Astronomy 74
Assignment 2

Your homework should be legible and comprehensive. This is great practice for presenting written material to scientists, supervisors, and customers! To be specific, you should:

1. write and draw diagrams neatly, or use a computer
2. clearly describe the steps to your solution, not just show the math
3. include a written narrative about the problem, your approach, and your conclusions
4. leave a half-page separation between ending one problem and starting the next, and use only one column

Note: You are allowed to make simplifying assumptions (though please state the assumption and the reason it's valid). This will be very useful for some problems!

10 pts

1. 21 cm is a transition of neutral hydrogen, sometimes called the "spin-flip" transition or hyperfine transition, occurs when the relative spins of the electron and proton swap from parallel to antiparallel or vice versa. The transitions are extremely rare, so even though the ISM is very diffuse, this transition can be considered to be in LTE. For this transition:

   - n_0 is the number density of atoms in the ground hyperfine state
   - n_1 is the number density of atoms in the excited hyperfine state
   - $g_0 = 2$
   - $g_1 = 6$
   - $A_{10} = 3 \times 10^{-15} s^{-1}$

   The excitation temperature (T_{ex}) is yet another temperature, is the temperature we'd measure if considering the population of energy levels relevant for the 21 cm transition. It is much larger than $T_* = h\nu_0/k$, which is the temperature corresponding with the frequency 21cm (ν_0 corresponds to the frequency of the 21cm emission).

   (a) How often does spontaneous emission of 21cm radiation occur for a given atom?

   (b) Approximately how many 21 cm transitions occur in a 1 cm^2 column 1 pc in length per second? Assume the number density of H atoms is 1 cm^{-3}, which is fairly typical (although in reality, the ISM has some much denser and some much less dense components). 1pc is 3×10^{18} cm.

   (c) In terms of the variables above, what is the intensity of a cloud that is **very optically thin** to 21cm radiation in terms of $\phi(\nu)$, A_{10}, n_1, and L where L is the distance through the cloud? There is no background light source, all the radiation is produced in the cloud.

   Using your result, explain how we can learn the column density of neutral hydrogen (N_H) by observing a spectrum of the 21 cm emission (which gives you intensity). Remember neutral hydrogen includes atoms in both the 0 and the 1 state, so you need to get the total of all neutral hydrogen from n_1 or N_1.

(d) What is the absorption coefficient α_ν in terms of $\phi(\nu)$, A_{10} (where 0 is the ground state and 1 is the excited state), ν_0 (frequency for 21 cm), T_{ex} (the excitation temperature), n_1 or n_0, and $T_* = h\nu_0/k$. Make sure the equation you're using includes stimulated emission.

(e) What is the optical depth for the warm neutral medium (WNM; $n_H = 0.3\mathrm{cm}^{-3}$; T=5000K) observing through 1kpc? For the cold neutral medium (CNM; $n_H = 30\mathrm{cm}^{-3}$; T=100K) through 50pc? You'll need to approximate the line profile $\phi(\nu)$ similarly to what we did in class for the Gl 436 example. For the width of the spectra line, $\Delta\nu$, you can use the velocity width Δv associated with a Maxwellian distribution of the appropriate temperature gas (Δv =1.5 km/s for the CNM at 100K, and Δv = 10 km/s for the WNM at 5000K).

2. This problem relates to the superposition of two polarized plane waves. Real astrophysical examples of polarization seem to generally be much messier than this example, but I believe that this example is at least not un-representative of a) some pulsar polarization and b) auroral emission from Jupiter and brown dwarfs.

10 pts (a) Two monochromatic waves propagating along the z-axis with the same frequency and same intensity are superposed. One wave is plane-polarized along the x-axis, and the other is right-handed circularly polarized. Find the Stokes parameters of the polarization of the superposed wave as a function of the phase difference ϕ between the electric field of the first wave and the x-component of the electric field in the second wave. You should start by writing the electric fields of the two monochromatic waves in exponential form.

(b) Assume that the two previous waves are the same except that they are quasi-monochromatic and independent. Find the polarization of the superposed wave, computing the Stokes parameters and the degree of polarization, and decompose it into a purely polarized and a purely unpolarized component.

3. (a) An electron is "wiggling" in one dimension such that its position is $x(t) = A\cos(\omega t)$.
10 pts What is the *average* power radiated by the particle? Note how it scales with the wiggle frequency ω.

(b) The classical Larmor's formula for radiation from an accelerated charged particle implies that the electron orbiting a proton in a hydrogen atom will radiate away its kinetic energy in a small fraction of a second, and the atom will collapse. This striking failure was one of the problems in classical physics that led to the development of quantum mechanics. A classical hydrogen atom consists of an electron in a circular orbit around a proton, with an orbital radius $r_0 = 5.3 \times 10^{-9}$ cm (the Bohr radius). Estimate the classical radiative lifetime t of such an atom.

4. A pulsar (discovered by Jocelyn Bell in 1967) is a rotating neutron star with a strong magnetic field, B_0. These might be trapped in place during the collapse of the neutron star, or grow through a dynamo mechanism or magnetorotational instability (see White et al., 2022). The magnetic axis of the neutron star is not aligned with its rotation axis, and so

10 pts there is a time-changing magnetic dipole, $\mathbf{m(t)}$, where $\mathbf{m}$ is the magnetic moment. The magnetic dipole is analagous to the electric dipole, with the replacement of $\mathbf{d}$ with $\mathbf{m}$. Assume the mass, radius, and surface magnetic field of the neutron star are M, R, and B_0. Let the angle between the magnetic and rotation axes be α, and the rotational angular velocity be ω.

(a) Find an expression for the radiated power P, in terms of ω, R, B_0, and α.

(b) The radiation extracts energy from the pulsar, resulting in the pulsar's rotation period increasing with time (it also heats the surrounding medium). If this is the only source of torque, find an expression for the spin-down timescale, τ of the pulsar. Assume the neutron star is a uniform solid body.

(c) First, re-write your equation for τ with all variables normalized, and in the units of years. That is, the equation should have a form: $\tau = X\mathrm{yr} \times (M/M_\odot)^n$ etc. Normalize to values of:

- $M = 1M_\odot$
- $R = 10^6$ cm
- $B_0 = 10^{12}$ G
- $\omega = 10^4$s
- $\alpha = 90\,\mathrm{deg}$

For a pulsar with the above values of M, R, B_0, and α, find P and τ for three values of ω: 10^4 s^{-1} (typical for newly formed pulsars), 10^3 s^{-1}, 10^2 s^{-1}