VOICE

RHS OPERATORS

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1 Krook source and sink

The following operator is a BGK term [1] that can act as a source or a sink of particles, momentum and energy. It has the following form:

$$S_{\text{BGK}}(x, v_a, t) = -\nu_\text{a}\mathcal{M}\left(f_a - g\right)$$

The ν_a coefficient sets the magnitude of the operator. It can be either constant or space and time dependent, see below. The mask function $\mathcal{M}(x)$ defines the spatial extent of the simulation box where the operator is active. It is a hyperbolic tangent function that takes a value of one inside a specific domain, and vanishes outside. This mask can have two different forms. The first one is such that the mask equals one inside the domain $[x_\ell, x_\text{r}]$. The Krook operator can then be used for instance as a source of particles active in this region. The mask function is thus defined in this case as

$$\mathcal{M}_{\text{sc}}(x) = \frac{1}{2}\left\{\tanh\left(\frac{x - x_\ell}{d}\right) - \tanh\left(\frac{x - x_\text{r}}{d}\right)\right\}. \tag{1}$$

The x_ℓ and x_r positions are defined using the total simulation box length L_x and the factor p_{mask} as

$$x_\ell = p_{\text{mask}}L_x; \quad x_\text{r} = (1 - p_{\text{mask}})L_x.$$

The second option is to have a mask of vanishing value in the domain $[x_\ell, x_\text{r}]$. The Krook operator can be used in this case as sink of particles that absorbs particles outside of this domain. The mask is then written as

$$\mathcal{M}_{\text{sk}}(x) = 1 - \frac{1}{2}\left\{\tanh\left(\frac{x - x_\ell}{d}\right) - \tanh\left(\frac{x - x_\text{r}}{d}\right)\right\}, \tag{2}$$

The d parameter controls the stiffness of the mask function. The target distribution function g is a maxwellian characterized by a constant density n_t and temperature T_t. It is written in normalized units as

$$g(v) = \frac{n_\text{t}}{\sqrt{2\pi T_\text{t}}}\exp\left(-\frac{v^2}{2T_\text{t}}\right).$$

Depending on the values given to n_t and T_t, the BGK operator will absorb or inject particles and energy. Two different versions of the operator exist: the ν coefficient can be either constant, or depend on the time and space variables.

Constant Krook operator: In the case of constant ν_a coefficients the differential equation $\partial_t f_\text{a} = S_{\text{BGK}}$ is analytical. Its solution is given by

$$f_a(t + dt) = g_\text{t} + (f_a(t) - g_\text{t})\exp\left(-\nu\mathcal{M}dt\right).$$

Table 1: Krook input parameters significance.

extent	stiffness	amplitude	density	temperature
p_{mask}	d	ν_{i}	n_{t}	T_{t}

Adaptive Krook operator: It is possible to chose a constant value of ν_{i} and to adapt ν_{e} at each timestep and spatial position such that

$$\nu_{\text{i}} = \text{cte}; \quad \nu_{\text{e}} = \nu_{\text{i}} \frac{n_{\text{i}} - n_{\text{t}}}{n_{\text{e}} - n_{\text{t}}}.$$

Note here that the n_a quantity is the density of species a, and thus depends on space and time. The $\partial_t f_{\text{a}} = S_{\text{BGK}}$ equation is no longer analytical and is solved with an explicit scheme (RK2 scheme for instance).

To verify the implementation of the operator a simple test is conducted, valid for both constant and adaptive ν_{a} coefficients. Integrating the $\partial_t f_{\text{i}} = S_{\text{BGK}}$ equation against the velocity variable yields

$$\partial_t n_{\text{i}} = -\nu_{\text{i}} \mathcal{M} \left(n_{\text{i}} - n_{\text{t}} \right),$$

which is an equation with analytical solutions. The input parameters of the code allowing for tuning the characteristics of the BGK operator are summarized in Table 1. Two additional parameters, namely `name` $\in \{$ "*source*", "*sink*" $\}$ and `type` $\in \{$ "*constant*", "*adaptive*" $\}$, are available. The first one allows for for choosing between the masks of Eq. (1) and Eq. (2). The second one defines the behavior of the ν_{a} coefficients: constant or adaptive.

2   Kinetic source

The source term developed hereafter is inspired by the one currently implemented in GYSELA [2, Appendix A]. It allows to inject independently density and energy. It has the following normalized expression

$$S_{\text{sc}}(x, v_s) = s_{\text{k}} \frac{\mathcal{M}_{\text{sc}}(x)}{\int_0^{L_x} \mathcal{M}_{\text{sc}}(x) \, dx} S_{\text{v}}(v_s) \tag{3}$$

The normalized velocity variable $v_s = v/v_{\text{th}_s}$ is simply denoted by v in the following. The mask function $\mathcal{M}_{\text{sc}}$ defines the spatial extent of the source. It has a hyperbolic tangent expression of the form

$$\mathcal{M}_{\text{sc}}(x) = \frac{1}{2} \left\{ \tanh \left(\frac{x - x_{\text{sc}}^{\text{l}}}{d_{\text{sc}}} \right) - \tanh \left(\frac{x - x_{\text{sc}}^{\text{r}}}{d_{\text{sc}}} \right) \right\}$$

Where x_{sc}^{l} and x_{sc}^{r} define the left and right position of the region inside which the source is active. The d_{sc} parameter controls the steepness of the source mask. The S_{v} term is written as

$$S_{\text{v}}(v) = \left\{ s_0 \left(\frac{3}{2} - \frac{v^2}{2T_{\text{sc}}} \right) + s_2 \left(-\frac{1}{2} + \frac{v^2}{2T_{\text{sc}}} \right) \right\} \frac{1}{\sqrt{2\pi T_{\text{sc}}}} e^{-\frac{v^2}{2T_{\text{sc}}}} \tag{4}$$

In Eq. 3 the L_x term stands for the simulation box length. The T_{sc} parameter defines the source temperature, which is constant in space and time. The s_0, s_2 and s_{k} are numerical inputs of the code that allows to define the properties of the source. In general we use $s_0 = 1$ so that the

magnitude of the source is controlled by the parameter s_k alone. In this case we have indeed $\int dx \int dv\, S_\mathrm{sc} = s_\mathrm{k}$. Using $s_0 = s_2 = 1$ we recover a Maxwellian source, i.e. in this case

$$S_\mathrm{v}(v) = \frac{1}{\sqrt{2\pi T_\mathrm{sc}}} e^{-\frac{v^2}{2T_\mathrm{sc}}} \tag{5}$$

Conversely, by taking $s_2 = 0$ the expression of S_v has a vanishing first moment, that is to say in this particular case $\int dv\, S_\mathrm{v} = 0$, the source injects energy but no particles. Whatever the values given to s_0 and s_2 the source is symmetric with respect to $v = 0$ therefore it does not inject any net momentum.

2.1 Derivation of the source

Let $S_\mathrm{v}(v)$ be a general source term. We decompose it on the Hermite polynomials basis. By doing this it becomes possible to adjust independently the amount of particles, momentum and energy that this source injects. For two functions f and g let us first introduce the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(y)g(y)e^{-y^2} dy \tag{6}$$

The first three Hermite polynomials are written as

$$H_0 = 1 \qquad\qquad |H_0|^2 = \sqrt{\pi}$$
$$H_1 = 2X \qquad\qquad |H_1|^2 = 2\sqrt{\pi}$$
$$H_2 = -2 + 4X^2 \qquad\qquad |H_2|^2 = 8\sqrt{\pi}$$

The Hermite polynomials form an orthogonal basis for the scalar product defined above, i.e. $\langle H_h, H_h' \rangle = \delta_{h,h'} |H_h|^2$. $\delta_{h,h'}$ is the Kronecker symbol that verifies $\delta_{h,h'} = 1$ if $h = h'$ and $\delta_{h,h'} = 0$ otherwise. Projecting the source term S_v on this basis gives

$$S_v(v) = \sum_{h=0}^{+\infty} c_h\, H_h\left(\frac{v}{\sqrt{2T_\mathrm{sc}}}\right) e^{-\frac{v^2}{2T_\mathrm{sc}}} \tag{7}$$

The c_h terms are real valued coefficients. We also introduced here the source temperature T_sc. We now take first three moments of this expansion to retrieve the particle, momentum and energy fluid sources associated with the S_v source term. Using the orthogonality of the Hermite basis we can show that these fluid sources are expressed as

$$\int_{-\infty}^{+\infty} dv\, S_\mathrm{v}(v) = \sqrt{2T_\mathrm{sc}} \sum_h \langle H_0, c_h H_h \rangle = \sqrt{2\pi T_\mathrm{sc}}\, c_0$$

$$\int_{-\infty}^{+\infty} dv\, v S_\mathrm{v}(v) = T_\mathrm{sc} \sum_h \langle H_1, c_h H_h \rangle = 2\sqrt{\pi} T_\mathrm{sc}\, c_1$$

$$\int_{-\infty}^{+\infty} dv\, \frac{1}{2} v^2 S_\mathrm{v}(v) = \sqrt{2\pi} T_\mathrm{sc}^{3/2} \left(2c_2 + \frac{1}{2}c_0\right)$$

Neglecting in the source expansion Eq. 7 all the terms for $h > 2$ we can obtain a source $\mathcal{S}_n$ that injects particles but no energy nor momentum by setting $c_1 = 0$ and $c_2 = -\frac{1}{4}c_0$. This source can be written as

$$\mathcal{S}_{\mathrm{n}}(v) = c_0 \left(\frac{3}{2} - \frac{v^2}{2T_{\mathrm{sc}}} \right) e^{-\frac{v^2}{2T_{\mathrm{sc}}}} \tag{8}$$

Conversely, a source that injects only momentum but no net particles nor energy is defined by $c_0 = 0$ and $c_2 = 0$. It is expressed as

$$\mathcal{S}_{\mathrm{u}}(v) = c_1 \sqrt{\frac{2}{T_{\mathrm{sc}}}} e^{-\frac{v^2}{2T_{\mathrm{sc}}}} \tag{9}$$

Lastly, a source that injects only energy can be obtained with $c_0 = 0$ and $c_1 = 0$. It is written as

$$\mathcal{S}_{\mathrm{h}}(v) = 2c_2 \left(-1 + \frac{v^2}{T_{\mathrm{sc}}} \right) e^{-\frac{v^2}{2T_{\mathrm{sc}}}} \tag{10}$$

We construct our source S_{v} using the three independent sources above as $S_{\mathrm{v}} = \mathcal{S}_{\mathrm{n}} + \mathcal{S}_{\mathrm{u}} + \mathcal{S}_{\mathrm{h}}$. By tuning the c_0, c_1 and c_2 parameter we can inject independently particles, momentum and energy in the plasma. In practice we always take $c_1 = 0$, i.e. the source does not inject any net momentum. Additionally by using $s_0 = \sqrt{2\pi T_{\mathrm{sc}}}c_0$ and $s_2 = \sqrt{2\pi T_{\mathrm{sc}}}c_2$ as input parameters in the code we are left with Eq. 4 for the expression of S_{v}.

References

[1] P. L. Bhatnagar, E. P. Gross, and M. Krook. "A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems". In: *Phys. Rev.* 94 (3 May 1954), pp. 511–525. DOI: 10.1103/PhysRev.94.511. URL: https://link. aps.org/doi/10.1103/PhysRev.94.511.

[2] Y. Sarazin et al. "Predictions on heat transport and plasma rotation from global gyrokinetic simulations". In: *Nuclear Fusion* 51.10 (Sept. 2011), p. 103023. DOI: 10.1088/0029-5515/51/10/103023. URL: https://doi.org/10.1088/0029-5515/51/10/103023.