# MTH 351: Homework 2

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1. Derivation of the Taylor Polynomial $P_{2n+1}(x)$ for $sin(x)$ centered at $x = 0$.

   A Taylor Polynomial is constructed by the expression $P_k(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + ... + b_k(x - a)^k$. Since the Taylor Polynomial is centered at the origin $a = 0$. The coefficients $b_n$ are given by $\frac{f^k(a)}{(k!)}$. The first four terms are calculated below.

$$b_0 = \frac{sin(0)}{0!} = 0 \tag{1}$$

$$b_1 = \frac{cos(0)}{1!} = 1 \tag{2}$$

$$b_2 = \frac{-sin(0)}{2!} = 0 \tag{3}$$

$$b_3 = \frac{-cos(0)}{3!} = \frac{-1}{6} \tag{4}$$

   From these first four coefficients and the fact the higher order derivatives will follow the same pattern it is clear that the Taylor Polynomial of $sin(x)$ will only include terms with odd degrees and their sign will flip every term. Using this, a general expression for each coefficient can be written as $b_n = \frac{-1^n}{(2n+1)!}$ and the Taylor Polynomial as $P_{2n+1} = \Sigma_{m=0}^{n} \frac{-1^m}{(2m+1)!} x^{2m+1}$ where $k = 2n + 1$.

2. Explanation of how to use identities to improve approximation.

   The two identities used to improve the Taylor's Series approximation of $sin(x)$ are $x = 2\pi * n + x_1$ and $sin(x) = -sin(x - \pi)$. The first identity is useful as $sin(x) = sin(x_1)$ since sine has a period of $2\pi$. Using the modulo function any value of $sin(x)$ can be computed using an x in the range $[0, 2\pi]$. This range can be shifted such that it is centered around $x = 0$ using the second identity, $sin(x) = -sin(x - \pi)$. The resulting range from shifting every x by $\pi$ is $[-\pi, \pi]$.

3. Explanation of how using identities reduces error.

   Taylor's Remainder Theorem gives the maximum error between the Taylor Polynomial and original function as $\frac{(x-a)^{(k+1)}}{(k+1)!} f^{(k+1)}(z)$ where $z$ is a number between $a$ and $x$. As $f^{(k+1)}(z)$ has to be between -1 and 1 the most significant term in increasing the error is $(x - a)^{k+1}$. From this term is it clear that the closer to $a$ $x$ is the smaller the error. The use of identities in the Taylor Polynomial reduced the range of $x$ from $[0, 4\pi]$ (given in assignment) to $[-\pi, \pi]$. This reduced the maximum distance between $x$ and $a$ from $4\pi$ to $\pi$. Using $n = 13$ (from the assignment) the maximum error reduces from $\frac{(4\pi)^{(27+1)}}{(27+1)!} \approx 19.67$ to $\frac{(\pi)^{(27+1)}}{(27+1)!} \approx 2.729 * 10^{-16}$.