RL Chap 2

Astr 74
Emily Boudreaux
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2.1 Review of E&M (P.51-54)

- Recall maxwells equations in free space (in a Vacume) and in some media (In The non-relativistic case)

- first in some media

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad D = \epsilon\vec{E}$$

charge density

current density

dielectric constant

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi\vec{j}}{c} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \qquad B = \mu\vec{H}$$

magnetic Permiability

- So then in free space (which is what we will almost always be working in during this class we note that $\epsilon = \mu = 1$ (in cgs units)

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi\vec{j}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

- These imply conservation of charge

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\left(\frac{4\pi\vec{j}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

we can split up these terms to simplify our lives $\rightarrow$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \underbrace{\left(\hat{i}\frac{\partial}{\partial x}\frac{4\pi\vec{j}}{c} + \hat{j}\frac{\partial}{\partial y}\frac{4\pi\vec{j}}{c} + \hat{k}\frac{\partial}{\partial z}\frac{4\pi\vec{j}}{c}\right)}_{\vec{\nabla}\cdot\vec{j}\left(\frac{4\pi}{c}\right)} + \vec{\nabla}\cdot\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}(\vec{\nabla}\,\vec{j}) + \vec{\nabla}\cdot\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 4\pi(\vec{\nabla}\,\vec{j}) + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}$$

0 in free-Space

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y)$$

- for a smooth and continuos function $f(x,y)$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}(\vec{\nabla}\,\vec{j})$$

Recall that the divergence of a curl is always 0

$$\boxed{0 = \vec{\nabla}\cdot\vec{j}}$$

$\leftarrow$ conservation of charge for a finite volume element

- P. 53 uses this to derive the Poynting Vector we will not go through that during class

$$\boxed{\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}}$$

$\leftarrow$ Poynting vector in free space in cgs units.

- The Poynting vector tells one the direction energy is propegating and how much energy passes through an area at a given time.

2.2 Plane Electromagnetic Waves (P. 55-59)

- We can derive the vector wave equation from maxwell's equation

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial E}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\frac{\partial^2 E}{\partial t^2}$$

note the vector identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\nabla(\underbrace{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} = -\frac{1}{c}\frac{\partial^2 E}{\partial t^2}$$

In free space this is 0

$$\boxed{0 = \frac{1}{c}\frac{\partial^2 E}{\partial t^2} - \nabla^2\vec{E}}$$

Vector Wave Equation

There is an identical equation for magnetic fields.

- Solutions take the form

wave vector $\vec{k} = k\vec{n}$

$$\vec{E} = \hat{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{B} = \hat{a}_2 B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

unit vector perpendicular to $\hat{a}_1$ and $\vec{S}$

- We can then use these to build up the most general solution to maxwell's equations in free space

$$i\vec{k}\cdot\hat{a}_1 E_0 = 0 \qquad i\vec{k}\cdot\hat{a}_2 B_0 = 0$$

$$i\vec{k}\times\hat{a}_1 E_0 = \frac{i\omega}{c}\hat{a}_2 B_0 \qquad i\vec{k}\times\hat{a}_2 B_0 = -\frac{i\omega}{c}\hat{a}_1 E_0$$

- from here we can relate E_0 and B_0 and then through those we can relate ω and k

$$E_0 = \frac{\omega}{kc}B_0 \qquad B_0 = \frac{\omega}{kc}E_0$$

$$E_0 = \left(\frac{\omega}{kc}\right)^2 E_0$$

$$\omega^2 = k^2c^2$$

$$\omega = kc$$

- However we can constrain $\omega, k > 0$

$$E_0 = \frac{kc}{kc}B_0 \quad \therefore \quad \boxed{E_0 = B_0}$$

- Recall that the Phase Velocity of a wave, $V_{ph} = \omega/k$, therefore the wave will Propegate at

$$V_{ph} = \frac{kc}{k} = c$$

- which makes as the phase velocity of light is c
  - note the group velocity is only c in free space.

- The Poynting Vector oscillates w/time; However, we will normally care more about the time averaged Poynting Vector and Energy densities.

$$E(t) = E_o e^{i\omega t} \qquad B(t) = B_o e^{i\omega t}$$

complex

- Let the Electric field and magnetic field strength oscillate as a function of time

- By Problem 2.1 the time average of Real components

$$\langle Re(E(t)) \cdot Re(B(t)) \rangle = \frac{1}{2} Re(\bar{E}_o B_o)$$

complex conjugate of E_o

Recall: $x = a + bi$
$\bar{x} = a - bi$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$|\vec{S}| = \frac{c}{4\pi} |\vec{E}||\vec{B}| \cancelto{1}{\sin\left(\frac{\pi}{2}\right)}$$

$$\langle S \rangle = \frac{c}{4\pi} \underbrace{\langle Re(E(t)) Re(B(t)) \rangle}_{\frac{1}{2} E_o B_o}$$

but we also know $E_o = B_o$
so $\frac{1}{2} E_o^2$ or $\frac{1}{2} B_o^2$

$$\boxed{\langle S \rangle = \frac{c}{8\pi} |E_o|^2 = \frac{c}{8\pi} |B_o|^2}$$

- However we also recall Eqn 2.11 which gives the energy density per unit volume

$$U_{field} = \frac{1}{8\pi} (E^2 + B^2)$$

- using our complex and oscillating E and B fields

$$\langle U_{field} \rangle = \frac{1}{8\pi} (\langle Re(E(t)) Re(E(t)) + Re(B(t)) Re(B(t)) \rangle)$$

$$\langle U_{field} \rangle = \frac{1}{8\pi} \left(\frac{1}{2} Re(\bar{E}_o E_o) + \frac{1}{2} Re(\bar{B}_o B_o) \right)$$

$$\langle U_{field} \rangle = \frac{1}{16\pi} (E_o^2 + B_o^2)$$

$$\boxed{\langle U_{field} \rangle = \frac{2E_o^2}{16\pi} = \frac{E_o^2}{8\pi} = \frac{B_o^2}{8\pi}}$$

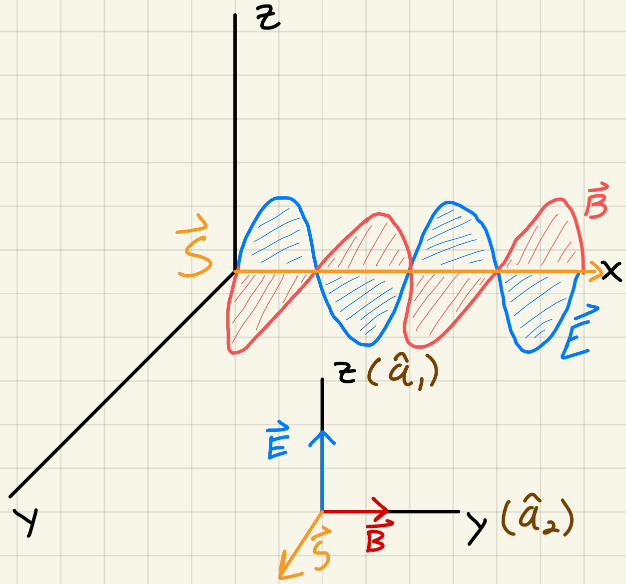
- finally then we can find the velocity of energy transport

$$\boxed{\frac{\langle S \rangle}{\langle u \rangle} = c}$$

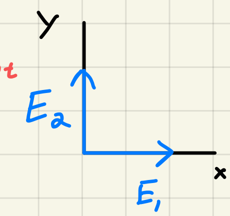
2.4 Polarization and Stokes Parameters (P.62-69)

Polarization: The orientation of oscillation

- In The Previous section we worked with linearly Polarized waves (i.e. The electric vector oscillates in the direction of $\hat{a}$, (along some Single unit Vector)



- The most General case of Polarization is found by considering the Superposition of 2 Solutions to the Vector Wave eqn. Which are perpendicular to one and other



$$\vec{E} = (E_1\hat{x} + E_2\hat{y})e^{-i\omega t} = \vec{E}_0 e^{-i\omega t}$$

- We can then ask what E_1 and E_2 are
- Generally these are some complex amplitudes

$$E_1 = \mathcal{E}_1 e^{i\phi_1} \quad E_2 = \mathcal{E}_2 e^{i\phi_2}$$

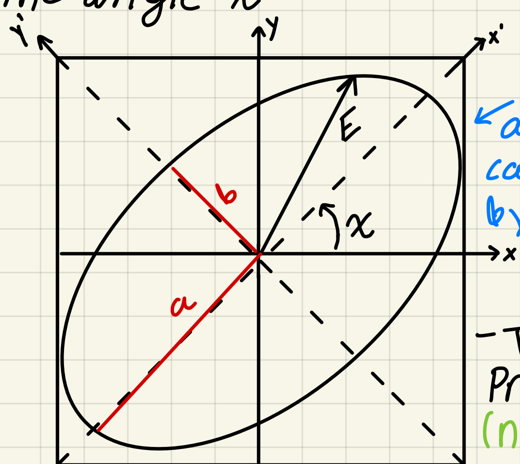
- Then We can select just the real component to find the Physical field in $\hat{x}$ and $\hat{y}$

$$E_x = \mathrm{Re}(E_1 e^{-i\omega t})$$
$$E_x = \mathrm{Re}(\mathcal{E}_1 e^{i\phi_1} e^{-i\omega t})$$
$$E_x = \mathrm{Re}(\mathcal{E}_1 e^{i(\phi_1 - \omega t)})$$
$$E_x = \mathcal{E}_1 \cos(\phi_1 - \omega t)$$

$\cos(x) = \cos(-x)$ therefore

$$\boxed{E_x = \mathcal{E}_1 \cos(\omega t - \phi_1),\ E_y = \mathcal{E}_2 \cos(\omega t - \phi_2)}$$

- The Book demonstrates that E_x and E_y Trace out an Ellipse tilted by some angle χ



In general a tilted ellipse can be described by

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t \quad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t \quad \left(-\tfrac{\pi}{2} \le \beta \le \tfrac{\pi}{2}\right)$$

- This ellipse will be Traced out by $\vec{E}$ as the Wave Propegates. When $\beta > 0$ The Trace will run clockwise (negative helicity) and for $\beta < 0$ the Trace will run counter clockwise (Positive helicity)

elliptical Polarization is most General ← Have helicity (handedness)

Special cases

$\beta = \pm\pi/4$: Then the ellipse becomes a circle (circularly Polarized)

$\beta = 0$ or $\pm\pi/2$: The ellipse collapses to a line (linearly Polarized) ← does not have helicity

Connecting E_x, E_y to this ellipse

- We need to rotate E'_x and E'_y to the x and y axis (through an angle χ)

$$E_x = \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t) \quad E_y = \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t - \sin\beta\cos\chi\sin\omega t)$$

- Now we can recognize that these have a very similar form to the other expression for E_x and E_y which we have found

trig ident. $\mathcal{E}_1 \cos(\omega t - \phi_1) = \mathcal{E}_0 (\cos\beta \cos\chi \cos\omega t + \sin\beta \sin\chi \sin\omega t)$

↳ $\mathcal{E}_1 \cos\omega t \cos\phi + \sin\omega t \sin\phi = \mathcal{E}_0 \cos\beta \cos\chi \cos\omega t + \mathcal{E}_0 \sin\beta \sin\chi \sin\omega t$

- set the two terms = one and other

$\mathcal{E}_1 \cancel{\cos\omega t} \cos\phi = \mathcal{E}_0 \cos\beta \cos\chi \cancel{\cos\omega t}$

$$\boxed{\begin{aligned} \mathcal{E}_1 \cos\phi_1 &= \mathcal{E}_0 \cos\beta \cos\chi \\ \mathcal{E}_1 \sin\phi_1 &= \mathcal{E}_0 \sin\beta \sin\chi \\ \mathcal{E}_2 \cos\phi_2 &= \mathcal{E}_0 \cos\beta \sin\chi \\ \mathcal{E}_2 \sin\phi_2 &= -\mathcal{E}_0 \sin\beta \cos\chi \end{aligned}}$$

going through the same procedure for both terms in Both E_x and E_y

- Generally we might know $\mathcal{E}_1$ ϕ_1 $\mathcal{E}_2$ and ϕ_2. If you do you can solve for $\mathcal{E}_0$ β and χ. We call forms of these solutions Stokes Parameters. These fully describe monochromatic elliptical Polarization.

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \qquad U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi$$

$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \qquad V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta$$

- alternate forms can be expressed by combining the above relations

$$\boxed{\mathcal{E}_0 = \sqrt{I} \qquad \sin 2\beta = \frac{V}{I} \qquad \tan 2\chi = \frac{U}{Q}}$$

I - Relates to the Flux

V - circularity Parameter ratio of a/b. If $V=0$ then the wave is linearly Polarized

U/Q - Orientation of ellipse to x axis. $U=Q=0$ is the condition for circular Polarization.

- All four Stokes Parameters are related

$$\boxed{I^2 = V^2 + U^2 + Q^2}$$

- for Quasi-monochromatic waves we define the stokes Parameters as a time average (These are a more general definition)

$$\boxed{\begin{aligned} I &\equiv \langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \rangle \qquad U \equiv \langle 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle \\ Q &\equiv \langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \rangle \qquad V \equiv \langle 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle \end{aligned}}$$

only fully correct when $\mathcal{E}_1$ $\mathcal{E}_2$ ϕ_1 and ϕ_2 are time independant

- finally, mostly in astro we deal with unpolarized light. We get Polarized light from scattering mostly (off dust for example) and from magnetic field interactions.

- The specific Reason we use stokes Parameters is Historical and basically boils down to How stoke's Had an observational setup built (i.e the orientation of Polarized Plates used).