SAha + Broadening

A74

Fall 2024

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9.5 Thermal distribution of Energy levels (P. 259-263)

Thermal Equilibrium

- generally population occupation is very hard to find.
- in Thermal Equilibrium However occupation is only a function of Temperature
  - in Thermal equilibrium Particles the highest entropy state and the Boltzmann maximizes entropy and is only a function of temperature
- The occupation of any given state is Proportional to $g e^{-\beta E}$ where $\beta = 1/kT$
- g is the degeneracy (also called the statistical weight)
  - The number of distinct quantum states sharing the same energy level. A good way to think of this is the number of ways which a particular energy can be reached.

- for the Purpose of this class we will limit or conversation to L-S coupling
  - coupling of the Spin angular momentum to the orbital angular momentum to get total angular momentum
- valid for light atoms

$$\underbrace{\vec{J}}_{\text{Total angular momentum}} = \underbrace{\vec{L}}_{\text{orbital angular momentum}} + \underbrace{\vec{S}}_{\text{spin angular momentum}}$$

- for L-S coupling we can find the degeneracy of an energy level $g = (2J+1)$

- The Boltzmann law

$$\underbrace{N_i}_{\text{number of atoms in state } i} = \frac{\overbrace{N}^{\text{Total number of atoms}}}{\underbrace{U}_{\text{Partion function}}} \underbrace{g_i}_{\text{degeneracy of state } i} e^{-\beta E_i}$$

$$N = \sum N_i$$

$$U = \sum g_i e^{-\beta E_i}$$

- at low temperatures $e^{-\beta E_i} \sim 1$ and only the ground state contributes to the Partition function, so $U = g_0$

- Typically one would compare two energy levels i and j so that

$$\boxed{\frac{N_i}{N_j} = \frac{g_i}{g_j} e^{[-\beta(E_j - E_i)]}}$$

for Hydrogen

$g = 2n^2$

$E_n = -13.6\,eV/n^2$

The Saha Equation

- feel free to derive yourself but we will not work through it in class

- The Saha equation tells you the relative occupation of diffrent ionization states.

$$\boxed{\frac{N_{i+1}}{N_i} = \frac{2U_{i+1}}{n_e U_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}}$$

ionization energy of ionization state i

number density of electrons in surranding media

example
- consider an electron ideal gas with number density of Particles n_e and Pressure P_e, then

$$P_e = n_e kT$$
$$n_e = \frac{P_e}{kT}$$

- Then plug into Sah

$$\frac{N_{i+1}}{N_i} = \frac{2kTU_{i+1}}{P_e U_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}$$

- at what temperature are ½ of hydrogen ionized if $P_e = 20\ N/m^2$?

$$\frac{1}{2} = \frac{N_{II}}{N_{Total}} = \frac{N_{II}}{N_I + N_{II}} = \frac{(N_{II}/N_I)}{1+(N_{II}/N_I)}$$

$$\frac{N_{II}}{N_I} = \frac{2kTU_{II}}{P_e U_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}$$

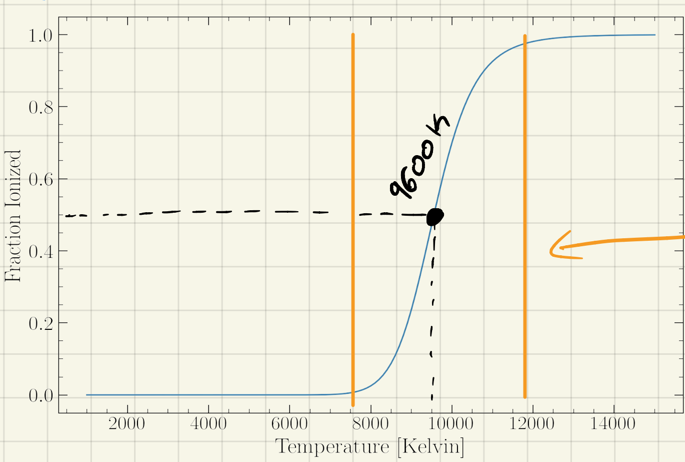
← Let $U_{II} = 1$ (no electrons), $U_I \sim 2$ $(U \sim g_2 \sim 2)$ then plug into CAS

Letting $N_{II}/N_I = \lambda$ and solving $\frac{1}{2} = \frac{\lambda}{1+\lambda}$ for (T)

$$\boxed{T \cong 9600K}$$

Temperature (roughly b/c of our sloppy handeling of Partition functions) where hydrogen emission Peaks.

Plot of $\frac{N_{II}}{N_{Total}}$ (what fraction are ionized)



Partial ionizon zone where large amounts of emission take Place

Emission Line Strength

- if we want an analytic way of understanding strengh of lines we need both Saha and Boltzmann

ex Consider a Balmer line (n=2)

$$\frac{N_2}{N_{Total}} = \frac{N_2}{N_{total}}\left(\frac{N_I}{N_I}\right)$$ ← almost all neutral hydrogen atoms are in n=1 or n=2, so $N_I \cong N_1 + N_2$

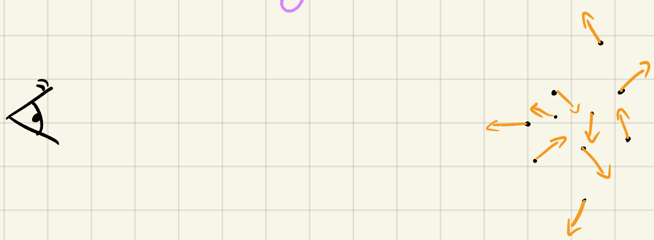
analytic expression for strength of Balmer A (Peak at 9900K)

$$\frac{N_2}{N_{total}} = \frac{N_2}{N_{Tot}}\left(\frac{N_I}{N_I}\right) = \frac{N_2}{N_{tot}}\left(\frac{N_I}{N_1+N_2}\right) = \left(\frac{N_2/N_1}{1+(N_2/N_1)}\right)\left(\frac{N_I}{N_{Tot}}\right) = \left(\frac{N_2/N_1}{1+(N_2/N_1)}\right)\left(\frac{1}{1+(\frac{N_{II}}{N_I})}\right)$$

10.6 Line Broadening (P. 287-291)

- Recall from section 1.6 (Einstien cœfficents) the line Profile $\phi(\nu)$
  - The shape of $\phi(\nu)$ is complicated. Here we will Treat 3 effects which Play a major roll in shaping $\phi(\nu)$

Doppler Broadining



- In each of these Particles rest frame they emmit at ν_0
- In an observer frame they emit at a shifted frequency as a function of their radial velocity

- We can capture the shif in frequency for some Particle moving along the line of sight, V_r

rest frame frequency

$$\Delta\nu = \frac{\nu_0 V_r}{c}$$

- The number of Particles with velocity V, $V_r < V < V_r + dV_r$

$$N = e^{-mV_r^2/2kT}\, dV_r$$

mass of Particle

- We also know that

$$V_r = \frac{c(\nu - \nu_0)}{\nu_0}$$

- Therefore

$$dV_r = \frac{c\,d\nu}{\nu_0}$$

$$N = e^{-mc^2(\nu-\nu_0)^2/2\nu_0^2 kT}\,(c/\nu_0)\,d\nu$$

- from here we get a Profile function

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}}\, e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2}$$

$$\boxed{\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}}$$

Doppler Broadening term

naútral Broadening

- We cannot have a Perfect constraint on Position and momentum. naútral Broadening originates from this

$$\Delta E\,\Delta t \sim \hbar$$

Time the state is occupied

Spread of energy in the state

- from this We get the "naútral line Profile (note how it is only a function of ν)

Lorentz Profile →

$$\boxed{\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}}$$

spontanious decay rate of atom in state n

Sum over all states n' of lower energy than n

$$\gamma = \sum_{n'} A_{nn'}$$

← Einsien A Coefficient

Collisional Broadening

- Because emission happens over some finite time, Δt, if Δt is somehow made smaller (say if the atom collides with another before emitting.) Then ΔE must grow to satisfy $\Delta E \Delta t \sim \hbar$

- assume collisions occure with an average frequency (average $\frac{1}{\text{time}}$ between collisions) of ν_{col} Then let, Γ be

$$\Gamma = \gamma + 2\nu_{col}$$

$$\boxed{\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2}}$$

- a reasonable ν_{col} estimate can be made by considering the number density, cross section, and thermal speed of the atoms

$$\nu_{col} \sim v_t n \sigma$$

- Collisional Broadening is most important when number density is high (High Pressure or Surface gravity)

  - in stars important for low-mass main sequence stars
  - extremly important for neutron stars and white dwarfs

- In addition to these three kinds of Broadening there is also Zeeman broadening and rotational Broadening.