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Bayesian Statistics I

Mathematical Exercises 2

Try to solve the problems before class. Don't worry if you fail, the important thing is trying.
You should not hand in any solutions.
This part of the course is not obligatory and is not graded.

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1. Feel the Bern. 

   (a) Let $x_1, ..., x_n \overset{\text{iid}}{\sim} \text{Bern}(\theta)$, with a $\text{Beta}(\alpha, \beta)$ prior for θ.
   Derive the predictive distribution for x_{n+1}.

   (b) You need to decide if you bring your umbrella during your daily walk. It has rained on two days during the last ten days, and you assess those ten days to be representative also for the weather today, the 11th day. Your utility for the action-state combinations are given in the table below. Assume a $\text{Beta}(1, 1)$ prior for θ. Compute the Bayesian decision.

   (c) How sensitive is your decision in (b) to the changes in the prior hyperparameters, α and β?

	Rainy	Sunny
Bring umbrella	10	20
Leave umbrella	-50	50

2. Campaign or no campaign - that is the question.

   (a) Let x_i be the number of sales of a product on month i. Let $x_1, ..., x_n \overset{iid}{\sim} N(\theta, \sigma^2)$ be the (approximate) distribution for the sales, and let $\theta \sim N(200, 50^2)$ a priori. Assume that $\sigma^2 = 25^2$ and that we have observed $n = 5$ and $\bar{x} = 320.4$. Compute the predictive distribution for x_6.

   (b) The company has the choice of performing a marketing campaign for their product. The marketing campaign costs $300 and is believed to increase sales by 20% compared to when no campaign is performed. The company sells the product for $p = 10$ dollar and the cost of producing the product is $q = 5$ dollar. There are no fixed production costs. Assume that the company's utility is described by $U(y) = 1 - \exp(-y/1000)$, where y is the total profit from sales in the next month. Should the company perform the marketing campaign? [Hint: the expected value of the exponential function of a normal random variable $S \sim N(\mu, \sigma^2)$ is $E(\exp(S)) = \exp(\mu + \sigma^2/2)$.]

3. Predictive distribution for a Poisson model

   (a) Do Exercise 13(a) in Chapter 2 of the course book. That is, assume that the number of fatal accidents on scheduled airline flights each year are independent with a Poisson(θ) distribution. Set a prior distribution for θ and determine the posterior distribution based on the data from 1976 through 1985, given below. Under this model, give a 95% predictive interval for the number of fatal accidents in 1986. You can use the normal approximation to the gamma and Poisson or compute using simulation.

   24, 25, 31, 31, 22, 21, 26, 20, 16, 22

4. Frequentist meltdown or Bayesian breakdown?

   (a) Let $x_1, ..., x_n \overset{iid}{\sim} \text{Uniform}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Let $\hat{\theta} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be an estimator of θ. Derive an expression for the (repeated) sampling variance of $\hat{\theta}$.

   (b) Derive the posterior distribution for θ assuming a uniform prior distribution. [Hint: Here it absolutely crucial to think about the support for the data distribution. Once you have observed some data, some θ values are no longer possible.]

   (c) Assume that you have observed three data observations: $x_1 = 1.1, x_2 = 2.09, x_3 = 1.4$. What would a frequestist conclude about θ? What would a Bayesian conclude? Discuss.

5. Who doesn't want to be Normal?

   (a) Let $x_1, ..., x_n \overset{iid}{\sim} \text{Bern}(\theta)$ and $\theta \sim \text{Beta}(\alpha, \beta)$ a priori. Find the posterior mode of θ.

   (b) Approximate the posterior distribution of θ by a normal distribution.

   (c) Assume now that you have the data $n = 6$ and $s = 1$. Plot the true posterior distribution and the normal approximation in the same graph. Assume a uniform prior for θ.

   (d) Redo the previous exercise, but this time with twice the data size: $n = 12$ and $s = 2$.

Have fun!

- Mattias