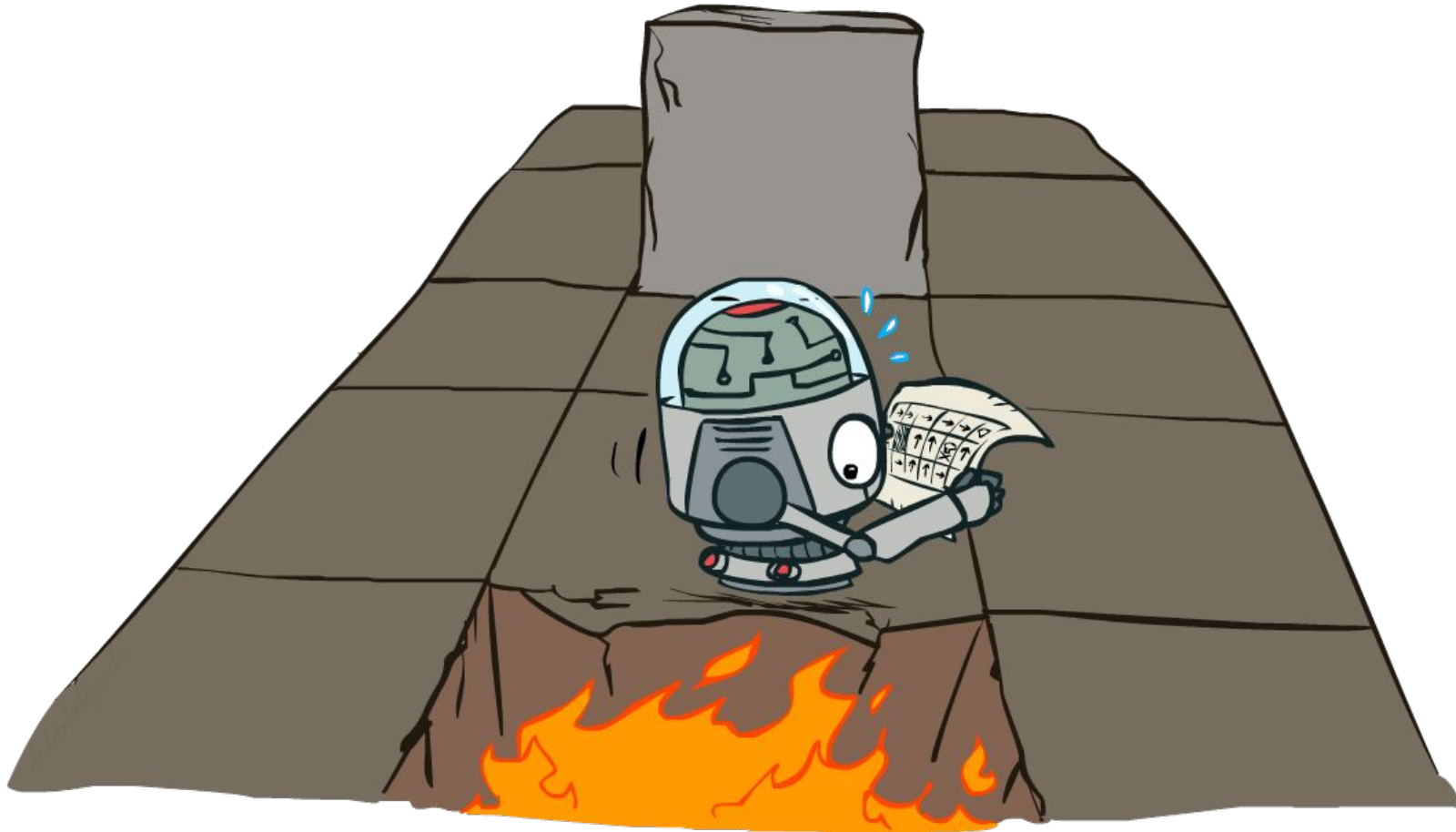


CS 4700: Foundations of Artificial Intelligence

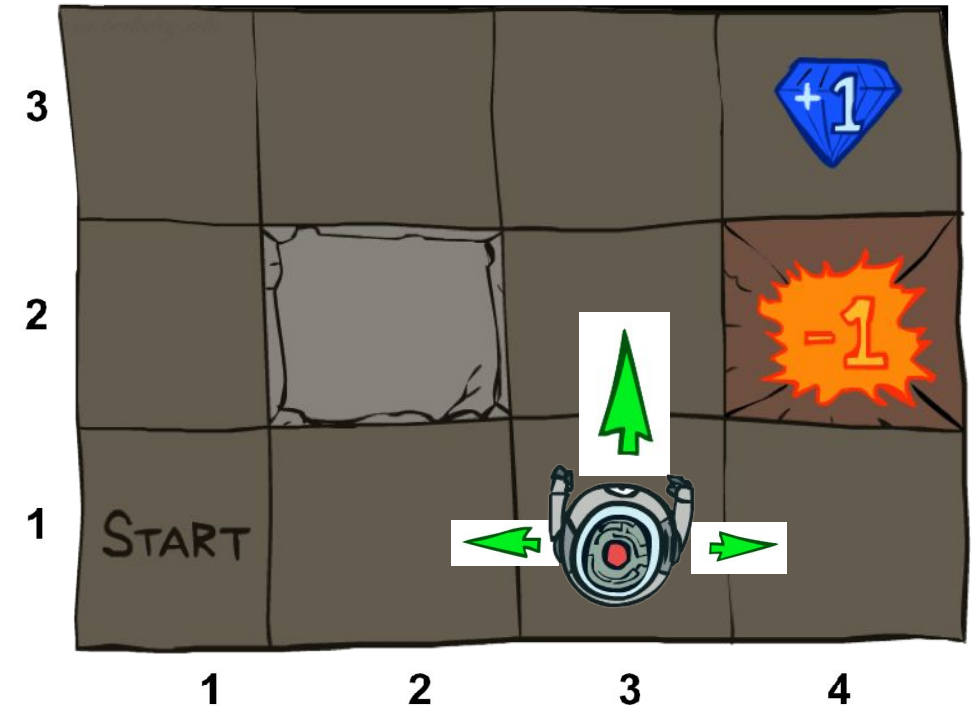
Markov Decision Processes II



Instructor: Kevin Ellis --- Cornell University

Example: Grid World

- A maze-like problem
 - The agent lives in a grid
 - Walls block the agent's path
- Noisy movement: actions do not always go as planned
 - 80% of the time, the action North takes the agent North
 - 10% of the time, North takes the agent West; 10% East
 - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
 - Small “living” reward each step (can be negative)
 - Big rewards come at the end (good or bad)
- Goal: maximize sum of (discounted) rewards



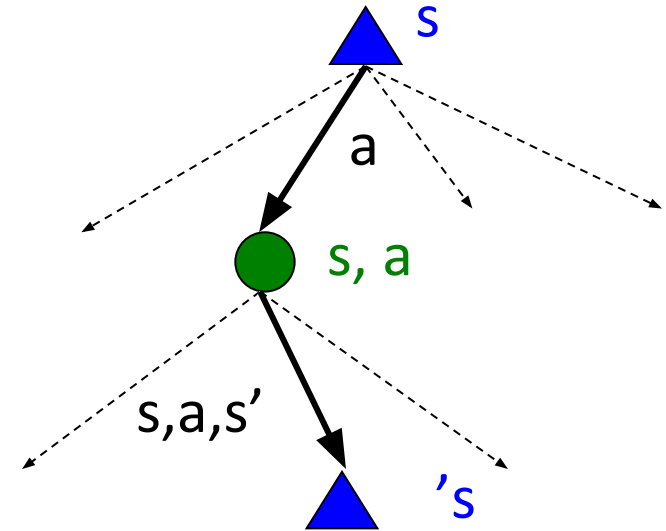
Recap: MDPs

- Markov decision processes:

- States S
- Actions A
- Transitions $P(s' | s, a)$ (or $T(s, a, s')$)
- Rewards $R(s, a, s')$ (and discount γ)
- Start state s_0

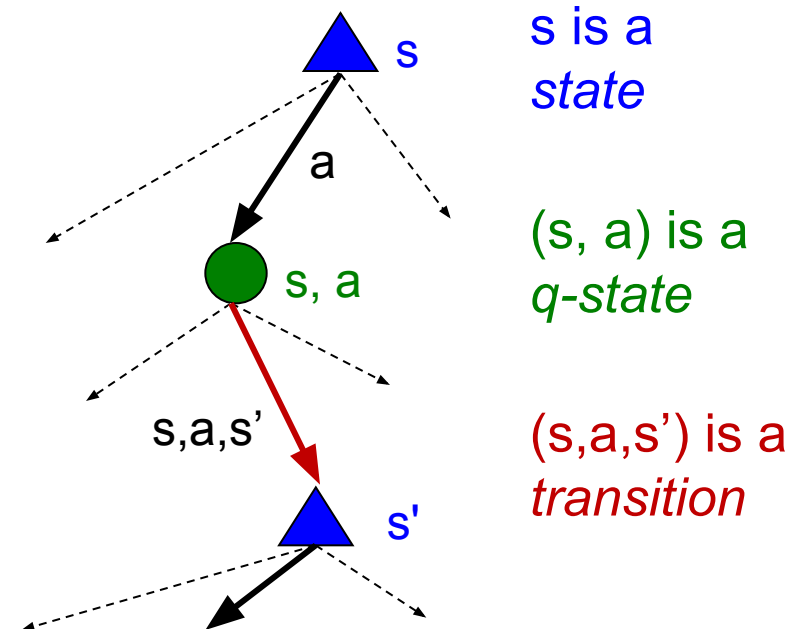
- Quantities:

- Policy = map of states to actions
- Utility = sum of discounted rewards
- Values = expected future utility from a state (max node)
- Q-Values = expected future utility from a q-state (chance node)



Optimal Quantities

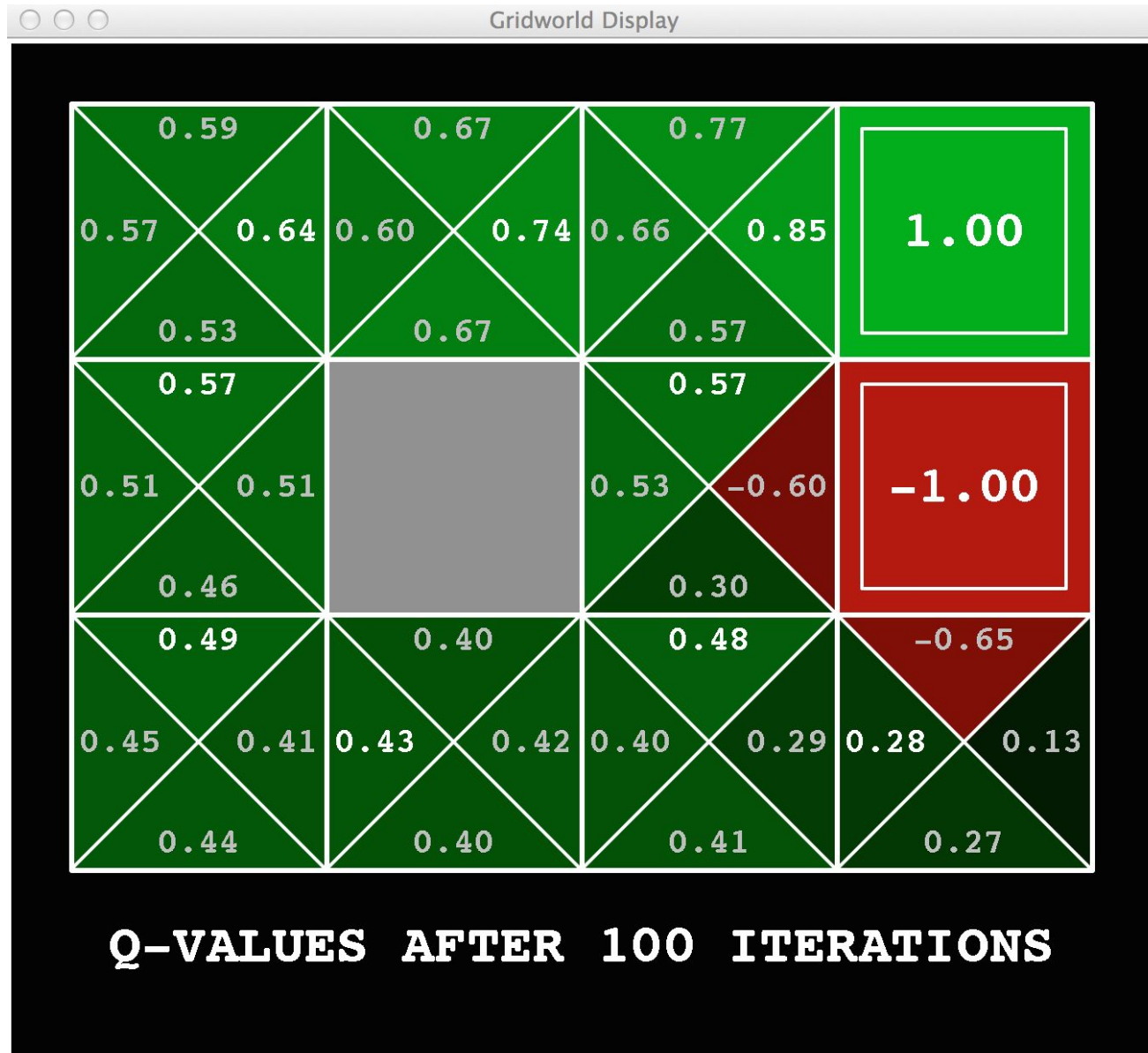
- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s



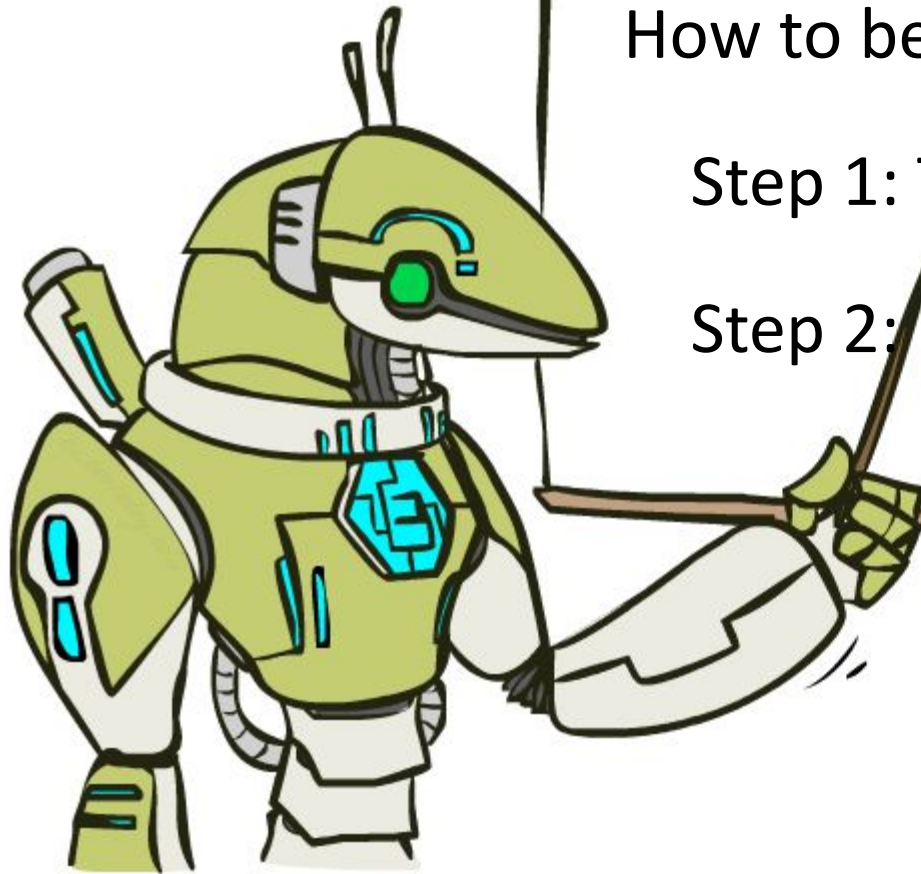
Gridworld Values V^*



Gridworld: Q^*



The Bellman Equations



How to be optimal:

Step 1: Take correct first action

Step 2: Keep being optimal

The Bellman Equations

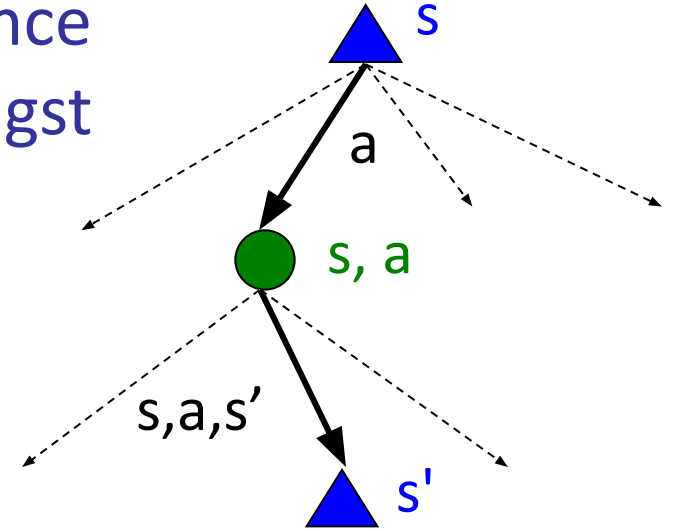
- Definition of “optimal utility” via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values

$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- These are the Bellman equations, and they characterize optimal values in a way we'll use over and over



Value Iteration

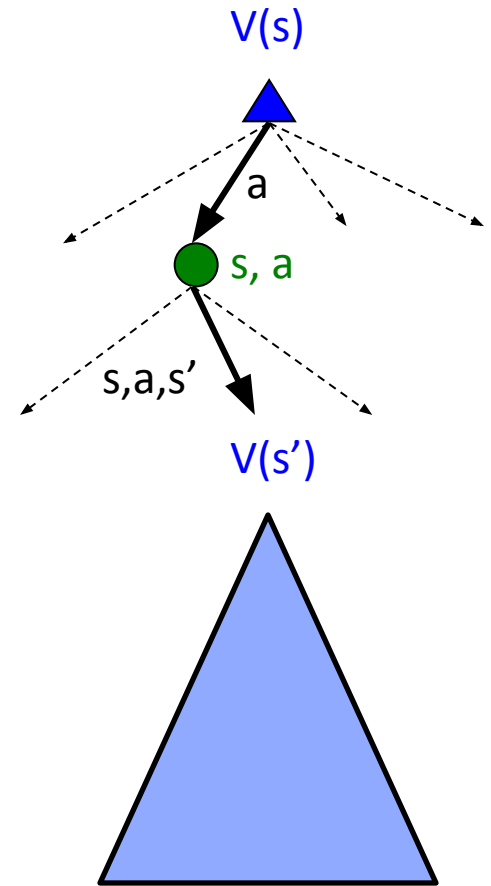
- Bellman equations **characterize** the optimal values:

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- Value iteration **computes** them:

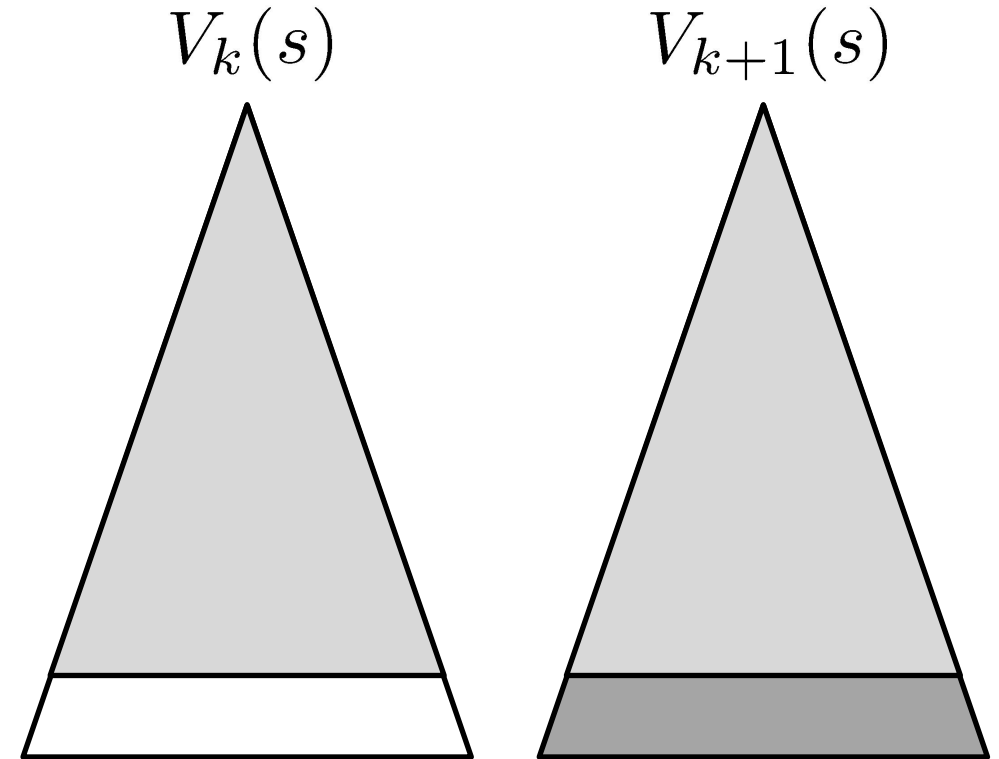
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Value iteration is just a fixed point solution method
 - ... though the V_k vectors are also interpretable as time-limited values

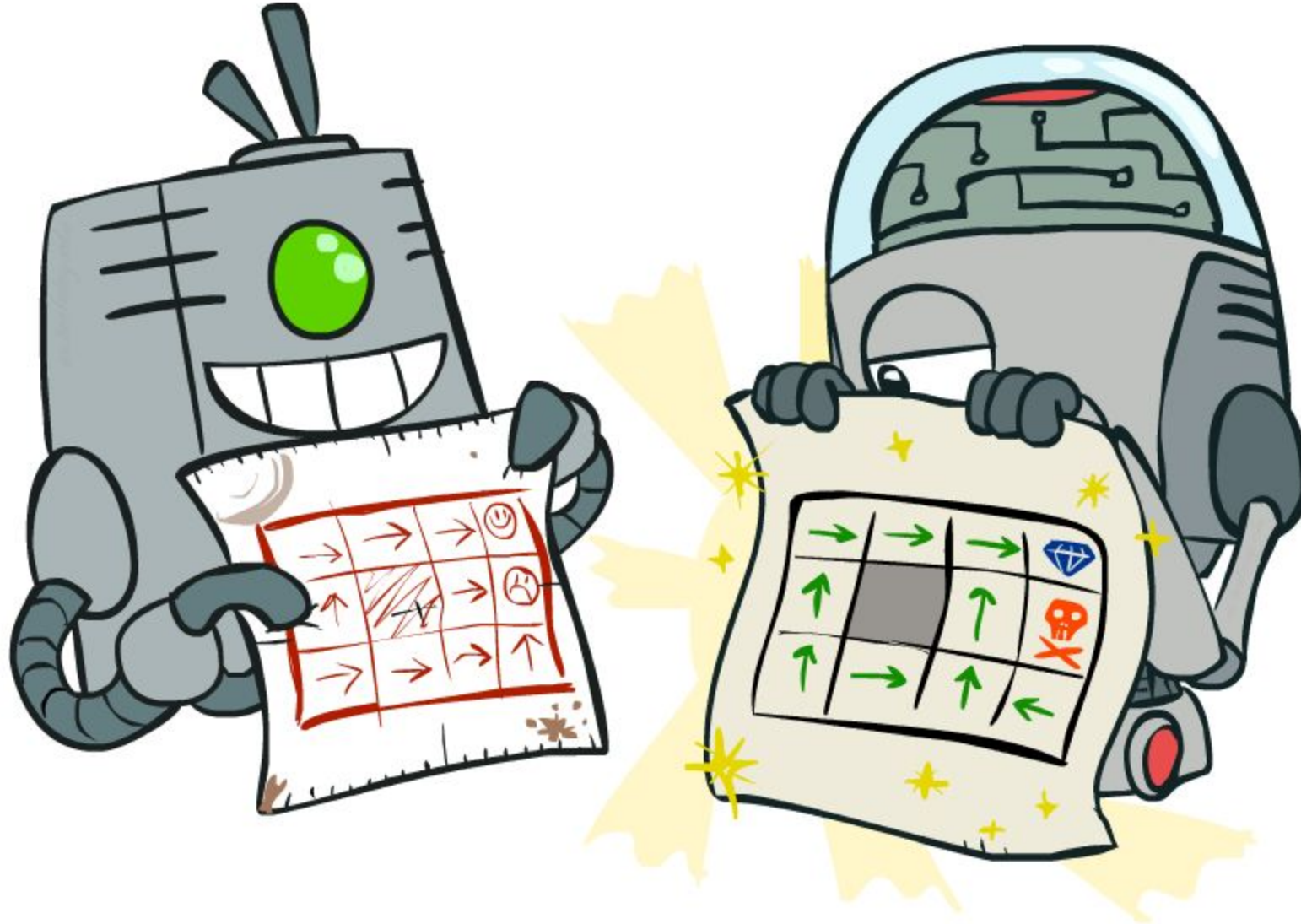


Convergence*

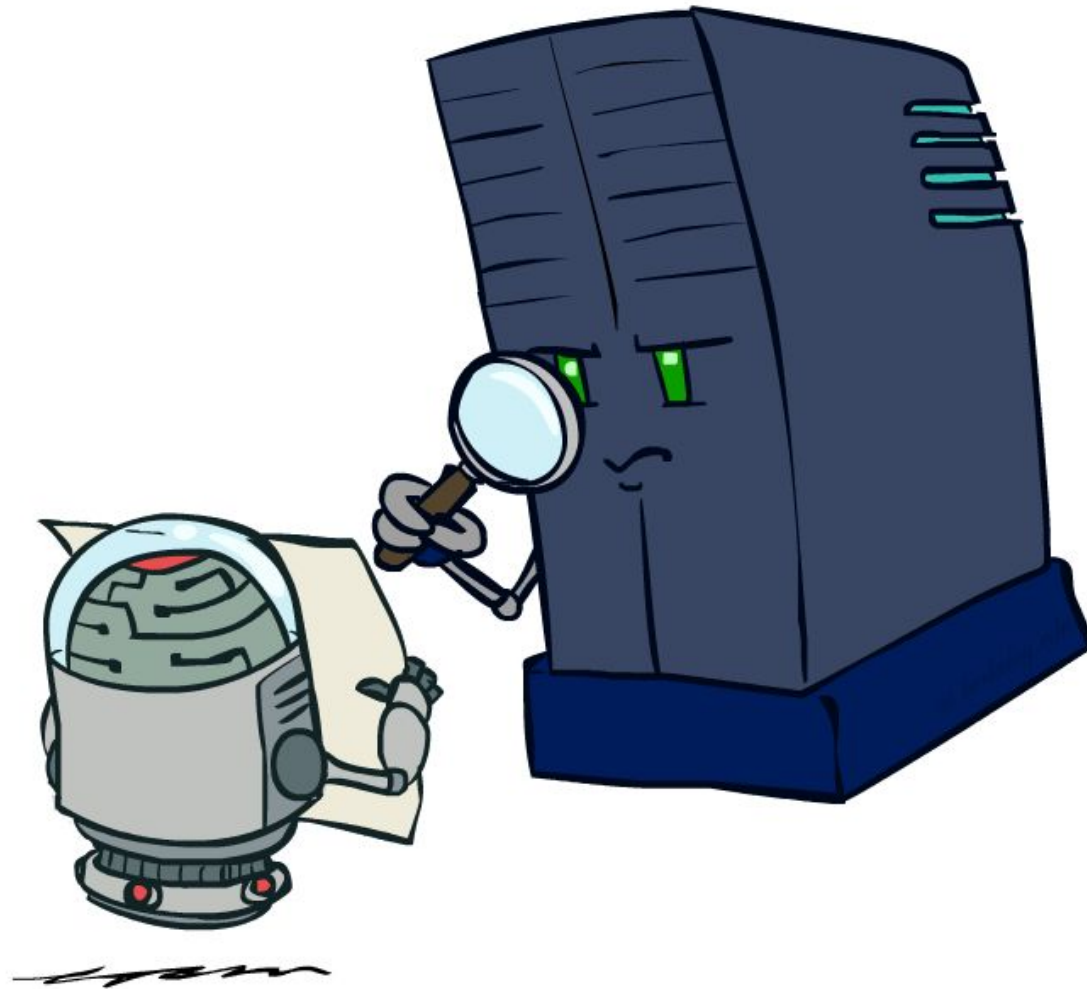
- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M , then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
 - Sketch: For any state V_k and V_{k+1} can be viewed as depth $k+1$ expectimax results in nearly identical search trees
 - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - That last layer is at best all R_{MAX}
 - It is at worst R_{MIN}
 - But everything is discounted by γ^k that far out
 - So V_k and V_{k+1} are at most $\gamma^k \max |R|$ different
 - So as k increases, the values converge



Policy Methods

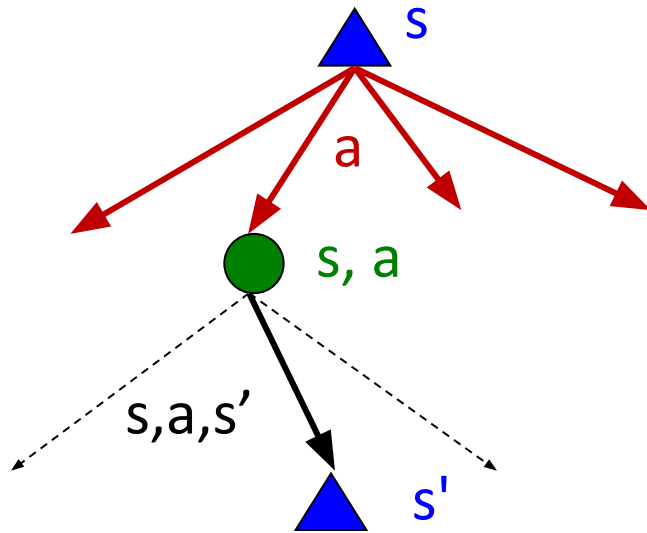


Policy Evaluation

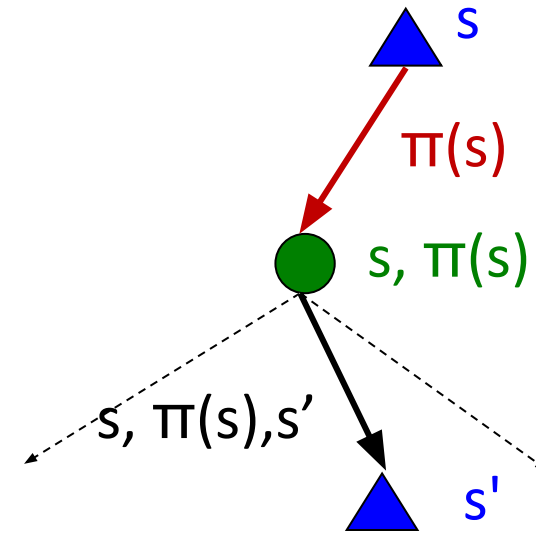


Fixed Policies

Do the optimal action



Do what π says to do

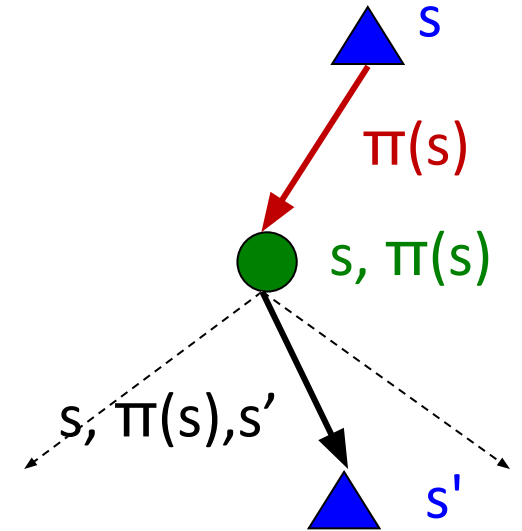


- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
 - ... though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s , under a fixed policy π :
 $V^\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')]$$



Example: Policy Evaluation

Always Go Right



Always Go Forward

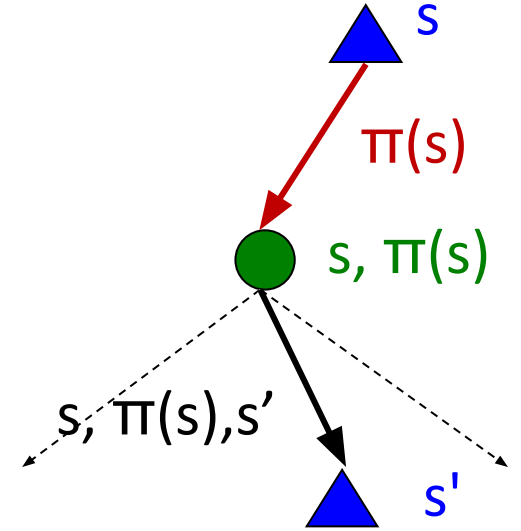


Policy Evaluation

- How do we calculate the V 's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

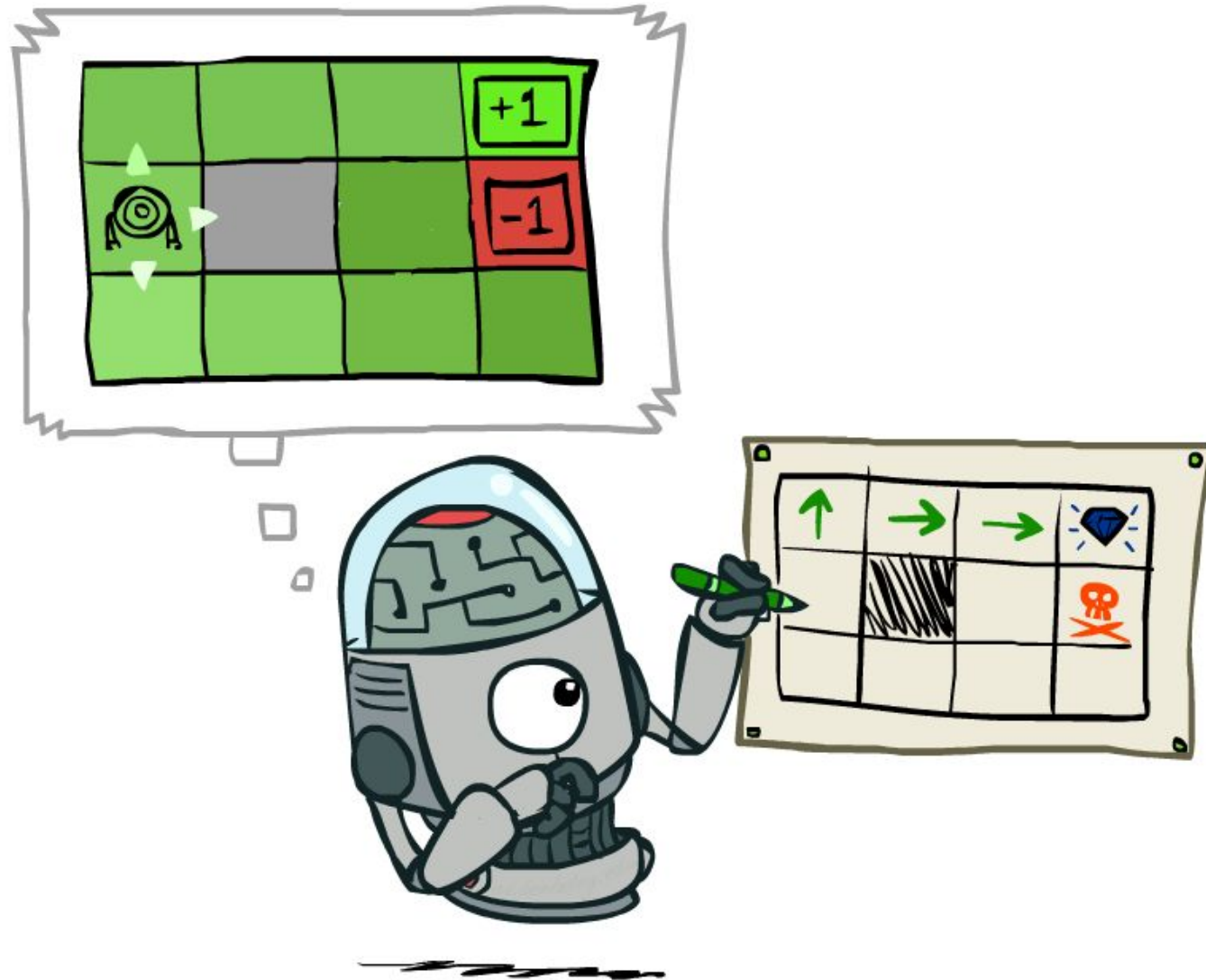
$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$



- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
 - Solve with Numpy/Matlab (or your favorite linear system solver)

Policy Extraction



Computing Actions from Values

- Let's imagine we have the optimal values $V^*(s)$
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)



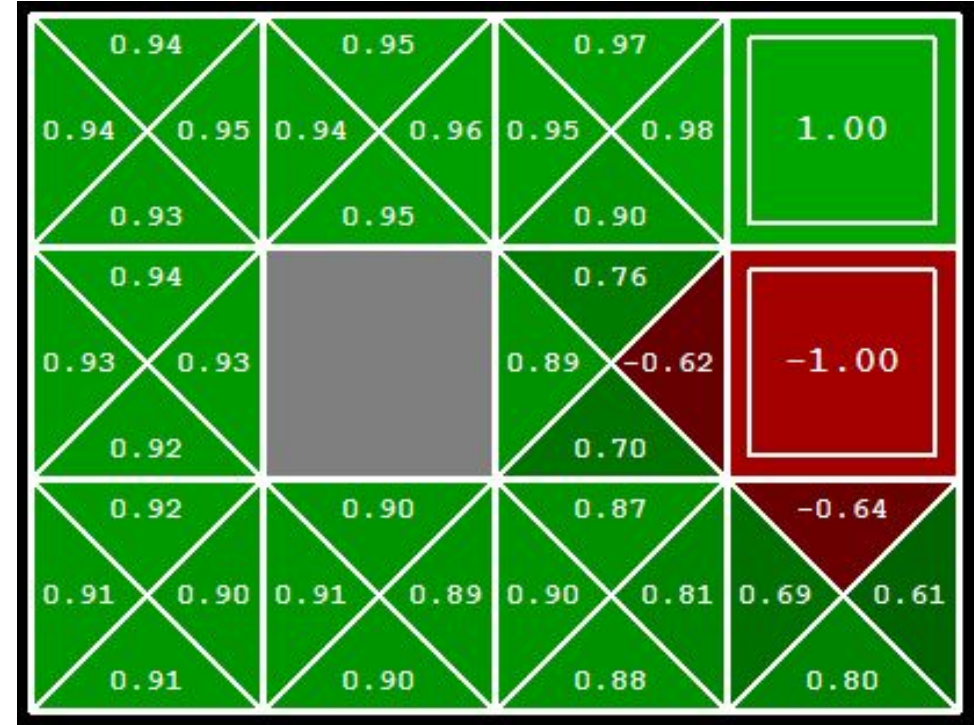
$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- This is called **policy extraction**, since it gets the policy implied by the values

Computing Actions from Q-Values

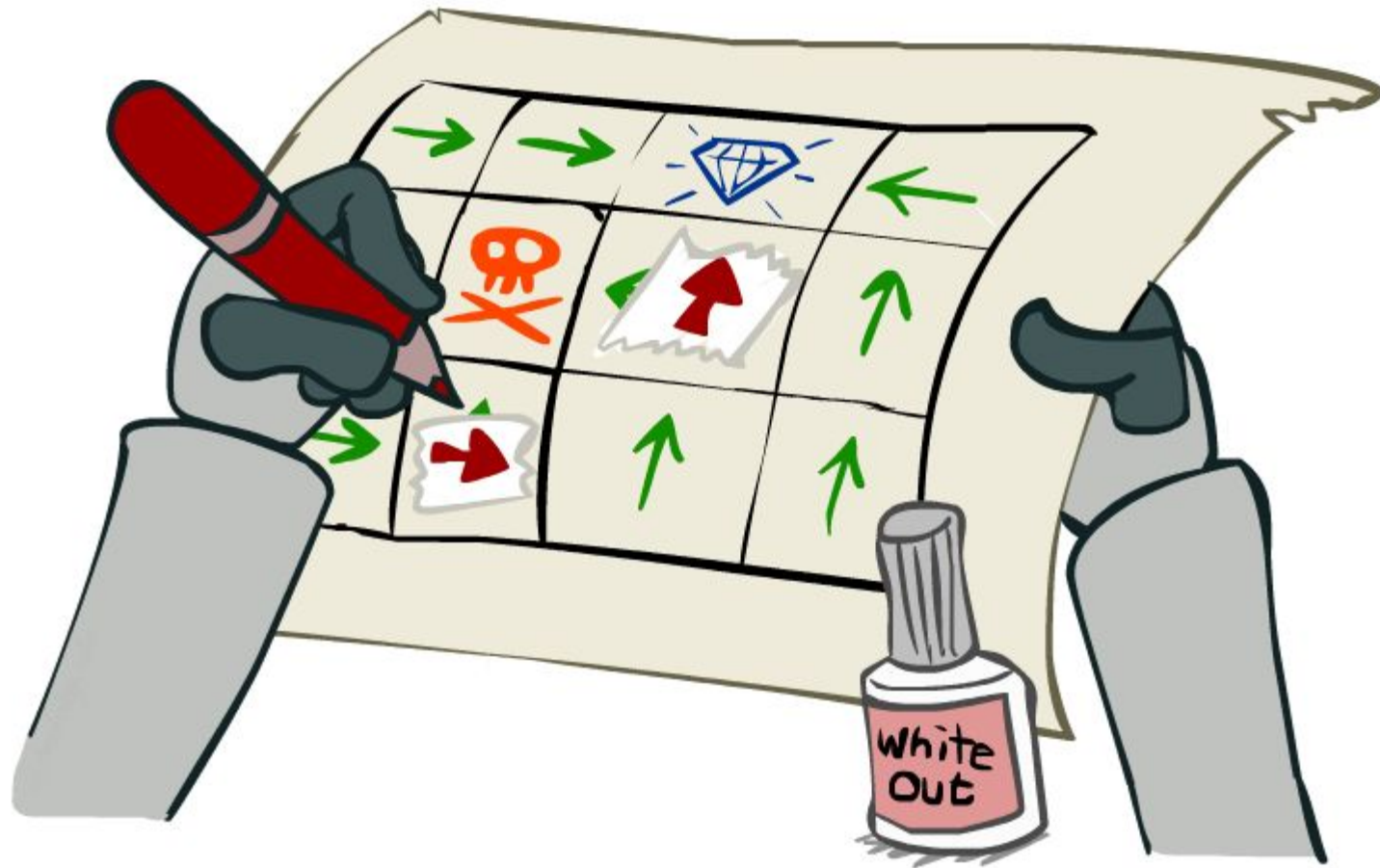
- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$



- Important lesson: actions are easier to select from q-values than values!

Policy Iteration

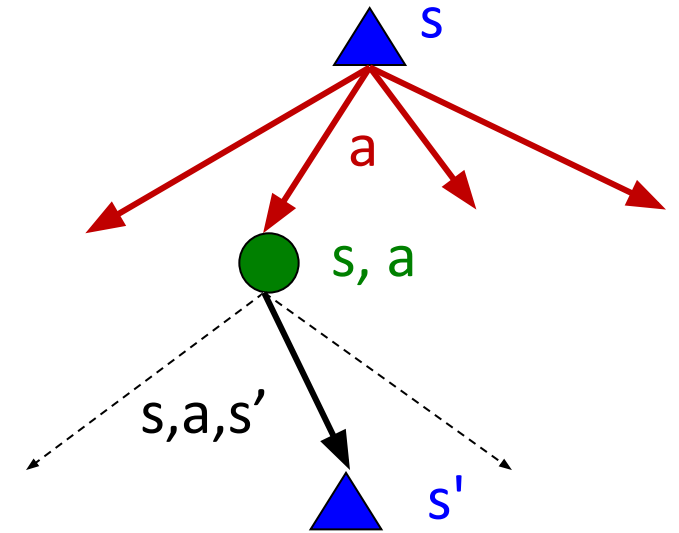


Problems with Value Iteration

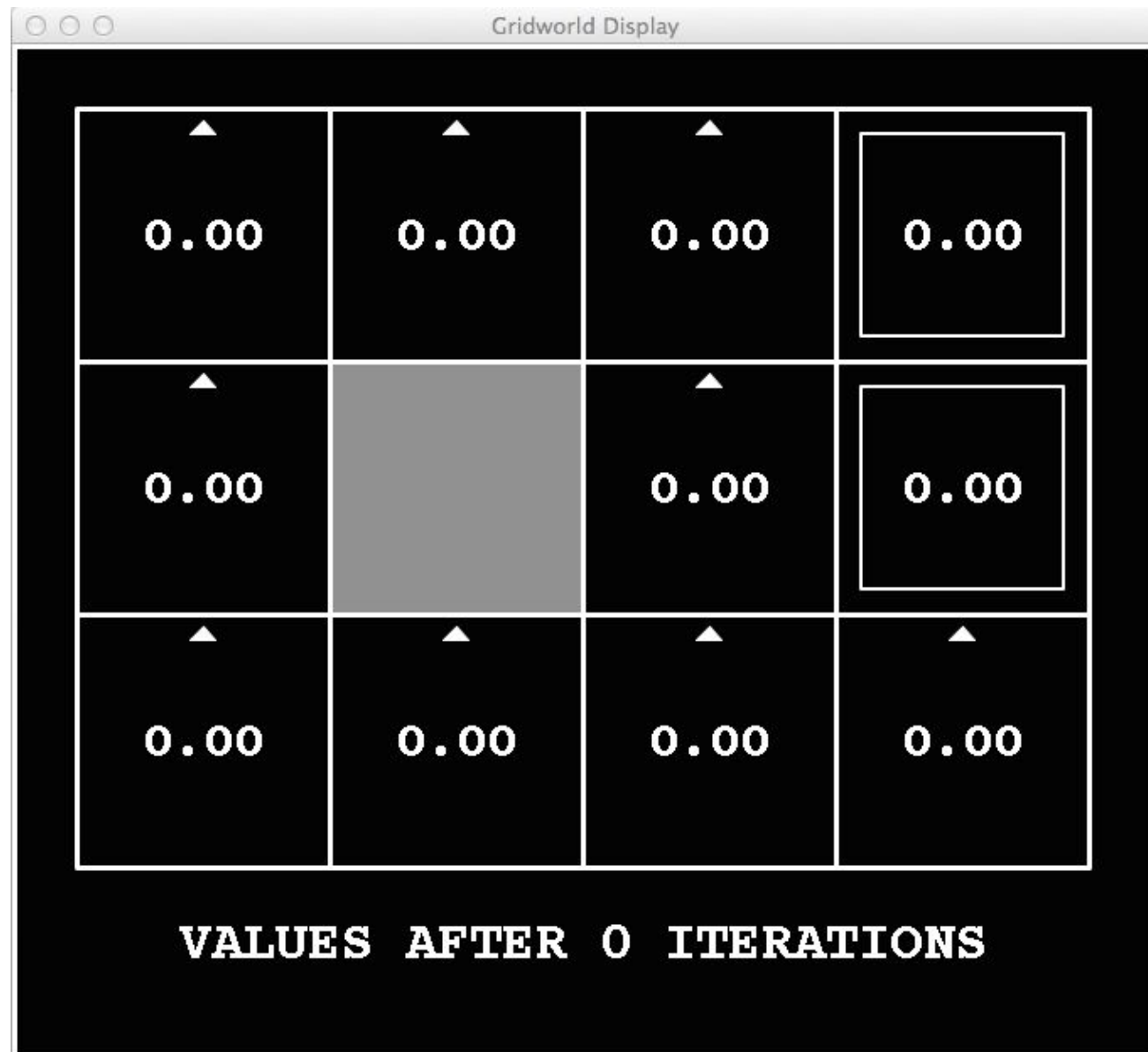
- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values

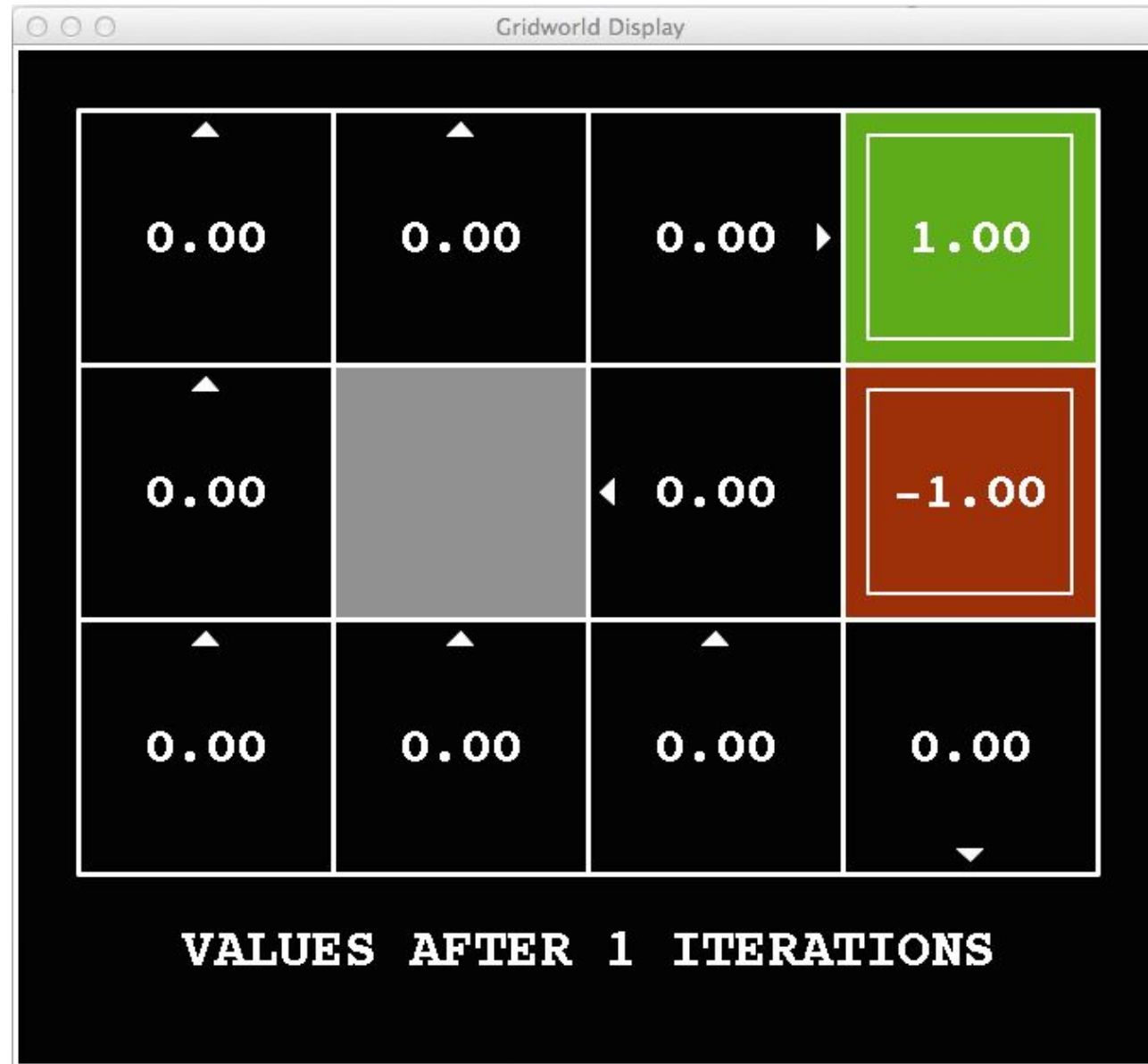


$k=0$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=1$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=2



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=3$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=4$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=7$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=8



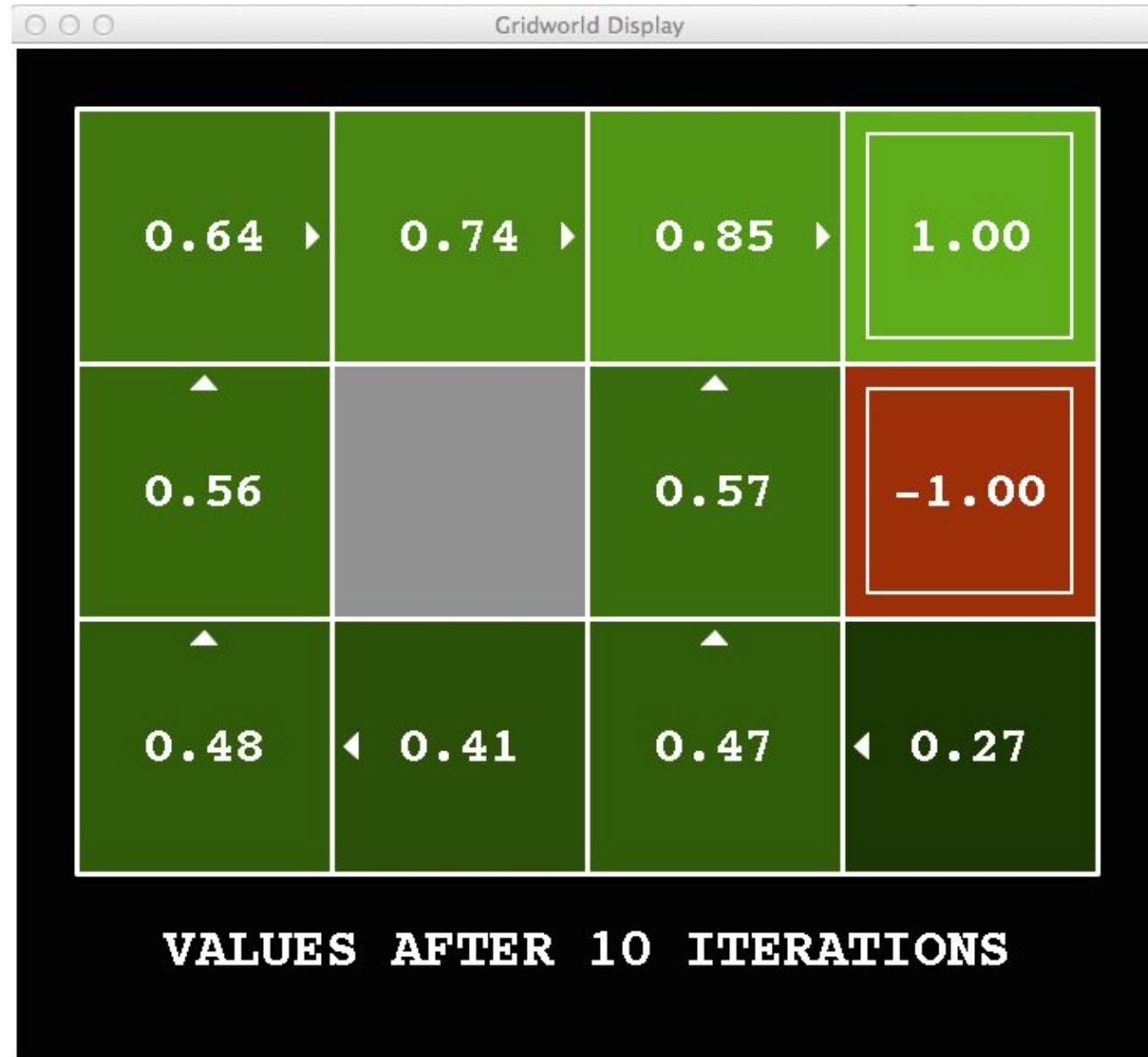
Noise = 0.2
Discount = 0.9
Living reward = 0

$k=9$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Iteration

- Alternative approach for optimal values:
 - **Step 1: Policy evaluation:** calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - **Step 2: Policy improvement:** update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges
- This is **policy iteration**
 - It's still optimal!
 - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') [R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:

$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

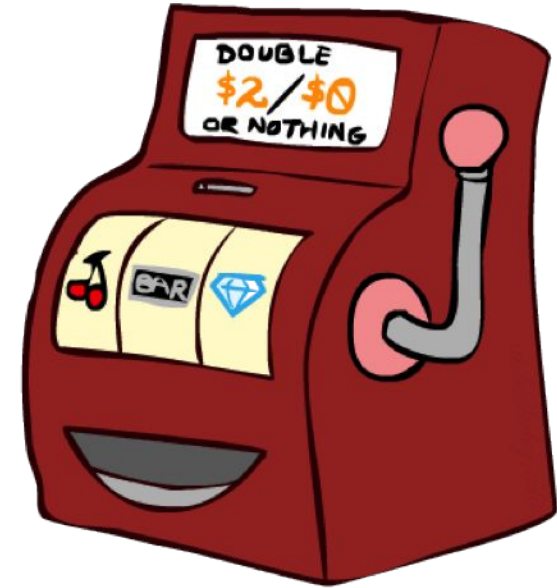
Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

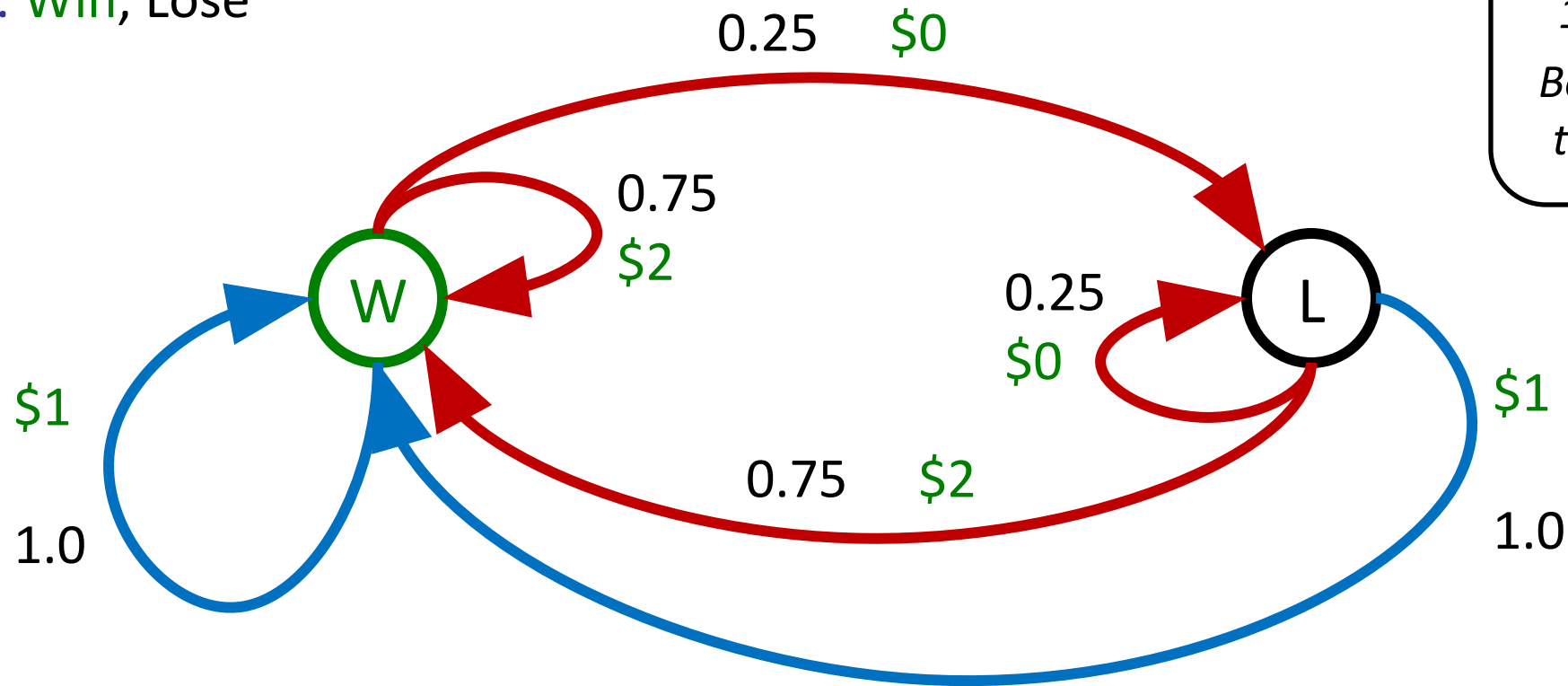
- So you want to....
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
 - They basically are – they are all variations of Bellman updates
 - They all use one-step lookahead expectimax fragments
 - They differ only in whether we plug in a fixed policy or max over actions

Double Bandits



Double-Bandit MDP

- Actions: *Blue*, *Red*
- States: *Win*, Lose



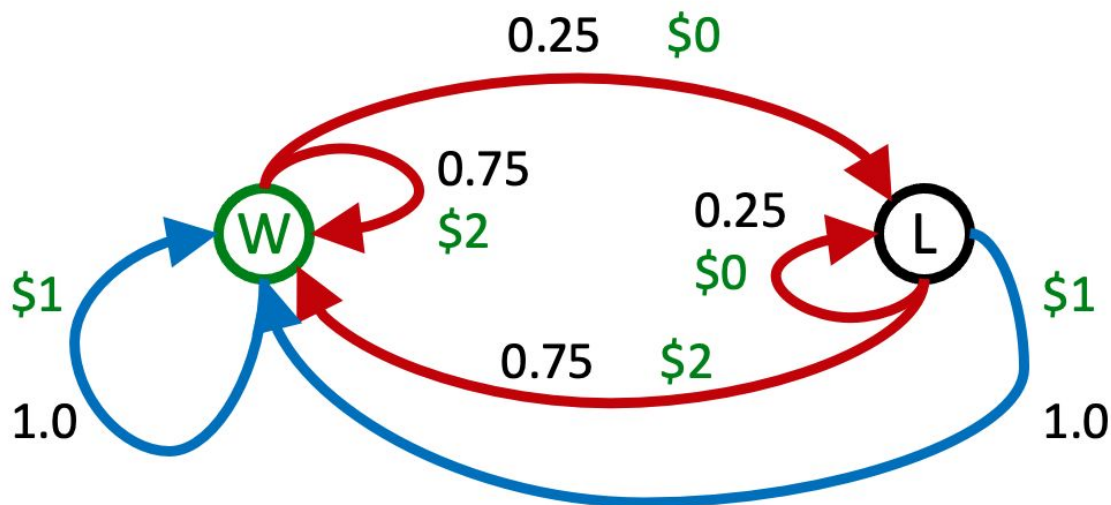
No discount
100 time steps
Both states have the same value

Offline Planning

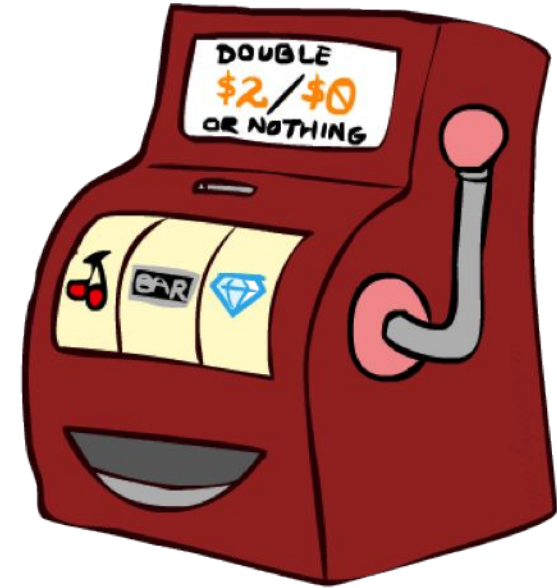
- Solving MDPs is offline planning
 - You determine all quantities through computation
 - You need to know the details of the MDP
 - You do not actually play the game!

No discount
100 time steps
Both states have the same value

	Value
Play Red	150
Play Blue	100



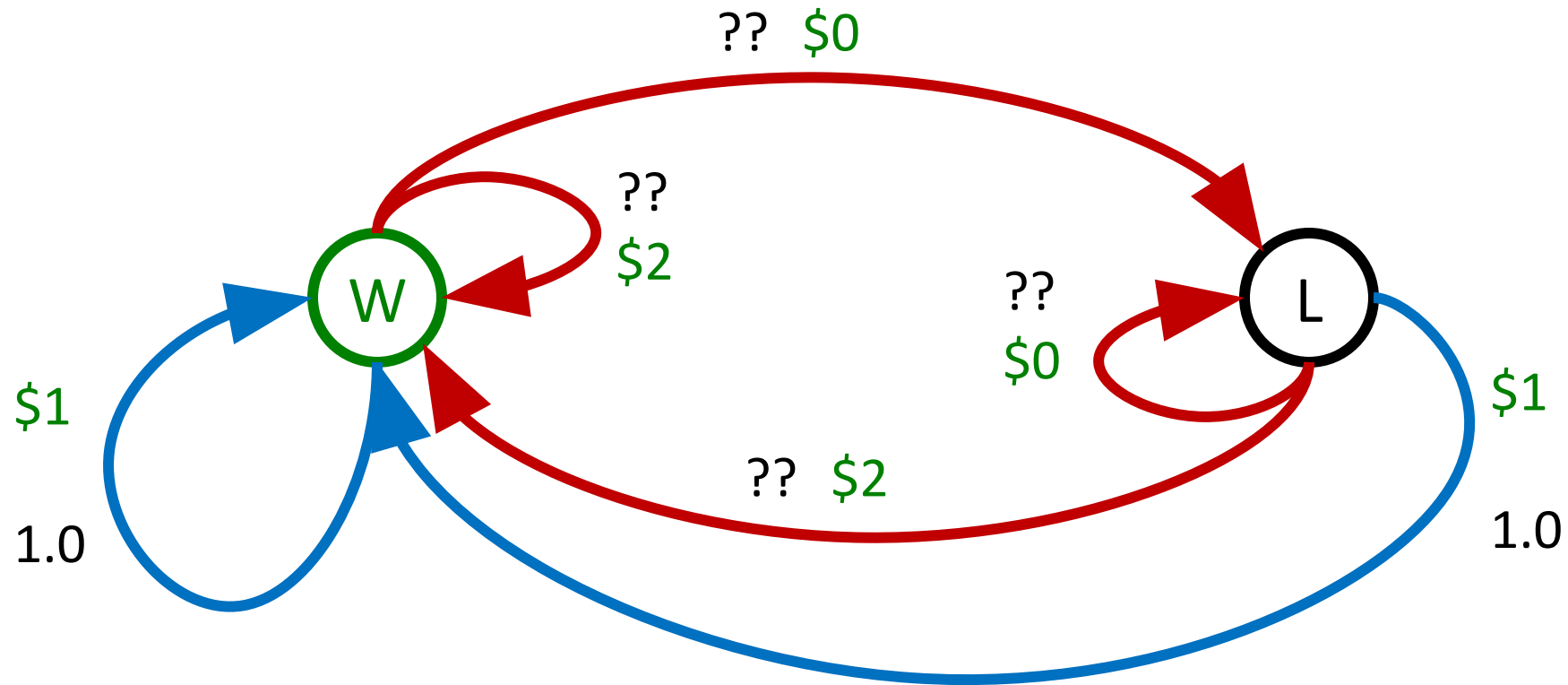
Let's Play!



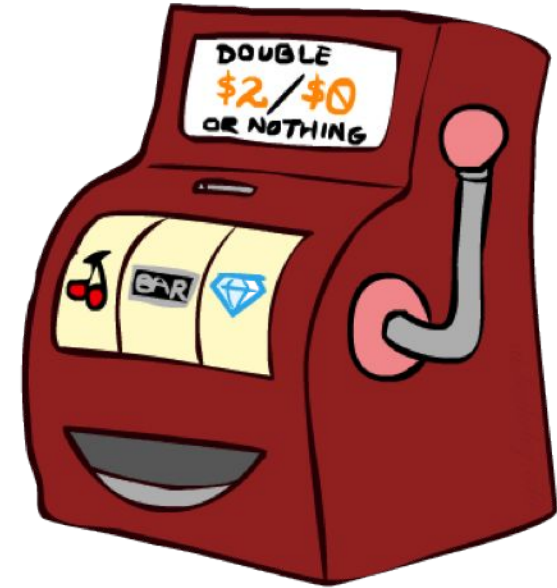
\$2 \$2 \$0 \$2 \$2
\$2 \$2 \$0 \$0 \$0

Online Planning

- Rules changed! Red's win chance is different.



Let's Play!



\$0 \$0 \$0 \$2 \$0
\$2 \$0 \$0 \$0 \$0

What Just Happened?

- That wasn't planning, it was learning!
 - Specifically, reinforcement learning
 - There was an MDP, but you couldn't solve it with just computation
 - You needed to actually act to figure it out
- Important ideas in reinforcement learning that came up
 - Exploration: you have to try unknown actions to get information
 - Exploitation: eventually, you have to use what you know
 - Regret: even if you learn intelligently, you make mistakes
 - Sampling: because of chance, you have to try things repeatedly
 - Difficulty: learning can be much harder than solving a known MDP



Next Time: Reinforcement Learning!

Asynchronous Value Iteration*

- In value iteration, we update every state in each iteration
- Actually, *any* sequences of Bellman updates will converge if every state is visited infinitely often
- In fact, we can update the policy as seldom or often as we like, and we will still converge
- Idea: Update states whose value we expect to change:
If $|V_{i+1}(s) - V_i(s)|$ is large then update predecessors of s

Interlude

