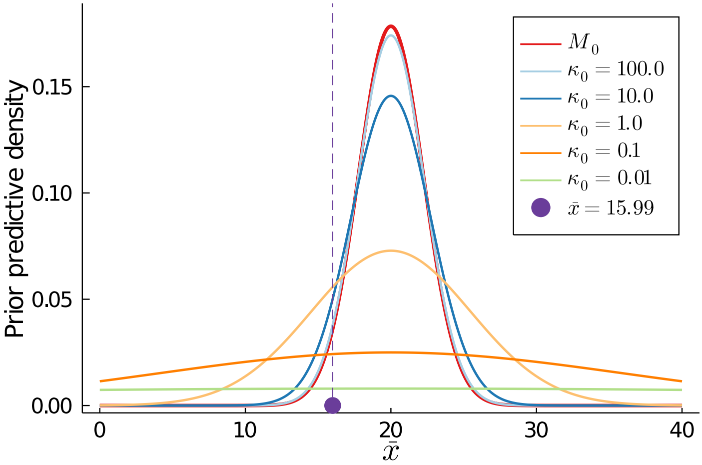
$$\log \frac{p(\bar{x}|M_0)}{p(\bar{x}|M_1)} = -\frac{1}{2}\log\left(\frac{\kappa_0}{\kappa_0 + n}\right) - \frac{n(\bar{x}-\mu_0)^2}{2\sigma^2}\left(\frac{n}{\kappa_0 + n}\right)$$

- $\kappa_0 \to \infty$ then $B_{01} \to 1$ (prior under M_1 is a point mass at 0)
- $\kappa_0 \to 0$ then $B_{01} \to \infty$ ($p(\bar{x}|M_1)$ is average $p(\bar{x}|\theta)$ wrt prior)

Internet speed data - Bayes factor



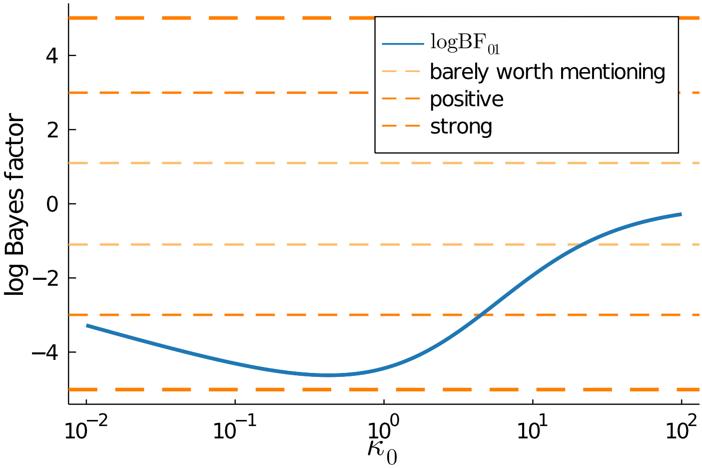
Internet speed data - prior predictive density



Vague priors for marginal likelihoods is a bad idea

- Smaller models always win when priors are very vague.
- **Improper priors** cannot be used for model comparison.

Internet speed data with $\bar{x} = 12$



Example: Geometric vs Poisson

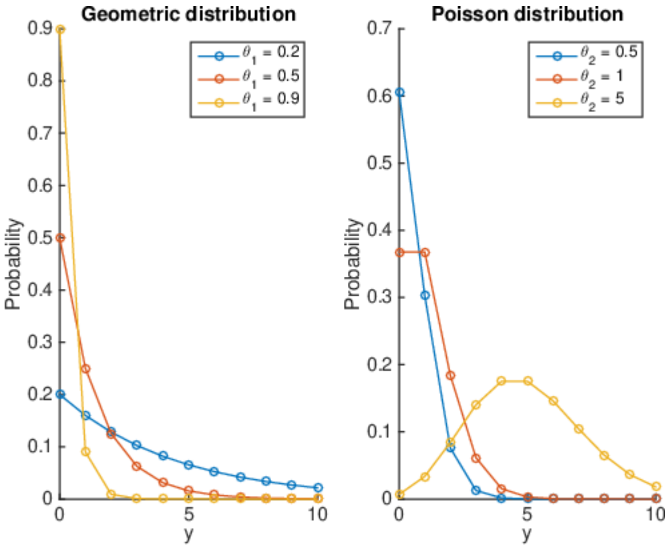
- Model 1 - **Geometric** with Beta prior:
  - $y_1, ..., y_n | \theta_1 \sim \mathrm{Geo}(\theta_1)$
  - $\theta_1 \sim \mathrm{Beta}(\alpha_1, \beta_1)$
- Model 2 - **Poisson** with Gamma prior:
  - $y_1, ..., y_n | \theta_2 \sim \mathrm{Poisson}(\theta_2)$
  - $\theta_2 \sim \mathrm{Gamma}(\alpha_2, \beta_2)$
- **Marginal likelihood** for M_1

$$
\begin{aligned}
p(y_1, ..., y_n | M_1) &= \int p(y_1, ..., y_n | \theta_1, M_1) p(\theta_1 | M_1) d\theta_1 \\
&= \frac{\Gamma(\alpha_1 + \beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} \frac{\Gamma(n + \alpha_1)\Gamma(n\bar{y} + \beta_1)}{\Gamma(n + n\bar{y} + \alpha_1 + \beta_1)}
\end{aligned}
$$

- **Marginal likelihood** for M_2

$$
p(y_1, ..., y_n | M_2) = \frac{\Gamma(n\bar{y} + \alpha_2)\beta_2^{\alpha_2}}{\Gamma(\alpha_2)(n + \beta_2)^{n\bar{y} + \alpha_2}} \frac{1}{\prod_{i=1}^{n} y_i!}
$$

Geometric and Poisson



Geometric vs Poisson

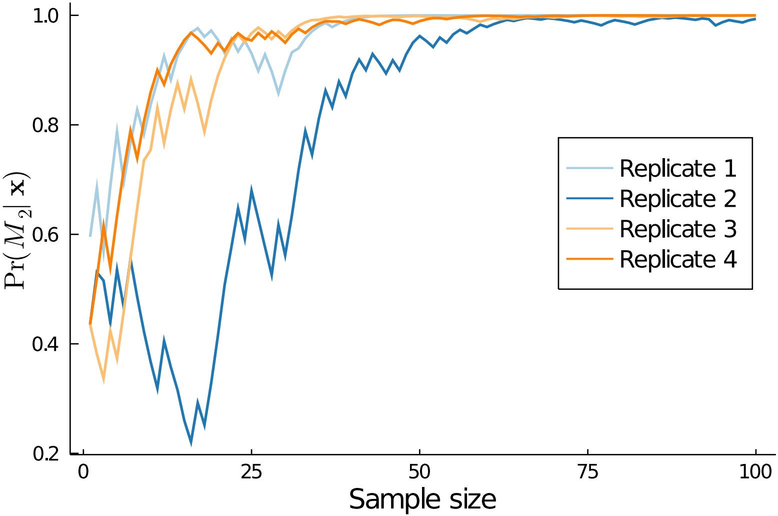
- Use priors to match prior predictive means:

$$E(y|M_1) = E(y|M_2) \quad \Longleftrightarrow \quad \alpha_1\alpha_2 = \beta_1\beta_2$$

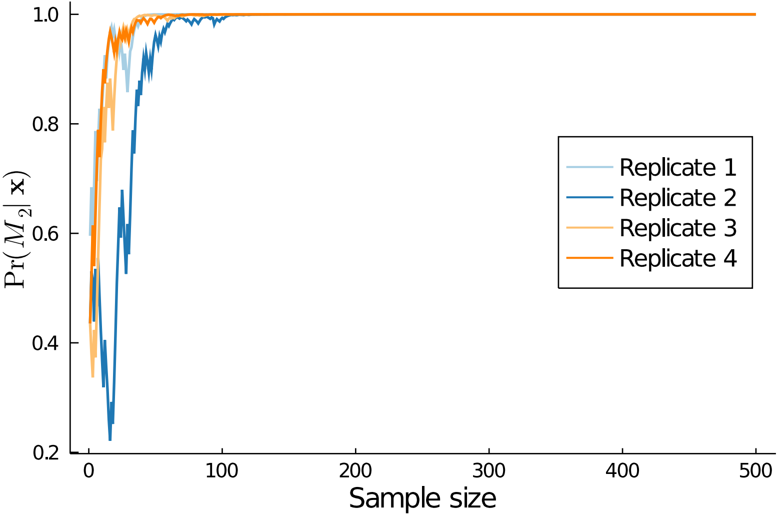
- Geometric model: $\alpha_1 = 10, \beta_1 = 20$.
- Poisson model: $\alpha_2 = 20, \beta_2 = 10$.

	$y_1 = 0,\ y_2 = 0$	$y_1 = 3,\ y_2 = 3.$
BF_{12}	4.54	0.29
$\Pr(M_1 \mid \mathrm{y})$	0.82	0.22
$\Pr(M_2 \mid \mathrm{y})$	0.18	0.78

Geometric vs Poisson for Pois(1) data



Geometric vs Poisson for Pois(1) data



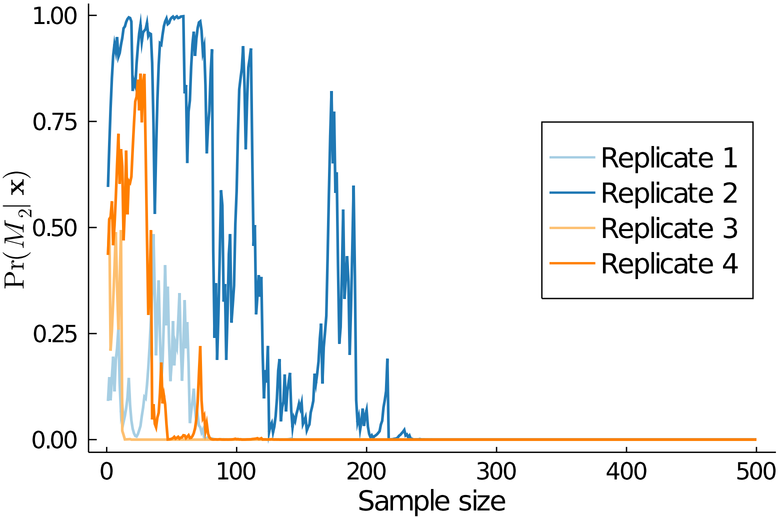
Asymptotic properties of marginal likelihood

- Set of compared models: $\mathcal{M} = \{M_1, ..., M_K\}$.
- $\mathcal{M}$**-closed**: data generating process $M^{\star}$ is in $\mathcal{M}$.
- $\mathcal{M}$-closed **consistency**:

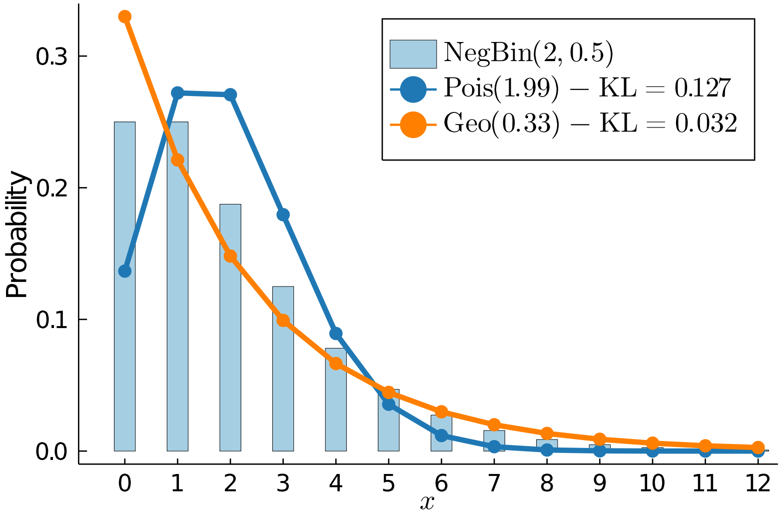
$$\Pr(M = M^{\star}|\mathrm{y}) \to 1 \quad \text{as} \quad n \to \infty$$

- $\mathcal{M}$**-open**: data generating process $M^{\star}$ is **not** in $\mathcal{M}$.
- $\mathcal{M}$**-open** is the realistic case.
- George Box: all models are false but some are useful.
- Where do posterior model probabilities go in $\mathcal{M}$**-open**?

Geometric vs Poisson for NegBin(2,0.5) data



Geometric vs Poisson for NegBin(2,0.5) data



Marginal likelihood is KL-consistent in $\mathcal{M}$-open

- **$\mathcal{M}$-open**: data generating process $M^{\star}$ is **not** in $\mathcal{M}$.
- **KL-consistency**: when $M^{\star} \notin \mathcal{M}$

$$\Pr\left(M = \tilde{M} | \mathrm{y}\right) \to 1 \quad \text{as} \quad n \to \infty,$$

- $\tilde{M}$ minimizes **KL divergence** between $p(\mathrm{y}|M)$ and $p(\mathrm{y}|M^{\star})$:

$$\mathrm{KL}(M^{\star}, M) = \int \log \frac{p(\mathrm{y}|M^{\star})}{p(\mathrm{y}|\hat{\theta}_M, M)} p(\mathrm{y}|M^{\star}) d\mathrm{y}$$

- $\hat{\theta}_M$ - model parameter that makes M as KL-close as possible to $M^{\star}$.

Model choice in multivariate time series[1]

- **Multivariate time series**

$$x_t = \alpha\beta' z_t + \Phi_1 x_{t-1} + ...\Phi_k x_{t-k} + \Psi_1 + \Psi_2 t + \Psi_3 t^2 + \varepsilon_t$$

- Need to choose:
  - **Lag length**, ($k = 1, 2.., 4$)
  - **Trend model** ($s = 1, 2, ..., 5$)
  - **Long-run (cointegration) relations** ($r = 0, 1, 2, 3, 4$).

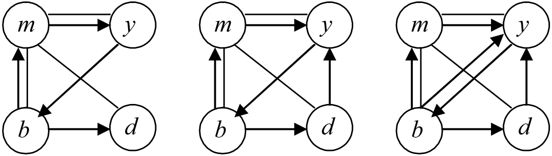
The most probable (k, r, s) combinations in the Danish monetary data.

k	1	1	1	1	1	1	1	1	0	1
r	3	3	2	4	2	1	2	3	4	3
s	3	2	2	2	3	3	4	4	4	5
$p(k, r, s \mid y, x, z)$	.106	.093	.091	.060	.059	.055	.054	.049	.040	.038

[1] Corander and Villani (2004). Statistica Neerlandica.

Graphical models for multivariate time series[2]

- **Graphical models** for multivariate time series.
- Zero-restrictions on the effect from time series i on time series j, for all lags. (**Granger Causality**).
- Zero-restrictions on inverse covariance matrix of the errors. Contemporaneous conditional independence.



$p(G|\mathbf{X}) = 0.0033$ $p(G|\mathbf{X}) = 0.0028$ $p(G|\mathbf{X}) = 0.0025$

[2] Corander and Villani (2004). Journal of Time Series Analysis.

Laplace approximation

- Taylor approximation of the log likelihood

$$\ln p(y|\theta) \approx \ln p(y|\hat{\theta}) - \frac{1}{2} J_{\hat{\theta},y} (\theta - \hat{\theta})^2,$$

so

$$p(y|\theta)p(\theta) \approx p(y|\hat{\theta}) \exp\left[-\frac{1}{2} J_{\hat{\theta},y} (\theta - \hat{\theta})^2\right] p(\hat{\theta})$$

$$= p(y|\hat{\theta}) p(\hat{\theta}) (2\pi)^{p/2} \left| J^{-1}_{\hat{\theta},y} \right|^{1/2}$$

$$\times \underbrace{(2\pi)^{-p/2} \left| J^{-1}_{\hat{\theta},y} \right|^{-1/2} \exp\left[-\frac{1}{2} J_{\hat{\theta},y} (\theta - \hat{\theta})^2\right]}_{\text{multivariate normal density}}$$

- **The Laplace approximation**:

$$\ln \hat{p}(y) = \ln p(y|\hat{\theta}) + \ln p(\hat{\theta}) + \frac{1}{2} \ln \left| J^{-1}_{\hat{\theta},y} \right| + \frac{p}{2} \ln(2\pi),$$

where p is the number of unrestricted parameters.

BIC

- **The Laplace approximation**:

$$\ln \hat{p}(y) = \ln p(y|\hat{\theta}) + \ln p(\hat{\theta}) + \frac{1}{2} \ln \left| J_{\hat{\theta},y}^{-1} \right| + \frac{p}{2} \ln(2\pi).$$

- $\hat{\theta}$ and $J_{\hat{\theta},y}$ can be obtained with **optimization/autodiff**.
- The **BIC approximation** assumes that $J_{\hat{\theta},y}$ behaves like $n \cdot I_p$ in large samples and the small term $\frac{p}{2} \ln(2\pi)$ is ignored

$$\ln \hat{p}(y) = \ln p(y|\hat{\theta}) + \ln p(\hat{\theta}) - \frac{p}{2} \ln n.$$

$\Pr(M_k|y)$ can be overfident - macroeconomics[3]

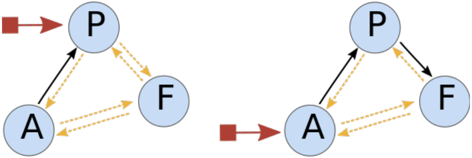
Table: Posterior model probabilities - Smets-Wouters DSGE model

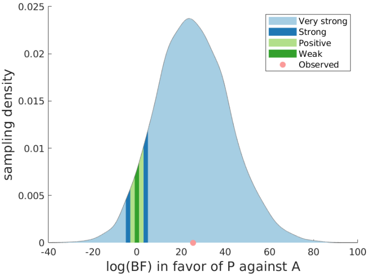
Base	M1	M2	M3	M4	M5	M6	M7	M8
0.01	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.00

[3] Oelrich et al (2020). When are Bayesian model probabilities overconfident?

Table: Posterior model probabilities - Dynamic Causal Models

A	F	P	AF	PA	PF	PAF
0.00	0.00	1.00	0.00	0.00	0.00	0.00





[4] Oelrich et al (2020). When are Bayesian model probabilities overconfident?

Marginal likelihood measures out-of-sample predictive performance

- The **marginal likelihood** can be **decomposed** as

$$p(x_1, ..., x_n) = p(x_1)p(x_2|x_1) \cdots p(x_n|x_1, x_2, ..., x_{n-1})$$

  a product of **intermediate predictive densities**

$$p(x_i|x_1, ..., x_{i-1}) = \int p(x_i|x_1, ..., x_{i-1}, \boldsymbol{\theta})p(\boldsymbol{\theta}|x_1, ..., x_{i-1})d\boldsymbol{\theta}$$

  and $p(\boldsymbol{\theta}|x_1, ..., x_{i-1})$ is the **intermediate posterior**.

- **Prediction of x_1** is based on the prior of $\boldsymbol{\theta}$. Sensitive to prior.
- **Prediction of x_n** uses almost all the data to infer $\boldsymbol{\theta}$. Not sensitive to prior when n is not small.

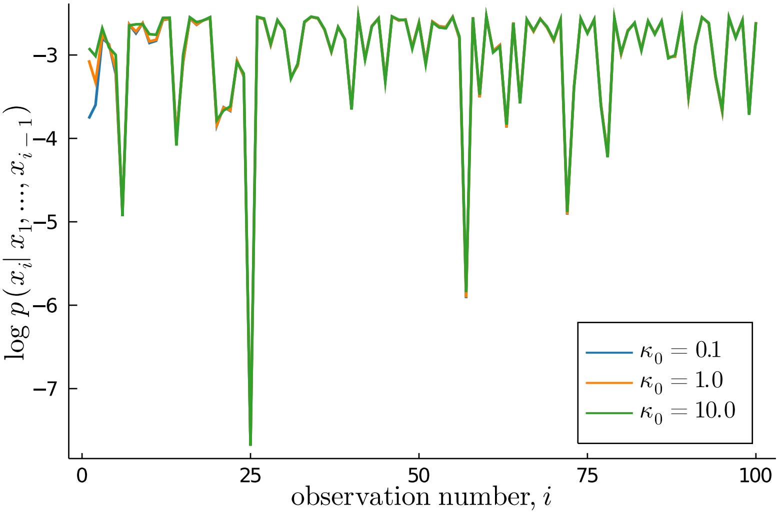
Normal example

- **Model**: $x_1, ..., x_n | \theta \sim N(\theta, \sigma^2)$ with σ^2 known.
- **Prior**: $\theta \sim N(0, \sigma^2/\kappa_0)$.
- **Intermediate predictive density** at time $i-1$

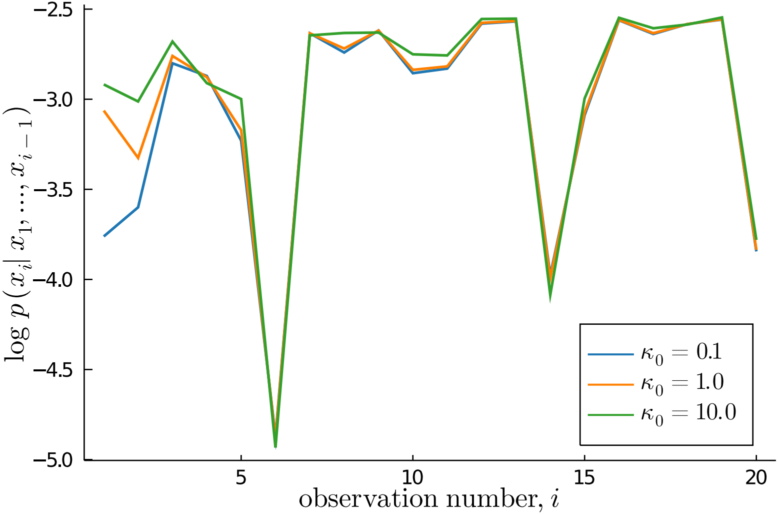
$$x_i | x_1, \ldots, x_{i-1} \sim \mathrm{N}\left(\mu_{i-1}, \sigma^2 \left(1 + \frac{1}{i-1+\kappa_0}\right)\right),$$

  where
  - $\mu_{i-1} = w_{i-1}\bar{x}_{i-1} + (1 - w_{i-1})\mu_0$
  - $\bar{x}_{i-1}$ is the sample mean of the first $i-1$ obs
  - $w_{i-1} = (i-1)/(i-1+\kappa_0)$
- $i = 1$, $x_1 \sim N\left[0, \sigma^2\left(1 + \frac{1}{\kappa_0}\right)\right]$ can be very sensitive to κ_0.
- Large i: $x_i | x_1, ..., x_{i-1} \overset{\text{approx}}{\sim} N\left(\bar{x}_{i-1}, \sigma^2\right)$, not sensitive to κ_0.

First observations are sensitive to κ_0



First observations are sensitive to κ_0 - zoomed



Log Predictive Score - LPS

- Reduce prior sensitivity: use n^* observations to train the prior.
- **(Log) Predictive (Density) Score (PS)**:

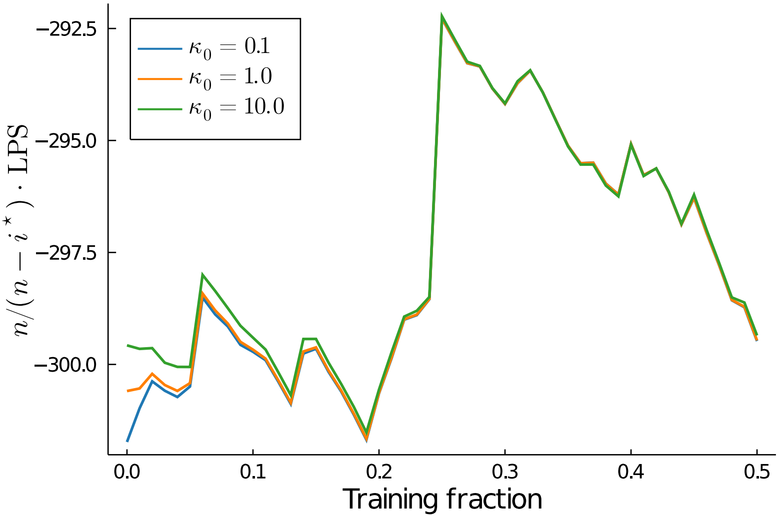
$$\underbrace{p(x_1)p(x_2|x_1)\cdots p(x_{n^*}|x_{1:(n^*-1)})}_{training}\ \underbrace{p(x_{n^*+1}|x_{1:n^*})\cdots p(x_n|x_{1:(n-1)})}_{test}$$

- Time-series: obvious which data are used for training.
- Cross-sectional data: training-test split by **cross-validation**:

$$\overbrace{1, 2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, n-1, n}^{n \text{ data observations}}$$

Split 1:	Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Split 2:	Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Split 3:	Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Split 4:	Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Split 5:	Fold 1	Fold 2	Fold 3	Fold 4	Fold 5

LPS not sensitive to κ_0



And hey! ... let's be careful out there

- Be especially **careful** with Bayesian model comparison when
  - The **compared models** are
    - very different in structure
    - severly misspecified
    - very complicated (black boxes).
  - The **priors** for the parameters in the models are
    - not carefully elicited
    - only weakly informative
    - not matched across models.
  - The **data**
    - has outliers (in all models)
    - has a multivariate response.