

R + L chapter 1

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1.1 fundamentals of Radiative transfer (P. 1-2)

- Electromagnetic radiation's wavelength is related to its frequency via

$$\text{Wavelength (m)} \rightarrow \lambda \nu = c \quad \begin{matrix} \text{Speed of light } (3 \cdot 10^8 \text{ cm s}^{-1} \\ \text{in a Vacume}) \end{matrix}$$

↑ frequency (Hz)

- The energy of a photon of a given frequency

$$\text{Photon energy (erg)} \rightarrow E = h\nu \quad \begin{matrix} \text{Planck's constant } (6.625 \cdot 10^{-27} \text{ erg s}) \end{matrix}$$

- The Temperature of a Photon of a given frequency

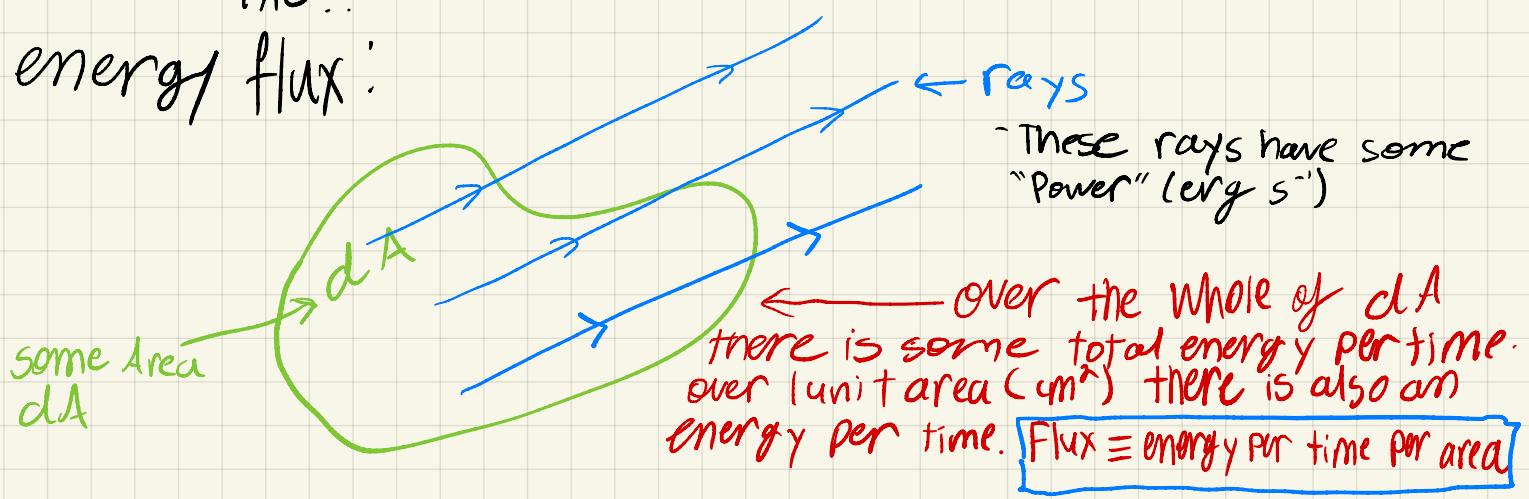
$$\text{Temperature (K)} \rightarrow T = \frac{E}{h} \quad \begin{matrix} \text{Boltzmann constant} \\ (1.38 \cdot 10^{-16} \text{ erg K}^{-1}) \end{matrix}$$

1.2 Radiative flux (P. 2 - 3)

- When the scale of the system, d , is much greater than the wavelength ($d \gg \lambda$) we can treat radiation as straight lines (rays)

- Radiative transfer theory all originates from this!!

- energy flux:



- flux from an isotropic source

- isotropic: emitting energy equally in all directions

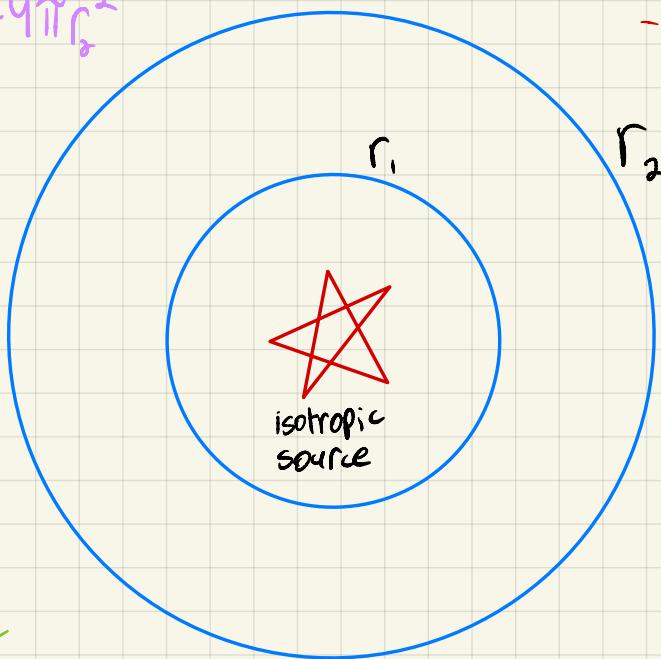
$$F(r_1) \cdot 4\pi r_1^2 = F(r_2) \cdot 4\pi r_2^2$$



$$F(r_1)r_1^2 = F(r_2)r_2^2$$



$$F(r_2) = \frac{F(r_1)r_1^2}{r_2^2}$$



- The total amount of energy through r_1 and r_2 must be equal

- This is nearly conservation of energy

- We can treat r_1 as the surface of the star, then r_2 as some distance, d , of a hypothetical observer

$$F_{obs} = \frac{F_* R_*^2}{d^2}$$

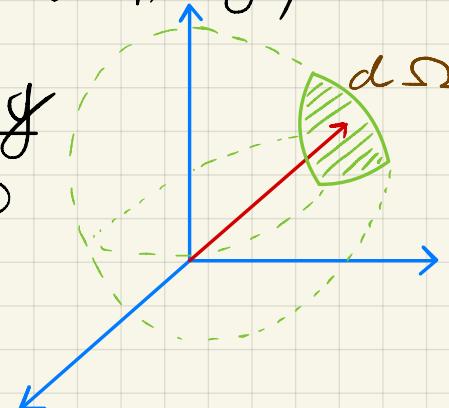
1.3 Specific Intensity and its moments (P.3 - 8)

- an individual ray carries effectively no energy

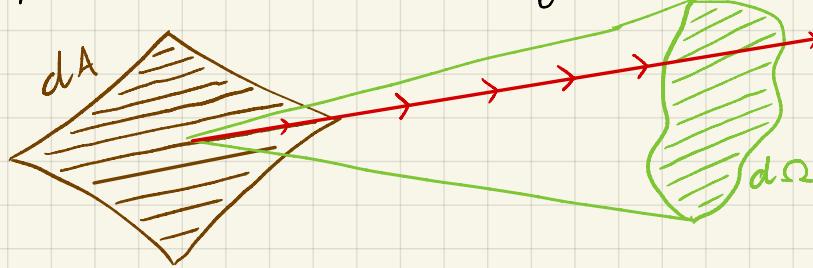
- We need to integrate over something

- Solid angle \rightarrow generalization of angle to 3D

- $\int d\Omega = 4\pi$ (There are 4π steradians in a sphere)



- The energy from dA subtending some solid angle $d\Omega$



- The energy over this solid angle is found by integrating all of the rays passing through.

- We also must recognize that Intensity may change w/ frequency.

The energy passing out of dA in the direction of $d\Omega$ $\rightarrow dE = I_\nu dA dt d\Omega d\nu$

↑
Specific intensity at frequency ν
[energy time⁻¹ area⁻¹ solid angle⁻¹ frequency⁻¹]

- Specific intensity is the amount of energy per unit time per unit area per unit solid angle per unit frequency.
 - note how this will change with orientation and location in source
 - your source may preferentially emit in 1 direction may be brighter in one place, and may emit more in one frequency

Specific Intensity Does Not Depend on Distance*

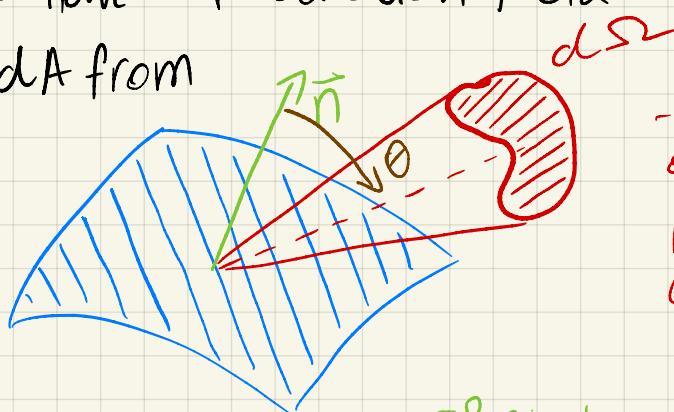
- Net flux

(Put another way, the specific intensity along a ray is constant)

* So long as we are in a vacume and ignoring gravitational redshift

- imagine we have a radiation field

- flux through dA from $d\Omega$



- $d\Omega$ is projected onto dA by $\cos \theta$ reducing the effective area

$$dF_\nu = I_\nu \overbrace{\cos \theta}^{\text{Projects area}} d\Omega$$

converts specific intensity to flux

$$F_\nu = \int I_\nu \cos \theta d\Omega$$

Radiative Energy Density

- energy per unit volume per unit frequency

- We start with energy density per unit solid angle $u_\nu(\Omega)$ such that

$$dE = u_\nu(\Omega) dA c dt d\Omega d\nu$$

- recall

$$dE = I_\nu dA d\Omega d\nu$$

- set these two equal

$$u_\nu(\Omega) dA c dt d\Omega d\nu = I_\nu dA d\Omega d\nu$$

$$u_\nu(\Omega) c = I_\nu \quad \boxed{u_\nu(\Omega) = \frac{I_\nu}{c}}$$

- We integrate over all solid angles

$$u_\nu = \int u_\nu(\Omega) d\Omega \quad \text{Let: } J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

\downarrow

$$d\nu = \frac{4\pi}{c} J_\nu$$

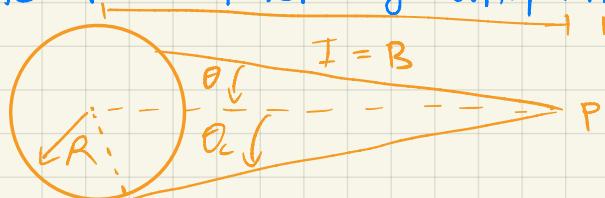
J_ν is called the mean intensity.

- Then the total radiation density is found by integrating u_ν over all frequencies

$$U = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

- We can derive the inverse square law from specific intensity

- consider a sphere of uniform brightness B



Recall $F = \int I \cos\theta d\Omega$

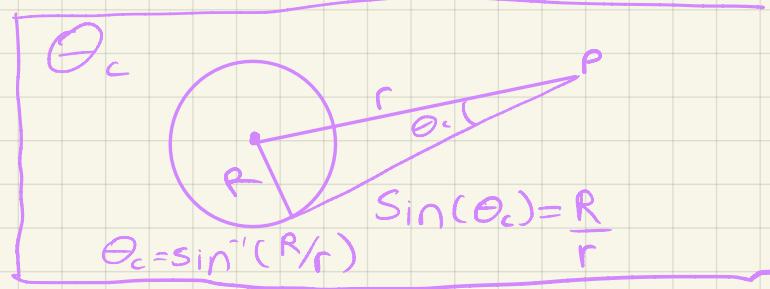
Then

$$F = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin \theta \cos \theta d\theta$$

integrate

$$F = \pi B (1 - \cos^2 \theta_c)$$

trig identity $(1 - \cos^2 \theta) = \sin^2 \theta$



- Therefore

$$F = \pi B \left(\frac{R}{r} \right)^2$$

← Note how the inverse square law fell out

- even though specific intensity is constant along a ray. The solid angle subtended by some area decreases with distance in such a way that the flux obeys an inverse square relation with distance

- consider letting $r = R$ C on the surface of the isotropic source

$$F = \pi B$$

1.4 Radiative Transfer (P.8 - 15)

Key Point: As a ray transmits through matter The specific intensity can change due to a number of factors. (emission, absorption, scattering, etc..)

Definitions
(for omission)

j - spontaneous emission coefficient

$$dE = jdV d\Omega dt$$

j_ν - Monochromatic $\stackrel{\curvearrowright}{j}$ volume differential

$$dE = j_\nu dV d\Omega dt d\nu$$

- generally j_ν and j_ν are dependent on direction of emission. However for an isotropic emitter we can integrate over all solid angles such that

$$j_\nu = \frac{1}{4\pi} P_\nu \quad \begin{matrix} \text{radiated power per unit} \\ \text{volume per unit frequency} \end{matrix}$$

- We could alternatively define some spontaneous emission coefficient which we will call emissivity

N.B. only for isotropic $\rightarrow dE = \epsilon_\nu \rho dV dt d\nu d\Omega \frac{1}{4\pi}$ fractional amount of energy radiated towards $d\Omega$

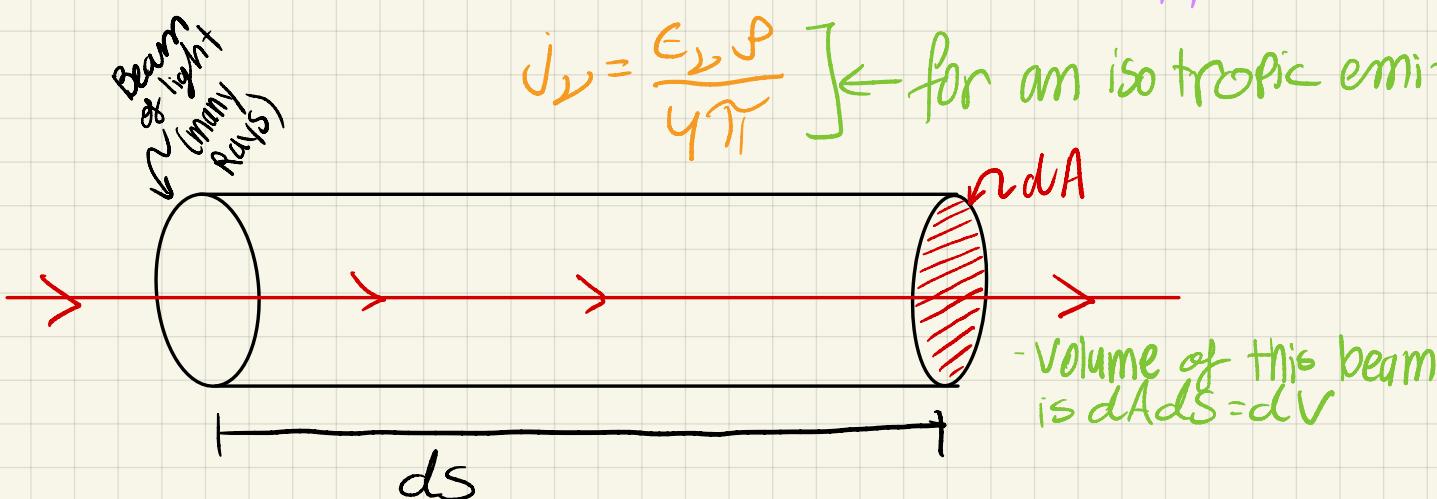
$\epsilon_\nu \rho dV dt d\nu$ mass density differential

$\frac{1}{4\pi}$ energy emitted spontaneously per unit frequency per unit time per unit mass [erg g⁻¹ s⁻¹ Hz⁻¹]

- Let

$$j_\nu d\nu d\Omega dt dV = \epsilon_\nu \rho dV dt d\nu \frac{1}{4\pi}$$

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi} \quad \begin{matrix} \text{for an isotropic emitter} \end{matrix}$$



- We can relate specific intensity to the emission coefficients by equating $dE(I_\nu)$ to $dE(j_\nu)$

differential $\rightarrow dI_\nu dA dt d\Omega d\nu = j_\nu dV d\Omega dt d\nu$

b/c we are thinking about adding energy to the system $dI_\nu = j_\nu ds$

- The change in specific intensity is the emission integrated over some column

Absorption

- Let the absorption coefficient be α_ν (units of cm^{-1}) such that

$$dI_\nu = -\alpha_\nu I_\nu ds$$

- consider some volume with randomly distributed particles having a number density n

- each particle has some effective cross sectional area of σ_ν

- The total area with the volume which can absorb is $N \cdot \sigma_\nu \Rightarrow n \cdot dA \cdot ds$

The total number of particles

This says any light hitting this area is absorbed.

$$dI_\nu dA d\Omega dt d\lambda = I_\nu (n \sigma_\nu dA ds) d\Omega dt d\lambda$$

$$dI_\nu = -n \sigma_\nu I_\nu ds$$

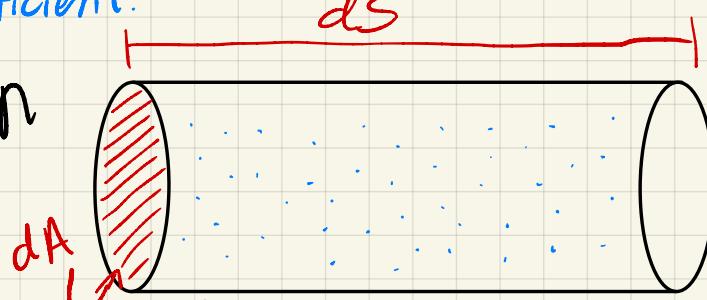
α_ν

- we often write α_ν in a slightly different way which lets us use mass density instead of number density

$$\alpha_\nu = \rho \gamma \kappa_\nu \quad \text{Opacity coefficient}$$

mass density

This is saying that the change in specific intensity is related to the initial specific intensity, the distance that ray travels over and some coefficient.



There are $n \cdot dA \cdot ds$ particles in this volume

- This is only a valid model if the cross section is small compared to the interparticle distance and the particles are randomly distributed

The Radiative transfer equation

- combining absorption and emission into one equation so we can trace how intensity changes as a ray transits a media

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

Solutions

- we will first consider 2 limiting cases

much of the rest of this class will be focused on finding various forms of α and j

NO EMISSION

NO ABSORPTION

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

$$\frac{1}{I_\nu} dI_\nu = -\alpha_\nu ds$$

$$I_\nu(s) = I_\nu(s_0) e^{-\int_{s_0}^s \alpha_\nu(s') ds'}$$

$$\frac{dI_\nu}{ds} = j_\nu$$

$$dI_\nu = j_\nu ds$$

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^s j_\nu(s') ds'$$

Optical depth

- we have been using length dimension s (which for example may be cm, m, ly, any dimension of length).
- We could parameterize distance differently, using something called optical depth, τ_ν .

$$d\tau_\nu = \alpha_\nu ds$$
$$\tau_\nu(s) = \int_{s_0}^s \alpha_\nu(s') ds' \quad \left[\begin{array}{l} \text{if we use this parameterization then} \\ \text{the transfer equation will be much simpler} \end{array} \right]$$

- a media is said to be optically thick when $\tau_\nu > 1$
need a better explanation of e-folding.
- optical depth is a e folding factor representing the number of photons scattered away from the ray.
- using optical depth we can write the transfer equation as

$$-\tau_\nu \text{ as } S_\nu \text{ are more often used than } \alpha_\nu, j_\nu, \text{ and } s$$
$$\frac{dI_\nu}{d\tau_\nu} = I_\nu + S_\nu \quad \left[\begin{array}{l} \text{source function} \\ S_\nu \equiv \frac{j_\nu}{L_\nu} \end{array} \right]$$

Solving the Radiative transfer equation

- Recall

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu + S_\nu$$

$$-\text{ multiply by a fancy 1} = \frac{e^{\tau_\nu}}{e^{\tau_\nu}}$$

$$\frac{e^{\tau_\nu}}{e^{\tau_\nu}} \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

Let t:

$$e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} = e^{\tau_\nu} (-I_\nu + S_\nu)$$

$$e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} = -e^{\tau_\nu} I_\nu + e^{\tau_\nu} S_\nu$$

$$Q = I_\nu e^{\tau_\nu}$$
$$S = S_\nu e^{\tau_\nu}$$

$$e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

Note how this is an expansion through the product rule of

$$\frac{d}{d\tau_\nu} e^{\tau_\nu} I_\nu = e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu$$

Recall we let $q = I_\nu e^{\tau_\nu}$

Recall we let $s = S_\nu e^{\tau_\nu}$

$$\left(\frac{dq}{d\tau_\nu} \right) = s$$

$$\int_{q(0)}^{q(\tau_\nu)} dq = \int_0^{\tau_\nu} s d\tau_\nu$$

This is now a differential equation we can solve!

make sure the limits are consistent!

$$q(\tau_\nu) = q(0) + \int_0^{\tau_\nu} s(\tau_\nu') d\tau_\nu'$$

) re-write using the def's
for q and s

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} s_\nu(\tau_\nu') d\tau_\nu'$$

The initial intensity attenuated by absorption

) integrated source term diminished by absorption

ex: constant source function S_ν

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} d\tau_\nu'$$

$\downarrow 1 - e^{-a}$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$$

$$I_2(\tau_2) = I_2(0)e^{-\tau_2} + S_2 - S_2 e^{-\tau_2}$$

$$I_2(\tau_2) = S_2 + e^{-\tau_2}(I_2(0) - S_2)$$

- Let's set $\tau_2 \rightarrow \infty$ and see what happens!

- Second term goes to 0!

$$\lim_{\tau_2 \rightarrow \infty} I(\tau_2) \rightarrow S_2$$

* for a source function independent of τ_2

- The source function is the quantity the intensity tries to approach given sufficient optical depth

Mean Free Path

- the average distance which a photon can travel before being absorbed.

Let

$$\langle \tau_2 \rangle = \alpha_2 l_2$$

Some quantity called
mean free path

- The probability that a photon travels one optical depth is

$$P = e^{-\tau_2}$$

- alternatively the mean optical depth traveled by a photon is 1

$$\langle \tau_2 \rangle = \alpha_2 l_2 = 1$$

$$\alpha_2 l_2 = 1$$

Recall:

$$l_2 = \frac{1}{\alpha_2} \rightarrow \therefore \alpha_2 = n \sigma_2$$

$$l_2 = \frac{1}{n \sigma_2}$$

$$\langle \tau_2 \rangle = \int_0^\infty \tau_2 e^{-\tau_2} d\tau_2 = 1$$

* for a homogeneous material this is more complex for the homogeneous case

- you will sometimes see local mean free path which is the mean free path in an inhomogeneous medium if it were to act homogeneously w/ local properties to the photon

1.5 Thermal Radiation (P. 15-27)

- Blackbody radiation (Electron population is only dependent on T)

- matter and radiation fields are in thermal equilibrium

- as opposed to thermal radiation

- consider an enclosure at temperature T with a very small hole in it.

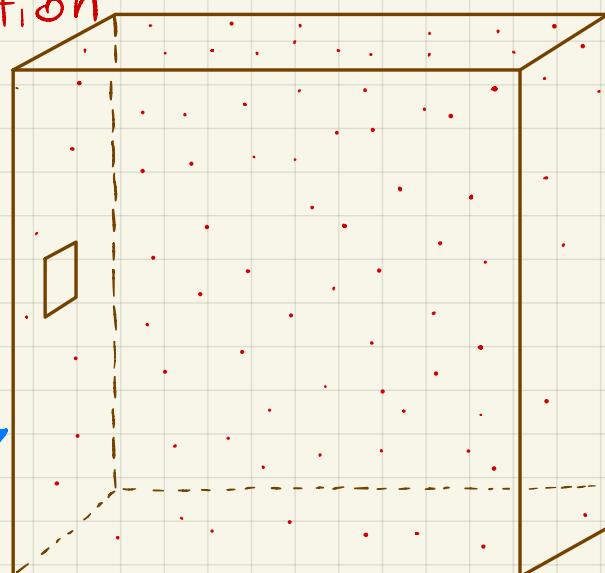
- I_ν is independent of the shape of the container.

- I_ν is solely a function of Temperature

$$I_\nu = B_\nu(T)$$

Temperature
(T)

↑ The Planck function



Kirchoff's Law for Thermal Emission

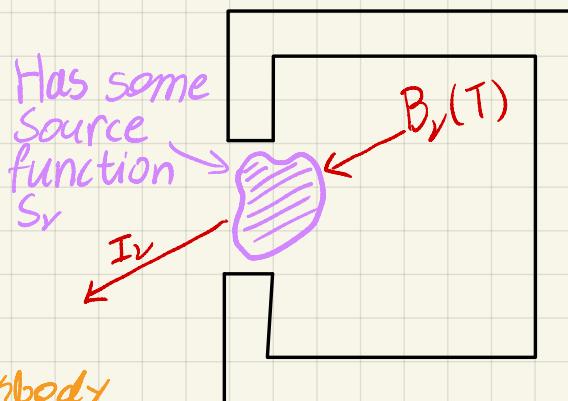
Recall

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu} (I_\nu(0) - S_\nu)$$

- Here $I_\nu(0)$ is $B_\nu(T)$

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu} (B_\nu(T) - S_\nu)$$

- But we know I_ν from a blackbody must purely be a function of T: $S_\nu = B_\nu(T)$



- Let a blackbody container w/ some material come to thermal equilibrium

- Thermal emission becomes blackbody radiation when transmitting through optically thick media

$$j_\nu = \alpha_\nu B_\nu(T) \quad \leftarrow \text{Kirchoff's law}$$

$$\frac{dI_\nu}{ds} = -I_\nu + B_\nu(T)$$

Transfer eqn. for thermal radiation

A Thermodynamic Treatment of Blackbody Radiation

Recall the first two laws of thermo...

$$1. dQ = dU + pdV \quad 2. dS = \frac{dQ}{T}$$

↑
 heat
 ↑
 total energy
 ↑
 entropy

Recall the momentum of an isotropic radiation field

$$P = \frac{1}{3} u \quad \text{The Total radiation density}$$

$$u = \frac{4\pi}{c} \int J_\nu dV \quad J_\nu = B_\nu(T)$$

derive...

$$\frac{dS}{T} = \frac{dU + pdV}{T}$$

u is energy density
so uV is total energy

(Substitution)

$$\frac{dS}{T} = \frac{d(uV) + \frac{1}{3} u dV}{T}$$

Product rule

$$\frac{dS}{T} = \frac{Vdu + udV + u/3 dV}{T}$$

Simplify

$$\frac{dS}{T} = \frac{Vdu}{T} + \frac{4udV}{3T}$$

$du = \frac{du}{dT} dT$

$$\frac{dS}{T} = \frac{VdUdT}{T} + \frac{4udV}{3T}$$

$$\left(\frac{\partial S}{\partial T} \right)_V = \frac{V}{T} \frac{dU}{dT}$$

$$\left(\frac{\partial S}{\partial V} \right)_T = \frac{4u}{3T}$$

$$\frac{1}{u} du = \frac{4dT}{T} \rightarrow \int \frac{1}{u} du = \int \frac{4}{T} dT$$

$$du = -\frac{4u}{3T} dt + \frac{4}{3} du$$

$$1 = -\frac{4u}{3T} dt + \frac{4}{3}$$

integration constant.

$$\ln u = \ln T + \ln a$$

$u(T) = aT^4$

*Stephan Boltzmann law

- Recall that for an isotropic radiation field $I_\nu = J_\nu$ and that

$$\alpha = \frac{4\pi}{c} \int J_\nu d\nu$$

- Therefore

$$aT^4 = \frac{4\pi}{c} \int J_\nu d\nu = \frac{4\pi}{c} \int B_\nu(T) d\nu = \frac{4\pi}{c} B(T)$$

- We can then solve for the integrated Planck function $B(T)$

$$\frac{4\pi}{c} B(T) = aT^4$$

$$B(T) = \frac{a c T^4}{4\pi}$$

- Recall

Deriving σ and a

- Done on Page 25

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3}$$

$$a = \frac{8\pi^5 k^4}{15c^3 h^3}$$

$$F = \gamma \nu B \quad (\text{eqn 1.14})$$

$$F = \frac{a c T^4}{4} \quad \text{Let } \sigma = \frac{a c}{4}$$

$$F = \sigma T^4$$

← most common form of
the Stephan-Boltzmann
law which you will see

Deriving the Planck Spectrum (Adapted From Phy 3700 @)
High Point University, Brad Barlow)

- remember what we want

is basically a distribution of energy over a run of ν

- so we need to count how much energy there is @ each ν

- recall 3D-wave equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

↓ cont... on next
page

- consider a cubic blackbody with side length L

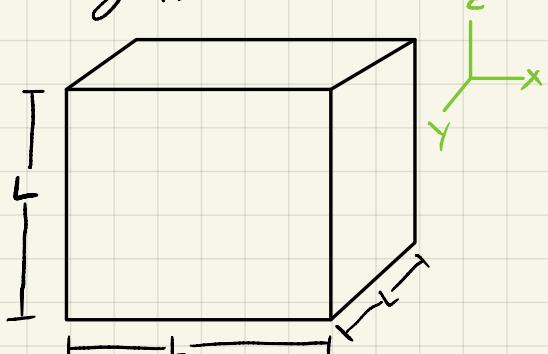
- we want to think about the radiation field inside this box

- The General Solution to the 3D wave eqn. is

$$E(x, y, z, t) = E(x)E(y)E(z)E(t)$$

Boundary

$$\begin{aligned} E=0 & @ x=0, L \\ & y=0, L \\ & z=0, L \end{aligned}$$



apply Boundary conditions to find

$$k_x = \frac{n_x \pi}{L}, \quad k_y = \frac{n_y \pi}{L}, \quad k_z = \frac{n_z \pi}{L}$$

we can then write $\rightarrow E = E_0 \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \sin(\omega t)$

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} E(x, y, z, t) \right) \quad \text{now we plug back into the wave eqn.}$$

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} E_0 \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \sin(\omega t) \right)$$

$$\frac{\partial^2 E}{\partial x^2} = -\frac{E_0 \pi^2 n_x^2}{L^2} \underbrace{\sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \sin(\omega t)}$$

Plug Partialials into wave equation

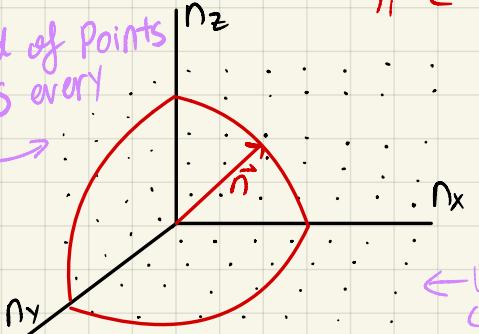
Same for all Partial derivatives so it will cancel out in the wave equation

$$\frac{E_0 \pi^2 n_x^2}{L^2} + \frac{E_0 \pi^2 n_y^2}{L^2} + \frac{E_0 \pi^2 n_z^2}{L^2} = \frac{E_0 \omega^2}{c^2}$$

$$\frac{\pi^2 (n_x^2 + n_y^2 + n_z^2)}{L^2} = \frac{\omega^2}{c^2}$$

$$n_x^2 + n_y^2 + n_z^2 = \frac{\omega L^2}{\pi^2 c^2}$$

This grid of points represents every possible mode



We now need to count the number of available energy states. It is convenient to work in some "n-space" such that

$$n = \sqrt{n_x^2 + n_y^2 + n_z^2} \quad \therefore n = \sqrt{\frac{\omega L^2}{\pi^2 c^2}}$$

Note that $n_i \in \mathbb{Z}^+$

\leftarrow we only consider cases where $n_i > 0$ b/c (first octant of an n-sphere)

- so then the total number of nodes, N , is

$$N = \frac{4}{3} \pi n^3 \left(\frac{1}{8}\right) (2)$$

↑ quantized volume of n -sphere ↑ diminish to just first octant where $n_i > 0$ ↑ each photon can be in 2 possible polarization states

- Now plug in the solution to the wave equation

$$N = \frac{1}{3} \pi \left(\frac{\omega L^2}{\pi^2 c^2} \right)^{3/2} \cdot (2\pi) = 2\pi \nu$$

$$N = \frac{\pi}{3} \left(\frac{4\pi^2 \nu^2 L^2}{\pi^2 c^2} \right)^{3/2}$$

↓ Total number of modes
 ↓ modes per unit frequency per unit volume

$$N = \frac{\pi}{3} \left(\frac{4\nu^2 L^2}{c^2} \right)^{3/2} = \frac{\pi}{3} \frac{8\nu^3 L^3}{c^3}$$

- The number of nodes per unit frequency

$$\frac{dN}{d\nu} = \frac{d}{d\nu} \left(\frac{8\pi L^3 \nu^3}{3 c^3} \right) = \frac{8\pi L^3 \nu^2}{c^3} \rightarrow \frac{(dN/d\nu)}{L^3} = \frac{8\pi \nu^2}{c^3}$$

- There are 4π steradians in a sphere so the modes per frequency per solid angle per unit volume (called the state density, ρ_s)

$$\rho_s = \frac{2\nu^2}{c^3}$$

- Now we need the average energy per state

- each state at a single frequency, ν , may have n photons so the energy is

$$E_n = nh\nu$$

- The average energy is a weighted sum of all possible energies ($n=0, 1, 2, \dots$) by the probability of that state being occupied.

- we then normalize this so it sums to 1 to be a probability

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\frac{E_n}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{E_n}{kT}}} \quad \text{let } \beta \equiv (kT)^{-1} \text{ so}$$

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}$$

- Recall from stat mech that the probability a state of energy E_n is occupied is proportional to

$$P_n \propto e^{-\frac{E_n}{kT}}$$

This simplifies to

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln \left(\sum_{n=0}^{\infty} e^{-\beta E_n} \right)$$

- We can then recognize $\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta nh\nu}$ as a geometric sequence which can take the form $(1 - e^{-\beta h\nu})^{-1}$
- So then the average energy per node

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln(1 - e^{-\beta h\nu})^{-1} = \frac{h\nu}{e^{h\nu/\beta} - 1} \quad \beta \equiv (kT)^{-1}$$

$$\bar{E} = \frac{h\nu}{e^{h\nu/kT} - 1} \quad \leftarrow \text{Bose-Einstein statistics with limitless particles}$$

- Finally the energy per solid angle per unit volume per frequency is

$$U_\nu(\Omega) dV d\nu d\Omega = \left(\frac{2\nu^2}{c^3}\right) \frac{h\nu}{e^{h\nu/kT} - 1} dV d\nu d\Omega$$

$$U_\nu(\Omega) = \frac{2h\nu^3/c^3}{e^{h\nu/kT} - 1}$$

Planck's law

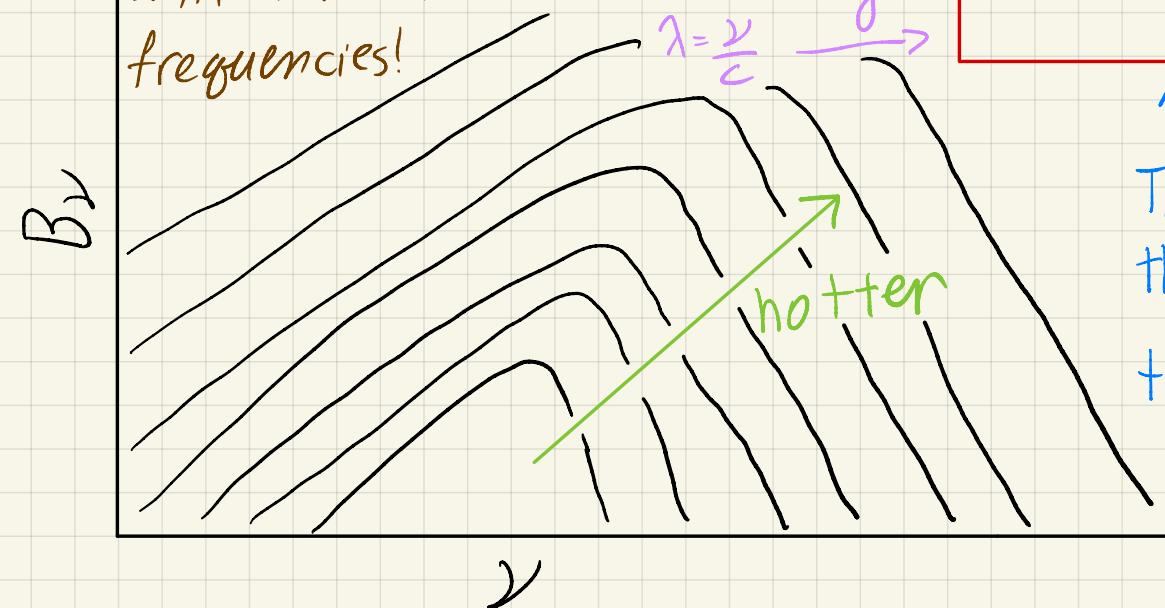
$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

- Recall $U_\nu(\Omega) = I_\nu / c$
and here $I_\nu = B_\nu$

- higher temp blackbodies emit more at all frequencies!

↑ note how this is not as simple as substituting $\lambda = \frac{\nu}{c}$

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}$$



↑ This is one of the most important things we will derive in this class.

Various Properties of the Planck law

- We will now investigate a number of useful limiting cases

$\hbar\nu \ll kT$ (Thermal effects dominate) (The Rayleigh-Jeans Law)

- To first order Series expansion

$$e^{\frac{h\nu}{kT}} - 1 \approx \frac{h\nu}{kT} + \mathcal{O}(n^2)$$

Planch's law

$$B(T) = \frac{2h\nu^3/c^2}{k\nu/kT}$$

$$B(T) = \frac{2\nu^2 kT}{c^2}$$

↑ The low frequency portion of the bb spectrum

$\hbar\nu \gg kT$ (Quantum effects dominate) (Wein's Law)

- The one in the denominator will be very small compared to the exponential so we can drop it

$$e^{\frac{h\nu}{kT}} - 1 \approx e^{\frac{h\nu}{kT}}$$

Planch's law

$$B(T) = \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}$$

↑ The high frequency portion of the bb spectrum

Monotonicity with temperature

- We already mentioned this but just to say it again. The hotter bb is entirely above any cooler bb spectrum. (you can note this by looking at the sign of $\frac{\partial B}{\partial T}$ and how it is always positive)

Wein's displacement law

- Note this is separate from Wein's law.

- relates the Peak frequency to T for B(T)
(or wavelength)

Solve this

$$\left[\frac{\partial B}{\partial \nu} \right]_{\nu=\nu_{\max}} = 0$$

$$B(T) = \frac{2h\nu^3/c^2}{e^{\frac{h\nu}{kT}} - 1}$$

$$\text{Let } \left. \frac{\partial B}{\partial \nu} \right|_{\nu=\nu_{\max}} = 0$$

$$\frac{\partial B}{\partial \nu} = \frac{2\nu^2 h (3Tk(e^{\frac{h\nu}{kT}} - 1) - \nu h(e^{\frac{h\nu}{kT}} - 1)e^{\frac{h\nu}{kT}})}{Tc^2 k (e^{\frac{h\nu}{kT}} - 1)^3}$$

$$\nu_{\max} = \frac{T k [W(-\frac{3}{e^3}) + 3]}{h}$$

$$\nu_{\max} = T k [W(-\frac{3}{e^3}) + 3]$$

Lambert W function

evaluate and select real component

- Wein's displacement law tells you the peak emission frequency for some temperature T of a blackbody

- Peak wavelength emission Wein's displacement \rightarrow does not happen @ the same law
Place as peak frequency emission $\star (\lambda_{\max} \nu_{\max} \neq C) \star$

$$\nu_{\max} \approx 2.82 T K$$

We can identify some characteristic Temperatures of the Planck Spectrum

1. Brightness temperature

- at a specific frequency
- The Temperature where the bb of that frequency has the same intensity as the observed Intensity

$$I_\nu = B_\nu(T_b)$$

- for low frequency, long wavelength work (such as radio)

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu \quad (\text{as long as } h\nu \ll kT)$$

$$T_b = T_b(0) e^{-\tau_\nu} + T(1 - e^{-\tau_\nu})$$

- consider the case when $\tau_\nu \rightarrow \infty$

Temperature
of the material.

$$T_b(\infty) = T_b(0) e^{-\infty} + T(1 - e^{-\infty}) = T_b e^{-\infty} + T - T e^{-\infty}$$

$$T_b(\tau_\nu \rightarrow \infty) = T$$

These Terms go to 0

↑ as a media where $h\nu \ll kT$ becomes optically thick its brightness temp approaches its true temp.

2. Color Temperature

- The Temperature corresponding to Peak emission of some material if that material were a Blackbody
 - Will give the correct temp for a blackbody emitter as well as some optically thin thermal emitters.
 - So long as $\tau_\nu \sim \nu_{\max}$ is fairly constant

3. Effective Temperature

- equate Total observed Flux to Flux from blackbody at T_{eff}

$$F_{\text{obs}} = \sigma T_{\text{eff}}^4$$

1.6 Einstein Coefficients (P. 27-33)

- Kirchhoff's law ($j_\nu = \alpha_\nu B_\nu$) implies there must be a fundamental relationship between emission and absorption
- Consider a system w/ 2 discrete energy levels, E and $E + h\nu_0$. There is some statistical likelihood of occupation, g_1 and g_2 for E and $E + h\nu_0$ respectively.

- We then have the processes to consider. Specifically these "Einstein coefficients" are related to the transition Probability per unit time.

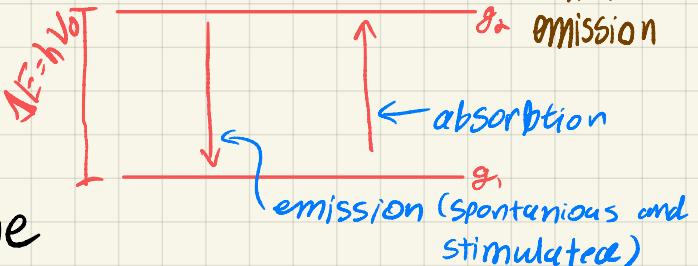
1. Spontaneous emission
2. Absorption
3. Stimulated emission

How to read these

A_{21} ← from energy level 2 → 1
Spontaneous emission

A - spontaneous Emission
B - Absorption and Stimulated

g_2 emission



A_{21} - Transition Probability Per unit time for spontaneous emission.

Absorption

- if an incident photon has energy = ΔE , in this example $h\nu_0$, then the System absorbs the Photon to transition from level 1 → 2

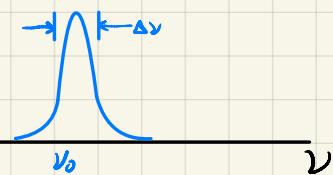
- photons do not have to be exactly at $h\nu_0$ to cause a transition. that is just the most probable frequency. The line profile, $\phi(\nu)$, gives us a probability distribution for the likely hood a photon causes the transition.

- we also need the density of photons at ν_0

$$\int_0^\infty \phi(\nu) d\nu = 1$$

$\phi(\nu)$

We will discuss in chap. 10 the how $\phi(\nu)$ comes about.



- Then we let

$B_{12} \bar{J}$ - Transition Probability per unit time for absorption

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu$$

Aside

- When J_ν slowly varies over $\Delta\nu$ then $\phi(\nu)$ approaches a δ-function at ν_0 . Therefore in many cases

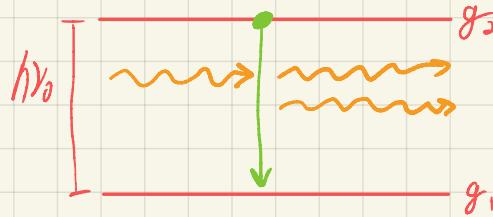
$$B_{12} J_{\nu_0} \sim B_{12} \bar{J}$$

$$B_{21} J_{\nu_0} \sim B_{21} \bar{J}$$

Stimulated Emission

- an incident Photon causes emission from the excited state in the same direction as the incident Photon.

$B_{21} \bar{J}$ - Transition Probability per unit time for stimulated emission.



Relationships between Einstein Coefficients

- if a system is in thermodynamic equilibrium then the number of $1 \rightarrow 2$ transitions must equal the number of $2 \rightarrow 1$ transitions. Therefore for some system in TE with n_1 and n_2 being the number density of atoms in states 1 and 2 respectively

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

$$\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1/g_2)e^{\hbar\nu/kT}(B_{12}/B_{21}) - 1}$$

Population Ratio:

The degeneracy:
The number of quantum states with the same energy

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{\frac{E_1 - E_2}{kT}}$$

$g_i \rightarrow$ The degeneracy of level i (statistical weight)

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{\frac{\hbar\nu}{kT}}$$

Recall:

- in TE $J_{21} = B_{21}$

- further the Planck function does vary slowly on the scale of a resonable $\Delta\nu$ so by the \bar{J} aside on the previous page

$$\bar{J} = J_{21} = B_{21}$$

$$\frac{2\hbar\nu^3/c^2}{e^{\hbar\nu/kT} - 1} = \frac{A_{21}/B_{21}}{(g_1/g_2)e^{\hbar\nu/kT}(B_{12}/B_{21}) - 1}$$

This equality needs to be temperature independent ($\bar{J} = B_{21}$ at all temperatures) this gives us 2 relationships

- Let the denominators and numerators equal each other

A_{21}/B_{21} (numerator)

$$\frac{2\hbar\nu^3}{c^2} = \frac{A_{21}}{B_{21}}$$

$$A_{21} = \frac{B_{21} 2\hbar\nu^3}{c^2}$$

B_{12} and B_{21} (denominator)

$$e^{\hbar\nu/kT} - 1 = \frac{g_1}{g_2} e^{\hbar\nu/kT} \frac{B_{12}}{B_{21}} - 1$$

$$\frac{g_1}{g_2} \frac{B_{12}}{B_{21}} = 1$$

$$g_1 B_{12} = g_2 B_{21}$$

for this to be true

we can see that

$$\frac{g_1}{g_2} \cdot \frac{B_{12}}{B_{21}} = 1$$

known as the detailed Balance relations

- note how these relations are only dependant on atomic properties and not on temperature (as is the case for Kirchoff's laws)

- Implies that these relations hold even when not in thermodynamic equilibrium!

- These relationships extend Kirchoff's laws to non thermal emission.

Thermo aside
- Recall the ratio of atoms in 2 States is given by the Boltzmann factor.

j_ν and α_ν in terms of A and B

- Let us define a **normal system** as one where lower energy states are more populated than higher energy states.
- We will also assume that emission is distributed w/ the same line profile as absorption ($\Phi(\nu)$)
- Both of these are generally good assumptions for astrophysical sources

$$dE = j_\nu dV d\Omega d\nu dt$$

- each atom is isotropically (see over 4π steradians) ν_0 of energy.

$$dE = \frac{h\nu_0}{4\pi} \underbrace{\Phi(\nu)}_{\text{relative}} \underbrace{n_2 A_{21}}_{\text{The probability of a transition happening at } \nu} dV d\Omega d\nu dt$$

The amount of energy released per transition per solid angle

$$j_\nu dV d\Omega d\nu dt = \frac{h\nu_0}{4\pi} \Phi(\nu) n_2 A_{21} dV d\Omega d\nu dt$$

$$j_\nu = \frac{h\nu_0}{4\pi} \Phi(\nu) n_2 A_{21}$$

- We can derive the absorption coefficient as a function of Einstein coefficients by recalling eqn. 1.66 and 1.67

1.66:

$$B_{12} \bar{J} = P_T$$

1.67:

$$\bar{J} = \int_0^\infty J_\nu \Phi(\nu) d\nu$$

$$dE = \frac{h\nu_0}{4\pi} n_1 B_{12} \Phi(\nu) I_\nu dV dt d\Omega d\nu$$

Energy per transition per unit solid angle.

Transition Probability for the Whole System Per unit time Per unit frequency Per unit Volume

- Assume dV is some cylinder such that $dV = d\lambda ds$
- Recall

$$I_\nu dA dt d\Omega d\nu = \frac{h\nu_0}{4\pi} n_1 B_{12} \Phi(\nu) I_\nu dA ds d\Omega d\nu$$

$$dE = I_\nu dA dt d\Omega d\nu$$

and

$$dI_\nu = \alpha_\nu I_\nu ds$$

↑ Intensity loss is proportional to the length of travel (ds) and the rate of intensity change per unit length

$$I = \frac{h\nu_0}{4\pi} n_1 B_{12} \Phi(\nu) ds$$

$$I - \frac{h\nu_0}{4\pi} n_1 B_{12} \Phi(\nu) ds \leftarrow \text{column length}$$

rate of energy change $\alpha_\nu = \frac{h\nu_0}{4\pi} n_1 B_{12} \Phi(\nu)$

- How do we add stimulated emission into this?

- Because stimulated emission is proportional to the same parameters as absorption it is more convenient to treat it as negative absorption

$$\alpha_\nu = \frac{h\nu_0}{4\pi} \phi(\nu) (n_1 B_{12} - n_2 B_{21})$$

- Recall the radiative transfer equation

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

$$\frac{dI_\nu}{ds} = -\frac{h\nu_0}{4\pi} \phi(\nu) (n_1 B_{12} - n_2 B_{21}) I_\nu + \frac{h\nu_0}{4\pi} \phi(\nu) n_2 A_{21}$$

- Recall the source function, S_ν ,

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

$$S_\nu = \frac{(h\nu_0/4\pi) \phi(\nu) n_2 A_{21}}{(h\nu_0/4\pi) \phi(\nu) (n_1 B_{12} - n_2 B_{21})}$$

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} \quad \text{however we also already know that } A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Generalized Kirchoff's law



$$S_\nu = \frac{2h\nu^3}{c^2} \left(\frac{g_2 n_1}{g_1 n_2} - 1 \right)^{-1}$$

$$B_{12} = \frac{g_2}{g_1} B_{21}$$

- We will look at 3 cases for these recent equations

I Thermal Emission (LTE)

- Matter is in equilibrium with itself but not with the radiation field then...

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{\frac{h\nu}{kT}}$$

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[1 - e^{-\frac{h\nu}{kT}} \right] \phi(\nu)$$

$$S_\nu = B_\nu(T)$$

2. Non-Thermal Emission

- Cases where the populations do not obey a Maxwell-Boltzmann distribution

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} e^{\frac{-E_1}{kT}}$$

3. Masers (Inverted Population)

- Normal Population when

$$\frac{n_1}{g_1} > \frac{n_2}{g_2}$$

- Inverted Population when

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}$$

- The absorption coefficient is (-) resulting in amplification of intensity along a ray

- This amplification can be very large. for example $\tau \sim 100$ cm can result in 10^{43} times amplification.

1.7 Scattering (P. 33-39)

- Whereas Thermal emission depends only on Temperature and not on the incident radiation field, scattering only depends on the incident radiation field.

Assumptions (more complex treatment in Chap. 7)

- Isotropic Scattering

- Scattered radiation is emitted uniformly over all solid angles.

- Coherent Scattering

- The total emitted Radiation at ν_0 = The Total absorbed radiation at ν_0

- Absorption + Emission Coefficients

emission coefficient $\rightarrow J_\nu = \sigma_\nu J_\nu$

$$d\tau = (\alpha_\nu + \sigma_\nu) ds$$

↑ Scattering coefficient (absorption)

- Transfer equation

$$\frac{dI_\nu}{ds} = -\sigma_\nu (I_\nu - J_\nu)$$

extinction coefficient

- Source function

$$S_\nu = \frac{J_\nu}{\sigma_\nu}$$

$$S_\nu = \frac{j_\nu J_\nu}{j_\nu} = J_\nu$$

$S_\nu = J_\nu$ ← for isotropic

coherent scattering

$$S_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

- It is no longer possible to use our early solution to the transfer eqn. b/c S_ν depends on I_ν

$$\frac{dI_\nu}{ds} = -\sigma_\nu \left(I_\nu - \frac{1}{4\pi} \int I_\nu d\Omega \right)$$

← intero-differential equation