TABLE 1 Project Timeline

25/10/22	First Project Meeting
	Developed analytics for classical toric code
	Made a start on the implementation in Julia
02/12/22	Project Card 1 Deadline
	Confirmed correct temperature profiles for the thermal bath method
	Compared thermal bath method with uniform and random initial states
	Implemented spin ice to benchmark against previous paper
	Resolved heat capacities in the Kubo method - wasn't accounting for the heat capacity of demons
	Fixed issue with periodic boundary conditions for the current density
01/01/23	Computed analytic expressions for the heat capacity and number of excitations, good fit to observations except for strange subsidiary peak
	Multithreaded the code
02/01/23	Project Meeting
	Silly errors - missing factors of 2 in definitions being 'per spin' or 'per vertex', factor of 2 in conductivity due to difference in lattice parameter in previous work
	Good agreement with Rau's paper but computationally bottlenecked – implemented parallelism across different histories for better convergence
	Spent some time working out how to use the TCM network
	Added the magnetic field term (incorrectly)
	Added functionality for arbitrary periodic lattices
	Worked out that the spin ice Hamiltonian is an antiferromagnetic Ising coupling(generally) for the toric code!
02/02/23	Project meeting with Claudio & Jonathan (PhD student)
	Ran some full-length simulations and confirmed the characteristic shape of the curves in the previous work, as well as a flat thermal diffusivity for the toric code!
09/03/23	Meeting to discuss analytics
24/03/23	Project meeting
	Determined the low-temperature dip in the analytic expression for the diffusivity is a result of the excitation density dropping below 1, related it to the low-energy expression for the density
	Discussed the possibility of correcting the deflection of the analytic D to 2 by using a mean-field symmetric exclusion process > predicts D exactly 1 for all T!
	Implemented spin swap dynamics to investigate the field
	Realised my error in implementing the field naively
	Developed the percolation ideas in the presence of a field over email with Claudio
17/04/23	Project meeting to discuss the percolation behaviour in a field, as well as wrapping up the project
19/04/23	Decided to investigate other lattices, both analytically and in simulations
02/05/23	Project Presentation
09/05/23	Project meeting with Jeffrey Rau (author of the previous paper): resolved the issue of the vanishing thermal diffusivity; discussed the unusual behaviour of kagome spin ice and decided on some further tests to confirm this; realised a correction to the model to include all excitations in the SEP part of the model
11/5/23	Project meeting with Jeffrey Rau, more discussion on the percolation problem, kagome spin ice, etc.
15/05/23	Project Deadline

* Deadline Dec: broad descⁿ & plan (? 2x check ?)
↳ open part III offices in TCM.
↳ renew TCM access → also ask ~ part III offices

* Rau in principle happy to discuss, esp w/. code.
↳ mix of using code from him

μcanonical own + revisiting analytics in new system. →? Demon bath. inherit.

SAFER?
analytical tractability probs harder ⇒ will need coarse-graining
↓
BUT can immediately apply Kubo approach w/. minor changes.

* 3x possible systems:
↳ 8-vertex, classical Heisenberg on weird lattice.

purely diffusive excitatⁿs. → comparison useful to tell difference.

need 4-body interactⁿ $\hat{H}$.
↓
v. straightforward
BUT 4-body ⇝ appendix + MC

currents ~ energy. → measure.

Kubo argument becomes more complex

MORE CLEAN RESU[LTS]
∵ uses spin-products on bonds
⇓
would be harder

↳ ⇒ have a look first then call w/. Rau.

* Possibly do some verificat^{n} vs. 'Rau's data'
OR vs. 'Rau's system + my code'

* Opt^{n}al: → In simpler case of diffusing monopoles can we ~~do~~ make analytical progress on this?

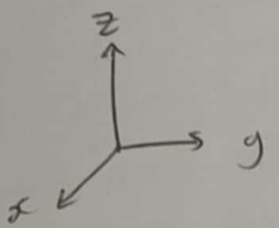
→ Ferromagnet in $B > 0^+$: $\rho_\downarrow \sim T$

-> get $\rho_\downarrow$ gradient & flow hot -> cold

↳ transfer entropy but not energy

↳ would have thermal conductivity BUT they're indep. $\Rightarrow \kappa = 0$?

↳ maybe worth looking @ it.

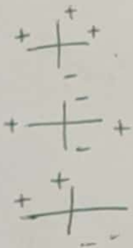


8-Vertex Model

$$H = -\lambda \sum_s \left(\prod_{i \in s} \sigma_i\right)$$

Look @ $T=0$ for simplicity:

GS for singles.



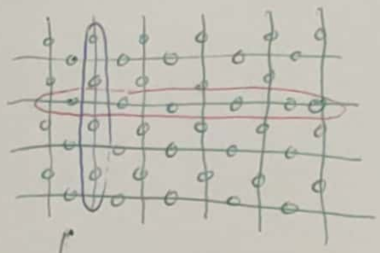
$\times 2$, $\times 2$, $\times 4$ $\Big\}$ $8\times$.

$$E_{GS} = -N\lambda$$

$$\# GS = (2^{N/2} - 1) \cdot 2^2$$

$(2^{N/2}-1)$: plaquette flips — local operators $\Rightarrow$ not topological

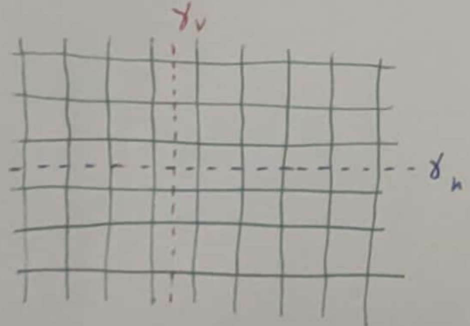
2^2: nonlocal line flips along bonds — # topological sectors.



$\sim \sigma_x$ operators but $/ \sigma_z$ doesn't work classically, i.e. $\nexists$ no dynamics @ $T=0$.

classical

We define now observables for paths $\sim$ axes of torus & on dual lattice

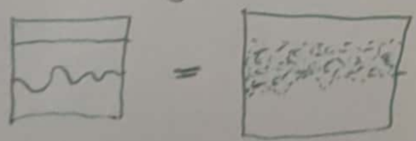


$$\Gamma_h = \prod_{i \in \gamma_h} \sigma_i = \pm 1$$

$$\Gamma_v = \prod_{i \in \gamma_v} \sigma_i = \pm 1$$

indep. of plaquetteflips ~~& nonlocal~~ BUT lineflips change its sign.

Can freely deform the γ (proof using fact that:



flip all spins between paths

$$\Rightarrow \Gamma_v \Gamma_v' = \prod_{s \text{ enclosed by } \gamma_v, \gamma_v'} A_s = +1$$

$A_s = \left(\prod_{i \in s} \sigma_i\right)$

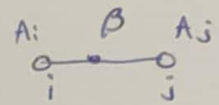
App. A : Energy Current

$$E[\underline{\sigma}, \underline{D}] = -\lambda \sum_i \prod_{\alpha \in i} \sigma_\alpha + \sum_\alpha D_\alpha$$

$\sigma_\alpha = \pm 1$, ~~$D_i \ldots$~~ $D_\alpha \in 4\lambda \mathbb{Z}$ $\because$ a single spin flip on a bond generates 2 prtcls e & $\bar{e}$ & changes the energy by $\pm 4\lambda$ ~~above the GS~~. (two diamonds change by $\pm 2\lambda$). in the GS only. In general, we can also move them freely $\Rightarrow \Delta E = 0, \pm 4\lambda$.

Suppose a single spin flip is proposed. As in Rau's paper, we have:

~~$\sigma_b' = \sigma_b(1 - 2\delta$~~ spin flip ~~site~~ bond β:



$$\sigma_\alpha' = \sigma_\alpha \left(1 - 2\delta_{\alpha\beta}\, \Theta(D_\alpha - \Delta E_\beta)\right)$$

$[-]$

$$D_\alpha' = D_\alpha - \delta_{\alpha\beta}\, \Theta(D_\alpha - \Delta E_\beta)\, \Delta E_\beta .$$

$$\Delta E_\beta = 2\lambda \sigma_\beta \Big(\sum_{i \ni \beta} \prod_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha\Big) \in \pm 4\lambda$$

We wish to define energy density

$$E = -\lambda \sum_i \prod_{\alpha \in i} \sigma_\alpha + \sum_i \frac{1}{2} \sum_{\alpha \in i} D_\alpha$$

works $\because$ each α shared between 2 sites.

$$= \sum_i \epsilon_i$$

$$\Delta E_\beta = \lambda (A_i + A_j - A_i' - A_j') = 2\lambda (A_i + A_j) \in \{0, 4\lambda, -4\lambda\}$$

$$\Rightarrow \epsilon_i = \frac{1}{2} \sum_{\alpha \in i} D_\alpha - \lambda \prod_{\alpha \in i} \sigma_\alpha$$

So...

$$\epsilon_i' = \frac{1}{2} \sum_{\alpha \in i} \underbrace{\left(D_\alpha - \delta_{\alpha\beta}\, \Theta(D_\alpha - \Delta E_\beta)\, \Delta E_\beta\right)}_{D_\alpha'}$$

$$- \lambda \prod_{\alpha \in i} \underbrace{\left\{ \sigma_\alpha \left(1 - 2\delta_{\alpha\beta}\, \Theta(D_\alpha - \Delta E_\beta)\right)\right.}_{\sigma_\alpha'}$$

$$= \frac{1}{2} \sum_{\alpha \in i} D_\alpha - \frac{1}{2} \sum_{\alpha \in i} \delta_{\alpha\beta}\, \Theta(\ldots)\, \Delta E_\beta$$

$$- \lambda \prod_{\alpha \in i} \sigma_\alpha \prod_{\alpha \in i} \left(1 - 2\delta_{\alpha\beta}\, \Theta(\ldots)\right)$$

$\epsilon \rightleftharpoons$

$$\epsilon_i' - \epsilon_i = -\frac{1}{2}\sum_{\alpha\in i} \delta_{\alpha\beta}\,\Theta(\dots)\,\Delta E_\beta$$

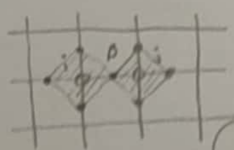
$$+ \lambda \prod_{\alpha\in i} \sigma_\alpha \Big[1 - \prod_{\alpha\in i}\big(1 - 2\delta_{\alpha\beta}\,\Theta(\dots)\big)\Big]$$

$$= -\lambda \sum_{\alpha\in i}\Big(\delta_{\alpha\beta}\,\Theta(\dots)\,\sigma_\beta \sum_{j\ni\beta}\prod_{\substack{\gamma\in j\\ \gamma\neq\beta}} \sigma_\gamma\Big)$$

$$+ \lambda \prod_{\alpha\in i} \sigma_\alpha \Big[1 - \prod_{\gamma\in i}\big(1 - 2\delta_{\gamma\beta}\,\Theta(\dots)\big)\Big]$$

Check for $\dot{j}_{ij}$:

Suppose we have



$A_i = 1, A_j = 1 \rightarrow \dot{j}_{ij} = 0$

$A_i = 1, A_j = -1 \Rightarrow \dot{j}_{ij} = -2\lambda$

$A_i = -1, A_j = 1 \Rightarrow \dot{j}_{ij} = +2\lambda$

$A_i = -1, A_j = -1 \Rightarrow \dot{j}_{ij} = 0$

antisymm
$\Rightarrow$ moves e prtcl
~~to~~ from i to j

$\Rightarrow \dot{j}_{ij} = E_e = +2\lambda$

Symm. $\Rightarrow$ spin-flip creates 2x
e prtcls @ i & j

Take $\beta \notin i$

$$\epsilon_i' - \epsilon_i = -\lambda(0) + \lambda\left(\prod_{\alpha\in i}\sigma_\alpha\right)\left(\overset{(1-1)}{\cancel{\beta \ni \prod_{\gamma\in i}(1-2\delta_{\gamma\beta})}}\right)$$

$$\cancel{= \lambda \prod_{\alpha\in i}\sigma_\alpha 2} = 0 \quad \text{as expected.}$$

Take $\beta \in i$

$$\epsilon_i' - \epsilon_i = -\lambda\sigma_\beta\,\Theta(\ldots)\sum_{j\ni\beta}\prod_{\substack{\gamma\in j\\ \gamma\neq\beta}}\sigma_\gamma$$

$$+\lambda\left(\prod_{\alpha\in i}\sigma_\alpha\right)\left(\cancel{1} - (\cancel{1} - 2\,\Theta(\ldots))\right)$$

$$= -\lambda\left(\sigma_\beta\sum_{j\ni\beta}\prod_{\substack{\gamma\in j\\ \gamma\neq\beta}}\sigma_\gamma\right)\Theta(\ldots)$$
$$+ 2\lambda\left(\prod_{\alpha\in i}\sigma_\alpha\right)\Theta(\ldots)$$

$$= -\lambda\left(\sum_{j\ni\beta}\prod_{\gamma\in j}\sigma_\gamma\right)\Theta(\ldots)$$
$$+2\lambda\left(\prod_{\alpha\in i}\sigma_\alpha\right)\Theta(\ldots)$$

let $\beta \in i, i'$

Only 2 sites for each bond.

$$= \lambda\left(\prod_{\alpha\in i}\sigma_\alpha - \prod_{\alpha\in i'}\sigma_\alpha\right)\Theta(\ldots)$$

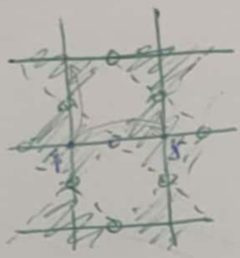
$$\dot{\epsilon}_i = -\sum_{j} j_{ij} \qquad \frac{\partial\epsilon_i}{\partial t} = -\nabla_i j \qquad \cancel{\nabla_i j = \sum_j j_{ij}}$$

$\Delta t = 1$

$$\Rightarrow \epsilon_i' - \epsilon_i = \lambda\,\Theta(D_\beta - \Delta E_\beta)\,\Theta_{\beta\in i}\left(\prod_{\alpha\in i}\sigma_\alpha - \prod_{\alpha\in i'}\sigma_\alpha\right)$$

$$= \lambda\,\Theta(D_\beta - \Delta E_\beta)\,\Theta_{\beta\in i}\left(2\left(\prod_{\alpha\in i}\sigma_\alpha\right) - \sum_{j\ni\beta}\left(\prod_{\alpha\in j}\sigma_\alpha\right)\right)$$

$$= -\sum_{j\in \mathrm{nn}(i)} j_{ij}$$



$$\Delta E = \Delta E_i + \Delta E_j \cancel{+ \Delta E_k + \Delta E_l}$$

$\gtrless$

$$\Delta E_i = \pm 2\lambda \quad \because$$

$$\frac{\Delta E_i}{\lambda} = (\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4})^f - (\text{''})^i \qquad \sigma^f_{i_1} = -\sigma^i_{i_1}$$

$$\cancel{= 2(\sigma}$$

$$\cancel{= (-\sigma^i_{i_1} - \sigma}$$

$$= +\sigma^i_{i_1}\left((\sigma_{i_2}\sigma_{i_3}\sigma_{i_4})^f + (\ldots)^i\right)$$

$$= 2\sigma^i_{i_1} = \pm 2$$

$$\prod_{n=1}^{N} a_n b_n = \prod_{n=1}^{N} a_n \prod_{n=1}^{N} b_n$$

$i, j \in \beta$ $\qquad A_i' = -A_i$

$$\Delta E_\beta = \lambda(A_i + A_j - A_i' - A_j')$$
$$= 2\lambda(A_i + A_j) \quad \square.$$

$$\cancel{\Delta E_\beta = -\lambda\Big[\sum_{i \in nn}}$$

$$\Delta E_\beta = -\lambda \sum_{i \ni \beta} \prod_{\alpha \in i} \sigma_\alpha' + \lambda \sum_{i \ni \beta} \prod_{\alpha \in i} \sigma_\alpha$$

$$= -\lambda \sigma_\beta' \sum_{i \ni \beta} \prod_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha + \lambda \sigma_\beta \sum \prod \sigma_\alpha \ldots$$

$$= -\lambda(\sigma_\beta' - \sigma_\beta) \sum \ldots$$

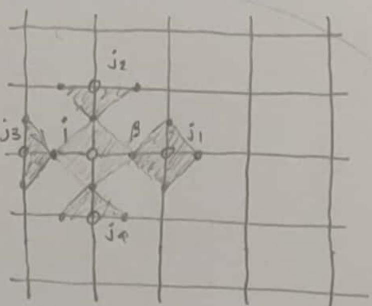
$$= +2\lambda\sigma_\beta\Big(\sum_{i \ni \beta} \prod_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha\Big)$$

$$= +2\lambda\sigma_\beta\left(\frac{1}{\sigma_\beta} A_i + \frac{1}{\sigma_\beta} A_j\right)$$

$$= 2\lambda(A_i + A_j)$$

We wish to relate $\epsilon_i' - \epsilon_i$ to some current flow between sites but it seems like all we have is :

$$\epsilon_i' - \epsilon_i = \begin{cases} 0 & \beta \notin i \\ \lambda \left(\prod_{\alpha \in i} \sigma_\alpha - \prod_{\alpha \in i'} \sigma_\alpha \right) \Theta(\ldots) & \beta \in i \end{cases}$$



We can identify each $\alpha \in i$ w/. some $j \in nn(i)$ just by following the bond!

In the above, we have

$$\epsilon_i' - \epsilon_i = \lambda \left(\overbrace{\prod_{\alpha \in i} \sigma_\alpha}^{A_i} - \overbrace{\prod_{\alpha \in j} \sigma_\alpha}^{A_j} \right) \Theta(\ldots)$$

$$= - \sum_{j \in nn(i)} j_{ij} = \lambda (A_i - A_j) \Theta(\ldots)$$

$$\underline{\dot{j}_{ij}^{(\beta)} = + \lambda (A_j - A_i) \Theta_{\beta \in j} \Theta_{\beta \in i} \Theta_{D_\beta \geq \Delta E_\beta}} \quad \left(\Delta E_\beta \in \{\pm 4, 0\} \right)$$

(for μcan, set $\Theta_{D_\beta \geq \Delta E_\beta} \longrightarrow \delta_{\Delta E_\beta, 0}$)

App. B

$\underline{J} = \frac{1}{2} \sum_{ij} (\underline{r}_j - \underline{r}_i) j_{ij}$ on the timestep w/. spin β flipped.

Now all is fine until we use $j_{ij} = -j_{ji}$. Does this hold? Trivially yes, which is good!

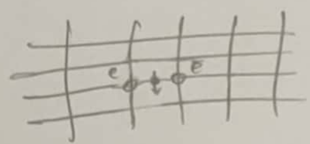
The rest looks fine!

all works as expected in monopole picture as well!

App. C

Now we want to estimate the dynamics of the ~~two~~ excitat^{n}s. We only have one type $\because$ classical 8-vertex (though in the Toric code the A_s & B_p excitat^{n}s are related by duality $\Rightarrow$ would be fine also).

So we want to consider a single excitat^{n}. i.e. an endpoint of a flipped A_s string on the lattice (not dual lattice)

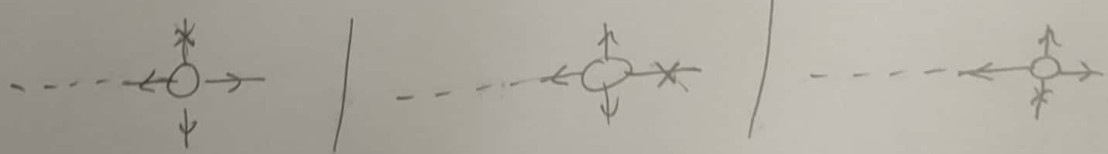


Why do we need to change anything?

We don't! The subdiffusive behaviour of excitat^{n}s in 2D spin ice is due to the disallowed hopping directions

$\hookrightarrow$ 1 direction ! allowed $\forall$ step & ~~disal~~ can always backtrack.

i.e:



assuming memoryless

Excitat^{n}s in 8-vertex are diffusive!

App. D

All holds the same? Fine up until we try to relate $\underline{P}$ to monopole pos^{n} $\underline{R}$.

Write local energy density $\epsilon_i = -\lambda \overbrace{\prod_{\alpha \in i} \sigma_\alpha}^{A_i} + \lambda$

Where we've shifted it so that $\epsilon_i = \lambda(1 - A_i) = \begin{cases} 0 & A_i = 1 \\ +2\lambda & A_i = -1 \end{cases}$

Note that unlike 2D spin ice, each spin flip creates <u>two</u> sites w/. $A = -1$ => <u>two</u> excitations

BUT this is nicer than the spin ice case ∵ only one type of excitatⁿ (no $Q = \pm 2$) and even if excitatⁿs aren't dilute then the sites still have @ most 1 excitatⁿ.

$$\Rightarrow \quad \underline{P} = \sum_{\substack{i \text{ s.t.} \\ A_i = -1}} \epsilon_i \underline{r}_i$$

Again, unlike the 2D spin ice case we don't need to worry ~ any kind of polarisatⁿ vector $\underline{d}$

$$\Rightarrow \quad \underline{P} = \lambda \sum_{i: A_i = -1}^{N_e} (1 - A_i) \underline{r}_i = +2\lambda \sum_{i: A_i = -1}^{N_e} \underline{r}_i$$

Assuming excitatⁿ motⁿ uncorrelated,

$$\langle |\underline{P}_t - \underline{P}_0|^2 \rangle_{eq} = (2\lambda)^2 \left\langle \sum_{i,j}^{N_e, N_e} (\underline{r}_i(t) - \underline{r}_i(0)) \cdot (\underline{r}_j(t) - \underline{r}_j(0)) \right\rangle_{eq} \quad \text{only } i=j \text{ matters}$$

$$\approx (2\lambda^2) N_e \langle |\underline{r}(t) - \underline{r}(0)|^2 \rangle_{eq.} \quad (\because \text{all identical})$$

$$\Rightarrow \quad \underbrace{\lim_{t \to \infty} \left[\frac{\langle |\underline{P}_t - \underline{P}_0|^2 \rangle_{eq}}{4t\delta t} \right]}_{} \approx (2\lambda)^2 N_e \underbrace{\lim_{t \to \infty} \left[\frac{\langle |\underline{r}_t - \underline{r}_0|^2 \rangle_{eq}}{4t\delta t} \right]}_{D}$$

$$\Rightarrow \quad K = \frac{K_{xx} + K_{yy}}{2} = \frac{1}{VT^2} (\,''\,) \approx \underline{\underline{\frac{(2\lambda)^2 N_e}{VT^2} D = K}}$$

So same result as in paper but more exact: only assumptⁿ is that monopoles are uncorrelated (I believe this <u>is</u> exactly true but I'll be hesitant and write "≈")

As a check, we have $E_{N_e} = +2\lambda N_e$ exactly

BUT we write $N_e \approx N e^{-\beta E_e} = N e^{-2\lambda/T}$.

N → nº sites

$\Rightarrow C \sim \frac{N_e}{V} \cdot \left(\frac{4\lambda}{T}\right)^2$

$K \approx \frac{N_e}{V} \cdot \left(\frac{4\lambda}{T}\right)^2 D$

$\sim DC$

which is good!

App. E

This all holds ∵ we still have $\Delta E_{spinflip} = \pm 4\lambda$

$\Rightarrow D_\alpha \in 4\lambda \mathbb{Z}_{\geq 0}$

$\Rightarrow C_{spin} = C_{demon} \frac{Var(E_{spin})}{C_{demon} T^2 - Var(E_{spin})}$ → Measurable.

$C_{demon} = N \left(\frac{4\lambda}{T}\right)^2 \frac{e^{4\lambda/T}}{(e^{4\lambda/T} - 1)^2}$

Important note from end of appendix:

We write $D = K/C_{spin}$ & neglect demon terms ∵ should be more physical ($\Delta \propto e^{-4\lambda/T}$ which is small)

Main Paper

$$T_\alpha = \frac{4\lambda}{\ln\left(1+\frac{4\lambda}{\langle D_\alpha\rangle}\right)}$$ still holds.

Just to check...

monopole step autocorrelat$^{\bar{n}}$ func$^{\bar{n}}$: should go w/. t^{-1} (diffus$^{\bar{n}}$ dist. goes w/. $t^{+1/2}$) if perfectly diffusive.

Typical runtimes are $\sim 10^5$ steps on $\sim 256^2$ lattices, but check!

$\hookrightarrow$ Careful of finite lattice size effects!

If ~~anything~~ everything is easier for the 8-vertex model! Esp. monopole dynamics but that was expected.

App. A (3)

$$\delta t\, \underline{J}(\sigma) = \sum_i{}' \underline{r}_i \left[\epsilon_i(\Omega(\sigma)) - \epsilon_i(\sigma)\right]$$

$[\nu\mu\sigma]$ cyclic

123 etc..

I want $\langle J^\nu p^\mu\rangle_{eq} - \langle J^\mu p^\nu\rangle_{eq} = ? = (\underline{J}\times\underline{p})^{\neq\mu,\nu}$

12 - 21 $= (\underline{J}\times\underline{p})_3 =$

23 - 32 $(\underline{J}\times\underline{p})_1$

31 - 13 $(\underline{J}\times\underline{p})_2$

$$p^\mu = \sum_i \underline{r}_i^\mu \epsilon_i$$

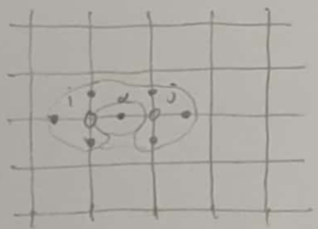
$\langle J^\nu p^\mu\rangle_{eq} - \langle J^\mu p^\nu\rangle_{eq} =$

$$\langle J^\mu P^\nu \rangle_{eq} = \langle \sum_i J^\mu r_i^\nu \epsilon_i \rangle_{eq}$$

Quick important Q: we know $\langle D_\alpha \rangle$ @ temp. T but what is $\langle \epsilon_\alpha \rangle$, i.e. energy/spin avg? This is more annoying than for 2D ice $\because$ spin avg & site avg. are a bit different.

We have $E_{spins} = -\lambda \sum_i \prod_{\alpha \in i} \sigma_\alpha = \sum \epsilon_i$

But we instead want ϵ_{α}



We could define

$$\epsilon_\alpha = \frac{1}{2}(\epsilon_i + \epsilon_j)$$

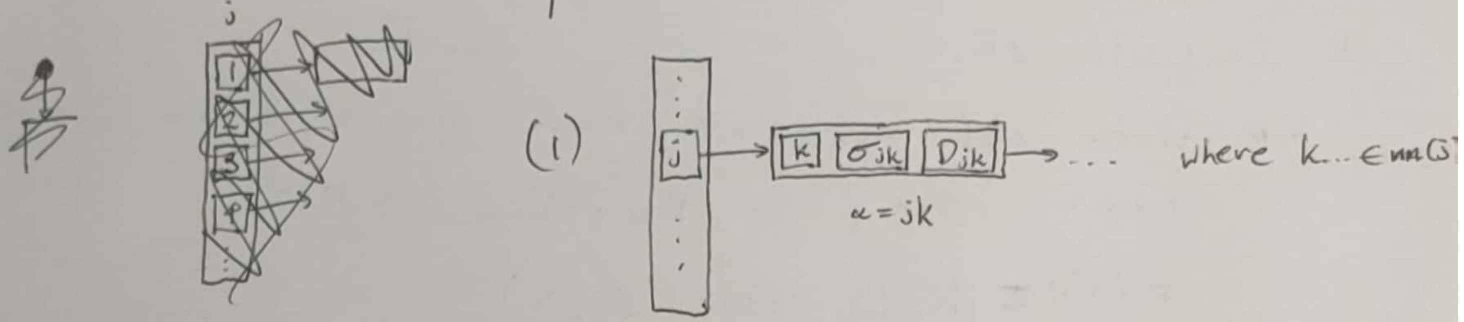
$$= -\frac{\lambda}{2} \sigma_\alpha \prod$$

This is very non-trivial $\Rightarrow$ not a useful endeavour.

Plan for the code:

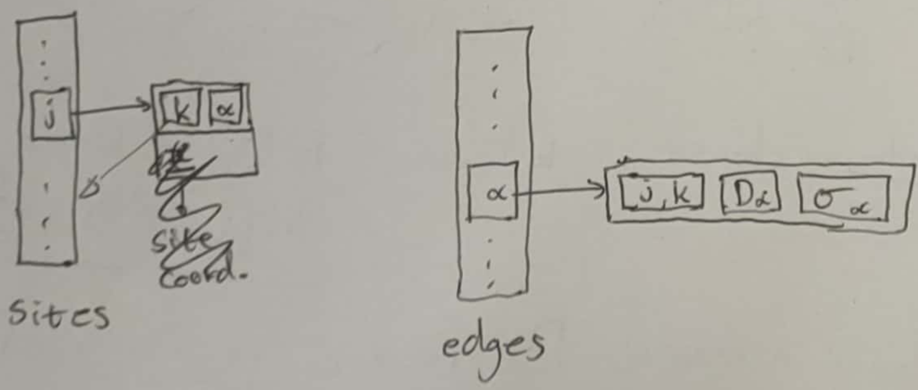
Store vertices as ~~a list~~ an array

Each vertex is a pointer to a new list:

(1) 

This works better than e.g. an adjacency matrix $\because$ we'd only have near-diagonal terms nonempty $\Rightarrow$ more efficient.

Should we also have a list of just edges?
Yes!

(2) 

This is good $\because$ can pick random edge to flip & then go to sites. But ~~other~~ accessing sites is important for calculating observables.

Having 2 lists also means we're not storing each edge twice like in (1).

need to decide on a prescriptn for bonds (or leave it to the user).

In general $\exists$ no good reason to order the sites on edges BUT given the geometry we want to set up, it certainly would: we need Cartesian direc.s

Have "coarse-graining" grid separate: each element holds some n° of vertices & bonds

So, the actual code needs to run like this:

Simulation — Demon bath
— μcanonical.

* Initialisat$^{\bar{n}}$.

↳ first set spins set in a ground state, then
which?

E_{demon} chosen s.t. $E_{spin} + E_{demon} \approx E(T)$

$$\cancel{E(T) \approx N_\alpha(\langle E_\alpha\rangle + \langle D_\alpha\rangle)}$$

$$\cancel{\approx N_\alpha\left(+ \frac{4\lambda}{e^{4\lambda/T}-1}\right)}$$

$Z = 1 + e^{-4\beta\lambda}$

$\Rightarrow \langle A_i \rangle = 1$

$$E(T) = N_i\left(\langle \epsilon_i^{spin}\rangle + \langle \epsilon_i^{demon}\rangle\right)$$

$\langle \epsilon_i^{spin}\rangle = -\lambda\langle A_i\rangle$

$$\epsilon_i = \frac{1}{2}\sum_{\alpha\in i} D_\alpha - \lambda\prod_{\alpha\in i}\sigma_\alpha$$

Z = nº bonds for each site.

$$\langle\epsilon_i\rangle = \frac{1}{2}\underbrace{\sum_{\alpha\in i}}_{z}\underbrace{\langle D_\alpha\rangle}_{\frac{4\lambda}{e^{4\lambda/T}-1}} - \lambda\underbrace{\left\langle\prod_{\alpha\in i}\sigma_\alpha\right\rangle}_{-N_i\lambda}$$

$\Rightarrow$ <u>initially</u> choose $E_{demon} = E(T) - E_{spin}$

$$= N_i\left(\langle\epsilon_i^{spin}\rangle + \frac{2\lambda z}{e^{4\lambda/T}-1} + \lambda\right)$$

i.e. randomly increment D_α from 0 in units of 4λ until ~~E_{demon}~~ $\sum_\alpha D_\alpha$ $\approx E_{demon}$.

↳ Perform a simple MH algo @ temp. T until thermalised (<u>canonical</u>)

Recall how the demon spin flip works: we accept moves iff $D_\alpha > \Delta E_\alpha := \Delta E(\sigma_\alpha \to -\sigma_\alpha)$

This lets us find a <u>local</u> temp. $T_\alpha = \dfrac{4k}{\ln\left(1 + \frac{4k}{\langle D_\alpha \rangle}\right)}$

Not possible in μcan

$\Downarrow$

for determining K we have: * thermal bath method
* Green–Kubo "

Anyway, let's look @ these two methods: ~~once~~

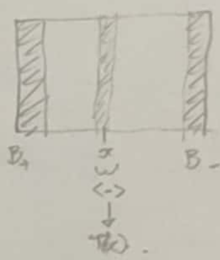
Thermal Bath

* Finite grid, PBCs in $\hat{e}_y$ & OBCs in $\hat{e}_x$.

↳ two ends $B_\pm$ held @ temp. diff ΔT

(to keep T const, $\alpha \in B_\pm$ are updated using standard <u>canonical</u> updates)

~~We find $T(x) \approx \frac{1}{L_y} \sum_{\alpha_y} \langle D_{\alpha_x \alpha_y} \rangle$~~



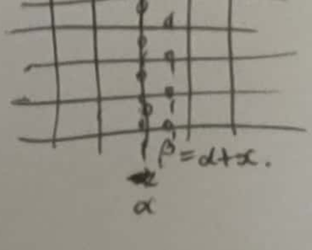
We find $T(x) = \langle D \rangle_{\text{over strip @ } x}$.

We <u>must</u> have $J^x = J^x_{B^+} = \frac{1}{2}\left(J^x_{B^+} - J^x_{B^-}\right)$

better statistics than

Where $J^x_{B_\pm}$ is an ~~strip~~ average over <u>all</u> bonds in $B_\pm$ (not just a strip).

We get $K = -J^x / (dT/dx) \rightsquigarrow K(T_\alpha) = \dfrac{-2J^x}{T_{i+x} - T_{i-x}}$



this is worth checking: no longer on-lattice! Should be fine?

Green-Kubo

* Setup as a few pages ago.

* Compute $j_{ii}(t)$ by summing $\dot{j}_{ii}^{(k)}$ over the sweep of N_α spins, updating the σ config after each step. Use that ~~to find~~ J. (or just T if mean)

* Green-Kubo: $$\cancel{K} = \frac{1}{N\langle T_\alpha^2\rangle} \sum_{\tau=0}^{\infty} \langle J^x_\tau J^x_0 \rangle \left(1 - \tfrac{1}{2}\delta_{\tau,0}\right)$$

(set $a = \delta t = k_B = 1$)

lattice spacing — timestep.

this is an autocorrelatn:
dominated by noise @ $\tau \to \infty$
BUT close to 0 $\Rightarrow$ maybe
try only including $\tau \in [0, 100)$
$\because$ decays quickly anyway

$\to$ play a bit w/. cutoff T & make sure indep. result

Also worth trying to look @ excitatn dynamics.

Can v. easily simulate this in a toy sim. but we <u>know</u> it should be diffusive

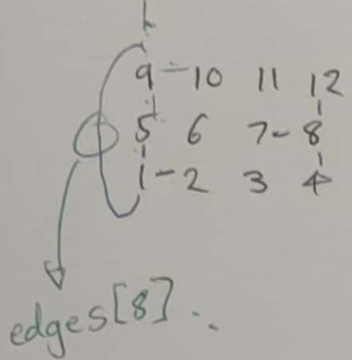
If you want to actually track this in the main sim, identify sites i where $A_i = -1$ & track them,

plotting $\overline{\langle |\underline{R}(t) - \underline{R}(0)|^2 \rangle (t)}$ ~~for each~~ (starting @ creatn & finishing @ annihalatn)
(overbar denotes avg. over all prtcl paths)

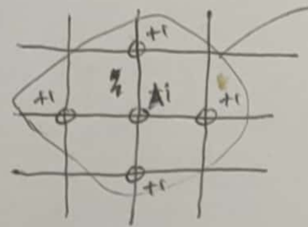
edges

Need to go back & worry abt space efficiency of Int64's, etc.

To initialise we need $\langle A_i \rangle$ in the canonical ensemble.



Well we have $A_i = \pm 1$ but there are many different ways this can happen. We need an approx^n: can't really consider a single site but consider:



+1 state if $A_i = +1$

for -1, pick from 4 ~~sites~~ edges.

$\Rightarrow$ energy cost 4λ

But this isn't it! Can also have 2 flips or 3 or 4!

# flips	Δenergy	# ways	A_i
0	0	1	+1
1	4λ	4	-1
2	$\cancel{8\lambda}$ 4λ	$4 \cdot 3 = 12$	+1
3	$\cancel{12\lambda}$ 8λ	4	-1
4	$\cancel{16\lambda}$ 8λ	1	+1

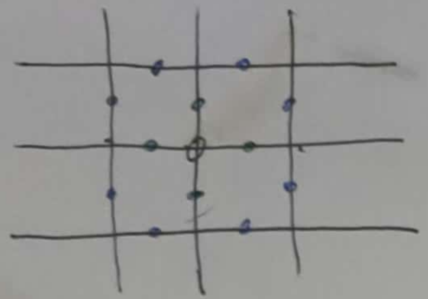
Careful! Check how many $A = -1$!

$$\langle A_i \rangle \cdot Z \approx 1 - 4e^{-4\lambda/T} + 12e^{-8\lambda/T} - 4e^{-12\lambda/T} + e^{-16\lambda/T}$$

ewww...

$$Z = \sum_{\text{states}} e^{-\beta\Delta\epsilon} \cdot \text{degeneracy}$$

$$\approx 1 + 4e^{-4\lambda/T} + 12e^{-8\lambda/T} + 4e^{-12\lambda/T} + e^{-16\lambda/T}$$



GS = loop gas.

$$Z = \sum_{\sigma} e^{-\beta E[\sigma]}$$

$$E = -\sum_i A_i$$

A_i .

$$\langle A_i \rangle = \sum_{\sigma} A_i \, e^{-\beta E[\sigma]}$$

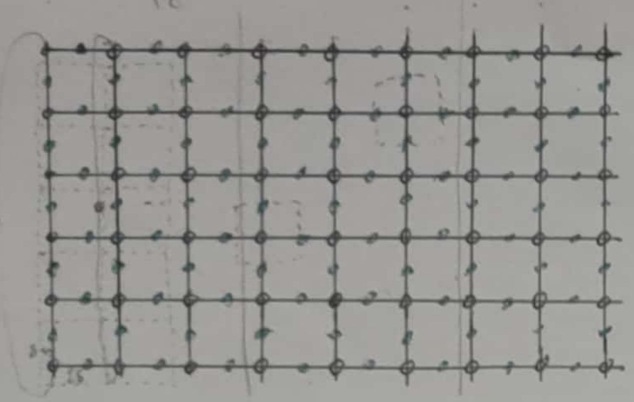
SHOULD just be able to consider the • edges ∵ equiv. to single-diamond approx$^{\bar{n}}$ for 2D spin ice. Perhaps this is not the same ∵ very different degeneracy (?)

If only green...

Start w/. gnd state
$A_i = +1$
$\Rightarrow$ ~~#~~ $\# \downarrow$ = even.
?

# flips	ΔE	# ways (^4C_n)	A_i
0	0	1	+
1	4λ	4	−
2	4λ	~~12~~ 6	+
3	8λ	4	−
4	8λ	1	+

Probs wrong!



Already need to use coarse-graining!

Need a cubic grid w/. lists of vertices & edges.

set $\delta = 0^+$

To make this easier for the square grid, we should order edges in a given site v.

$L_x \times L_y$ vertices

PBC edges

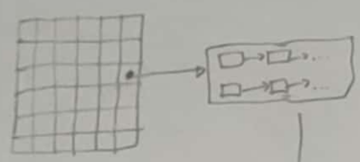
i.e. $V.E = [+\tfrac{1}{2}x, +\tfrac{1}{2}y, -\tfrac{1}{2}x, -\tfrac{1}{2}y]$

$\Rightarrow L_x \cdot (L_y - 1) + (L_x - 1) \cdot L_y + L_x$

$= 2L_xL_y - \cancel{L_x} - L_y + \cancel{L_x}$

then just drop last two.

We need



Construct when generating ~~lattice~~ graph initially!

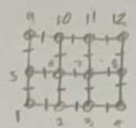
→ from ~~SQ~~ SquareLattice function.

Just int arrays. ⟹ need 2D array of tuples of ints.

e.g. for 4×3, $|V| = 4 \cdot 3 = 12$

$|E| = 24 - 3 = 21$

17



1, 2, 16.

For calculating $J^{x}_{B_\pm} = \frac{1}{L_y} \sum_{i \in B_\pm} \langle \Delta \epsilon_i \rangle$

be careful. This is in the heat bath regions

$\Rightarrow$ no demon contrib^{n} here

$\Rightarrow \epsilon_i = -\lambda A_i$

flip β.

$\Rightarrow \Delta \epsilon_i = \lambda (A_i - A_i')$

$$A_i' = \begin{cases} A_i & i \notin \beta \\ -A_i & i \in \beta \end{cases}$$

$\Rightarrow \Delta \epsilon_i^{(\beta)} = 2 \lambda A_i \, \Theta_{i \in \beta}$

Then we get $\Delta E^{(\beta)} = \sum_i \Delta \epsilon_i^{(\beta)}$

$= 2\lambda (A_{\beta 1} + A_{\beta 2})$

where βj = j^{th} vertex neighbouring β

as we know!

The above is correct but completely useless!

Really the paper means we should calculate

$J^{x}_{B_\pm} = \frac{1}{L_y} \langle \Delta E_{B_\pm} \rangle$

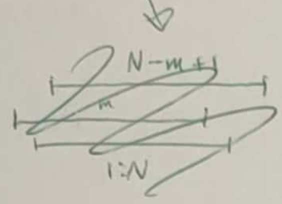
just any energy change in region

ΔE_i was dumb notation, didn't mean site energy at all.

$$\boxed{\begin{array}{l} \text{for } m \text{ in } 0:N-1 \\ \quad Q \mathrel{-}= D[m] - D[N-m+1] \\ \quad S_1[m] = Q \,/\, N-m \end{array}}$$

$$\Downarrow \; m \to m+1$$

for m in $1:N$
$\quad Q \mathrel{-}= D[m] - D[N-m+2]$



for m in $1:N$
$\quad$ for k in $1:N-m$
$\quad\quad S1[m] \mathrel{+}= D[k] + D[k+m-1]$
$\quad S1[m] \mathrel{/}= N-m+1$

$$S1[m] = \frac{1}{N-m+1} \sum_{k=1}^{N-m} \underbrace{D[k]}_{S1[m]^{(1)}} + \underbrace{D[k+m-1]}_{S1[m]^{(2)}}$$

$$S1[m+1] = \frac{1}{N-m} \sum_{k=1}^{N-m-1} D[k] + D[k+m]$$

$$= \frac{1}{N-m} \left(\sum_{\ell=1}^{N-m-1} D[\ell] + \sum_{\ell=2}^{N-m} D[\ell+m-1] \right) \quad \ell = k+1$$

$$= \frac{1}{N-m}\Big(SI[m]^{(1)} - D[\cancel{\ }N-m] + SI[m]^{(2)} - D[m] \Big)$$

$$SI[m+1] = \frac{N-m+1}{N-m}\, SI[m] - \frac{1}{N-m}\big(D[m] + D[N-m] \big)$$

$$SI[1] = \frac{1}{N\,\cancel{\ }} \sum_{k=1}^{N-1} \big(D[k] + D[k] \big)$$

$$= \frac{2}{N} \sum_{k=1}^{N-1} D[k].$$

---

p- cell Hamiltonian on general cell complex is:

$$\hat{H}_p = -J_{p-1} \sum_{s\in\Delta_{p-1}} A_s - J_{p+1} \sum_{\mu\in\Delta_{p+1}} B_\mu - \Gamma_p \sum_{\sigma\in\Delta_p} Z_\sigma \quad + h.c.$$

$$A_s = \prod_{\sigma\in V(s)<\Delta_p} Z_\sigma \quad , \quad B_\mu = \prod_{\sigma\in\partial\mu} X_\sigma$$

Now when making the Hamiltonian classical as for the Toric code we set: $X \to 0$ $\quad Z \to$ some values on the p-cells that takes values in eigenvalues of Z, i.e. in $\{1, e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2\pi i(N-1)/N}\}$

N such eigenvalues, hence $\mathbb{Z}_N$ gauge theory.

e.g. for the $\{0,1\}$-cell complex we have

$$\hat{H}_1 = -J_0 \sum_{s\in\Delta_0} A_s - J_2 \underbrace{\cancel{\sum_{\mu\in\Delta_2} B_\mu}}_{\to 0 \,\because\, \text{no 2-cells.}} - \underbrace{\cancel{\Gamma_1 \sum_{\sigma\in\Delta_1} Z_\sigma}}_{\text{string tension term}}$$

We don't care ~ this for now $\Rightarrow$ set $\Gamma_p = 0$

We are also working w/. $N=2$ by default rather than w/. the N-clock algebra.

The big statement then is this:

The distinct, linearly-indep. ground states of $\hat{H}_p$ are labelled by homology classes $H_p(\Delta, \mathbb{Z}_N)$ w/. Δ the cell complex.

& note $H_p(\Delta, \mathbb{Z}_2) \cong H_{d-p}(\Delta, \mathbb{Z}_2)$ Poincarè Lemma

or any field so no torsion.

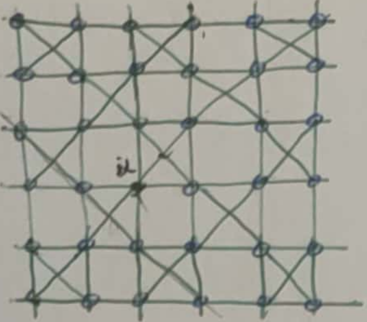
for a $\underbrace{\text{d-dim}^{\text{n}}\text{al}}_{\text{largest cell type.}}$ cell complex

$$\hat{H}_0 = 0 - J_1 \sum_{\mu \in \Delta_1} B_1 + 0$$

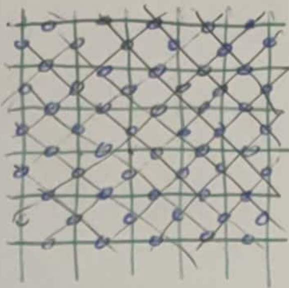
obviously same physics as $\hat{H}_1$

=> same G.S. degeneracy.

2D [illegible] as in paper.



$\simeq$



$$\hat{H} = \frac{J}{2}\sum_{\alpha} \sum_{\beta \in m(\alpha)} \sigma_\alpha \sigma_\beta$$

$$= J \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta .$$

$$\hat{H} = +\lambda \sum_i \left(\sum_{\alpha \in i} \sigma_\alpha \right)^2$$

$$\parallel$$

$$(\sigma_{i+x} + \sigma_{i-x} + \sigma_{i+y} + \sigma_{i-y})^2$$

equivalent ground state spaces.

$$H_i = \cancel{\sigma_{i+x}^2} + \ldots \rightarrow O(1) = 4\lambda$$

$$+ 2\lambda\big(\sigma_{i+x}\sigma_{i-x} + {}''_{y} + \sigma_{i+x}\sigma_{i+y} + \sigma_{i+x}\sigma_{i-y} + \ldots \big) = J N_\alpha$$

$H = -\lambda \sum_i A_i$

$$H_i = 2\lambda \sum_{\substack{\text{all possible} \\ \text{pairs } \alpha \neq \beta \\ \in i}} \sigma_\alpha \sigma_\beta + 4\lambda N_i$$

$\Delta E_\beta = +\lambda(\Delta A_i + \Delta A_{i'})$

pair

each $\wedge(\alpha,\beta)$ only appears once in sum over i $\Rightarrow$ w/ extra edges it's just $\langle\alpha\beta\rangle$.

$\Rightarrow 2\lambda = J$

$$A_i = \Big(\sum_{\alpha \in i} \sigma_\alpha \Big)^2$$

$$\Delta A_i = \Big(\sum \sigma_\alpha - 2\sigma_\beta\Big)^2 - A_i$$

$$= \cancel{4 - 4\sigma_\beta A_i}$$

$$= 4 - 4\sigma_\beta \underbrace{\sum_\alpha \sigma_\alpha}_{\pm\sqrt{A_i}}$$

$$\hat{H} = \sum_i A_i \qquad A_i = \Big(\sum_{\alpha \in i} \sigma_\alpha \Big)^2 .$$

$$\epsilon_i = \frac{1}{2}\sum_{\alpha \in i} D_\alpha - \lambda \Big(\sum_{\alpha \in i} \sigma_\alpha \Big)^2$$

flip β $\Rightarrow$ $D_\alpha \rightarrow D_\alpha - \delta_{\alpha\beta}\,\Theta(\cancel{D_\alpha - \Delta E_\beta})\,\Delta E_\beta$

$\sigma_\alpha \rightarrow \sigma_\alpha (1 - 2\delta_{\alpha\beta}\Theta)$

$$\epsilon_i' - \epsilon_i = \frac{1}{2}\sum_{\alpha \in i} D_\alpha - \frac{1}{2}\sum_{\alpha \in i} \delta_{\alpha\beta}\,\Theta\,\Delta E_\beta$$

$$+ \lambda \Big(\sum_{\alpha \in i} \sigma_\alpha - 2\Theta \sum_{\alpha \in i} \sigma_\alpha \delta_{\alpha\beta} \Big)^2$$

$\Rightarrow \quad \dot{J}_{ij}^{(\beta)} = 2\lambda \sigma_\beta (\Sigma_i - \Sigma_j) \theta$

$$\Delta E^{(\beta)} = -4\lambda \left(\sigma_\beta (\Sigma_i + \Sigma_j) - 2 \right)$$

$$\Sigma_i \in \{-4, \dots, +4\}.$$

$\Rightarrow$ Can also simulate 2D spinice (albeit rotated) in my setup.

Define $B_i = \sum_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_i$

$\Rightarrow \quad \Delta E^{(\beta)} = -4\lambda (B_i + B_j - 2)$ ewww...

$$\dot{J}_{ij}^{(\beta)} = -2\lambda (B_j - B_i) \theta$$

for $H = J \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta = \lambda \sum_i \underbrace{\left(\sum_{\alpha \in i} \sigma_\alpha \right)^2}_{A_i = C_i^2} + O(1)$

w/. $2\lambda = J$.

Is this right? In paper we're told:

$\Delta E^{(\beta)} = -2J\sigma_\beta \sum_{\langle \alpha\beta \rangle} \sigma_\alpha$ on dual lattice

$$= -2J\sigma_\beta \left(\sum_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha + \sum_{\substack{\alpha \in j \\ \alpha \neq \beta}} \sigma_\alpha \right)$$

$$= -2J\sigma_\beta (B_i + B_j)$$

$$= -2J\sigma_\beta (C_i - \sigma_\beta + C_i - \sigma_\beta)$$

$$= -4\lambda (\sigma_\beta (C_i + C_j) - 2) = -4\lambda (B_i + B_j - 2)$$

① $\beta \notin i$

$$\Rightarrow \epsilon_i' \cancel{+\epsilon_i} = \frac{1}{2}\sum_{\alpha\in i} D_\alpha - \lambda\Big(\sum_{\alpha\in i}\sigma_\alpha\Big)^2$$
$$= \epsilon_i$$

② $\beta \in i$ $\qquad \beta \in \{i, i'\}$

$$\epsilon_i' = \frac{1}{2}\sum_{\alpha\in i} D_\alpha - \frac{1}{2}\theta\,\Delta E_\beta$$
$$+ \lambda\Big(\sum_{\alpha\in i}\sigma_\alpha - 2\theta\sigma_\beta\Big)^2$$

$$= \underbrace{\frac{1}{2}\sum_{\alpha\in i} D_\alpha + \lambda\Big(\sum_{\alpha\in i}\sigma_\alpha\Big)^2}_{\epsilon_i} - \frac{1}{2}\theta\,\Delta E_\beta$$

$$\cancel{\lambda(2\theta\sigma_\beta)^2} \qquad \theta^2 = \theta. \qquad \sigma^2 = 1$$

$$+ 4\lambda\theta^2 - 4\lambda\theta\sigma_\beta\sum_{\alpha\in i}\sigma_\alpha$$

$$\epsilon_i' - \epsilon_i = -4\lambda\theta\sigma_\beta\sum_{\alpha\in i}\sigma_\alpha + 4\lambda\theta^2 - \frac{1}{2}\theta\,\underline{\Delta E_\beta}.$$

$\cancel{\Delta E_\beta = }$

$$\Delta E_\beta = +\lambda(\Delta A_i + \Delta A_j)$$
$$= +\lambda(4 - 4\sigma_\beta \Sigma_i + 4 - 4\sigma_\beta A_{i'})$$
$$= -4\lambda(\sigma_\beta(\Sigma_i + \Sigma_{i'}) - 2)$$

$$\epsilon_i' - \epsilon_i = -4\lambda\sigma_\beta\Sigma_i + \cancel{4\lambda} \cancel{-4\lambda} + 2\lambda\sigma_\beta(\Sigma_i + \Sigma_j)$$
$$= 2\lambda\sigma_\beta(\Sigma_j - \Sigma_i) = -\sum_{\langle ij\rangle} J_{ij}$$

So I'm right!

$$H = \lambda \sum_i \Big(\underbrace{\sum_{\alpha \in i} \sigma_\alpha}_{A_i} \Big)^2 = \underbrace{\frac{\lambda}{2} \sum_{\langle \alpha \beta \rangle} \sigma_\alpha \sigma_\beta + \text{const.}}_{\text{on new lattice w/. extra edges.}}$$

$$\left.\begin{aligned} \Delta E^{(\beta)} &= -4\lambda \sigma_\beta (B_i + B_j) \\ j_{ij}^{(\beta)} &= -2\lambda (B_j - B_i)\, \theta \end{aligned}\right\}$$ v. similar to 8-vertex in this form!

$\Big(i, j \in \beta.$ & $B_i = \sum_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha$

Now the question is: ① $B_i \in \{ \dots ? \}$.

② $\Delta E^{(\beta)} \in$ " ?

③ $j_{ij}^{(\beta)} \in$ " ?

$\sigma_\alpha \in \{\pm 1\}$

~~① $B_i = \sum_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha \in \{-3, \dots, +3\}$~~

1	1	1
1	1	-1
1	-1	-1
-1	-1	-1

① $B_i = \sum_{\substack{\alpha \in i \\ \alpha \neq \beta}} \sigma_\alpha \in \{\pm 1, \pm 3\}$

② B_i & B_j share no σ's

$\Rightarrow \quad B_i + B_j \in \{0, \pm 2, \pm 4, \pm 6\}$

$\qquad B_i - B_j \in \underbrace{\{0, \pm 2, \pm 4, \pm 6\}}_{X}$

$\Rightarrow \quad \Delta E^{(\beta)} \in -4\lambda X = \cancel{4} -2J X$

$\qquad j_{ij}^{(\beta)} \in -2\lambda X = -J X$

1	3	4	-2
1	-3	-2	4
-1	3	2	-4
-1	-3	-4	2
1	-1	0	2
1	1	2	0
3	-3	0	6
3	3	6	0
-3	-3	-6	0
-1	-1		0

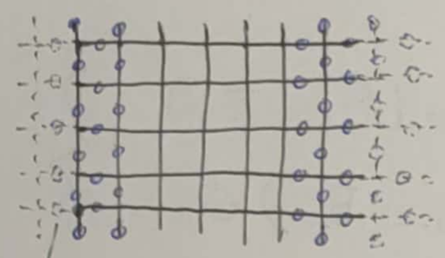
Note: ΔE quantised in units of $\pm 4J = \pm 8\lambda$

Δj_{ij} " " $\pm 2J = \pm 4\lambda$.

Realisation: We <u>need</u> a rough boundary @ the heat bath edges.

Shouldn't actually make a difference ∵ can. dynamics there.

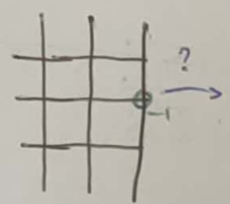
Why? It allows $A=-1$ prtcls to disappear into bath region.



flipping a spin here generates/annihilates • prtcl

↳ can then diffuse in/out by flipping next spin.

→ BUT can't get here if no bath!



Not a problem for us though ∵ ∃ bath on either side.

Also not a ~~pb~~ problem <u>in</u> bath ∵ canonical dynamics so although the excitatns get stuck in the baths they can be removed easily ∵ $\Delta E<0$ to kill them in cold bath.

We wish to make further progress on implementing 2D spin ice in our framework.

What remains? * Energy changes are now in units of 8λ rather than 4λ

$\Rightarrow$ important for demons etc.

$\Rightarrow$ throughout code, replace 4λ w/. $\Delta E\_scale = \begin{cases} 4\lambda & \text{8-vertex} \\ 8\lambda & \text{6-vertex.} \end{cases}$

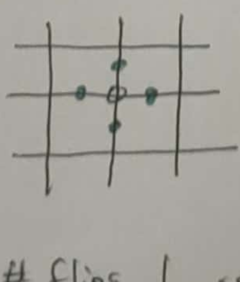
~~* We need a B function (this is sufficient, if we need $\hat{A}$ it's just $\tilde{A} = (B+\sigma_\beta)^2$ also easy to implement~~

* We need to calculate $\Delta E^{(\beta)}$ & j_{ij} differently depending on the case.

* Change to thermalisat$^{\text{n}}$.

$\langle \epsilon_i^{spin} \rangle = \langle \lambda A_i \rangle$

To find this, proceed as before: single tetrahedron approx$^{\text{n}}$



$A_i \in \{0, 4, 16\}$.

~~Start in all +1 state $\Rightarrow$ energy of 16λ~~

Start in ground state $\Rightarrow 2\uparrow, 2\downarrow$

$\Rightarrow \frac{4!}{2!2!} = 6 = {}^4C_2$

# flips	ΔE	
		$8?$

* Excitat$^{\bar{n}}$s

$\pm 1, \pm 1, \pm 1, \pm 1$

$$A_i \in \{0, \underbrace{2^2}_{p}{}^{\text{single}}, \underbrace{4^2}_{q}{}^{\text{double}}\} \qquad A_i = C_i^2$$

Rau et al. show $E = 2\lambda \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta$

has energy $E = 4\lambda \sum_I Q_I^2 - 2\lambda N$

$$Q_I = \frac{1}{2}(-1)^I \sum_{\alpha \in I} \sigma_\alpha$$

$$= \frac{1}{2}(-1)^I C_i$$

$$\Rightarrow E = \lambda \sum_i A_i - 2\lambda N$$

this is the constant we neglected earlier when showing equivalence of 2D ice & 6-vert

The fusion rules for these prtcls are interesting! A q can be seen as 2 x p w/ antibinding energy 8λ (each p has 4λ energy) when sharing the site.

energetic cost to combining 2xp into a q.

Don't care ~ this for diffus$^{\bar{n}}$ ∵ no fusion allowed

⇒ unless I create q prtcls manually they won't arise in diffus$^{\bar{n}}$ expt.

if they do, need to track p's & q's separately.

I keep running into one key problem:
How to approximate $\langle \epsilon_i^{spin} \rangle = \langle \pm \lambda A_i \rangle$ for a single diamond/site.

How many possible states for a single vertex?

State	Energy	A	$\tilde{A}$
$\mid\mid\mid\mid$		+1	16
$\mid\mid\mid\bar{\mid}$		−1	4
$\mid\mid\bar{\mid}\mid$		−1	4
$\mid\mid\bar{\mid}\bar{\mid}$		+1	0
$\mid\bar{\mid}\mid\mid$		−1	4
$\mid\bar{\mid}\mid\bar{\mid}$		+1	0
$\mid\bar{\mid}\bar{\mid}\mid$		+1	0
$\mid\bar{\mid}\bar{\mid}\bar{\mid}$		−1	4
$\bar{\mid}\mid\mid\mid$		−1	4
$\bar{\mid}\mid\mid\bar{\mid}$		+1	0
$\bar{\mid}\mid\bar{\mid}\mid$		+1	0
$\bar{\mid}\mid\bar{\mid}\bar{\mid}$		−1	4
$\bar{\mid}\bar{\mid}\mid\mid$		+1	0
$\bar{\mid}\bar{\mid}\mid\bar{\mid}$		−1	4
$\bar{\mid}\bar{\mid}\bar{\mid}\mid$		−1	4
$\bar{\mid}\bar{\mid}\bar{\mid}\bar{\mid}$		+1	16

This is completely ignoring neighbouring vertices unlike what I did before.

$\Rightarrow$ 8-vertex: $E = -\lambda A$ $(\lambda = 1)$

$$Z = 8e^{+\lambda/T} + 8e^{-\lambda/T}$$

$$\sum_{\text{state}} A_* e^{+\lambda A/T} = 8e^{+\lambda/T} - 8e^{-\lambda/T}$$

In Ray, $\tilde{Z} = Z e^{4\lambda/T}$ ∵ relative to ground state energy $\Rightarrow$ extra $-\lambda N = -4\lambda$ in Hamiltonian (we neglected this).

6-vertex: $E = +\lambda \tilde{A}$ $(\lambda = 1)$

$$Z = 6 + 8e^{-4/T} + 2e^{-16/T}$$

$$\sum_{\text{state}} \tilde{A} e^{-\lambda \tilde{A}/T} = 0 + \frac{8 \cdot 4}{32} e^{-4/T} + \frac{2 \cdot 16}{32} e^{-16/T}$$

So correct! $\Leftarrow$

$$\Rightarrow \quad \langle A \rangle = \frac{e^{\lambda/T} - e^{-\lambda/T}}{e^{\lambda/T} + e^{-\lambda/T}} = \tanh(\lambda/T)$$

$$\langle \hat{A} \rangle = \frac{\cancel{32}\,16\,(e^{-4/T} + e^{-16/T})}{\cancel{6}\,3 + \cancel{8}\,4 e^{-4/T} + \cancel{2} e^{-16/T}}$$

$$= 16\, \frac{e^{-4/T} + e^{-16/T}}{3 + 4e^{-4/T} + e^{-16/T}}$$

$\cancel{T_{NO}: \quad T_{Two}:}$

@ large T, $\langle \hat{A} \rangle \approx 16\, \dfrac{2 - \frac{4}{T} - \frac{16}{T}}{\cancel{32}\, 8 - 2 \cdot \frac{16}{T}}$

$$\approx 16\, \frac{\cancel{2}\,1 - \cancel{20}\,10/T}{\cancel{8}\,4 - \cancel{32}\,16/T}$$

$$\approx \cancel{16}\,4 \left(1 - \frac{16}{T}\right)\left(\cancel{4}\,1 + \frac{4}{T}\right)$$

$$\cancel{\approx 16\left(2 - \frac{34}{T}\right)}$$

$$\approx 4\left(1 - \frac{12}{T}\right) \longrightarrow 4$$

Note: $\langle \tilde{A} \rangle \xrightarrow{T \to \infty}\!\!\!\!\!\!/\;\; 16$! Interesting!

$\qquad\qquad \longrightarrow 4$

@ small T, $\langle \tilde{A} \rangle \to 0$ as we'd expect.

even though q excitat^ns accessible @ T large, because of the extra binding energy the # p particles dwarfs them.

... I think?

Suppose we considered

$$\cancel{\langle \sqrt{\hat{A}} \rangle}\; E = \lambda \sum_i \sqrt{|\hat{A}|}$$

$$= \lambda \sum_i \left| \left(\sum_{\alpha \in i} \sigma_\alpha \right) \right|$$

so $\exists$ no binding cost.

Then...

$$Z = 6 + 8e^{-2/T} + 2e^{-4/T}$$

$$\sum \sqrt{\hat{A}}\, e^{-\lambda\sqrt{\hat{A}}/T} = \frac{8 \cdot 2}{16} e^{-2/T} + \frac{2 \cdot 4}{8} e^{-4/T}$$

$$\langle \sqrt{\hat{A}} \rangle = 4\, \frac{8e^{-2/T} + e^{-4/T}}{3 + 4e^{-2/T} + e^{-4/T}}$$

$$\approx \begin{cases} T \to \infty & \sim 4 \\ T \to 0^+ & 0 \end{cases}$$

as more expected ∴ no binding

Rau writes

$$\tilde{Z}_{1\diamond} = 8 + 2e^{-6\beta J} + 6e^{2\beta J}$$
(tetrahedron = 4 spins)

Whence $E_{1\,spin} = \frac{1}{2} \hat{E}_{1\diamond} = \frac{1}{2} \cdot -\frac{1}{Z_{1\diamond}} \frac{\partial}{\partial \beta} \tilde{Z}_{1\diamond}$

$$= 6J \frac{e^{-6J\beta} - e^{2J\beta}}{\tilde{Z}_{1\diamond}}$$

above gnd state.

avoids 2x counting ∵ each spin shared by 2 $\diamond$ $\left(2 \times \frac{1}{4} \times E_{1\diamond}\right)$

So now in our version we have

$$Z_{1\diamond} = 6 + 8e^{-4\lambda\beta} + 2e^{-16\lambda\beta} = 6 + 8e^{-2J\beta} + 2e^{-8J\beta}$$

$$\Rightarrow E_{1\,spin} = \frac{1}{2} \cdot -\frac{1}{Z_{1\diamond}} \frac{\partial}{\partial \beta} Z_{1\diamond}$$

$$= \frac{1}{2}\, 16\lambda \frac{e^{-4\lambda\beta} + e^{-16\lambda\beta}}{Z_{1\diamond}}$$

$$= 8J \frac{e^{-2J\beta} + e^{-8J\beta}}{Z_{1\diamond}}$$

$$\tilde{Z}_1 = Z'_1 e^{-4\lambda\beta} = Z_1 e^{-2J\beta}$$

$$\tilde{E}_1 = \frac{1}{2} \cdot -\frac{1}{\tilde{Z}} \frac{\partial}{\partial \beta}\left(Z e^{-2J\beta}\right)$$

$$= -\frac{e^{-2J\beta}}{2\tilde{Z}}\left(\frac{\partial Z}{\partial \beta} - 2J\tilde{Z}\right) = E_1 + J$$

makes sense!

So the takeaway is for <u>us</u> we set 0 energy for $\tilde{A}_i = 0 \ \forall i \in$ vertices

$\Rightarrow$ $E_{spin} = 0$ initially $\because$ GS

$$\underbrace{\langle E_{1\,spin}(T)\rangle}_{\langle\lambda\tilde{A}\rangle} \approx 16\lambda \frac{e^{-4\lambda/T} + e^{-16\lambda/T}}{6 + 8e^{-4\lambda/T} + 2e^{-16\lambda/T}}$$

$$= \langle E_{Rau\,1\,spin}\rangle + \underbrace{2\lambda}_{J}$$

So that $N_i\langle LHS\rangle_1 = N_i\langle RHS\rangle + \underbrace{N_i J}$

$N_i J = 2N_i\lambda = N_\alpha\lambda$ for PBCs

accounts for difference in zero of energy (lower by $-JN$ in Rau's paper) $\because$ additive const.

$$\langle E_{1\,spin}\rangle = \frac{\pm\lambda\langle A\rangle}{2} \ ?$$

seems dodgy but it's what Rau did... ~~Isn't~~ ~~Don't~~

$$\langle\lambda\tilde{A}\rangle \approx \begin{cases} T\to 0^+ & 16\lambda\,\dfrac{1+1}{6+8+2} = 2\lambda \\ T\to\infty & \end{cases}$$

so $\tilde{A} \sim 1$

works for Rau $\because$ $N_\alpha = N_i$

Not for us! We <u>don't</u> need the factor of $\frac{1}{2}$!

$$\langle\lambda\tilde{A}\rangle \approx \begin{cases} T\to 0^+ & 0 \\ T\to\infty & 4\lambda \Rightarrow \langle\tilde{A}\rangle = 4 \end{cases}$$

bit odd: $\langle\tilde{A}\rangle = 4$ but $\tilde{A} \in \{0, \pm 4, 16\}$. $\Rightarrow$ q pert drowns out

Require same for 8-vertex, i.e. $\langle \sigma_1\rangle = \frac{1}{2}\tanh(\lambda/T)$

$\langle LHS \ldots \tanh(\lambda/T)$

$\delta E = 4, 8$

Problem: $\tilde{A} \sim n^\circ$ n.n.s unlike A.

$\Rightarrow$ in heat bath regions the energy drops sharply.

No problem for Rau et al $\because$ their lattice has no partial tetrahedra.

Is this an issue?

~~NOT~~ if $E_{strip} \sim$ const. in main sample.

actually should have some profile w/. T

$\longrightarrow$ ~~It's~~ roughly const. @ $16 \cdot L_y$ for $x \sim$ in sample (bit of mess @ ∂s)

$W: L_x - W - 1$ is okay

extra unit outside bath broken?

small problem to fix for later.

Only affects certain x? ~~initial~~ config

Why could this be? Only real soln. is the

$\underline{240/15 = 16}$

currently sample is in maximal energy state w/. all spins aligned. (except in baths).

WTF? ~~Should only happen~~ There's no forcing for that, it's an antiferromagnet! (frustrated).

↳ $\tilde{A} = 16$ everywhere in sample after long time...

↳ Taking $\lambda \to -\lambda$ in $\Delta E^{(\beta)}$ instead gives us the desired antiferromagnetic oscill$^{\underline{n}}$ in $\hat{A}$?

↳ if in G.S, $A_i = 0 \ \forall i$

$\to \Delta E^{(\beta)} = +8\lambda$ as expected.

$\to$ implies two errors:

* missing sign somewhere $\because$ system trying to maximise energy subject to demons ($\Rightarrow$ not much happens)
* weird T dependence.

If we take $\lambda \to -\lambda$ what do we get?

FUCK.

I had σ instead of $(-1)^\sigma$ somewhere...
(in ΔE_flip).

* Code doesn't seem happy w/. $\langle \hat{A} \rangle$ as we found it ... ⇒ maybe look @ that again.

* On the plus side, having fixed this, the ∂ ~~effects~~ effect for $\hat{A}$ in the bath's is no longer a problem.

À la Rau, $E(T) = \sum_\alpha \left(\langle \epsilon^{spin}_\alpha \rangle + \langle \epsilon^{demon}_\alpha \rangle \right)$ correct for any graph

$$= \sum_j \left((\,'' \,)_j^{\;= \pm \lambda \langle A_i \rangle} + (\,''\,)_j^{\;= \frac{1}{2}\sum_{\alpha \in i} \langle D_\alpha \rangle} \right)$$

$$= N_\alpha \left(\pm \frac{\lambda}{2} \langle A \rangle + \frac{\delta E}{e^{\delta E / T} - 1} \right)$$

$N_\alpha = 2N_j$ if square.

$$E_D = E(T) - E^{spin}_{GS} \qquad E^{spin}_{GS} = \begin{cases} -\lambda N_j = -\frac{\lambda}{2} N_\alpha & 8-v \\ 0 \quad (-N_\alpha J = -4N_j\lambda) & 6-v \end{cases}$$

we know
$$\langle \epsilon^{spin}_j \rangle = \begin{cases} -\lambda \langle A_j \rangle = -\tanh(\lambda\beta) \\ +\lambda \langle \tilde{A}_j \rangle = +16\lambda \dfrac{e^{-4\beta\lambda} + e^{-16\beta\lambda}}{3 + 4e^{-4\beta\lambda} + 2e^{-16\beta\lambda}} \end{cases}$$

$$= +12\lambda \frac{e^{-12\beta\lambda} + e^{+4\beta\lambda}}{4 + e^{-12\beta\lambda} + 3e^{+4\beta\lambda}} + 4\lambda$$

$$\Rightarrow E_D = \begin{cases} \sum_j \left(-\tanh(\lambda\beta) + \dfrac{z_j}{2} \dfrac{\delta E}{e^{\delta E/T} - 1} \right) + \lambda N_j \\ \sum_j \left(16\lambda \dfrac{e^{-4\beta\lambda} + e^{-16\beta\lambda}}{3 + 4e^{-4\beta\lambda} + 2e^{-16\beta\lambda}} + '' \right) + 0 \end{cases}$$

① $\sum_{\alpha} O_\alpha = \frac{1}{2} \sum_j \left(\sum_{\alpha \in j} O_\alpha \right)$

② $\sum_j O_j = \sum_\alpha \left(\sum_{j \in \alpha} \frac{1}{z_j} O_j \right)$

e.g. take $O = 1$: ① $E = \frac{1}{2} \sum_{v \in V} z_v$

② $V = \sum_{e \in E} \sum_{v \in e} \frac{1}{z_v}$

& let $z_v = z \;\forall v \in V$ $\Rightarrow$ $E = \frac{z}{2} V$

$V = \frac{2}{z} E$

So... $\overset{\epsilon^P_\alpha}{\overset{\shortparallel}{\sum_\alpha D_\alpha}} = \frac{1}{2} \sum_j \overbrace{\left(\sum_{\alpha \in j} D_\alpha \right)}^{2\epsilon^P_j}$

$\sum_j \epsilon^\sigma_j = \sum_\alpha \underbrace{\left(\sum_{j \in \alpha} \frac{1}{z_j} \epsilon_j^\sigma \right)}_{\epsilon_\alpha}$

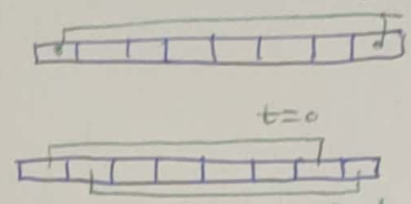
$\Rightarrow$ for $E(T)$ it's better to work in the former (vertex) picture!

$$K \propto \sum_{\tau=0}^{T-1} \langle J_{\tau+t} J_t \rangle_t \, a_\tau$$

$T = 8$

$t=0$

$\tau = T-1 = 7$



$t=0$

$\tau=6$

$t=1$

$\tau=5$

$t = 0, 1, 2$

$t \in [0, T-1-\tau]$

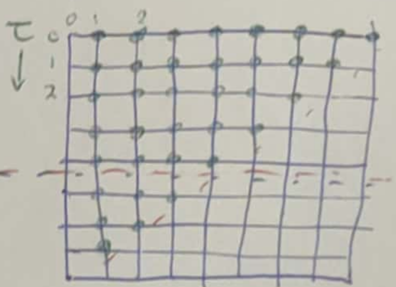
<u>except</u> t <u>1-indexed</u> $\Rightarrow$ $t \in [1, T-\tau]$

$$K \propto \sum_{\tau=0}^{\min\{T_{max}, T-1\}} \sum_{t=1}^{T-\tau} \underbrace{\frac{J_{\tau+t} J_t \, a_\tau}{T-\tau}}_{b_{t\tau}}$$

$$= \sum_{\tau=0}^{T-1} \sum_{t=1}^{T-\tau} b_{t\tau}$$

$\min(T_{max}, T-t)$

$$= \sum_{t=1}^{T} \sum_{\tau=0}^{T-t} b_{t\tau}$$



$t=1$ $\quad \tau = 0 \ldots T-1$

$t=2$ $\quad \tau = 0 \ldots T-2$

$$K \propto \sum_{t=1}^{T} \sum_{\tau=0}^{T-t} \left(\frac{1}{T-\tau} J_{\tau+t} J_t \, a_\tau \right)$$

$$a_\tau = \begin{cases} 1 & \tau \neq \\ 1/2 & \tau = \end{cases}$$

$= 1 - \frac{1}{2}\delta_{\tau}$

$$= \left\langle \sum_{\tau=0}^{T-t} \frac{1}{T-\tau} J_{\tau+t} J_t \, a_\tau \right\rangle_t$$

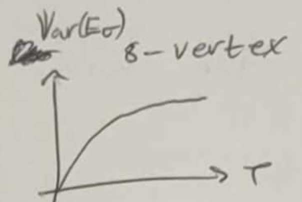
<u>can</u> ~~bootstrap~~ if we ~~forget~~ using ~~the fft~~ autocorr function.

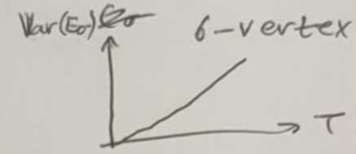
<u>Summary of testing for Kubo</u>

$T_{\mu can} \sim 2\, T_{demon}$
∵ thermalisatⁿ step.

* Get ~~$\mathcal{J}(t)$~~ $A(t) = \langle J(t) J(0) \rangle - \langle J(0) \rangle^2 \sim e^{-t/\tau}$
as expected but temp. dependence wrong!

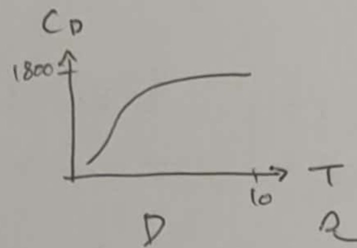
* For C_σ, we have: . . .

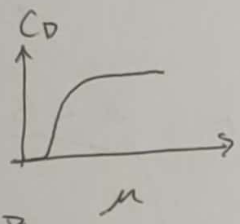




(μcan a bit noisier & messy @ T~0 ∵ few changes for short therm time)

$Var(E_\sigma) \sim$ same for μcan & demon
$\Rightarrow$ difference is in C_D?

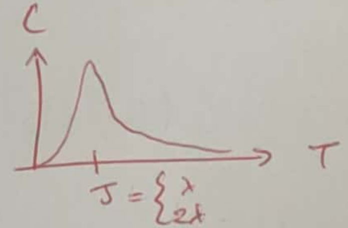




only difference!

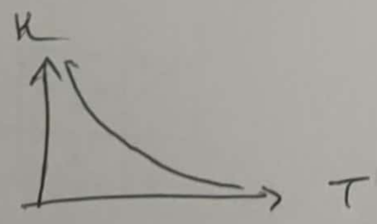
BUT ... why are we using C_D for μcan?

fixed! We now have



BUT μcan does worse @ low temps

and



} <u>WRONG!</u>

↳ fix: <u>un</u>-normalise the autocorrelatⁿ!

Check: in Rau's paper $j_{\alpha\beta}^{(\gamma)} = \underline{2\lambda\Theta\sigma_\alpha\sigma_\beta(\delta_{\beta\gamma} - \delta_{\alpha\gamma})}$

Hard to compare w/ ours...

$$\dot{j}_{ij}^{(\gamma)} = -2\lambda\,\theta(B_j - B_i)$$

We know $\Delta\epsilon_i^{(\gamma)} = -\sum_j \dot{j}_{ij}^{(\gamma)}$ & likewise $\Delta E^{(\gamma)} = -\sum_{\alpha\beta} \dot{j}_{\alpha\beta}^{(\gamma)}$

$$\Delta E^{(\gamma)} = \sum_i \Delta\epsilon_i^{(\gamma)}$$

where $\underline{i \sim j}$ & $\underline{\alpha\sim\beta}$ on resp. lattices.

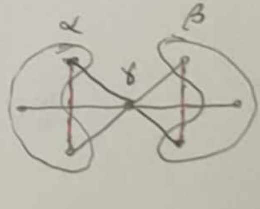
$$\Delta E_{①}^{(\gamma)} = -2\lambda\Theta \sum_{\langle i,j\rangle} (B_j - B_i) = -2\lambda\Theta \sum_{ij}$$

$$\cancel{= -2\lambda N_v \Theta}$$

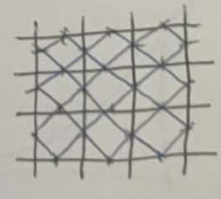
$$\Delta E_{②}^{(\gamma)} = -2\lambda \sum_{\langle\alpha,\beta\rangle} \sigma_\alpha\sigma_\beta(\delta_{\beta\gamma} - \delta_{\alpha\gamma})$$

$$\dot{j}_{ij}^{(\gamma)} = -2\lambda\,\Theta(B_j - B_i) \quad \text{for} \quad i \underset{\gamma}{—} j$$

$$\dot{j}_{\alpha\beta}^{(\gamma)} = 2\lambda\,\Theta\underbrace{(\sigma_\alpha - \sigma_\beta)}_{=0\ @\ \alpha=\beta}\sigma_\gamma \quad \text{for} \quad \alpha\ i \underset{\gamma}{—} j\ \beta \quad \sim$$



$$\Downarrow$$



$$\Delta E^{(\gamma)} = -\sum_\alpha \sum_{\beta\in nn(\alpha)} \dot{j}_{\alpha\beta}^{(\gamma)}$$

$$= -\sum_\alpha \dot{j}_{\alpha\gamma}^{(\gamma)}$$

$\beta \in nn(\alpha)$

$\Rightarrow$ pick one of red bonds.

$$i \overset{\gamma}{\sim} j$$

$$\Delta E^{(\gamma)} = 2\lambda\Theta \sum_i \sum_{j\in nn(i)} B_j - B_i = 2\lambda\Theta(B_{\gamma1} - B_{\gamma2})$$

$$\Delta E^{(\gamma)} = -\cancel{2\lambda\Theta\sigma_{\gamma1}\sigma_{\gamma2}\epsilon\,8} - 2\lambda\Theta\sum_\alpha\sum_{\beta} \sigma_\alpha\sigma_\beta(\delta_{\beta\gamma} - \delta_{\alpha\gamma})$$

Problem

$$\underline{J} = \frac{1}{2}\sum_{ij}{}' j_{ij}(\underline{r}_i - \underline{r}_j) = \sum_{\alpha}{}' j_{12}^{(\alpha)}(\underline{r}_1^{(\alpha)} - \underline{r}_2^{(\alpha)})$$

BUT this includes bonds α between opposite ends of sample $\because$ PBCS

$\Rightarrow$ dominant contrib$^{\underline{n}}$ from these terms $\therefore \propto L$.

$\to$ get overall $\underline{J} \hat{\propto} L$.

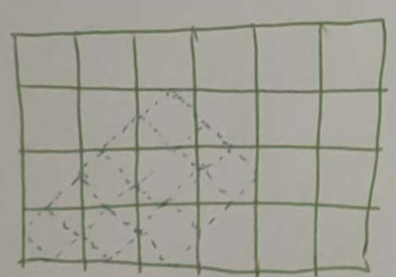
$\Rightarrow$ problematic scaling

$\Rightarrow$ use OBCs?

(problem for defining 6-vertex GS)

$\downarrow$

TO FIX.



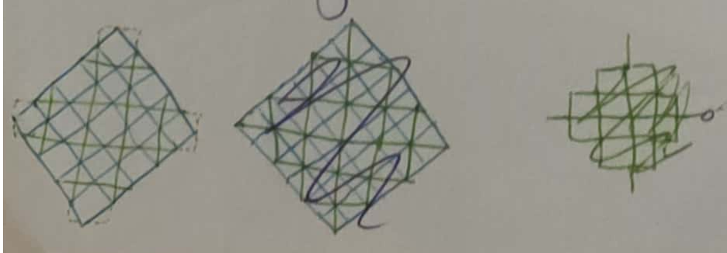
ALSO OBCs a problem for 6-vertex $\because$ $\tilde{A}_{corner} \in \{0, 4\}$

$\tilde{A}_{edge} \in \{4, 9\}$

$\tilde{A}_{bulk} \in \{0, 4, 16\}$.

$\Rightarrow \nexists$ no GS of 6-vertex w/. edges like these...

Rau gets around this à la:



~~Now~~, $\cancel{A_{corner} \in \{\}}$ $\cancel{A_{edge} \in \{\}}$

Be ~~care~~ careful w/. duality @ boundaries ...

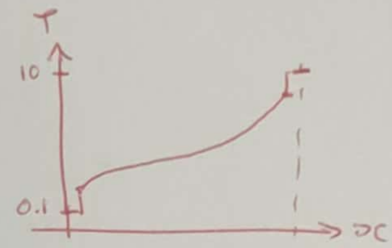
Now, on our green lattice every site has 4 spins!

Solution: Still use PBCs <u>BUT</u> need a funcⁿ to find actual displacement $\underline{r}_1 - \underline{r}_2$ in T^2 rather than in an unrolled T^2.

Problems

* J depends on L_x, W for bath method (not on L_y)

* @ ends of bath, we have:



∴ fails to thermalise @ ends?

For larger bath this less of an issue, but worth being aware it affects the K values @ ends ∴ $T_{i+x} - T_{i-x}$ v. large! ⟶ Maybe consider excluding these edges cases?

Done!

$\left\{ \begin{array}{l} \text{These problems are not present in} \\ \text{the Kubo method code!} \end{array} \right\}$.

$J \propto \frac{W^2}{L_x^2}$? Nope!

Scaling as a function of W/L_x is reasonable physically

$$J_\pm = \frac{1}{L_y} \overbrace{\cancel{\sum_{\alpha \in B_\pm}}}^{\sim \text{Why}} \overbrace{\cancel{\langle \Delta E^{(\alpha)} \rangle}}^{\sim \text{const.}}$$

$$\sim \cancel{L_x}\, W$$

Why would $\Delta E^{(\alpha)}$ scale as W/L_x^2 ???...

Or maybe $J \propto \frac{W^2}{(L_x - W)^2}$?

25 x 5

W	J
[illegible]	
10	0.11
9	
8	
7	0.0625
6	
5	0.0486
4	
3	

Key point: κ varies with W & L_x.

L_x	W	J
15	3	0.074
20	4	0.057
25	5	0.048
30	6	0.042

$W/L_x = 1/5$

$L_x = 5W$

$\Rightarrow$ not a func^n of $\left(\frac{W}{L_x}\right)$:(as we might expect physically...

Very strange...

Problem: For bath method, J depends on L_x, W non-trivially (roughly indep. of L_y).

Note that we've defined:

$$J^x_{B_\pm} = \langle \Delta E_{B_\pm} \rangle / L_y = \langle \sum_{\substack{\alpha \in B_\pm \\ \alpha \in \text{flips}(t)}} \Delta E^{(\alpha)} \rangle / L_y$$

α must be flipped spins in Metropolis updates.

$$= \frac{1}{L_y} \sum_{\substack{\alpha \in B_\pm \\ \cancel{\alpha \in \text{flips}}}} \langle \Delta E^{(\alpha)} \rangle$$

So we ~~ave~~ sum up avg. energy changes on each ~~site~~ edge then divide by L_y.

$\#\ \text{edges}_\pm = |B_\pm| = W L_y$

$\Rightarrow$ should be indep...

$L_x \times L_y$	W	J^x
15×15	5	~ 0.11
25×25	"	~ 0.05
20×20	"	~ 0.07
20×15	"	~ 0.07
15×25	"	~ 0.11
15×20	"	~ 0.11
25×15	"	~ 0.05
$\cancel{25}\ 15 \times 15$	3	~ 0.07
$\cancel{20}\ 15 \times 15$	4	$\sim$

$\Rightarrow J^x \sim$ indep. of L_y

$\sim 1/L_x^2$.

Suppose $J^x \sim 1/L_x^2$

$\ln J^x \sim -2 \ln L_x$

~~$J \sim e^{-L_x}$~~

~~$\ln J \sim -L_x$~~

$J \sim 1/L_x^2$? Why??

We should have $E \sim \frac{\delta E}{2} \cdot N_{site}\, e^{-\delta E/2T}$

$$\delta E = \begin{cases} 4\lambda & \text{each monopole has } 2\lambda \\ 8\lambda = 4J & \text{" } \quad \text{" } 2J = 4\lambda \end{cases}$$

$$\cancel{\Rightarrow C_V \sim \frac{\partial E}{\partial T}\Big|_V \sim \left(\frac{\delta E}{2T}\right)^2 N_{site}}$$

@ peak, $C \sim \cancel{k_B} \left(\frac{\delta E}{2T}\right)^2 \frac{N_1}{V} \sim \frac{N_{site}\cancel{k_B}}{V} \left(\frac{\delta E}{2T}\right)^2 e^{-\delta E/2T}$

$$x = \frac{\delta E}{2T} \Rightarrow C \sim \frac{N_{site}\cancel{k_B}}{V} x^2 e^{-x}$$

$$\Rightarrow \text{max @ } x = 2 \Rightarrow T = \frac{\delta E}{4} = \begin{cases} \lambda \\ 2\lambda = J \end{cases}$$

$C(T = \delta E/4) \sim \frac{N_{site}}{V} \cdot 0.541$ for spin

Also $K \sim \left(\frac{\delta E}{2T}\right)^2 \frac{N_1}{V} D$

We then get $C_{demon} \sim N_{edge} \left(\frac{\delta E}{T}\right)^2 \frac{e^{\delta E/T}}{(e^{\delta E/T} - 1)^2}$

$$\sim \underbrace{N_{edge}\left(\frac{\delta E}{T}\right)^2}_{N_e'} e^{-\delta E/T}$$

$$\Rightarrow \frac{Var(E_{spin})}{T^2} \sim \frac{N_e' N_s e^{-3\delta E/2T}}{N_e' e^{-\delta E/T} + N_s e^{-\delta E/2T}}$$

$$= \frac{N_e' N_s e^{-\delta E/T}}{N_e' e^{-\delta E/2T} + N_s}$$

$$\sim \frac{N_e' \cancel{N_s}}{\cancel{N_s}} \frac{e^{-\delta E/T}}{1 + \frac{N_e}{N_s} e^{-\delta E/2T}}$$

$$\sim N_e' e^{-\delta E/T} \left(1 - \frac{N_e'}{N_s} e^{-\delta E/2T}\right)$$

$$\sim N_e e^{-\delta E/T} \left(\frac{\delta E}{T}\right)^2.$$

$$K_{Bath} \sim \frac{-2 \langle J^x \rangle \text{ const.}}{T_{i+x} - T_{i-x}}$$

$\Rightarrow \quad K = K_{max}$ when $|T_{i+x} - T_{i-x}|$ min.

$$\Rightarrow \quad \frac{dT}{dx} \text{ min.}$$

---

eh.

Assume $K \sim \cancel{\langle J^2 \rangle / T^2}$ $\langle J_x^2 \rangle / T^2$ holds @ peak, then note that ?

<u>WHY</u> a factor of 2 out?

Implicat$^{\overline{n}}$: Missing factor of <u>δE somewhere...</u>

$$Z = 16 \cosh(\lambda / T) = \sum_{\alpha} e^{+\lambda A_\alpha / T}$$

$$\Rightarrow \langle A_\alpha \rangle = \frac{1}{Z} \frac{\partial Z}{\partial(\lambda/T)}$$

$$=$$

$$Z = 6 + 8e^{-4\lambda/T} + 2e^{-16\lambda/T} = \sum_{\alpha} e^{-\lambda A_\alpha / T}$$

$$\Rightarrow \langle A \rangle = \frac{1}{Z} \frac{\partial Z}{\partial(-\lambda/T)} = \frac{1}{Z} \left(32e^{-4\lambda/T} + 32e^{-16\lambda/T} \right.$$

$$= 16 \frac{-4 \quad -16}{3 \quad 8^{-4} \quad 1^{-16}}.$$

Jackknife

$$\langle O \rangle_{\text{bin } n} = f(\text{bin } n)$$

$$\langle O_{\neq n} \rangle = \frac{1}{N_B - 1} \sum_{m \neq n} \langle O \rangle_{\text{bin } m}$$

$$\langle O \rangle = \frac{1}{N_B} \sum_m \langle O_{\neq m} \rangle$$

$$\sigma^2_{\langle O \rangle} = \frac{N_B - 1}{N_B} \sum_m \left(\langle O_{\neq m} \rangle - \langle O \rangle \right)^2$$

$$= (N_B - 1)^2 \cdot \frac{1}{N_B(N_B - 1)} \quad ''$$

inflation factor

Bootstrap

$$\langle O_n \rangle = \frac{1}{N_B} \sum_m^{N_B} \langle O \rangle_{\text{random bin}}$$

$$\langle O \rangle = \frac{1}{N_R} \sum_m^{N_R} \langle O_m \rangle$$

choose these N_R from the N_B randomly & w/ replacem

$$\sigma^2_{\langle O \rangle} = \frac{1}{N_R - 1} \sum_m^{N_R} \left(\langle O_m \rangle - \langle O \rangle \right)^2$$

for $N_R \Rightarrow N_B - 1$, $N_B \Rightarrow N_B - 1$

this kinda looks like Jackknife

In my code I have

eliminates error $\to O(N^{-2})$.

Can't find a source for this! → Can only find $\langle O' \rangle = N f(\text{all}) - (N-1)\langle O \rangle$

$$\langle O' \rangle = \frac{1}{N - M} \left(N f(\text{all data}) - M \langle O \rangle \right)$$

→ Key point: bias corrections ~ negligible compared to error ⇒ often okay to ignore

$$\sigma^2_{\langle O' \rangle} = \frac{1}{M(M-1)} \sum_n \left(\langle O \rangle_{\text{bin } n} - f(\text{all data}) \right)^2$$

$$= \frac{M-1}{M} \sum_n \left(\langle O \rangle_{\neq n} - \cancel{f(\text{all data})}\ \langle O \rangle \right)^2$$

~~6PaB2~~ $\quad$ ~~$r = N\ell$~~ $\quad N - R = W$

~~$\hat{\Theta} = \frac{1}{N-W}\left(N t_0 - W t_i\right)$~~

$W = N - 1$?

Got it! Bad notation in the paper!

$$\hat{\Theta} = \frac{1}{W}\left(N t_0 - (N-W) t_i\right)$$

is the actual result by Quenouille.

AND this is referring to t_0 to an estimator on the whole sample

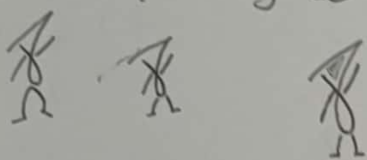
& t_i an estimator on subsamples of $N-W$, i.e. using the Jackknife binning.

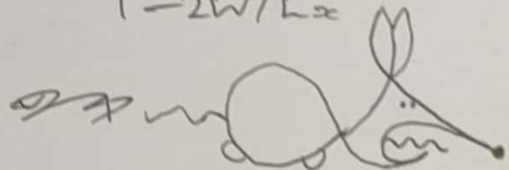
Let's be a bit more careful for J.
In the bath regions, we in fact get acceptance only @ a rate $e^{-\beta\Delta E}$ (again, indep. of W, L_x/L_y => okay).

We keep $T_+ - T_- = \Delta T$ const.

$$\Rightarrow \quad K \sim W\left(1 - \frac{2W}{L_x}\right) \quad \forall T$$

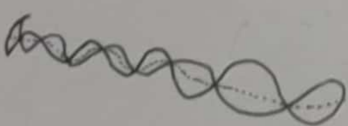
$\because$ $J \sim W$ & $T_{i+1} - T_{i-1} \sim \dfrac{1}{1-2W/L_x}$





$\Rightarrow$ $K \sim W$ for $W \ll L_x$

L_x	W	
50	5	$\leadsto$ $K_{max} \sim 0.5 - 0.6.$
50	10	$\sim$ ~ 0.5



<u>Waste of time</u>

~~Intere.~~
Max @
$\sim 0.2 - 0.25$
in Kubo.

Dependences of J, ΔT: <u>cancel</u>
$\Rightarrow$ K indep of W, L_x, L_y
$\Rightarrow$ <u>JUST</u> a factor of 2 out?

Let's consider T large.

Then we have that <u>every</u> proposal is accepted.

$\Rightarrow$ We expect $\Delta E_{B^\pm}(t) =$ avg. energy of a spin flip $\cdot N_E \cdot \frac{N_{E \in B^\pm}}{N_E}$

$$N_{E \in B^\pm} = L_y \cdot W$$

$$\Rightarrow \quad \Delta E_{B^\pm}(t) = \Delta E_{avg} \cdot L_y \cdot W$$

$$\Downarrow$$

$$\cancel{\Delta E_{avg}} = \Delta E_{flip} = \begin{cases} 6: & -4\lambda\, \sigma_p(B_{i+B} \\ 8: & \lambda(A_i - A_j) \end{cases}$$

Could find avg. value in 1-vertex approx^{n} BUT key point is it shouldn't scale

$$\Rightarrow \quad \Delta E_{B^\pm} \propto L_y W. \quad \& \quad J_{B^\pm} \approx \frac{\langle \Delta E_{B^\pm}}{L_y}$$

Now for the T part. This depends on the <u>whole</u> lattice. Writing $\frac{\Delta T}{\Delta x} = \frac{T_+ - T_-}{(L_x - 2W)}$

we get $T_{i+x} - T_{i-x} \sim \frac{T_+ - T_-}{1 - 6W/L_x)}$

$$\Rightarrow \quad K \sim W \cdot \frac{1 - 2W/L_x}{T_+ - T_-}$$

What are we seeing?

We <u>should</u> have $K \sim \delta E$ $\because$ excitat^{ns} carry energy $\delta E/2$! $K \sim \left(\frac{\delta E}{2T}\right)^2 e^{-\delta E/2T}$ D

↳ consistent w/. peaks @ $\delta E/\cancel{4} = \begin{cases} 1 & (8) \\ 2 & (6) \end{cases}$

So where's the problem in our code?

We're summing up the ΔE_{flip} fine, no missing factors of δE...

<u>Aha!</u> It's in the <u>temperature</u> estimates!

<u>Wait</u> no. I add back in the δE when finding the demon energies.

So that's not it... ⁐̈

That's <u>literally</u> all it could be!

T or J?

$$T_k = \frac{\delta E}{\ln(1 + \delta E/\langle D_k \rangle)} \Bigg\} ?$$

I'm confident in T $\because$ ~ correct endpoints ⇒ can assume it <u>must</u> be in J.

We want

$K > 0$ & $\frac{dT}{dx} > 0 \quad \Rightarrow \quad J < 0$

$J = \frac{J_{hot} - J_{cold}}{2} \qquad J_{hot} = -J_{cold}$

$\Rightarrow$ Should have $J_{hot} < 0$ & $\Rightarrow J_{cold} > 0$

BUT we instead observe the opposite

$\Rightarrow$ our calc$^{\underline{n}}$ of $J_\pm$ is going wrong somewhere...

Currently we're saying $J_\pm = \frac{\sum_{\alpha \in \pm} \cancel{t} \langle \Delta E_\alpha \rangle}{L_y}$

$= \langle \Delta E_\pm \rangle / L_y.$

i.e. sum up all ΔE's in $\pm\ \forall t$ & then avg.

I suspect the def$^{\underline{n}}$ should be $J_\pm = -\frac{1}{L_y} \langle \Delta E_\pm \rangle$

~~i.e. minus the energy ~~increase~~ gain~~

we'll use this for now

i.e.: $\Delta E : \ldots \begin{cases} < 0 : & \text{heat flow out} = -\Delta E \\ > 0 : & \text{heat flow in} = +\Delta E \end{cases}$

We want J = heat flow out of hot $-$ heat flow in to cold? Idkkk...

Anyway! Scaling :^ :^ :^ ...

Key observation: varying the sampling strip width N_s from $2 \mapsto 1$ **does** change the peak by the same factor.

$N_s \to 1$ fixes it?

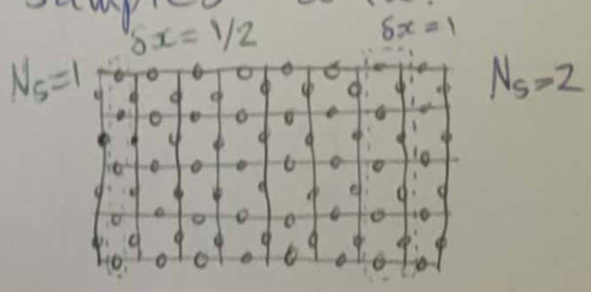
Of fucking course it does!

I've said $\left.\frac{dT}{dx}\right|_{x_i} \approx \frac{T_{i+1} - T_{i-1}}{2}$

BUT I've assumed $\delta x = x_{i+1} - x_i \ \forall i$ is (a) constant (not v. general but probs reasonable) and (b) $= 1$!

$\Rightarrow x_{i+1} - x_{i-1} = \begin{cases} N_s = 1 : & 2 \\ N_s = 2 : & 1 \end{cases}$

Not true $\because N_s = 2$ samples à la:



Naïve diffusivity

e hop on lattice of bond length a

↳ can hop by flipping one of 4 edges in its ∂^+.

$$\Rightarrow \quad a_M = a, \quad \delta t_M = \delta t/4$$

In ~~2~~ d dims, $\langle |\underline{R}(t)|^2 \rangle \sim 2dDt$

$$\sim \frac{a_M^2}{\delta t_M} t = \frac{4a^2}{\delta t} t$$

$$\Rightarrow \quad \underline{\underline{D_8 \sim \frac{2a^2/\delta t}{d}}} \sim \frac{2}{d} \overset{d=2}{\sim} 1$$

$\Rightarrow$ 8-vertex should have $D_{max} \sim 1$

In 6-vertex, we get (à la Rau, including ice rules), $a_M = a$, $\delta t_M = \delta t/3$

$$\rightarrow \quad \langle |\underline{R}(t)|^2 \rangle \sim \frac{7}{9} \frac{a_M^2}{\delta t_M} t = \frac{7}{3} \frac{a^2}{\delta t} t$$

$$\Rightarrow \quad D_6 \sim \frac{7a^2/6\delta t}{d} \sim \frac{7}{6d} \overset{d=2}{\sim} \frac{7}{12}$$

Julia question.

Suppose I have an array:

$$\text{Vertices} = \begin{array}{cccc} 1 & 2 & j & N_v \\ \boxed{\text{Vertex}} & \boxed{\text{Vertex}} & \cdots & \end{array} = v$$

Can I create a new list

$$\cancel{\boxed{\text{Vertex } 3}} \quad L = \boxed{\boxed{v[3]} \; \boxed{v[1]} \; \boxed{v[22]} \; \cdots} \,?$$

↳ Key req.-s: ① Modifying any element of L modifies it in v (i.e. not copies of $v[j]$)

② L is not ~~expen~~ more expensive in memory than instead having

$$L' = \boxed{\boxed{3} \; \boxed{1} \; \boxed{22} \; \cdots}$$

Works! But hard to debug ∵ circular references...

→ maybe use later

Julia question.

Suppose I have an array:

$$\text{Vertices} = \begin{array}{ccccc} 1 & 2 & & j & N_v \\ \hline \boxed{\text{Vertex}} & \boxed{\text{Vertex}} & \cdots & & \end{array} = v$$

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$$L' = \boxed{\boxed{3} \; \boxed{1} \; \boxed{22} \; \cdots}$$

Works! But hard to debug ∵ circular references...

→ maybe use later

Okay. Next up, what should C really be?

I'm getting $C \sim$ few 100 @ its peak value.

For μcan, $C_{spin} = \dfrac{\mathrm{Var}(E_{spin})|_T}{T^2}$

C_{peak} occurs @ $T = \delta E/2$

$$\Rightarrow C^{peak}_{spin} = 4\left(\frac{2}{\delta E}\right)^2 \underbrace{\mathrm{Var}(E_{spin}(T=\delta E/2))}_{\sigma^2_{\delta E/2}}$$

For demon, $C_{spin} = C_D \dfrac{\mathrm{Var}(E_{spin})/T^2}{\cancel{\mathrm{Var}E_{spin}}\; C_D - \mathrm{Var}(E_{spin})/T^2}$

$$C_D = N_E\left(\frac{\delta E}{2T}\right)^2 \frac{e^{\delta E/2T}}{(e^{\delta E/2T}-1)^2}$$

$$\Rightarrow \cancel{C^{peak}_{spin}} = C^{peak}_D \frac{1}{C^{peak}_D\left(\frac{\delta E}{2}\right)^2 \frac{1}{\sigma^2_{\delta E/2}} - 1}$$

$$C^{peak}_D = N_E\left(\frac{\delta E/2}{\delta E/2}\right)^2 \frac{e^1}{(e^1-1)^2} = N_E \frac{e}{(e-1)^2}$$

$$\Rightarrow C_{spin} = \frac{C_D\, \sigma^{-2}_T}{N_E(\delta E/2)^2 \frac{e^{\delta E/2T}}{(e^{\delta E/2T}-1)^2} - \sigma_T^{2}}$$

$$\Rightarrow C^{peak}_{spin} = C^{peak}_D \frac{\sigma^2_{\delta E/2}}{N_E\left(\frac{\delta E}{2}\right)^2 \frac{e}{(e-1)^2} - \sigma^2_{\delta E/2}}$$

$$= N_E \frac{e}{(e-1)^2} \frac{\sigma_{\delta E/2}^2}{N_E\left(\frac{\delta E}{2}\right)^2 \frac{e}{(e-1)^2} - \sigma_{\delta E/2}^2}$$

$$= \frac{\sigma_{\delta E/2}^2}{(\delta E/2)^2 - \frac{(e-1)^2}{e}\frac{1}{N_E}\sigma^2_{\delta E/2}}$$

$$= \frac{1}{\frac{(\delta E/2)^2}{\sigma^2_{\delta E/2}} - \frac{(e-1)^2}{e N_E}}.$$

$$\cancel{\sigma^2} \quad \sigma^2 = \mathrm{Var}(E_{spin})$$
$$= \mathrm{Var}(-\sum_i \lambda A_i)$$
$$= \lambda^2 \sum_i \underbrace{\mathrm{Var}(A_i)}_{\alpha}$$
$$= N_V \lambda^2 \alpha = \mathcal{O}(1)$$

$A_i \in \pm 1$

PBCs $\Rightarrow$ $N_E = 2N_V$

$\delta E/2 \sim \lambda$

$\Rightarrow \sigma^2 = N_E \lambda^2 (2\alpha)$

$\cancel{\delta E = f\lambda = \lambda \begin{cases} 4 & (8) \\ 8 & (6) \end{cases}}$

$$C_{spin}^{peak} \sim \frac{N_E}{\frac{f^2}{2\alpha} - \frac{(e-1)^2}{e}}$$

$\frac{\delta E}{2\lambda} = f = \begin{cases} 2 & (8) \\ 4 & (6) \end{cases}$

$\Rightarrow$ $\cancel{f}$ $C_{spin}^{peak} \sim N_e / (\frac{f^2}{2\alpha} - 1.\cancel{0}861\ldots)$ ~ 1

$\Rightarrow$ need to divide by N_E?

Why $C \sim$ factor $\frac{1}{2}$ for demon compared to ucan?

We know $C_s^p / N_E \sim \begin{cases} \frac{1}{2/\alpha - 1} \\ \frac{1}{8/\alpha - 1} \end{cases}$

$\alpha < 1 \Rightarrow$ always +ve.

<u>Worst</u> case is $\alpha = 1$ $\Rightarrow C_s^p / N_E \sim \begin{cases} 1.0 \\ 0.14 \end{cases}$

$$\hat H_n = -\lambda_{n-1}\sum_{i\in\Delta_{n-1}} \hat A_i - \lambda_{n+1}\sum_{p\in\Delta_{n+1}} \hat B_p + h.c. - h_n \sum_{\alpha\in\Delta_n}\hat Z_{\alpha}$$

$n=1$: $$\hat H_1 = -\lambda_0 \sum_i \hat A_i - \lambda_2 \sum_p \hat B_p + h.c. - h_1 \sum_{\alpha} \hat Z_\alpha$$

$n=0$: $$\hat H = -0 - \lambda_1 \sum_\alpha \hat B_\alpha + h.c. - h_0 \sum_i \hat Z_i$$

$$\hat B_\alpha = \hat Z_{\alpha_1} \hat Z_{\alpha_2}$$

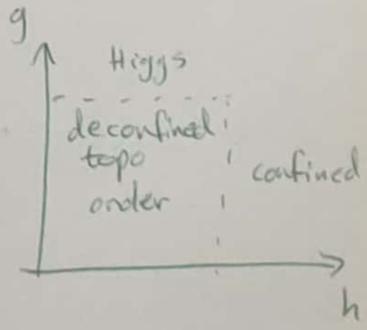
has excitat^ns when $B_\alpha < 0$ -> where one of the spins is flipped.

So we get

$$\delta \hat H_n = -\cancel{k} \sum_{\alpha\in\Delta_n} (h_n \hat Z_n + g_n \hat X_n)$$

electric string tension

hopping amplitude /fugacity for e excitat^ns.



duality is: replace k-cells in Δ w/. (d-k)-cells in $\Delta^\vee$ → dual

i.e. (if space orientable or N (in $\mathbb{Z}_N$) $= 2$)

$$\partial(\sigma_p^\vee) = (\nu(\sigma_p))^\vee \in \Delta^\vee_{d-p-1}$$

also swaps $g \leftrightarrow h$ & $e \leftrightarrow m$ etc.

Note topo order destroyed @ finite T usually

BUT for $d=4$, $\hat{H}_2$ is self-dual

$\Rightarrow$ no prtcl excitat^ns

defects are closed strings $\Rightarrow$ density $\propto$ exp. suppressed in length $\Rightarrow$ don't

avoids factor of 2 $\therefore$ $\hat{A}+\hat{A}^\dagger$ Herm.

$$\hat{A}_i = \prod_{\alpha\in\partial^+ i} \hat{Z}_\alpha$$

Toric Code $\to$ 8-Vertex

$$\hat{Z}_\alpha = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \ddots & \\ & & & & \omega^{N-1}\end{pmatrix}\frac{1}{2}$$

$\lambda_m \in \mathbb{R}$ WLOG.

$$\hat{H}_n = -\lambda_{n-1}\sum_{i\in\Delta_{n-1}} \hat{A}_i - \lambda_{n+1}\sum_{p\in\Delta_{n+1}} \hat{B}_p + \text{h.c.} - \sum_{\alpha\in\Delta_n}(h_n\hat{Z}_\alpha + g_n\hat{X}_\alpha) + \text{h.c.}$$

* Keep X's or Z's? For $d=2$, $n=1$ this is equivalent $\because$ site-plaquette duality. (say Z here)
* Replace with classical variables

$$H_n = -\lambda_{n-1}\sum_{i\in\Delta_{n-1}} A_i - h_n\sum_{\alpha\in\Delta_n}\sigma_\alpha \qquad A_i = \left(\prod_{\alpha\in\partial^+ i}\tfrac{1}{2}(\sigma_\alpha+\sigma_\alpha^*)\right) = \prod_{\alpha\in\partial^+ i}\cancel{2}\,\mathrm{Re}(\sigma_\alpha)$$

$$= -\lambda_{n-1}\sum_{i\in\Delta_{n-1}}\prod_{\alpha\in\partial^+ i}(2\,\mathrm{Re}\,\sigma_\alpha) - h_n\sum_{\alpha\in\Delta_{n-1}}\sigma_\alpha$$

If n is annoying, feel free to pick the X's then rotate 90°.

Next

Given $\hat{H}_n = -\sum_{i \in \Delta_{n\pm1}} (\lambda_{n-1} A_i + \lambda_{n+1} B_i)$

We still need to combine A & B!

For concreteness use Δ_{n-1} terms.

& ~~B~~ $\tilde{B}_i \propto \sum_{\alpha \in \partial^+ i} \sum_{p \in \partial^+ \alpha} \tilde{B}_p$

$$H_1 = -\lambda_0 \sum_i \tilde{A}_i - \lambda_2 \sum_p \tilde{B}_p$$

$$\sum_i \tilde{A}_i = \sum_\alpha \Big(\underbrace{\sum_{i \in \partial^+ \alpha} \frac{1}{|\partial^+ i|} \tilde{A}_i}_{\tilde{A}_\alpha} \Big)$$

$$\sum_\alpha \tilde{A}_\alpha = \sum_p \Big(\underbrace{\sum_{\alpha \in \partial p} \frac{1}{|\partial^+ \alpha|} \tilde{A}_\alpha}_{\tilde{A}_p} \Big)$$

$$\tilde{A}_p = \sum_{\alpha \in \partial p} \sum_{i \in \partial \alpha} \frac{\tilde{A}_i}{|\partial^+ i| \, |\partial^+ \alpha|}$$

For a graph (in 2D) $d=2$ $\quad |\partial^+ i| = ?$ $\quad |\partial^+ \alpha| = 2$

$= 4$

square lattice

The other way,

$$\cancel{\tilde{B}_p} \quad \hat{B}_i = \sum_{\alpha \in \partial^+ i} \sum_{p \in \partial^+ \alpha} \frac{\tilde{B}_p}{|\partial p| \, |\partial \alpha|}$$

For a graph in 2D $d=2$, $\quad |\partial p| = ?$ $\quad |\partial \alpha| = 2$

$= 4$

↓ Take $\mathbb{Z}_2$ order.

$$H_n = -\lambda_{n\mp1} \sum_{i\in\Delta_{n\mp1}} A_i - h_n \sum_{\alpha\in\Delta_n} \sigma_\alpha \qquad A_i = \prod_{\substack{\alpha\in\{\partial^+ i,\, \partial i\} \\ n-1 \quad n+1}} \cancel{2}\,\mathrm{Re}(\sigma_\alpha)$$

$\cancel{2}\,\sigma_\alpha$

~~Don't like this factor of 2...~~

	Δ_0	Δ_1	Δ_2
~~$\lvert\partial^+\rvert$~~	~~$\in\mathbb{Z}$~~	~~2 if $d=2$.~~	
~~$\lvert\partial\rvert$~~	~~0~~		

Alternative view: Define $\hat{W} = (\hat{X} + \hat{Z})/2$

$$\hat{H}_n = -\lambda_{n-1} \sum_{i\in\Delta_{n-1}} \hat{A}_i - \lambda_{n+1} \sum_{p\in\Delta_{n+1}} \hat{B}_n + h.c. - \sum_{\alpha\in\Delta_n} (h_n \hat{Z}_n + g_n \hat{X}_n) + h.c.$$

$$\tilde{X} = \tfrac{1}{2}(\hat{X} + \hat{X}^{+})$$

$$\text{〃} \quad Z$$

$$\hat{H}_n = -\lambda_{n-1} \sum_{i\in\Delta_{n-1}} \Big(\prod_{\alpha\in\partial^+ i} \tilde{Z}_\alpha \Big) - \lambda_{n+1} \sum_{p\in\Delta_{n+1}} \Big(\prod_{\alpha\in\partial p} \tilde{X}_\alpha \Big)$$
$$- \sum_{\alpha\in\Delta_n} (h_n \tilde{Z}_n + g_n \tilde{X}_n)$$

~~IF I can uniquely identify Δ_{n-1} & Δ_{n+1} terms.~~

Thm: Using $\displaystyle\sum_{a\in\Delta_n} O_p = \sum_{b\in\Delta_{n\pm1}} \Big(\sum_{a\in\partial^\mp b} O_a / |\partial^\pm a| \Big)$

$$E = \sum_{\alpha} \epsilon_\alpha \qquad \epsilon_\alpha = \sum_{i \in \partial^*_\alpha} \frac{1}{z_i} D_i - \lambda A_\alpha$$

$$\epsilon_\alpha' = \sum_{i \in \partial\alpha} \frac{1}{z_i} \left(D_i - \delta_{ij} \Theta \Delta E^{(j)} \right) - \lambda A_\alpha + 2\lambda \Theta A_\alpha \delta_{j \in \partial\alpha}$$

$$\epsilon_\alpha' - \epsilon_\alpha = 2\lambda \Theta A_\alpha \delta_{j \in \partial\alpha} - 2\lambda \sum_{i \in \partial\alpha} \frac{\Theta \delta_{ij}}{z_i} \sum_{\beta \in \partial^+_j} A_\beta$$

$$= 2\lambda\Theta \left[A_\alpha \delta_{j\in\partial\alpha} - \sum_{\beta \in \partial^+ j} \sum_{i \in \partial\alpha} \delta_{ij} \frac{1}{z_i} A_\beta \right]$$

$$\delta_{j\in\partial\alpha} / z_j$$

$$= \begin{cases} j \in \partial^+\alpha & 2\lambda\Theta \left[A_\alpha - \sum_{\beta \in \partial^+_\alpha} \frac{1}{z_j} A_\beta \right] \\ j \notin \partial^+\alpha & 0 \end{cases}$$

$$\varepsilon_i = D_{\cancel{j}i} - \frac{\lambda}{2} \sum_{\alpha \in \partial^+_i} A_\alpha \qquad \Delta E^{(j)} = 2\lambda \sum_{\alpha \in \partial^+_j} A_\alpha$$

$j =$ flipped spin

$$\varepsilon_i' = D_i' - \frac{\lambda}{2} \sum_{\alpha \in \partial^+_i} A_\alpha'$$

$$= D_i - \delta_{ij}\Theta \Delta E^{(j)} - \frac{\lambda}{2} \sum_{\alpha \in \partial^+_i} A_\alpha (1 - 2\Theta \delta_{j \in \partial\alpha})$$

$$\varepsilon_i' - \varepsilon_i = -\delta_{ij}\Theta\Delta E^{(j)} + \lambda\Theta \sum_{\alpha \in \partial^+_i} A_\alpha \underbrace{\delta_{\alpha \in \partial^+_j}}_{\substack{i=j \\ \text{OR } i \overset{\alpha}{-} j}}$$

$$= \sum_{\alpha \in \partial^+_j} \delta_{\alpha \in \partial^+_i}$$

$$= -\delta_{ij}\Theta 2\lambda \sum_{\alpha \in \partial^+_j} A_\alpha + \dots$$

$$= \Theta\lambda \sum_{\alpha \in \partial^+_j} A_\alpha (\delta_{\alpha \in \partial^+_i} - 2\delta_{ij})$$

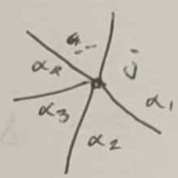
$$= \begin{cases} \substack{j \neq i \\ j \not\vdash i} & 0 \\ j = i & -z_j \lambda A_\alpha = -z_i \lambda \\ \substack{j \neq i \\ j - i} & z_i \lambda \cancel{\Theta} A_\alpha \end{cases}$$

$$\hat{H} = -\lambda \sum_\alpha A_\alpha \qquad A_\alpha = \prod_{i \in \partial\alpha} \sigma_i$$

$$= \sigma_{\alpha_1} \sigma_{\alpha_2}$$

$$E = -\lambda \sum_\alpha A_\alpha + \sum_i D_i$$

I want to flip spin j



$$A_{\alpha_1} \rightarrow -A_{\alpha_1}$$
etc.

$$\Rightarrow \Delta E^{(j)} = 2\lambda \sum_{\alpha \in \partial^+ j} A_\alpha$$

$$A'_\alpha = A_\alpha \left(1 - 2\Theta\, \delta_{\substack{\alpha \in \partial^+ j \\ j \in \partial\alpha}}\right)$$

$$\sigma_i' = \sigma_i (1 - 2\delta_{ij}\Theta)$$

$$D_i' = D_i - \delta_{ij}\Theta\, \Delta E^{(j)}$$

$$E = \sum_i \epsilon_i \quad \text{where} \quad \epsilon_i = D_i - \lambda \underbrace{\sum_{\alpha \in \partial^+ i} \frac{1}{2} A_\alpha}_{\sim A_i}$$

$$= D_i - \frac{\lambda}{2} \sum_{\alpha \in \partial^+ i} A_\alpha$$

$\Rightarrow$ We <u>can</u> rotate:

$$\alpha \hat{X} + \beta \hat{Z} \longrightarrow \gamma \hat{Z} \qquad \text{w/: } \gamma^2 = \alpha^2 + \beta^2$$

($\because$ unitary)

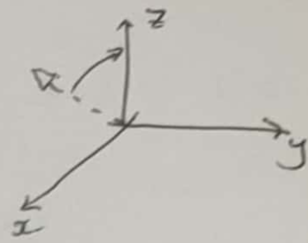
$\downarrow$

rotat^{n} of

$\theta = \arctan(-\alpha/\beta)$

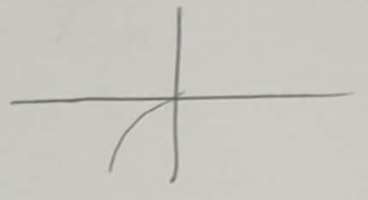
$\sim + \hat{e}_y$

Propopse $\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \underline{a} = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}$



⊛

$$\Rightarrow \left[0 + \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} \cos\theta + \begin{pmatrix} \beta \\ 0 \\ -\alpha \end{pmatrix} \sin\theta \right] = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}$$



$$\Rightarrow \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$$

$\alpha, \beta > 0$

$$\alpha + \beta \tan\theta = 0 \qquad \tan\theta = \left(-\alpha/\beta\right)$$

$$\beta \cos\theta - \alpha \sin\theta = \gamma$$

$$(c \quad s) \begin{pmatrix} \alpha c + \beta s \\ \beta c - \alpha s \end{pmatrix} = \cancel{s\gamma} \quad s\gamma$$

$$\cancel{\alpha c^2 + \beta s c + \beta s c - \alpha s^2 = s\gamma}$$

$$\cancel{\alpha c + 2\beta s \quad \alpha s t = t\gamma}$$

$$\alpha + \beta t = 0 \qquad \alpha = -\beta t$$

$$\beta c - \alpha s = \gamma$$

$$\beta c + \beta s t = \gamma$$

$$\beta c^2 + \beta s^2 = \gamma c$$

$$\beta = \gamma c \qquad \cos\theta = \beta/\gamma$$

$$\boxed{\tan\theta = -\alpha/\beta}$$

$$\Rightarrow \sin\theta = -\alpha/\gamma$$

$$\Rightarrow \boxed{\gamma^2 = \alpha^2 + \beta^2}$$

$\prod_{\alpha \in \partial^+ i} Z_\alpha =$

$$e^{-i\hat{n}\cdot\sigma\frac{\theta}{2}}\, \underline{a}\cdot\underline{\sigma}\, e^{+i\hat{n}\cdot\sigma\frac{\theta}{2}}$$

$$= \cancel{[\hat{n}\underbrace{(\hat{n}\cdot a)}_{a_\perp} + (}$$

$$= [\underline{a}_\parallel + \underline{a}_\perp \cos\theta + (\hat{n}\times\underline{a})\sin\theta]\cdot\underline{\sigma}$$

Step 1: set $\lambda_{n-1} = \lambda_{n+c}$ etc.

I want $\hat{R}_n(\theta) = e^{-i\hat{n}\cdot\sigma\frac{\theta}{2}} = \mathbb{1}\cos(\theta/2) - i\hat{n}\cdot\sigma\sin(\theta/2)$

$\cancel{\sigma_j \to R}$

$\hat{\sigma}_j \longrightarrow \hat{R}\,\hat{\sigma}_j\,\hat{R}^+$

$\alpha\sigma_1 + \beta\sigma_3 \to \gamma\sigma_3$?

$\underline{a} = \begin{pmatrix}\alpha\\0\\\beta\end{pmatrix}$

$\underline{a}_\parallel = \hat{n}(a\cdot\hat{n})$

$\underline{a} = \underline{a}_\parallel + \underline{a}_\perp$

We want: $\cancel{[\hat{n}(\begin{pmatrix}\alpha\\0\\\beta\end{pmatrix}\cdot\hat{n}) + (}$

(*) $[\hat{n}(a\cdot\hat{n}) + (a - \hat{n}(a\cdot\hat{n}))\cos\theta + (\hat{n}\times a)\sin\theta] = \begin{pmatrix}0\\0\\\gamma\end{pmatrix}$

$$\hat{Z} = \sum_{n=0}^{N-1} |n\rangle\langle n|\, e^{2\pi i n/N}$$

$$\hat{Z}^{+} = \sum_{n=0}^{N-1} |n\rangle\langle n|\, e^{-2\pi i n/N}$$

For a generic graph,

$$\sum_{\alpha\in\Delta_1} O_\alpha = \overset{\frac{1}{|\partial\alpha|}}{\overset{\shortparallel}{\frac{1}{2}}} \sum_{j\in\Delta_0}\left(\sum_{\alpha\in\partial^+ j} O_\alpha\right)$$

$$\sum_{j\in\Delta_0} O_j = \sum_{\alpha\in\Delta_1}\left(\sum_{j\in\partial\alpha} \underset{\frac{1}{|\partial^+ j|}}{\underset{\shortparallel}{\frac{1}{z_j}}}\, O_j\right)$$

$$\sum_{\alpha} O_\alpha = \sum_{j}\left(\sum_{\alpha\in\partial^+ j} \frac{1}{|\partial\alpha|}\, O_\alpha\right)$$

$\partial = \partial^-$

$\partial^+ = \partial^+$

$$\sum_{j} O_j = \sum_{\alpha}\left(\sum_{j\in\partial\alpha} \frac{1}{|\partial^+ j|}\, O_j\right)$$

$$\sum_{p\in\Delta_n} O_p = \sum_{q\in\Delta_{n\pm1}}\left(\sum_{p\in\partial^\mp q} \frac{1}{|\partial^\pm p|}\, O_\alpha\right)$$

$$\hat{H} = \hat{H}_0 - \hat{B} f(t)$$

$\downarrow$

$$\overline{\langle \hat{A}(t) \rangle} = \overline{\langle \hat{A}(t) \rangle_0} + \int_{-\infty}^{t} d\tau \, \chi(t-\tau) f(\tau)$$

$$\chi(t-\tau) = i \overline{\langle [\hat{A}^I(t), \hat{B}^I(\tau)] \rangle_0}$$

$\Downarrow$ classical

$$\chi(t-\tau) = \overline{\langle \{A(t), B(\tau)\} \rangle_0}$$

$$J_\mu = K_{\mu\nu} \partial_\nu T$$

$$K_{GK} = \frac{1}{k_B T^2} \lim_{t \to \infty} \int_0^t d\tau \lim_{V \to \infty} \frac{1}{V} \langle \underline{J}(\tau) \cdot \underline{J}(0) \rangle$$

$$K_{ij} = \frac{V}{k_B T^2} \int_0^\infty d\tau \, \langle q_i(0) q_j(\tau) \rangle$$

$$\frac{\partial \varepsilon_i}{\partial t} = -\sum_j j_{ij}$$

$$\underline{q} = \frac{1}{V} \frac{d}{dt} \underbrace{\sum_i \underline{r}_i \varepsilon_i}_{\underline{P}} = \frac{1}{V} \frac{d\underline{P}}{dt}$$

$$j_{ij} = -j_{ji}$$

$$= -\frac{1}{V} \sum_{ij} j_{ij} \underline{r}_i$$

$$\Rightarrow j_{ij} \underline{r}_i = \frac{1}{2}(j_{ij} - j_{ji}) \underline{r}_i$$

$$= \dots$$

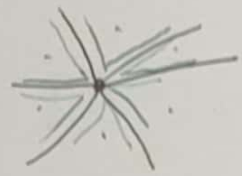
$$= -\frac{1}{V} \underline{J}$$

$$\hat{H} = -\alpha \sum_i \prod_{\alpha \in \partial^+ i} \tilde{Z}_\alpha - \beta \sum_p \prod_{\alpha \in \partial p} \tilde{X}_\alpha$$

I want to try & get $\sum_p \prod_{\alpha \in \partial p} \rightsquigarrow \sum_i \prod_{\alpha \in \partial^+ i}$

$$\sum_p \prod_{\alpha \in \partial p} \tilde{X}_\alpha = \sum_\alpha \left(\sum_{p \in \partial^+ \alpha} \frac{1}{|\partial p|} \prod_{\beta \in \partial p} \tilde{X}_\beta \right)$$

$$= \sum_i \left(\underbrace{\sum_{\alpha \in \partial^+ i} \sum_{p \in \partial^+ \alpha}}_{\sim\, \alpha \in \partial^{+2} i} \frac{1}{|\partial p||\partial \alpha|} \prod_{\beta \in \partial p} \tilde{X}_\beta \right)$$



Can't really make progress :(

We want to estimate K_{max} for the 8-vertex vs 6-vertex model.

$$\dot{j}_{ij}^{(6)} \in \{0, \pm 4\lambda, \pm 8\lambda, \pm 12\lambda\}$$

$$\dot{j}_{ij}^{(8)} \in \{0, \pm 2\lambda\}$$

$$\cancel{\underline{J} = \underbrace{\frac{1}{2}\sum_{ij}}_{N_v^2} \underbrace{(\underline{r}_j - \underline{r}_i)}_{\sim a = 1} \underbrace{\dot{j}_{ij}}_{\sim\lambda} \sim N_v^2 \lambda}$$

$$K_{ij} = \frac{1}{VT^2} \sum_{\tau=0}^{\tau_{max} \sim \infty} \langle J^i(\tau + t) J^j(t) \rangle_t \left(1 - \frac{\delta_{\tau 0}}{2}\right)$$

$$\underline{J} = \frac{1}{2} \sum_{ij} (\underline{r}_j - \underline{r}_i) \dot{j}_{ij}$$

$$= \underbrace{\sum_{\alpha}}_{N_E} \underbrace{(\underline{r}_{\alpha 2} - \underline{r}_{\alpha 1})}_{\sim a} \underbrace{\dot{j}_{\alpha(1 \to 2)}}_{\lambda} \sim \lambda N_E$$

$$\sim \frac{1}{VT^2} \lambda^2 N_E^2 \cancel{\mathcal{O}(1)}$$

$$\sim \frac{\lambda^2}{T^2} N_E$$

$\delta E/2$ = above.

$$K \sim \left(\frac{\delta E}{4T}\right)^2 \frac{N_e}{V} D$$

$N_e \sim N_v e^{-\delta E/2T}$

$V \overset{?}{\sim} N_E \overset{?}{\sim} 2N_v$ } same for 6 & 8

$$\sim \left(\frac{\delta E}{4T}\right)^2 e^{-\delta E/2T} D$$

$$\sim \left(\frac{\delta E}{2T}\right)^2 e^{-\delta E/2T} D$$

$D \sim$ const. in T excluding T small

$\Rightarrow$ peaks @ $\frac{\delta E}{2T} \sim 2 \Rightarrow T \sim \frac{\delta E}{4} = \begin{cases} (6) & 2\lambda \\ (8) & 1\lambda \end{cases}$ as seen

$\Rightarrow K_{max} \sim 4e^{-2}$ for both? $\qquad D \sim \begin{cases} (6) & 7/12 \\ (8) & 1 \end{cases}$

Different tack. $K = DC \sim C$ if $D \sim$ const.

$D^{(6)} < D^{(8)} \Rightarrow$ should have $K^{(6)}_{max} < K^{(8)}_{max}$

↳ lower-energy e excitat$^{\bar{n}}$s but more mobile & ↑'d pop$^{\bar{n}}$ @ given T

Given $K^{(6)}_{max} \sim 0.25$ this implies $K^{(8)}_{max}$ being 0.4 is more on the money than 0.125.

$$K^{(8)}_{max} \sim K^{(6)}_{max} \cdot \frac{1}{7/12} \sim \underbrace{K_{max}}_{0.25} \frac{12}{7} \sim \underline{0.43}$$

$\Rightarrow$ missing factor is in Kubo

$C^{(8)}_{max} \sim 0.2$ for both methods. & $C^{(6)}_{max} \sim 0.25$ "

$K^{(6)}_{max} \sim 0.20$

$$\frac{K}{DC} = 1$$

$$K^{(8)}_{max} \sim K^{(6)}_{max} \frac{D^{(8)}}{D^{(6)}}$$

This is all kinda useless...

$$\cancel{\sum_{ij} A_{ij} = \sum_{\alpha\beta} \left(\sum_{\substack{i\in\partial\alpha \\ j\in\partial\beta}} \frac{1}{|\partial^+ i||\partial^+ j|} A_{ij} \right)}$$

$A_{ij} = 0$ unless $\exists\ \gamma$ s.t. $i \overset{\gamma}{—} j$

$$\sum_{ij} A_{ij} = \sum_{i} \sum_{j\in nn(i)} A_{ij}$$

$A_{ij} = A_{(ij)}$

$$= 2 \sum_{\langle ij \rangle} A_{ij}$$

$$= 2 \sum_{\underbrace{\langle i \overset{\alpha}{—} j \rangle}_{\alpha}} A_{\alpha}$$

$$\Rightarrow \sum_{ij} \frac{1}{2} (\underline{r}_j - \underline{r}_i)\, j_{ij} = \underline{\sum_{\alpha} (\underline{r}_{\alpha_j} - \underline{r}_{\alpha_i})\, j_{\alpha_j \alpha_i}} \quad \Rightarrow \text{correct } \ddot{\smile}$$

$i \overset{\beta}{\text{———}} j$ $\qquad \Delta E^{(\beta)}$ ~~$= + \lambda \left[\left(\sum_{\alpha \in \partial^+ i} \sigma_\alpha - 2\sigma_\beta \right. \right.$~~

$$= + \lambda \left[\Delta \left(\tilde{A}_i + \tilde{A}_j \right) \right]$$

$$E = \lambda \sum_i \tilde{A}_i$$

$$\tilde{A}_i = \left(\sum_{\alpha \in \partial^+ i} \sigma_\alpha \right)^2 = \tilde{C}_i^2 \qquad \Delta A_i = \left(\sum_{\alpha \in \partial^+ i} \sigma_\alpha - 2\sigma_\beta \right)^2 - \left(\sum \cdots \right)^2$$

$$\sigma_\beta \rightarrow -\sigma_\beta \qquad = \left(\tilde{C}_i - 2\sigma_\beta \right)^2 - \tilde{C}_i^2$$

$$= 4 \sigma_\beta^2 - 4 \sigma_\beta \tilde{C}_i \qquad (\sigma_\beta^2 = 1)$$

$$\Rightarrow \Delta E^{(\beta)} = 4\lambda \left(2 - \sigma_\beta \left(\tilde{C}_i + \tilde{C}_j \right) \right)$$

Define $\tilde{B}_j = \tilde{C}_j - \sigma_\beta$

$$\Rightarrow \Delta E^{(\beta)} = 4\lambda \left(2 - \sigma_\beta \left(\tilde{B}_i + \tilde{B}_j + 2\sigma_\beta \right) \right)$$

$$= -4\lambda \sigma_\beta \left(\tilde{B}_i + \tilde{B}_j \right)$$

$$E = \lambda \sum_i \tilde{C}_i^2 + \sum_\alpha D_\alpha \qquad \sum_\alpha \cdots = \sum_i \sum_{\alpha \in \partial^+ i} \frac{1}{|\partial \alpha|} \cdots$$

$$= \sum_i \epsilon_i = \sum_i \left(\lambda \tilde{C}_i^2 + \frac{1}{2} \sum_{\alpha \in \partial^+ i} D_\alpha \right)$$

$$\Delta D_\alpha = -$$

$$\Delta \epsilon_i = - \sum_j j_{ij}$$

$$\Delta E^{(\beta)} = - \sum_{ij} j_{ij} \quad \sum_\alpha \Delta D_\alpha \qquad i' \overset{\beta}{\text{———}} j'$$

$$= 2\lambda \sum_{ij} \left(\tilde{B}_j - \hat{B}_i \right) \Theta_i \Theta_j$$

$$= 2\lambda \left(B_j' - B_i' \right) + 2\lambda \left(B - 0 \right.$$

$$\dot{\mathcal{J}} = \frac{1}{2} \sum_{ij} \left(\underline{r}_j - \underline{r}_i \right) j_{ij}$$

$$= \frac{1}{2} \sum_i \sum_{j \in nn(i)} \left(\underline{r}_j - \underline{r}_i \right) j_{ij}$$

j_{ij} only nonzero if $j \in nn(i)$

$$C_D = N_E \left(\frac{\delta E}{T}\right)^2 \frac{e^{\delta E/T}}{(e^{\delta E/T} - 1)^2}$$

£ £ £ €

$$C_\sigma = \frac{\sigma_E^2 C_P}{C_D T^2 - \sigma_E^2}$$

α α

β $\delta\ \partial\ \partial^+$ süß

$$\sim \left(\frac{\delta E}{T}\right)^2 \frac{e^{-\delta E/T}}{(1 - e^{-\delta E/T})^2}$$

@ max, $T \sim \frac{\delta E}{4}$.

$$\sim \left(\frac{\delta E}{T}\right)^2 e^{-\delta E/T}\left(1 + 2e^{-\delta E/T} + \dots\right)$$

Whereas $K \sim D\left(\frac{\delta E}{2T}\right)^2 e^{-\delta E/2T}$ has max @ $\frac{\delta E}{2T} \sim 2$

Demon bath gives peak $\sim \underline{0.3^{(6)}}$ as expected

& $\sim (0.4^{(8)}_{-0.5})$

$\Rightarrow$ roughly in ratio $\frac{12}{7}$ as expected.

$\Rightarrow$ factor of $\frac{1}{2}$ in Kubo method?

$$K_{ab} = \cancel{\frac{1}{N_E T^2}} \frac{1}{VT^2} \sum_{\tau=0}^{\infty} \left\langle J^a_{\tau+t} J^a_t \right\rangle_t \left(1 - \frac{\delta_{\tau 0}}{2}\right)$$

Two options: * Missing factor in J

* $V = N_V$ or N_E?

$$\underline{N_E = 2N_V}$$

$$\underline{J} = \underbrace{\sum_\alpha}_{N_E} \underbrace{(\underline{r}_{\alpha 1} - \underline{r}_{\alpha 2})}_{\sim a \sim 1} \underbrace{J_\alpha}_{\sim \lambda}$$

$\sim N_E \lambda$

<u>Opinion</u> : * the factor of 2 is <u>unlikely</u> to be in the Kubo method

$$\therefore \quad K = \frac{1}{N_E T^2} \sum_{\tau=0}^{\infty} \langle J_{t+\tau} J_t \rangle_t \left(1 - \frac{\delta_{\tau 0}}{2}\right)$$

- Can't be $\sqrt{2}$ in J really
- T is correct
- Could be $2\times$ counting sth?

* $\Rightarrow$ more likely in Demon dynamics, possibly in J or Δx?

$$K_i = -J \frac{2\Delta x}{T_{i+1} - T_{i-1}}$$

$$\Delta x = \begin{cases} 1 & N_s = 1 \\ 1/2 & N_s = 2 \end{cases}$$

$$A_\alpha' = A_\alpha(1 - 2\Theta\,\delta_{j\in\partial\alpha}) \qquad \begin{array}{l}\sigma_i' = \sigma_i(1 - 2\delta_{ij}\Theta)\\ D_i' = D_i - \delta_{ij}\Theta\,\Delta E^{(j)}\end{array}$$

$$\Rightarrow \epsilon_i' = D_i - \delta_{ij}\Theta\,\Delta E(j) - \frac{\lambda}{2}\sum_{\alpha\in\partial^+ i}(A_\alpha - 2A_\alpha\Theta\,\delta_{j\in\partial\alpha})$$

$$\Rightarrow \epsilon_i' - \epsilon_i = \Theta\lambda\sum_{\alpha\in\partial^+ i} A_\alpha\,\delta_{j\in\partial\alpha} - 2\Theta\lambda\sum_{\alpha\in\partial^+ j} A_\alpha\,\delta_{ij}$$

3 cases: where non-zero:

① $i=j$

$$\Delta\epsilon_i^{(j)} = \Theta\lambda \underbrace{\sum_{\alpha\in\partial^+ j}}_{\text{coord}^{n}\text{ of } j} A_\alpha - 2\Theta\lambda\sum_{\alpha\in\partial^+ j} A_\alpha$$

$$= -\Theta\lambda\sum_{\alpha\in\partial^+ j} A_\alpha = -\Theta\lambda\sum_{k \mid j-k} A_{(j-k)}$$

② $i\neq j$

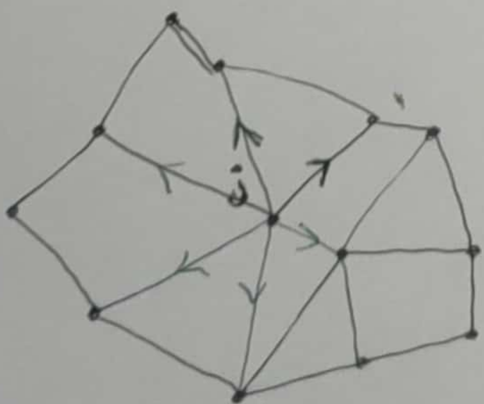
$$\Delta\epsilon_i^{(j)} = +\Theta\lambda\, A_\beta = \Theta\lambda\, A_{(i-j)}$$

③ $i\neq j$

$i \not- j$

$$\Delta\epsilon_i^{(j)} = 0$$

So...



initially GS { Suppose all $\sigma = +1$ initially $\Rightarrow$ all $A = +1$

λA_β energy flows <u>out</u> along each bond $\beta\in\partial^+ j$

$\Delta E = \sum_i \Delta\epsilon_i = 0$? <u>As expected</u> $\therefore$ μ can.

$$\Delta \epsilon_i^{(k)} := - \sum_{j} \dot{j}_{ij}^{(k)}$$

$$\Rightarrow \quad \dot{j}_{ij}^{(k)} = -\Big(\delta_{ik} \left(- \Theta \lambda A_{(i-j)}\right) + \delta_{jk} \left(+ \Theta \lambda A_{(i-j)}\right) \Big)$$

$$\boxed{\dot{j}_{ij}^{(k)} = \Theta \lambda A_{(i-j)} \left(\delta_{ik} - \delta_{jk} \right)}$$

as in
Rau's paper
w/. $J \to -\lambda$

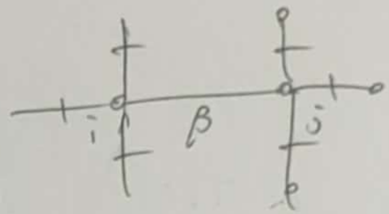
$$\Delta \epsilon_k^{(k)} = - \sum_{j} \dot{j}_{kj}^{(k)} = - \Theta \lambda \sum_j A_{(k-j)}$$

$$\Delta \epsilon_i^{(k)} \overset{k=j_1}{=} - \sum_{j} \dot{j}_{ij}^{(k)} = + \Theta \lambda A_{(i-j_1)}$$

$$\boxed{J := \frac{1}{2L_y}\left(\langle \Delta E_h \rangle - \langle \Delta E_c \rangle\right)}$$

6-Vertex: I'm getting $\Delta E_{flip} \in \{0, \pm 4, \pm 8, \pm 12, \pm 16\}$.

$B_i = \sum_{\substack{\alpha \in \partial^+ i \\ \alpha \neq \beta}} \sigma_\alpha \in \{\pm 1, \pm 3\}$ $\qquad \sigma_\alpha \in \{\pm 1\}$.



$\Rightarrow \Delta E^{(\beta)} = -4\lambda\, \sigma_\beta \left(\{\pm 1, \pm 3\} + \{\text{''}\}\right)$

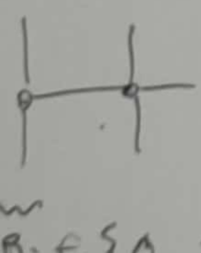
$= \pm 4\lambda \{0, \pm 2, \pm 4, \pm 6\}$ $\qquad$ + + + $\quad$ 3

$\qquad$ + − + $\quad$ 1

$= \{0, \pm 8\lambda, \pm 16\lambda, \pm 24\lambda\}$.

EXCEPT on a boundary (closed)

1.5 $\quad$ 11.5 $\qquad\qquad$ 1 $\quad$ 20



$B_i \in \{0, \pm 2\}$. $\qquad\qquad B_i, B_j \in \{0, \pm 2\}$

$B_j \in \{\pm 1, \pm 3\}$ $\qquad\qquad \Downarrow$

$\Downarrow$

$\Delta E \in \pm 4\lambda \{\pm 1, \pm 3, \pm 5\}$ $\qquad \Delta E \in \pm 4\lambda\{0, \pm 4\lambda\}$

$\in \{\pm 4\lambda, \pm 12\lambda, \pm 20\lambda\}$ $\qquad \in \{0, \pm 16\lambda\}$.

Not an issue for K_{bath} even though it appears in J $\because$ just energy in/out.

$$K = \frac{1}{N_E T^2} \sum_{\tau=0}^{\tau_{max}} \frac{1}{t_{max}-\tau} \sum_{t=1}^{t_{max}-\tau} \underbrace{J_{t+\tau} J_t (\ldots)_\tau}_{\alpha_{t\tau}}$$

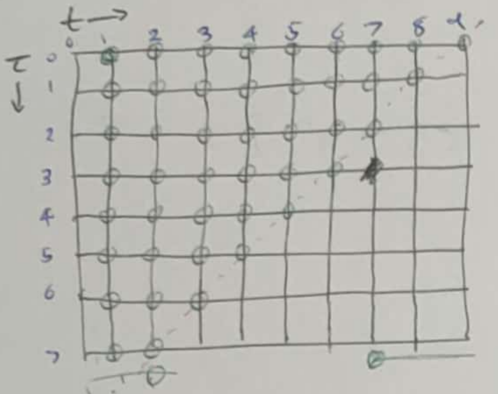
$\tau_{max} = 7$

$t_{max} = 9$.

$\cancel{t(\tau)7}$

$t_{upper}(\tau) = t_{max} - \tau$

$\tau_{upper}(t) = \min(\tau_{max}, t_{max} - t)$



$t=1$	0
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	7

$w = 3$

$\min(t_{max} - t, \tau_{max})$

$$\sum_{\tau=0}^{\tau_{max}} \sum_{t=1}^{t_{max}-\tau} a_{t\tau}$$

$$= \sum_{t=1}^{t_{max}} \sum_{\tau=0}^{\min(\tau_{max}, t_{max}-t)}$$

$$\Rightarrow K = \frac{1}{N_E T^2} \sum_{t=1}^{t_{max}} \underbrace{\left(\sum_{\tau=0}^{\min(t_{max}-t, \tau_{max})} \frac{J_{t+\tau} J_t \left(1 - \frac{\delta_{\tau 0}}{2}\right)}{t_{max} - \tau} \right)}_{\beta_t}$$

$\beta_t =$

Now expand

$$\sum_i \beta_i \epsilon_i(\underline{\sigma}) \approx \beta \sum_i \epsilon_i(\underline{\sigma}) - \beta^2 (k_B \underline{\nabla} T) \cdot \sum_i \underline{r}_i \epsilon_i(\underline{\sigma}) + \cdots$$

$$= \beta E(\underline{\sigma}) - \underbrace{\beta^2 (k_B \underline{\nabla} T) \cdot \underline{P}}_{\phi}$$

$$\Rightarrow \langle J^\mu \rangle = \frac{\sum_\sigma J^\mu(\underline{\sigma}) e^{-\beta E(\underline{\sigma}) - \beta^2(\ldots)}}{\sum_\sigma e^{-\cdots}}$$

$$\approx \langle J^\mu \rangle_{eq} + \left(\langle J^\mu \phi \rangle_{eq} - \langle J^\mu \rangle_{eq} \langle \phi \rangle_{eq} \right)$$

$$\langle J^\mu_{eq} \rangle = 0 \Rightarrow \langle J^\mu \rangle = \langle J^\mu \phi \rangle_{eq}.$$

$$\Rightarrow K_{\mu\nu} = \frac{1}{k_B T^2} \frac{1}{V} \left[-\langle J^\mu P^\nu \rangle_{eq} + \delta t \sum_{\tau=1}^{\infty} \langle J^\mu_\tau J^\nu_0 \rangle_{eq} \right]$$

USE $-\delta t \langle J^\mu J^\nu \rangle_{eq} = \langle J^\mu P^\nu \rangle_{eq} + \langle J^\nu P^\mu \rangle_{eq}$

to get symm. cpt-s.

$$\underline{P} = \sum_i \underline{r}_i \varepsilon_i$$

$$= \sum_\alpha \left(\sum_{i \in \partial\alpha} \frac{1}{|\partial^+ i|} \underline{r}_i \varepsilon_i \right)$$

$$\underline{r}_\alpha := \left(\sum_{i \in \partial\alpha} \underline{r}_i \right) / |\partial\alpha|$$

$$E = -\lambda \sum_\alpha \sigma_{\alpha_1} \sigma_{\alpha_2} + \sum_i D_i = \sum_i \epsilon_i$$

$$= -\lambda \sum_\alpha \underbrace{\prod_{i \in \partial\alpha} \sigma_i}_{A_\alpha} + \sum_i D_i$$

$\frac{1}{2}$ → true in <u>any</u> Δ ~~of~~

$$\Rightarrow \epsilon_i = D_i - \lambda \sum_{\alpha \in \partial^+ i} \cancel{\frac{1}{|\partial\alpha|}} A_\alpha$$

$$= D_i - \frac{\lambda}{2} \sum_{\alpha \in \partial^+ i} A_\alpha$$

$$\Delta E_{(j)} = -\lambda \Delta \left(\sum_{\alpha \in \partial^+ j} A_\alpha \right)$$

$$= 2\lambda \sum_{\alpha \in \partial^+ j} A_\alpha$$

$$D_i' = D_i - \delta_{ij} \Theta \Delta E_{(j)}$$

$$\sigma_i' = \sigma_i (1 - 2\delta_{ij} \Theta)$$

$$\Rightarrow A_\alpha' = \sigma_{\alpha_1}' \sigma_{\alpha_2}'$$

$$= \sigma_{\alpha_1} \sigma_{\alpha_2} (1 - 2\delta_{j\alpha_1} \Theta)(1 - 2\delta_{j\alpha_2} \Theta)$$

$$= A_\alpha \Big(1 - 2 \underbrace{(\delta_{j\alpha_1} + \delta_{j\alpha_2})}_{\delta_{j \in \partial\alpha}} \Theta\Big)$$

$$\underline{P} = \sum_i \underline{r}_i \epsilon_i$$

$$\langle J^\mu P^\nu \rangle_{eq} = \sum_{ijk} \underbrace{\frac{1}{2}(r_j^\mu - r_i^\mu) \cdot r_k^\nu}_{\frac{(r_j^\mu - r_i^\mu) r_k^\mu}{2}} \underbrace{\langle j_{ij} \epsilon_k \rangle_{eq.}}_{2 A_{[ij]k}}$$

$$\langle j_{ij} \epsilon_k \rangle_{eq} = \sum_{\underline{\sigma}} P_{eq}(\underline{\sigma}) \, j_{ij}(\underline{\sigma}) \epsilon_k(\underline{\sigma})$$

$$-\sum_j j_{ij}(\underline{\sigma}) = \frac{\partial \epsilon_i(\underline{\sigma})}{\partial t} = \frac{\epsilon_i(\Omega(\underline{\sigma})) - \epsilon_i(\underline{\sigma})}{\delta t}$$

$$\langle J^\mu P^\nu \rangle_{eq} = \sum_{ijk} (r_j^\mu r_k^\nu - r_i^\mu r_k^\nu) A_{[ij]k}$$

$$" \quad \overline{\cancel{A}} \; \overset{\mu \leftrightarrow \nu}{- - -} = \sum_{ijk} \begin{pmatrix} r_j^\mu r_k^\nu - r_i^\mu r_k^\nu \\ - r_j^\nu r_k^\mu + r_i^\nu r_k^\mu \end{pmatrix} A_{[ij]k}$$

$$= \sum_{ijk} (r_k^\mu -$$

nope

f

Cont. eqn:

$$-\sum_j j_{ij}(\underline{\sigma}) = \frac{\partial \epsilon_i(\underline{\sigma})}{\partial t} = \frac{\epsilon_i(\Omega(\underline{\sigma})) - \epsilon_i(\underline{\sigma})}{\delta t}$$

$$\delta t\, \underline{J}(\underline{\sigma}) = -\frac{1}{2}\delta t \sum_{ij} (\underline{r}_i - \underline{r}_j)\, j_{ij}(\underline{\sigma})$$

$$= -\delta t \sum_{ij} \underline{r}_i\, j_{ij}(\underline{\sigma})$$

$$= \sum_i \underline{r}_i \left(\epsilon_i(\Omega(\underline{\sigma})) - \epsilon_i(\underline{\sigma}) \right) \qquad (\Rightarrow \langle \underline{J} \rangle_{eq} = \underline{0}$$

$$J^\mu J^\nu = \frac{1}{\delta t^2} \sum_{ij} r_i^\mu r_j^\nu \Big(\left[\epsilon_i(\underline{\sigma}')\epsilon_j(\underline{\sigma}') + \epsilon_i(\underline{\sigma})\epsilon_j(\underline{\sigma}) \right] - \left[\epsilon_i(\underline{\sigma}')\epsilon_j(\underline{\sigma}) + \epsilon_i(\underline{\sigma})\epsilon_j(\underline{\sigma}') \right] \Big)$$

$$\langle J^\mu J^\nu \rangle = -\Big\langle \sum_{ij} r_i^\mu r_j^\nu \Big(\epsilon_j(\underline{\sigma}) \left[\epsilon_i(\underline{\sigma}') - \epsilon_i(\underline{\sigma}) \right] + \underset{--}{\overset{i \leftrightarrow j}{}} \Big) \Big\rangle$$

$$= -\delta t \frac{1}{\delta t} \left(\cancel{\sum_i} \langle P^\mu J^\nu \rangle + \langle P^\nu J^\mu \rangle \right)$$

Problem w/. Hall effect in antisymm part of $\mathbb{K}$.

$$\cancel{\frac{\partial \epsilon}{\partial t} \neq \underline{\nabla} \cdot \underline{j} =}$$

purely an energy current

$$\frac{\partial \epsilon}{\partial t} = - \underline{\nabla} \cdot \underline{j} = - \partial_\mu \left(- K_{\mu\nu}(T) \partial_\nu T_\nu \right)$$

$$c(T) \frac{\partial T}{\partial t} = K_{\mu\nu} \partial_\mu \partial_\nu T + \frac{\partial K_{\mu\nu}}{\partial T} \partial_\mu T \partial_\nu T$$

<u>only</u> dep. on $\mathbb{K}_{(\mu\nu)}$

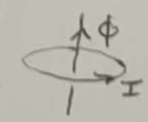
$\Rightarrow$ ONLY see Hall effects on <u>boundaries</u>

Kubo formula includes "energy magnetisation/polarisation"

$P = \sum \underline{r}_i \epsilon_i$ which is defined only up to an arb. func^{n} of T & other parms

The paper goes on to say the lack of definition is due to the absence of macroscopic <u>energy</u> currents in eq^{m}

↳ Contrast w/. electrical case: can define $\sigma_{[\mu\nu]}$

∵ torus geometry allows measurement of ~~mag~~ <u>macroscopic</u> current flow w/o edges ( magnetic flux $\Rightarrow$ current)

Not so in $\mathbb{K}_{[\mu\nu]}$ case! $\rightarrow$ can only be measured w/. an edge/gradient.

$\Rightarrow$ can <u>only</u> measure K relative to <u>another</u> material.

Suppose it <u>should</u> be N_v rather than N_E.

Then it is the $\underline{6}$-vertex model which is out by a factor of 2.

Doesn't really help :(.

$$\epsilon_i = \frac{1}{2}\sum_{\alpha\in\partial^+ i} D_\alpha + \lambda \underbrace{\Big(\sum_{\alpha\in\partial^+ i} \sigma_\alpha\Big)}_{A_i}{}^2$$

$$\Delta D_\alpha = -\Delta E^{(\beta)} \ominus \delta_{\alpha\beta}$$

$$\cancel{A_i' = \delta_{\beta\in\partial^+ i} \Big(\sum_{\alpha\in\partial^+ i} \sigma_\alpha - 2\ominus\sigma_\beta\Big) + A_i}$$

$$\cancel{= \delta_{\beta\in\partial^+ i}}$$

$$A_i' = \Big(\sum_{\alpha\in\partial^+ i} \sigma_\alpha - 2\,\delta_{\beta\in\partial^+ i}\ominus\sigma_\beta\Big)^2$$

$$= A_i + \Big(4 - 4\sigma_\beta\sum_{\alpha\in\partial^+ i} \sigma_\alpha\Big)\ominus\delta_{\beta\in\partial^+ i}$$

$$\Delta A_i = 4\ominus\delta_{\ldots}\Big(\sigma_\beta - \sum_{\alpha\in\partial^+ i}\sigma_\alpha\Big)\sigma_\beta$$

$$= -4\,\sigma_\beta B_i \ominus \delta_{\ldots}$$

$$\Rightarrow \Delta\epsilon_i = -\frac{1}{2}\Delta E^{(\beta)}\ominus\delta_{\beta\in\partial^+ i} - 4\lambda\sigma_\beta B_i \underbrace{\ominus\,\delta_{\beta\in\partial^+ i}}_{\ominus}$$

$$\Delta\epsilon_i/\ominus = 2\lambda\sigma_\beta(B_i + B_j) - 4\lambda\sigma_\beta B_i$$

$$= 2\lambda\sigma_\beta(B_j - B_i)$$

$$\epsilon_i = \lambda A_i^2 + \frac{1}{2}\sum_{\alpha\in\partial^+ i} D_\alpha$$

$$\Delta E^{(\beta)} = -4\lambda\sigma_\beta(B_i + B_j$$

$$i \overset{}{\underset{\beta}{\circ\!\!-\!\!\circ}} j$$

$$D_\alpha' = D_\alpha - \Delta E^{(\beta)}\,\Theta\,\delta_{\alpha\beta}$$

$$\sigma_\alpha' = \sigma_\alpha(1 - 2\Theta\,\delta_{\alpha\beta})$$

$$\Rightarrow \epsilon_i' = \lambda\sum_{\alpha\in\partial^+ i}\sigma_\alpha - 2\lambda\Theta\sum_{\alpha\in\partial^+ i}\sigma_\alpha\,\delta_{\alpha\beta}$$

$$+ \frac{1}{2}\sum_{\alpha\in\partial^+ i} D_\alpha - \frac{1}{2}\Theta\,\Delta E^{(\beta)}\sum_{\alpha\in\partial^+ i}\delta_{\alpha\beta}\,.$$

$$\cancel{\delta\epsilon_i^{*} = -2\lambda\Theta\,\sigma_\beta\,\delta_{\beta\in\partial^+ i}} + 2\lambda\Theta\,\sigma_\beta(B_{\beta_1} + B_{\beta_2})\,\delta_{\beta\in\partial^+ i}$$

$$= (\Theta\,\delta_{\beta\in\partial^+ i})\,[$$

$$\Delta\epsilon_i = \Theta\,\delta_{\beta\in\partial^+ i}\;2\lambda\sigma_\beta(B_j - B_i) \qquad \delta_{i,j\in\partial\beta}$$

$$= -\sum_j j_{ij}$$

$$\Rightarrow \dot{j}_{ij}^{(\beta)} = -2\lambda\sigma_\beta(B_j - B_i)\,\Theta\,\delta_{i,j\in\partial\beta}$$

$\Rightarrow$ missing σ_β in our code?

perhaps responsible for factor of 2?
In combo w/. N_V vs N_E controve[rsy]
this could work...

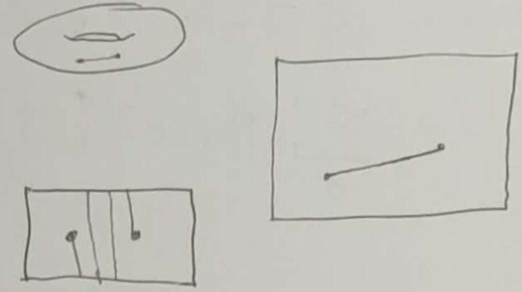
Shift E so $E(\ell) = 4\lambda\ell$ WLOG.

BUT what is Δ_ℓ?

Okay. First, the endpoints. To pick 2ℓ pts for e prtcls, $\Delta_{\text{ends},\ell} = \underbrace{N_v \cdot (N_v - 1) \cdot \dots}_{\# 2\ell}$

$$= \frac{{}^{N_v}P_{2\ell}}{r}$$

$\because$ order matters (can choose to ignore order if done self-consistently)

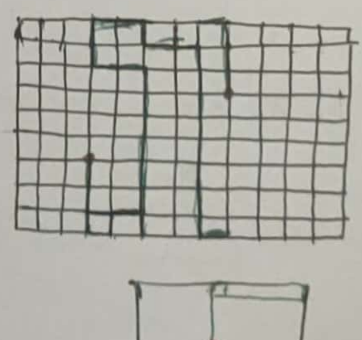


Now the strings. Ono.

~~On a 2D lattice~~ suppose

On a lattice, start with the minimal connecting line. We can get any other line by flipping plaquettes $\because$ can't be contractible.

Even if loop is wound around torus, we can make it contractible! IF winding # is even

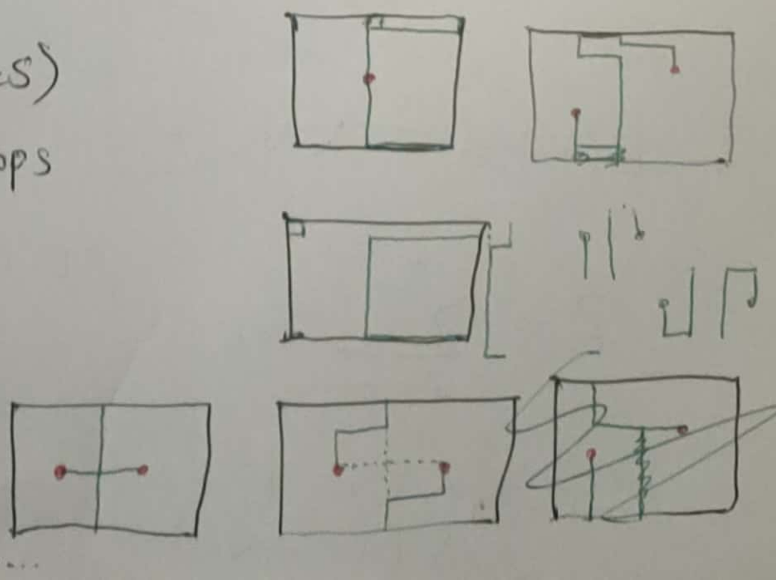


SO we have (as for the GS) 2^2 sectors $\because$ non-cont. loops

$$\Rightarrow \Delta_{\text{line for } 2e} = 2^2 \cdot \sum_{m=0}^{N_p}$$

weird

Surely we only care ~ plaquette flips next to loop...



Toric code = example of quantum double model.
(Abelian)

apparently, topo deg. $= 4^g = 2^{2g} = 2^{b_1}$

& same for excited states?

nth betti n°: b_n
= n° of ...
$\begin{cases} n=0 & \text{connected cpt-s} \\ n>0 & \text{n-dimensional holes.} \end{cases}$

$N_p = N_v - N_e/2$

$\text{deg} = \text{topo deg} \cdot 2^{N_p - 1}$

flipping all plaq-s
= flipping all spins twice

b_k = rank of H_k
(or dim of $H_k(X, V)$ over vector space)

In 1st excited state, energy is $(4 - N_v)\lambda$
& two sites have $A = -1$

↳ There are as many such terms as GS deg $\cdot$ # ways to pick the two sites, i.e. $\binom{N_v}{2}$

Where we in the quantum case we'd have both e & m prtcls => also have to pick whether to excite 2e $\binom{N_v}{2}$ or 2m $\binom{N_p}{2}$.

$$\Delta_\ell = \underbrace{2^{N_p - 1} \cdot 2^{b_1}}_{\Delta_0} \cdot \binom{N_v}{2\ell}$$

$$E_\ell = (4\ell - \cancel{N_v})\lambda$$

arb. const. => discard it.

$$Z = \sum_{\sigma} e^{-\beta E(\sigma)} = \sum_{\ell=0}^{N_v/2} \Delta_\ell e^{-\beta E_\ell}$$

$$= \Delta_0 \sum_{\ell=0}^{N_v/2} \binom{N_v}{2\ell} \left(e^{-4\beta\lambda}\right)^\ell$$

Let's take $\beta \gg \frac{1}{\lambda}$ i.e. each term smaller

$\Rightarrow$ can take $\ell \ll N_v/2$

$\cancel{\binom{N_v}{2\ell}} =$

$m = 2\ell$

$\ell = 0, 1, 2, \dots, N_v/2$

$2\ell = 0, 2, 4, \dots$

$$Z = \Delta_0 \sum_{\ell=0}^{N_v/2} \binom{N_v}{2\ell} \left(e^{-2\beta\lambda}\right)^{2\ell}$$

$$\cancel{\binom{N}{2\ell} \sim \frac{1}{\sqrt{4\pi\ell}} \left(\frac{Ne}{2\ell}\right)^{2\ell} e^{-\frac{4\ell^2}{2N}}}$$

$$\frac{N_v!}{2!(N_v-2)!}$$

$$\frac{N_v(N_v-1)}{2}$$

$$Z \sim \Delta_0 \left[1 + \frac{N_v(N_v-1)}{2} e^{-4\beta\lambda} + \dots\right]$$

We know $\binom{N}{k} = \binom{N-1}{k-1} + \binom{N-1}{k}$ x

~~We want~~ $\binom{N}{2\ell} = \frac{N!}{2\ell!(N-2\ell)!}$

$$(1-(-1)^\ell)$$

$\cancel{Z = \Delta_0}$

$p = 2\ell = \text{n}^{\circ}$ e excitat$^{\text{ns}}$

$$Z = \Delta_0 \sum_{p=0}^{N_v/4} \underbrace{\tfrac{1}{2}\left(1 + (-1)^p\right)}_{\substack{=1 \text{ if } p \text{ even} \\ =0 \text{ if } p \text{ odd}}} \binom{N_v}{p} \left(e^{-2\beta\lambda}\right)^p$$

$$= \frac{(1+(-1)^{p})^p}{2^p} \ ?$$

problem. :)

$$\sum_{\underline{\sigma}_\ell} e^{-\sum_i \beta_i \epsilon_i(\underline{\sigma})} = \frac{1}{(2\ell)!} \sum_{i_1 \neq i_2 \neq \dots \neq i_{2\ell}} e^{-2\lambda(\beta_{i_1} + \beta_{i_2} + \dots \beta_{i_{2\ell}})}$$

For $\beta_i = \beta \;\forall i$, this is equal to $\binom{N}{2\ell} e^{-4\beta\lambda\ell}$

$$\frac{1}{(2\ell)!} \sum_{i_1 \neq i_2 \neq \dots \neq i_{2\ell}} = {}^N P_{2\ell} / (2\ell)! = {}^N C_{2\ell} = \binom{N}{2\ell}$$

$$Z = \Delta_0 \sum_{\ell=0}^{N_v/2} \underbrace{\frac{1}{(2\ell)!} \sum_{i_1 \neq \dots \neq i_{2\ell}} e^{-2\lambda(\beta_{i_1} + \dots + \beta_{i_{2\ell}})}}_{f_\ell}$$

$\beta_i - \beta_j \approx -\beta^2 (\underline{r}_i - \underline{r}_j) \cdot \underline{\nabla} T$

$\underline{\nabla} T$ const. across lattice

$T_i = T + \underline{r}_i \cdot \underline{\nabla} T$

~~$\beta_i - \beta_j + \beta_j$~~?

$$f_\ell = \sum_i \sum_{i_2 \neq \dots \neq i_{2\ell}} e^{-2\lambda(\beta_i + \beta_{i_2} + \dots \beta_{i_{2\ell}})}$$

~~$\beta_{i_n} \approx \beta_i - \beta^2 (\underline{r}_{i_n} - \underline{r}_i) \cdot \underline{\nabla} T \Rightarrow \beta_i + \sum_{n=2}^{2\ell} \beta_{i_n} = 2\ell\beta_i - \beta^2 \underline{\nabla} T \cdot \sum_n (\underline{r}_{i_n} - \underline{r}_i)$~~

~~$\Rightarrow f_\ell = \sum_i \sum_{i_2 \neq \dots i_{2\ell}} e^{-}$~~

~~$\beta_i + \sum_{n=2}^{2\ell} \beta_{i_n} = 2\ell\beta_i - \beta^2 \underline{\nabla} T \cdot \left(\left(\sum_n \underline{r}_{i_n}\right) - (2\ell-1)\underline{r}_i \right)$~~

$$P_0(\underline{\sigma}) = \frac{1}{Z_0} e^{-\sum_i \beta_i \epsilon_i(\underline{\sigma})}$$

$$\beta_i \xrightarrow{t \to \infty} \beta$$

$$P_{eq}(\underline{\sigma}) = \frac{1}{Z_{eq}} e^{-\beta E(\underline{\sigma})}$$

I want to use $Z_{eq} = \Delta_0 \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\beta\lambda l} = \sum_{\underline{\sigma}} e^{-\beta E(\underline{\sigma})}$

BUT can't $\therefore$ β needs to depend on the site!

That's an issue.

$$Z_0 = \Delta_0 \sum_{\underline{\sigma}} e^{-\sum_i \beta_i \epsilon_i(\underline{\sigma})}$$

$$= \Delta_0 \sum_{l=0}^{N_v/2} \underbrace{\sum_{\underline{\sigma}_l} e^{-\sum_i \beta_i \epsilon_i(\underline{\sigma}_l)}}_{\text{configs of each } l\text{-sector}}$$

$\vdots$

$4\lambda \updownarrow$ —— $l=2$

$4\lambda \updownarrow$ —— $l=1$

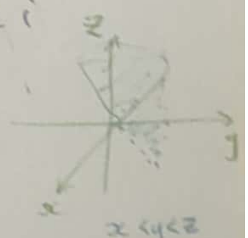
—— $l=0$

~~Suppose we want to place~~

First define $\epsilon_i = -\lambda A_i$ (exclude demons)

BUT we want $\epsilon_i = 0$ @ $A_i = 1$

$$\Rightarrow \epsilon_i = \lambda(1 - A_i) = \begin{cases} 0 & \text{otherwise.} \\ 2\lambda & \text{if excitat}^{\text{n}} \text{ on site } i \end{cases}$$



We create e prtcls @ sites i & j

$$\Rightarrow \cancel{Z_0} \quad \sum_{\underline{\sigma}_1} e^{-\sum_i \beta_i \epsilon_i(\underline{\sigma}_1)} = \Delta_0 \underbrace{\sum_{i \neq j} e^{-2\lambda(\beta_i + \beta_j)}}$$

$$\sum_{\underline{\sigma}_0} = \Delta_0$$

$i \leftrightarrow j$ same state!

if $\beta_i = \beta \; \forall \, i$,

$$= \underbrace{\sum_{i \neq j}}_{\binom{N_v}{2}} e^{-4\beta\lambda}$$

~~Should be $i < j$~~

$H \longrightarrow H - ?$ $\qquad$ $\underline{J} = -K \underline{\nabla} T.$

To make progress we need to introduce local temp. fluctuat^{ns}:

$$\sum_i \beta_i \epsilon_i(\underline{\sigma}) \approx \beta \sum_i \epsilon_i(\underline{\sigma}) - \beta^2 \underline{\nabla} T \cdot \underline{P}$$
$$= \beta\left(E(\underline{\sigma}) - \beta \underline{P} \cdot \underline{\nabla} T\right)$$

$$\underline{P} = \sum_i \underline{r}_i \epsilon_i(\underline{\sigma})$$

$$H \longrightarrow H - \underbrace{\beta \underline{P}}_{\text{disp.}} \cdot \underbrace{\underline{\nabla} T}_{\text{applied force}} \quad \text{as } \underline{\text{variables}}$$

In eq^{m}, @ $T=0$, $\langle \underline{P} \rangle_{eq} = \sum_{\underline{\sigma}} P_{eq}(\underline{\sigma}) \sum_i \underline{r}_i \epsilon_i(\underline{\sigma}) = 0$

$\Rightarrow$ it's an okay displacement?

Recall that given a disp. x & force f,

$$\langle (x - \bar{x})^2 \rangle = \frac{k_B T}{f'(\bar{x})}$$

$$\underline{P} = -\sum_{ij} \underline{r}_i j_{ij}(\underline{\sigma})$$
$$= +\sum_{ij} \underline{r}_i j_{ij}(\underline{\sigma})$$

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} \qquad {}^N P_k = \frac{N!}{(N-k)!} \qquad {}^N P_2 = N(N-1)$$

$${}^N P_3 = N(N-1)(N-2)$$

$${}^N C_2 = \frac{N(N-1)}{2}$$

$${}^N C_3 = \frac{N(N-1)(N-2)}{6}$$

$$\sum_{x<y<z} =$$

All I need is $\langle J_\tau^\mu J_0^\nu \rangle_{eq}$.

$$P_{eq}(\sigma) = \frac{1}{Z_{eq}} e^{-\beta E(\sigma)}$$

$$Z_{eq} = \Delta_0 \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\lambda\beta l}$$

$$\underline{J} = \frac{1}{2} \sum_{ij} (\underline{r}_j - \underline{r}_i)\, j_{ij}$$

$$= -\sum_{ij} \underline{r}_i \underbrace{j_{ij}}_{= \delta_{ij}\left(\epsilon_i(\Omega(\sigma)) - \epsilon_i(\sigma)\right)}$$

$\cancel{J_\tau^\mu J_0^\nu =}$

$\cancel{J_\tau^\mu J_0^\nu =}$

$$E(\sigma) = \sum_i \epsilon_i(\sigma)$$

$$\langle J_0^\mu \rangle = \frac{1}{Z_{eq}} \sum_\sigma J_0^\mu e^{-\beta E(\sigma)}$$

$$= -\frac{1}{Z_{eq}} \sum_{i \,\&\, j} \underline{r}_i \sum_{\sigma} j_{ij}(\sigma)\, e^{-\beta \sum_i \epsilon_i(\sigma)}$$

$$= -\frac{1}{Z_{eq}} \sum_{i} \underline{r}_i \sum_{\sigma} \Big(\epsilon_i(\Omega(\sigma))\, e^{-\beta \sum_i \epsilon_i(\sigma)} - \underline{\epsilon_i(\sigma) e^{-\beta \sum_i \epsilon_i(\sigma)}} \Big)$$

$\cancel{= \frac{1}{Z_{eq}}}$

$$= \langle \underline{P}^\sigma \rangle_{eq} - \langle \underline{P}^{\Omega(\sigma)} \rangle_{eq} = 0 \quad \because P_{eq} \text{ inv. under } \Omega.$$

$$\langle J^{\mu}_{\tau} J^{\nu}_{0} \rangle_{eq} Z_{eq} = \sum_{\sigma} J^{\mu}_{\tau} J^{\nu}_{0} \, e^{-\beta E(\sigma)}$$

$$= \cancel{\sum_{ij} \underline{r}_i^{\mu} \, r_j^{\nu}}$$

$$= \sum_{ij} \underline{r}_i \otimes \underline{r}_j \sum_{\sigma} \left(\epsilon_i(\Omega^{\tau+1}(\sigma)) - \epsilon_i(\Omega^{\tau}(\sigma)\right) \cdot \left(\epsilon_j(\Omega(\sigma)) - \epsilon_j(\sigma)\right) e^{-\beta E(\sigma)}$$

$$= \sum_{ij} \underline{r}_i \otimes \underline{r}_j \sum_{\sigma} e^{-\beta E(\sigma)} \Big[\epsilon_i(\Omega^{\tau+1}(\sigma))\epsilon_j(\Omega(\sigma)) + \epsilon_i(\Omega^{\tau}(\sigma))\epsilon_j(\sigma) - \epsilon_i(\Omega^{\tau+1}(\sigma))\epsilon_j(\sigma) - \epsilon_i(\Omega^{\tau}(\sigma))\epsilon_j(\Omega(\sigma))$$

$\cancel{\langle \epsilon_i(\tau)} \epsilon$ We assume $P_{eq}(\Omega^t(\sigma)) = P_{eq}(\sigma)$ $\therefore \begin{pmatrix} E \text{ const.} \\ \therefore \mu \text{can} \end{pmatrix}$

$$\langle \epsilon_i(t)\, \epsilon_j(t') \rangle_{eq.} = \frac{1}{Z_{eq}} \sum_{\sigma} \epsilon_i(\Omega^{t}(\sigma))\, \epsilon_j(\Omega^{t'}(\sigma))\, e^{-\beta \sum_k \epsilon_k(\sigma)}$$

$$\sum_{\sigma} a_{\sigma} = \Delta_0 \sum_{l=0}^{N_v/2} \frac{1}{(2l)!} \sum_{i_1 \dots i_{2l}} a_{i,\text{ occupied}}$$

$$\Rightarrow \sum_{\sigma} \epsilon_i(\Omega^t(\sigma))\, \epsilon_j(\sigma) e^{-\beta E} = \Delta_0 \sum_{l=0}^{N_v/2} \frac{1}{(2l)!} \sum_{\underbrace{n_1 \dots n_{2l}}_{\underline{n}}} e^{-4\beta\lambda l} (2\lambda)^2 \delta_{j \in \underline{n}} \cdot \delta_{i \in \Omega^t(\underline{n})}$$

non zero iff $\exists$ prtcl @ j a $t=0$ & @ i @ t
(prtcls can annihilate.)

First let's tackle $\underline{\delta_{j\in \underline{n}}}$

Suppose $j = n_1$, then the rest are unfixed.

$2l$ choices

$\Rightarrow$ ~~$\sum$ ...~~

$$\text{LHS} = \Delta_0 \sum_{l=0}^{N_v/2} \frac{(2\lambda)^2}{(2l-1)!} \sum_{\underline{n}_2,\dots n_{2l}} e^{-4\beta\lambda l}\, \delta_{i\in\Omega^t(j,\underline{n})}$$

Now impose ~~μcan~~ μcan dynamics: $\dot{\epsilon}_i \neq 0\ \forall i$ but $\underline{\dot{E} = 0}$

$\Rightarrow$ no annihilatns. $\Rightarrow$ don't need to worry about
taking two $\underline{n}_m$ to the same site.

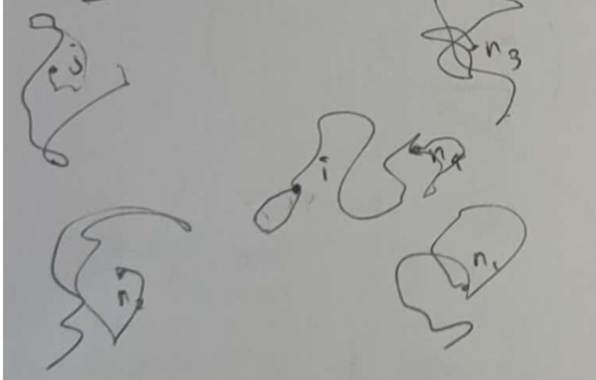
$^N P_K = \begin{Bmatrix} N \\ K \end{Bmatrix}$

$\underline{n}_2 = n_2 \dots n_{2l}$

$$\sum_{\underline{n}_2,\neq\dots,\underline{n}_{2l}} \delta_{i\in\Omega^t(j,\underline{n}_2)} = ?$$

only 1 prtcl can hit i after t steps

~~$\exists$ $2l$ choices out of a total~~



$\therefore$ $\Rightarrow$ can split into sum.

$$? = \cancel{\delta_i} \sum_{\underline{n}_2\neq\dots n_{2l}} \left(\delta_{i\in\Omega^t(j)} + \delta_{i\in\dots} + \dots\right)$$

$$= \sum_{\underline{n}_2} \delta_{i\in\Omega^t(j)} + \sum_{\underline{n}_2}\sum_m \delta_{i\in\Omega^t(n_m)}$$

~~$= {}^{N_v}P$~~ ~~$\frac{(N_v-2(l-1))!}{(N_v-2l)!}$~~ ~~$N$~~

~~$= {}^{N_v-1}P_{2l-1}$~~

$$= \begin{Bmatrix} N_v - 1 \\ 2l-1 \end{Bmatrix} \delta_{i\in\Omega^t(j)} + \sum_m \left(\begin{Bmatrix} N_v-1 \\ 2l-2 \end{Bmatrix} \sum_{\substack{n_m \\ \neq \text{others}}} \delta_{i\in\dots} \right.$$

$$\delta_{i\in\Omega^t(j)} \quad \sum_{\substack{n_m \\ \neq\text{others}}} \delta_{i\in\Omega^t(n_m)} = ?$$

$$= \langle \delta(\underline{r}_j(t) - \underline{r}_i(0)) \rangle \quad , \quad = \cancel{\frac{1}{a^2}\int d^2r\, \delta(\underline{r}(0) - \underline{r}_i)}$$

$$= \delta(\underline{r}_j(t) - \underline{r}_i)$$

$$=$$

$$\Omega^t(n_m) = i$$
$$\Rightarrow n_m = \Omega^{-t}(i)$$

~~BUT~~ Ω ~~irrev.~~

$$\sum_{\substack{n_m \\ (\neq j, \neq n_3)}} \delta_{\Omega^t(n_m) = i} = \frac{1}{a^2}\int d^2 \langle \underline{r} \rangle\, \delta(\underline{r}_m(t) - \underline{r}_i)$$

$$\| \qquad \approx \frac{1}{a^2}\int d^2\underline{r}\, \delta(\underline{r}(t) - \underline{r}_i)$$

$$\sum_{\langle n_m \rangle}$$

i ~~is fixed $\Rightarrow \Omega^{-t}(i)$ is also fi~~

$\Omega^t(\cdot)$ irreversible BUT μ can $\Rightarrow \exists$ an inverse for the particles $\because$ no annihilat$^{\bar{n}}$.

$\Rightarrow$ can write $n_m = i' = \Omega^{-t}(i)$ otherwise get 0 $\because \epsilon_i$ uncorrelated w/. ϵ_j

$$? = \begin{Bmatrix} N_v - 1 \\ 2\ell - 1 \end{Bmatrix} \delta_{j=i'} + \begin{Bmatrix} N_v - 1 \\ 2\ell - 2 \end{Bmatrix} \sum_m \Big(\underbrace{\sum_{\langle n_m \rangle} \delta_{n_m = i'}}_{} \Big)$$

only nonzero if $j \neq i'$
$n_{(\neq m)} \neq i'$

$\Rightarrow$ for 2^{nd} req,

$n_2, \ldots \neq i', j$ $\qquad \sum_{n_2} \sum_m \delta_{n_m = i'}$

$$\sum_m \Big(\sum_{n_{\neq m}} \sum_{\langle n_m \rangle} \delta_{n_m = i'} \Big)$$

" iff. $n_{\neq m} \neq i'$, $j \neq i'$

$$? = \begin{Bmatrix} N_v - 2 \\ 2\ell - 1 \end{Bmatrix} \delta_{j=i'} + \begin{Bmatrix} N_v - 2 \\ 2\ell - 2 \end{Bmatrix} \Big(\sum_{\langle m \rangle} \Big) \delta_{j \neq i'}$$

$$\downarrow 2\ell - 1$$

$$= \begin{Bmatrix} N_v - 2 \\ 2\ell - 1 \end{Bmatrix} \delta_{j=i'} + (2\ell - 1) \begin{Bmatrix} N_v - 2 \\ 2\ell - 2 \end{Bmatrix} \delta_{j \neq i'}$$

$$\langle \epsilon_i(t)\epsilon_j(0)\rangle_{eq} Z_{eq} = \Delta_0 \sum_{l=0}^{N_v/2} (2\lambda)^2 \left[\binom{N_v-2}{2l-1} \delta_{j=i'} + \binom{N_v-2}{2l-2} \delta_{j\neq i'} \right] e^{-4\beta\lambda l}$$

$\delta_{j\neq i'}$

$i'=j$ $i'\neq j$

fix one prtcl @ $i'=j$ $\Rightarrow$ others unfixed

~~fix two prtcls @ $i'\neq j$~~

fix 2 prtcls @ i' & j $\Rightarrow$ others unfixed.

$$Z_{eq} = \Delta_0 \sum_{l=0} \binom{N_v}{2l} e^{-4\beta\lambda l} = \Delta_0 \tilde{Z}$$

$$\langle \epsilon_i(t)\epsilon_j(0)\rangle_{eq} = \cancel{\frac{4\lambda^2}{\tilde{Z}}\sum}$$

t here meant $(\delta t)t$ really but take $\delta t=1$

$$= 4\lambda^2 \frac{\delta_{ii'}}{\tilde{Z}} \sum_{l=0}^{N_v/2} \left[\binom{N_v-2}{2l-1} \delta_{j=i'} + \binom{N_v-2}{2l-1} \delta_{j\neq i'} \right] e^{-4\beta\lambda l}$$

valid?

Δ

~~B~~

$$\langle J^\mu_\tau J^\nu_0\rangle_{eq} = \sum_{ij} r_i \otimes r_j \Big(\langle \epsilon_i(\tau+1)\epsilon_j(1)\rangle_{eq} + \langle \epsilon_i(\tau)\epsilon_j(0)\rangle_{eq} - \langle \epsilon_i(\tau+1)\epsilon_j(0)\rangle_{eq} - \langle \epsilon_i(\tau)\epsilon_j(1)\rangle_{eq} \Big)$$

$$= \sum_{ij} \underbrace{r_i \otimes r_j}_{\alpha_{ij}} \Big(\langle \epsilon_i(\tau)\epsilon_j(0)\rangle_{eq} + \langle \epsilon_i(\tau)\epsilon_j(0)\rangle_{eq} - \langle \epsilon_i(\tau+1)\epsilon_j(0)\rangle - \langle \epsilon_i(\tau-1)\epsilon_j(0)\rangle_{eq} \Big)$$

time transl. for ϵ_i's

symm.

$$\sum_{ij} \alpha_{(ij)} \langle \epsilon_i(t)\epsilon_j(0)\rangle_{eq} = ?$$

$$? = \frac{4\lambda^2}{\tilde{Z}} \sum_{l=0}^{N_V/2} e^{-4\beta\lambda l} \sum_{ij} \underline{\alpha}_{ij}\, \delta_{\exists i'} \left(\binom{N_V-2}{2l-1} \delta_{j=i'} + \binom{N_V-2}{2l-2} \delta_{j\neq i'} \right)$$

$i' \xrightarrow{\Omega^t} i$

$j = i'$

$\Downarrow$

$j_t = i$

$i' = \Omega^{-t}(i)$ Where we've assumed an inverse exists, i.e. it's reachable from elsewhere (otherwise just get 0).

$\Downarrow$

i' is unique

$\therefore$ μcan dynamics.

Could imagine a site i where $\Omega(\underline{\sigma}_{\text{con}:i}) = \Omega(\Omega_{\underline{\sigma}+\text{co}:i})$

hence the $\delta_{\exists i'}$ (even if $i = i'$ that's fine!)

$\cancel{i''} = i'?$

$$? = \underbrace{\frac{4\lambda^2}{\tilde{Z}} \sum_{l} e^{-4\beta\lambda l}}_{} \sum_{i'j} \underline{\alpha}_{ij} \left(\ldots \delta_{j=i'} + \ldots \underset{1-\delta_{j=i'}}{\overset{\shortparallel}{\delta_{j\neq i'}}} \right)$$

$i' = j$

$i' = j \rightsquigarrow i$

$$= \cancel{(\ldots) \left[\sum_{j} \underline{\alpha}_{\Omega^t(j),j} \binom{N_V-2}{2l-1} \right.} +$$

$$= (\ldots)_l \sum_{i'j} \underline{\alpha}_{ij} \left[\underbrace{\binom{N_V-2}{2l-2}}_{A_l} + \underbrace{\left(\binom{N_V-2}{2l-1} - \binom{N_V-2}{2l-2} \right)}_{B_l} \delta_{j=i'} \right]$$

$$= (\ldots)_l \left[A_l \sum_{ij} \underline{\alpha}_{ij} + B_l \sum_{j} \underline{\alpha}_{\Omega^t(j)\,j} \right]$$

$$\delta_{j \neq i'} = 1 - \delta_{j = i'}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$n = N_v - 2$$
$$k = 2\ell - 1$$

$$\binom{n}{k}\delta + \binom{n}{k-1}(1-\delta)$$

$$= \binom{n}{k-1} + \delta\left(\binom{n}{k} - \binom{n}{k-1}\right)$$

$$= \binom{n}{k-1}\left[1 + \delta\left(\frac{n-k+1}{k} - 1\right)\right]$$

$$= \binom{n}{k-1}\left[1 + \delta\left(\frac{n+1-2k}{k}\right)\right]$$

$$= \binom{N_v-2}{2\ell-2}\left[1 + \delta \left(\frac{N_v - 1 - 2(2\ell-1)}{2\ell-1}\right)\right]$$

$$\cancel{\binom{n}{k-1} =}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$= \frac{n-k+1}{k}\frac{n!}{(k-1)!(n-k+1)!}$$
$$= \frac{n-k+1}{k}\binom{n}{k-1}$$

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

$$= \frac{N_v - 1 - 4\ell + 2}{2\ell - 1}$$

$$\frac{N_v + 1 - 4\ell}{2\ell - 1}$$

$$\binom{n}{k} - \binom{n}{k-1} = \binom{n+1}{k} - 2\binom{n}{k-1}$$

$$= \binom{n}{k-1} + \delta\binom{n+1}{k} - 2\delta\binom{n}{k-1}$$

$$= \binom{N_v-2}{2\ell-1}(1-2\delta) + \binom{N_v-1}{2\ell-1}\delta$$

$$\binom{n}{k-1} \Big/ \binom{n}{k} = \frac{k}{(1+n)-k} = \frac{k}{1+n}\left(1 - \frac{k}{n+1}\right)^{-1}$$

$$\simeq \frac{k}{1+n}\left(1 + \frac{k}{n+1} - \dots\right)$$

$$\Rightarrow \cancel{\binom{n}{k}} \quad \binom{n}{k} - \binom{n}{k-1}$$

$$\simeq \binom{n}{k}\left[1 - \frac{k}{1+n}\left(1 + \frac{k}{n+1} - \dots\right)\right]\Big\}$$

$r_i^{\mu} r_j^{\nu}$ $\qquad$ $r_{\pi(i)}^{\mu} r_j^{\nu}$

$\sim r_j^{\mu} r_j^{\nu}$ $\qquad$ LT^{-2} $\quad$ L

L^2T^{-2}

$\underline{x}_\tau \cdot \underline{a}_0$

Focus on $\parallel$ cpt.

Torque!

$r_j^{\prime} r_j^{\prime}$ $\qquad$ $r_j^{\prime} r_j^{\prime}$ $\qquad$ $\underline{x}_0 \cdot \underline{a}_\tau$

$$(x_\tau - x_{\tau-1}) - (x_{\tau+1} - x_\tau)$$

$$\dot{x}_{\tau-1} - \dot{x}_\tau = -\ddot{x}_\tau$$

$$\frac{Z_{eq}}{4\lambda^2\Delta_0}\langle J^{\mu}_{\tau} J^{\nu}_{0}\rangle_{eq} = \sum_{l=0}^{N_v/2} e^{-4\beta\lambda l} B_l \sum_{\underline{p}} (\underline{p}(0))^{\mu} (-\ddot{\underline{p}}(\tau))^{\nu}$$

particle trajectory

$$-\langle J^{\mu}_{\tau} J^{\nu}_{0}\rangle_{eq} = \left(\frac{\Delta_0}{Z_{eq}}(2\lambda)^2 \sum_{l=0}^{N_v/2} e^{-4\beta\lambda l}\, 2l B_l\right)\langle \underline{x}^{\mu}_{0}\, \ddot{\underline{x}}^{\nu}_{\tau}\rangle$$

$\|$ trans-inv.

$\langle \dot{x}^{\mu}_{0}\, \dot{x}^{\nu}_{\tau}\rangle$

$$\Rightarrow \cancel{\sum_i \underline{x}_i(0) \cdot -2\ddot{\underline{x}}_i(\tau)}$$

$$\Downarrow$$

~~We know~~ $\cancel{\rho(\underline{x}|t) = \frac{A}{\sigma} e^{-|\underline{x}|^2/2\sigma} \qquad \sigma = \sqrt{2Dt}}$

$$\cancel{\sum_i \underline{x}_i(0) \cdot \ddot{\underline{x}}_i(t)} \longrightarrow \cancel{\overset{\#\,\text{prtcls}}{\widetilde{2l}} \langle \underline{x}(0) \cdot \ddot{\underline{x}}(t) \rangle}$$

$$\cancel{= 2l \frac{A}{\sigma} \int d^d\underline{x} \; \underline{x}(0) \cdot \ddot{\underline{x}}(t)\, e^{-|\underline{x}|^2/2\sigma}}$$

$$\rho_{2l}(\underline{x}|0) = \sum_{i=1}^{2l} \delta(\underline{x} - \underline{x}_i(0))$$

We know that $\rho_1(\underline{x}|0) = A\delta(\underline{x})$ has soln.

$$\rho_1(\underline{x}|t) = A \frac{1}{\sqrt{4\pi Dt}} e^{-|\underline{x}|^2/4Dt}. \qquad \sigma^2 = 2Dt$$

$$= \frac{A}{\sqrt{2\pi}\,\sigma} e^{-|\underline{x}|^2/2\sigma^2}$$

$$\Rightarrow \rho_{2l}(\underline{y}|0) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{i=1}^{2l} e^{-|\underline{y} - \underline{x}_i(0)|^2/2\sigma^2},$$

<u>Then</u> $\sum_i \underline{x}_i(0) \cdot \ddot{\underline{x}}_i(t)$

Ω : $\underline{\sigma} \longrightarrow \left.\begin{array}{l} * \text{ propose edge } \alpha \\ * \text{ flip if } \Delta E = 0 \end{array}\right\} \longrightarrow \underline{\sigma}'$

BUT! I don't need to know Ω exactly, it's enough to know what happens to prtcls.

For 8-vertex, ...

each timestep: $*$ pick N_E edges → only those adjacent to prtcls will actually flip → moves prtcl along edge.

$*$ flip if $\Delta E = 0$

Assuming prtcls ~~diffuse~~ dilute, we can neglect collisions (~elastic ∵ no annihilatⁿ)

⇒ on avg, each timestep we move a particle by ~~4 δt~~ $\underset{\substack{= \\ 4 \\ \text{(assumed const.)}}}{z}$ steps in random walk on v lattice. (lattice parm = a = 1)

$$\Rightarrow \quad D = \frac{a^2}{2d\,\delta t_{step}} = \frac{a^2}{2d\,\delta t/4} \overset{\substack{a=1 \\ \delta t=1}}{=} \frac{2}{d}$$

~~obeys~~ has $P(\underline{x}|t)$ obeying $\dfrac{\partial P}{\partial t} = D\nabla^2 P$

↓ ~~diffus.~~ dilute approxⁿ.

nᵒ density $\rho(\underline{x}, t)$ ∝ prob. density.

$$\sum_{ij} \underline{\alpha}_{ij} \langle \epsilon_i(t)\, \epsilon_j(0) \rangle_{eq} = \frac{4\lambda^2}{\tilde{Z}} \sum_l e^{-4\beta\lambda l} \Big(A_l \sum_{ij} \underline{\alpha}_{ij} + B_l \sum_j \alpha_{\Omega^{+}(j)\, j} \Big)$$

~~unjustified assumpt^{n}~~

$$\cancel{\approx \frac{4\lambda^2}{\tilde{Z}} \sum_l e^{-4\beta\lambda l} A_l \sum_{ij} \underline{\alpha}_{ij}}$$

$\cancel{j = \Omega^{+}(i)}$

$\because$ dynamics - indep.

$$\langle J^{\mu}_{\tau} J^{\nu}_{0} \rangle_{eq} = 2(\,'')_{\tau,0} - (\,'')_{\tau+1,0} - (\,'')_{\tau-1,0}$$

A_l terms cancel!

confusing il

$$\cancel{\approx \frac{4\lambda^2}{\tilde{Z}} \sum_l e^{-4\beta\lambda l} B_l \sum_{ij}}$$

$i_\tau = \Omega^{+}(i)\rho$

let $\cancel{i_\tau = \Omega^{+}(i)}$ now

$$= \frac{4\lambda^2}{Z_{eq}} \Delta_0 \sum_l e^{-4\beta\lambda l} B_l \sum_i \left(2\alpha_{i i_\tau} - \alpha_{i_{\tau+1} i} - \alpha_{i_{\tau-1} i} \right)$$

$$\cancel{= \frac{4\lambda^2}{Z_{eq}} \Delta_0 \sum_l e^{-4\beta\lambda l} B_l \sum_r \left(2\underline{r}(\tau) \ldots \right.}$$

Also note $\cancel{B_l \neq 0 \text{ iff}}$ $\underline{\alpha}_{i i_\tau}$ should be accompanied by a δ_{prtcl} @ i_0 & i_τ

$$\langle J^{\mu}_{\tau} J^{\nu}_{0} \rangle_{eq} = \frac{4\lambda^2}{Z_{eq}} \Delta_0 \sum_{l=0}^{N/2} e^{-4\beta\lambda l} B_l\, \delta_{\epsilon_i = 2\lambda} \sum_i r_i^{\nu} \left(\underline{r}^{\mu}_{i_\tau} - \underline{r}_{i_{\tau+1}} - \underline{r}_{i_{\tau-1}} \right)^{\mu}$$

$$\| $$

$$r_0^{\nu} \left((\underline{r}_\tau - \underline{r}_{\tau-1}) - (\underline{r}_{\tau+1} \ldots \right.$$

Ω^t is <u>such</u> a pain to work with

$\Rightarrow$ I <u>have</u> to make an approx$^{\bar{n}}$ here.

$\sum_{ij}(\delta_{j=i'})$ is <u>only</u> satisfied on $\mathcal{O}(N_v)$ terms which is small compared to the $\mathcal{O}(N_v^2)$ for $\sum_{ij}(1)$

~~It's also~~ $\cancel{\mathcal{L}\sum_j \underline{\alpha}_{\Omega^t(j)j}}$ ~~only has significant value if~~

$\underline{\alpha}_{ij} = \underline{r}_i \otimes \underline{r}_j$

Crucially, $B_\ell = \binom{N_v-2}{2\ell-1} - \binom{N_v-2}{2\ell-2}$

$= \binom{n}{k} - \binom{n}{k-1}$

$\approx \binom{n}{k}\left(1 + \mathcal{O}\left(\frac{k}{n+1}\right)\right)$ } low temp. $(\beta \gg 1)$ $\Rightarrow$ $\ell \sim k \ll n$

$\Rightarrow B_\ell$ <u>not</u> small, in fact $A_\ell \approx \frac{k}{k+1} B_\ell$

I'm <u>trying</u> to neglect the $j = i'$ terms. BUT

I can't see how, $A_\ell \sum_{ij} \sim B_\ell \sum_j$

$A_\ell n^2 \sim B_\ell n$

$\Rightarrow$ only diff. can be in summands:

$\underline{\alpha}_{ij}$ vs. $\underline{\alpha}_{\Omega^t(j)j}$

$\underline{r}_i \otimes \underline{r}_j$ $\quad$ $\underline{r}_{\Omega^t(i)} \otimes \underline{r}_j$

<u>ONLY</u> way to justify binning 2nd term is to say

Nope! No valid reason!

$$\rho(y|t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-|\underline{y} - \underline{x}(0)|^2/2\sigma^2}$$

$\because$ fix $\underline{x}(0)$ WLOG. $\because$ trans-inv.

$\underline{x}(0) \cdot \langle \ddot{\underline{x}}(\tau) \rangle$

$\cancel{\langle \dot{x}^2 \rangle = 2\langle x \dot{x} \rangle = 2D}$

$\cancel{\int d^2 y \, \rho(y|t) \ddot{y}}$ $\quad \cancel{\langle x_t^2 \rangle = \sigma = 2Dt}$

$\Rightarrow$

$\cancel{\langle x(\tau) \rangle}$

If $\exists$ Langevin eqn. then $\quad m\langle \ddot{\underline{x}} \rangle = 0$

$m \ddot{\underline{x}} = \eta$.

$x_0^1 \ddot{x}_\tau^1$

$$x_0^\nu (2x_\tau - x_{\tau+1} - x_{\tau-1})^\mu \simeq - x_0^\nu \ddot{x}_\tau^\mu$$

$$x_0^\nu (\dot{x}_{\tau-1} - \dot{x}_t)^\mu$$

$\overset{?}{=}$ trans-inv.

Valid? Doubtful.
Maybe true $\because$ summed over all sites w/. prtcls?

$(x_1^\nu - x_0^\nu) \dot{x}_\tau^\mu$

$\dot{x}_0^\nu \dot{x}_\tau^\mu$.

$\cancel{\dot{x}_\tau} = \underline{x}_0 e^{-\gamma}$

$\frac{\gamma}{m} = \frac{D}{k_B T}$?

$\langle \dot{x}(\tau) \cdot \dot{x}(0) \rangle \propto e^{-\gamma t}$

$\gamma \propto D/T$

$$\langle J^{\mu}_{t} J^{\nu}_{0} \rangle_{eq} = \frac{4\lambda^2}{Z_{eq}} \Delta_0 \sum_{\ell} B_\ell e^{-4\beta\lambda\ell} \cdot 2\ell \cdot A e^{-\gamma t}$$

$$B_\ell = \binom{N_v - 2}{2\ell - 1}\left(1 + G\left(\frac{2\ell - 1}{N_v - 1}\right)\right)$$

when $T \sim N$?

$$B_\ell = \cancel{\binom{n}{k}}$$

$$= \binom{n}{k} - \binom{n}{k} \cdot \frac{k}{(n+1)-k}$$

$$= \binom{n}{k}\left[1 - \frac{k}{(n+1)-k}\right]$$

$$\propto \frac{n+1-2k}{n+1-k}$$

$n = N_v - 2$

$k = 2\ell - 1$

$$= \frac{N_v - 1 - 4\ell + 2}{N_v - 2 + 1 - 2\ell + 1}$$

$m = 2\ell$

$= 0, 2, 4, \ldots N_v$

$$= \frac{N_v - 2m + 1}{N_v - m}$$

$$= \cancel{1 + \frac{1}{N_v - 2m}}$$

$\Rightarrow$ ~~fails~~ @

$$= \cancel{1 + \frac{1-m}{N_v - m}}$$

$\cancel{(N_v + 1 - 2m)(N_v}$

We have $m \leq N_v$

$\Rightarrow$ bottom always 0

But top is 0

for $N_v \sim 2m$

$\Rightarrow 4\ell \sim N_v$

$\Rightarrow$ @ temp. where

$4\ell \sim N_v$

$\Rightarrow e^{-N_v\beta}$

$\Rightarrow T \sim N_v$

$$x_0^{\nu}\left(2x_\tau^{\mu} - x_{\tau+1}^{\mu} - x_{\tau-1}^{\mu}\right) = x_0^{\nu}\left(x_\tau^{\mu} - x_{\tau-1}^{\mu}\right) - x_0^{\nu}\left(x_{\tau+1}^{\mu} - x_\tau^{\mu}\right)$$

$\|$ P_{eq} inv. under $\sigma \to \Omega(\sigma)$

$$\cancel{x_8} \quad x_1^{\nu}\dot{x}_\tau^{\mu} - x_0^{\nu}\dot{x}_\tau = \underline{\underline{\dot{x}_0^{\nu}\dot{x}_\tau^{\mu}}}$$

$$\langle \dot{x}_0^{\nu}\dot{x}_\tau^{\mu}\rangle_{\text{prtcls}}$$

$$\langle |\dot{\underline{x}}|^2\rangle = \frac{d k_B T}{m} = \frac{c/m^2}{2\cancel{m}\gamma}$$

$$\Rightarrow c = (2 d k_B T)\, m^2\gamma$$

$$m\ddot{\underline{x}} = -m\gamma\dot{\underline{x}} + m\underline{\xi}$$

$$\langle \underline{\xi}\rangle = \underline{0}$$

$$\langle \underline{\xi}(t)\cdot\underline{\xi}(t')\rangle = c\,\delta(t-t')\,\cancel{/}m^2$$

$$\langle \xi^{\mu}(t)\,\xi^{\nu}(t')\rangle = \frac{c}{d}\,\delta_{\mu\nu}\,\delta(t-t')\,/m^2$$

has soln. $\dot{\underline{x}}(t) = \dot{\underline{x}}(0)e^{-\gamma t} + \int_0^t dt'\,\underline{\xi}(t')\,e^{-\gamma(t-t')}$

$$\Rightarrow \langle \dot{x}^{\nu}(0)\dot{x}^{\mu}(t)\rangle = \langle \dot{x}_0^{\nu}\dot{x}_0^{\mu}e^{-\gamma t}\rangle_\xi$$

$$+ \int_0^t dt'\,\cancel{\langle \xi^{\mu}(t')\rangle_\xi}^{\,0}\, e^{-\gamma(t-t')}\,\dot{x}_0^{\nu}$$

$$= \langle \dot{x}_0^{\mu}\dot{x}_0^{\nu}\rangle_\xi\, e^{-\gamma t} \qquad \gamma = ?$$

$$D = \frac{k_B T}{m\gamma} \qquad = \langle \dot{x}_0^{\mu}\dot{x}_0^{\nu}\rangle_\xi\, e^{-\cancel{\frac{t}{\beta m \gamma}}} \qquad m\gamma = \frac{k_B T}{D}\ ?$$

But what is $\langle \dot{x}_0^{\mu}\dot{x}_0^{\nu}\rangle$? $\quad 2dDt$

$$\dot{x}_0^{\mu} \sim \frac{a_{step}}{\delta t_{step}}$$

We expect " $= \delta_{\mu\nu}\dfrac{k_B T}{m}$

$$\sim \frac{4a}{\delta t} \quad\Rightarrow\quad \langle \dot{x}^{\nu}(0)\dot{x}^{\mu}(t)\rangle = \delta_{\mu\nu}\frac{k_B T}{m}e^{-\frac{k_B T}{mD}t}$$

$$\sum_{ij} \underline{\alpha}_{ij} \langle \epsilon_i(t)\, \epsilon_j(0) \rangle_{eq} = \frac{\Delta_0}{Z_{eq}}\, 4\lambda^2 \sum_{\ell=0}^{N\nu/2} e^{-4\beta\lambda\ell} \Big[A_\ell \sum_{ij} \underline{\alpha}_{ij} + \delta_{i\,i_0 t}\, B_\ell \sum_{j} \alpha_{j_t j} \Big]$$

if $j \not\to i$ (another prtcl $\to i$)

if $j \to j_t = i$

$= 0$ unless $\exists$ prtcl @ j @ 0 + @ i @ t $\Rightarrow$ prtcls @ i', j @ $t=0$

o.e. prtcls @ i, j_t @ t.

only 2nd term is t-dependent

$$\langle J^\mu_t\, J^\nu_0 \rangle_{eq} = -\frac{\Delta_0}{Z_{eq}}\, 4\lambda^2 \sum_{\ell=0}^{N\nu/2} e^{-4\beta\lambda\ell}\, B_\ell\, \underbrace{\sum_{j_0} \Big(2\underline{\alpha}_{j_t j_0} - \alpha_{j_{t+1} j_0} - \alpha_{j_{t-1} j_0} \Big)^{\mu\nu}}_{\text{indep. of } \ell}$$

$$\underline{\alpha}_{j_t j_0} = (\underline{r}^\mu_{j_t} \otimes \underline{r}^\nu_{j_0})$$

only sum over j where $\exists$ prtcls @ $t=0$

$$\sum_j \alpha_{\Omega^t(j)\, j}$$

$$=$$

$$\sum_{j} \Big(2 r^\mu_j(t)\, r^\nu_j(0) - r^\mu_j(t+1)\, r^\nu_j(0) - r^\mu_j(t-1)\, r^\nu_j(0) \Big)$$

$$\cancel{\sum_{j} \big(2 r^\mu_j(t) - r^\mu_j(t+1) \big)}$$

$$=$$

$$\sum_{\underline{r}} \big(2\underline{r}(t) - \underline{r}(t+1) - \underline{r}(t-1) \big)^\mu\, \underline{r}(0)^\nu$$

$$(A + B + C + \dots)(\bar{A} + \bar{B} + \bar{C} + \dots)$$

$$= A\bar{A} + B\bar{B} + \dots + A\bar{B} + B\bar{A} + \dots$$

If i has no $i' \to i$ then: $\langle \epsilon_i(t)\, \epsilon_j(0) \rangle = 0$

$\therefore$ summing over <u>all</u> initial configs.

Also require $\sigma_{t=0}$ has a prtcl @ j

$\Rightarrow$ only sum over configs w/o a particle @ j, i'

$$\binom{N_v}{2l} \frac{2l}{N_v} \frac{\cancel{2l-1}}{\cancel{N_v-1}} \left(1 - \frac{2l-1}{N_v-1}\right)$$

$$= \frac{N!}{k!(N-k)!} \frac{k}{N} \left(\frac{N-k}{N-1}\right)$$

$$= \frac{(N-1)!}{(k-1)!(N-k-1)!} \frac{1}{N-1} = \frac{(N-2)!}{(k-1)!(N-k-1)}$$

$$= \frac{(N-2)!}{(k-1)!(N-2-(k-1))!}$$

$$= \binom{N-2}{k-1}$$

$$\binom{N}{k} \frac{k}{N} \frac{k-1}{N-1} = \frac{N!}{k!(N-k)!} \frac{k}{N} \frac{k-1}{N-1}$$

$$= \frac{(N-2)!}{(k-2)!(N-k)!} = \binom{N-2}{k-2}$$

$$\sum_{ij} \alpha_{ij} \langle \epsilon_i(t) \epsilon_j(0) \rangle_{eq}$$

$$= 4\lambda^2 \frac{\Delta_0}{Z_{eq}} \sum_{l=0}^{N_v/2} e^{-4\beta\lambda l} \sum_{ij^+} \alpha_{ij} \left(C_l + D_l \delta_{i=j^+}\right)$$

$$C_l = \binom{N_v-2}{2l-2} \qquad D_l = \binom{N_v-2}{2l-1}$$

$l=1$ Case

$$\langle \epsilon_i(t)\epsilon_j(0)\rangle = \frac{\Delta_0}{Z} e^{-4\beta\lambda} \sum_{m\neq n} \frac{1}{2!}(2\lambda)^2 (\delta_{i^-=n}+\delta_{i^-=m})\cdot(\delta_{j=n}+\delta_{j=m})$$

$$= \frac{\Delta_0}{2Z} e^{-4\beta\lambda}(2\lambda)^2 \sum_{m\neq n}\Big((\delta_{i^-=n}\delta_{j=n}+\delta_{i^-=m}\delta_{j=m}) + (\delta_{i^-=n}\delta_{j=m}+\delta_{i^-=m}\delta_{j=n})\Big)$$

$$= \frac{\Delta_0}{2Z} e^{-4\beta\lambda}(2\lambda)^2 \Big[\delta_{i^-=j}\begin{Bmatrix} N_v-1 \\ 1\end{Bmatrix}\cdot 2 \; \cancel{+\,\delta_{i^-\neq j}\begin{Bmatrix} N_v \\ 2\end{Bmatrix}\cdot 2} \; + \delta_{i^-\neq j}\cdot 2 \Big]$$

$\sum_{m\neq n}\delta_{i^-=n}\delta_{j=m}$

$\sum_{mn}\delta_{i^-=n}\delta_{j=m}\delta_{i^-\neq j}$

$$= \frac{\Delta_0}{Z}(2\lambda)^2 e^{-4\beta\lambda}\big[\delta_{i^-=j}(N_v-1)+\delta_{i^-\neq j}\big]$$

$$= \frac{\Delta_0}{Z}(2\lambda)^2 e^{-4\beta\lambda}\big[1+(N_v-2)\,\delta_{i^-=j}\big]$$

$\delta_{i^-=j} = \delta_{i=\Omega^t(j)}$

1: const. (time-indep, any other prtcls @ i & j on correct times)

$\delta_{i=\Omega^t(j)}$: demands $\exists$ prtcl $\underline{v}(t)$ from $\underline{r}_i \xrightarrow{\Omega^t} \underline{r}_j$

$$\langle \epsilon_{\Omega^t(j)}(t)\epsilon_j(0)\rangle = \frac{\Delta_0}{Z}(2\lambda)^2 e^{-4\beta\lambda}(N_v-1)$$

$$\langle \epsilon_{\neq\Omega^t(j)}(t)\epsilon_j(0)\rangle = \frac{\Delta_0}{Z}(2\lambda)e^{-4\beta\lambda}$$

$$\Rightarrow \sum_{ij}\alpha_{ij}\langle\ldots\rangle_{ij} = \frac{\Delta_0}{Z}e^{-4\beta\lambda}(2\lambda)^2\sum_{ij}\big(\alpha_{ij}+(N_v-2)\alpha_{ij}\,\delta_{i=\Omega^t(j)}\big)$$

$$= \frac{\Delta_0}{Z}(\ldots)\Big(\sum_{ij}\alpha_{ij}+(N_v-2)\sum_j\alpha_{\Omega^t(j)j}\Big)$$

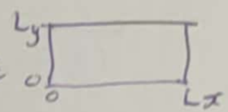
$$\beta_i = \frac{1}{T + \underline{r}_i \cdot \underline{\nabla} T} \approx \frac{1}{T}\left(1 - \frac{1}{T}\underline{r}_i \cdot \underline{\nabla} T\right) = \beta - \beta^2 \underline{r}_i \cdot \underline{\nabla} T$$

$$\Rightarrow f_\ell = \sum_{i_1 \neq \ldots \neq i_{2\ell}} e^{-4\lambda\beta\ell}\, e^{2\lambda\beta^2 \underline{\nabla} T \cdot \sum_{n=2}^{2\ell} \underline{r}_{i_n}}$$

$$= e^{-4\lambda\beta\ell} \sum_{i_1 \neq \ldots \neq i_{2\ell}} \prod_{n=2}^{2\ell} e^{2\lambda\beta^2 \underline{\nabla} T \cdot \underline{r}_{i_n}}$$

$$= \cancel{e^{-4\lambda\beta\ell} \int \left(\frac{d^3 \underline{r}_1}{a^3}\right) \cdots \left(\frac{d^3 \underline{r}_{2\ell}}{a^3}\right) e}$$

$$= e^{-4\lambda\beta\ell}\, I^{2\ell}$$

$\underline{r} \in$ 

$$I = \frac{1}{a^2} \int d^2 \underline{r}\; e^{\overbrace{2\lambda\beta^2 \underline{\nabla} T}^{\underline{K}} \cdot \underline{r}}$$

$$= \frac{1}{a^2} \int_0^{aL_x} dx \int_0^{aL_y} e^{K_x x}\, e^{K_y y}$$

$$= \frac{1}{a^2} \frac{1}{K_x K_y} \left(e^{K a L_x} - 1\right)\left(e^{K a L_y} - 1\right)$$

$$= \frac{\left(e^{2\lambda\beta^2 L_x \partial_x T} - 1\right)\left(e^{2\lambda\beta^2 L_y \partial_y T} - 1\right)}{(2\lambda\beta^2)^2 (\partial_x T)(\partial_y T)} \qquad a = 1$$

$$\Rightarrow Z = \Delta_0 \sum_{\ell=0}^{N_v/2} \left(\overbrace{I^2 e^{-4\lambda\beta}}^{r}\right)^\ell \frac{1}{2\ell!}$$

$$= \cancel{B_0 \frac{r^{N_v/2+1}}{r}}$$

$$\cancel{Z = \Delta_0 \frac{1 - r^{N_v/2+1}}{1 - r}}$$

$$r = I^2 e^{-4\lambda\beta}$$

$$I = \frac{1}{(2\lambda\beta^2)^2 T_x T_y} \left(e^{2\lambda\beta^2 L_x T_x} - 1\right)\left(e^{\dots y} - 1\right)$$

For T_x, T_y small $\quad I \approx \dfrac{(2\lambda\beta^2 L_x T_x)(\dots)_y}{(2\lambda\beta^2)^2 T_x T_y}$

$$= \cancel{2\lambda\beta^2}\, \underbrace{L_x L_y}_{N_v}$$

$$Z = \Delta_0 \sum_{l=0}^{N_v/2} \frac{1}{(2l)!} r^l$$

$$\approx \Delta_0 \sum_{l=0}^{N_v/2} \frac{1}{(2l)!} (\cancel{2\lambda\beta^2} N_v)^{2l} e^{-4\lambda\beta l}$$

$\dfrac{N_v!}{(N_v - 2l)!} \sim N_v^{2l}$ so agrees with other form if we take $T_x, T_y \to 0$.

Idk where to go from here...

$$\langle J^\mu_t \rangle = \langle J^\mu_0 \rangle + \sum_{\tau=0}^{t-1} \sum_{\underline{\sigma}} J^\mu_\tau(\underline{\sigma}) \left[P_1(\underline{\sigma}) - P_0(\underline{\sigma}) \right]$$

$$P_1(\underline{\sigma}) = \sum_{\underline{\sigma}'} \delta_{\Omega(\underline{\sigma}'),\underline{\sigma}} \frac{e^{-\Sigma_i \beta_i \epsilon_i(\sigma')}}{Z_1} \qquad Z_0 = Z_1 ?$$

$$= P_0(\underline{\sigma}) \sum_{\sigma'} \delta_{\Omega(\sigma'),\sigma} \, e^{\Sigma_i \beta_i (\epsilon_i(\underline{\sigma}) - \epsilon_i(\sigma'))}$$

$$\epsilon_i(\underline{\sigma}) = \epsilon_i(\Omega(\sigma'))$$

Cont. eqn: $\sum_i \beta_i \epsilon_i(\sigma') = \sum_i \beta_i \epsilon_i(\Omega(\sigma')) + \delta t \sum_{ij} \beta_i j_{ij}(\sigma')$

$$P_1(\sigma) = P_0(\sigma) \sum_{\sigma'} \delta_{\Omega(\sigma'),\sigma} \, e^{-\delta t \sum_{ij} \beta_i j_{ij}(\sigma')}$$

$$= P_0(\sigma) \sum_{\sigma'} \delta_{\Omega(\sigma'),\sigma} \, e^{-\frac{\delta t}{2} \sum_{ij} (\beta_i - \beta_j) j_{ij}(\sigma')}$$

$\beta_i \approx \beta - \beta^2 \underline{r}_i \cdot \underline{\nabla} T$

$j_{ij} = -j_{ji}$

$\beta_i - \beta_j \approx -\beta^2 (\underline{r}_i - \underline{r}_j) \cdot \underline{\nabla} T$

$$= P_0 \sum_{\sigma'} \delta_{\dots} \, e^{+\frac{1}{2} \delta t \beta^2 \underline{\nabla} T \cdot \sum_{ij} (\underline{r}_i - \underline{r}_j) j_{ij}(\sigma')}$$

$$= P_0 \sum_{\sigma'} \delta_{\dots} \, e^{-\beta^2 \underline{\nabla} T \cdot \underline{J}(\underline{\sigma}') \delta t}$$

$$Z/\Delta_0 = \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\beta\lambda l}$$

For β small, we have:

$$\binom{N_v}{2l-1} = \frac{N_v!}{(N_v-2l+1)!(2l-1)!}$$

$$Z/\Delta_0 \approx \sum_{l=0}^{N_v/2} \binom{N_v}{2l} (1 - 4\beta\lambda l)$$

$$\binom{N_v}{2l} 4\beta\lambda l = \frac{N_v!}{(N_v-2l)!\,(2l)!} 2\beta\lambda(2l)$$

$$= (N_v-2l+1) \binom{N_v}{2l} 2\beta\lambda$$

Thm: $\cancel{\sum_{l=0}^{N_v/2}} \; \cancel{\approx \binom{N_v}{2l}}$

Thm: $\displaystyle\sum_{l=0}^{N_v/2} \binom{N_v}{2l} = \sum_{l=1}^{N_v/2} \binom{N_v}{2l-1}$

$$Z/\Delta_0 \approx \cancel{2\beta} \sum_{l=0}^{N_v/2} \binom{N_v}{2l} - 2\beta\lambda \sum_{l=0}^{N_v/2} \binom{N_v}{2l-1} (N_v - 2l + 1)$$

$$\langle J^\mu_t \rangle_{\Omega} = \langle J^\mu_0 \rangle_{\Omega} + \sum_{\tau=0}^{t-1} \langle J^\mu_{\tau+1} - J^\mu_\tau \rangle$$

$$= \langle J^\mu_0 \rangle + \sum_{\tau=0}^{t-1} \sum_{\underline{\sigma}} \left(J^\mu_{\tau+1}(\underline{\sigma}) - J^\mu_\tau(\underline{\sigma}) \right) P_0(\underline{\sigma})$$

$$= \langle J^\mu_0 \rangle + \sum_{\tau=0}^{t-1} \sum_{\underline{\sigma}} J^\mu_\tau(\underline{\sigma}) \left(P_1(\underline{\sigma}) - P_0(\underline{\sigma}) \right)$$

$$P_1(\underline{\sigma}) = \sum_{\underline{\sigma}'} P_0(\underline{\sigma}') \; \delta_{\Omega(\underline{\sigma}'), \underline{\sigma}}$$

$$\langle J^\mu_t \rangle = \langle J^\mu_0 \rangle - \delta t\, \beta^2 \sum_{\nu} \nabla_\nu T \sum_{\tau=1}^{t} \langle J^\mu_\tau J^\nu_0 \rangle_{eq}.$$

$$\parallel$$

$$\cancel{\langle J^\mu_{eq} \rangle}^{\,=0} + \beta \left(\langle J^\mu \phi \rangle_{eq} - \cancel{\langle J^\mu \rangle_{eq}}^{\,=0} \langle \phi \rangle_{eq} \right)$$

$$\beta^2 \underline{\nabla} T \cdot \langle J^\mu \underline{P} \rangle_{eq}.$$

I want ~~to~~ find $\langle J^\mu_\tau J^\nu_0 \rangle_{\underline{eq}}$ (and maybe $\langle J^\mu P^\nu \rangle_{eq}$)

$$= \sum_{\underline{\sigma}} P_{eq}(\underline{\sigma})\, J^\mu_\tau J^\nu_0$$

$$J^\mu = \frac{1}{2} \sum_{ij} (\underline{r}_j - \underline{r}_i)\, j_{ij}$$

$$\langle J^\mu \rangle = \frac{1}{Z_0} \sum_{\underline{\sigma}} J^\mu(\underline{\sigma})\, e^{-\sum_i \beta_i \epsilon_i(\underline{\sigma})}$$

$$\underline{P} = \sum_i \underline{r}_i \epsilon_i$$

$$\binom{n}{2\ell}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \binom{n-2}{k-2} + \binom{n-2}{k-1} + \binom{n-2}{k-1}$$

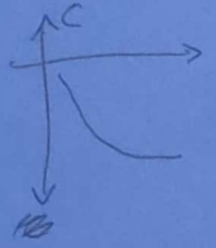
Because the dynamics are μcan overall,

$$E_{spin} + E_{demon} = E_{cs} = -\lambda N_v$$

For the purposes of getting $Var(E_{spin})$ we can just find $Var(E_{demon})$ instead!

Doesn't seem to help: we have $C<0$

&



$-$ve iff $C_D T^2 - \sigma_E^2 < 0$

$$\frac{\sigma_E}{T^2} > \underline{\underline{C_D}}$$

$$\cancel{\binom{n}{n} = \frac{n!}{n!(n-n)!}}$$

$$\left\{ {n \atop 0} \right\} = \frac{n!}{0!} = n!$$

$$\left\{ {n \atop n} \right\} = n!$$

$$\left\{ {n \atop n} \right\} = 1$$

$$\left\{ {n \atop n-1} \right\} \neq n$$

$$\left\{ {1 \atop 1} \right\} = 1$$

$$\left\{ {n \atop 0} \right\} = \frac{n!}{(n-0)!} = 1$$

$$\left\{ {1 \atop 0} \right\} = 1$$

$$\left\{ {n \atop n} \right\} = \frac{n!}{(n-n)!} = n!$$

$$\delta^2 = \sum_n^4 \sum_m^4 \hat{e}_n \cdot \hat{e}_m \sim 2D \cdot \underbrace{4}_{\#hops}$$

$D \sim 1$

$d = 2$

$$\sim \underline{8}$$

$\cancel{\langle \dot{r}_n^i \dot{r}_n^j \rangle = x}$

$$\langle r(t) r(s) \rangle - \cancel{\langle r(t) \rangle \langle r(s) \rangle} = \sqrt{\frac{t}{s}} \qquad 1 \leq t \leq s$$

$$\Rightarrow \delta \text{ things} = \sqrt{\frac{1}{t+1}} + \sqrt{\frac{0}{t}} - \sqrt{\frac{0}{t+1}} - \sqrt{\frac{1}{t}}$$

$$= \cancel{\sqrt{\frac{t-(t+1)}{t(t+1)}}}$$

$$= \sqrt{\frac{1}{t}} - \sqrt{\frac{1}{t+1}} = \sqrt{\frac{1}{t(t+1)}}$$

$$\rho_{s,t} = \frac{\min(s,t)}{\sqrt{st}} = \frac{\gamma_{st}}{\sqrt{\gamma_{ss}\gamma_{tt}}}$$

~~δ thing = ρ~~

$$\mathbb{E}(x_{t+\tau} x_t) \quad \because \mathbb{E}(x_t) = \mathbb{E}(\overset{\text{const.}}{x_0} + \cancel{\sum_{\tau=1}^{t} W_\tau}) = \mathbb{E}(x_0) = \text{const.}$$

$-\text{const.}$

$\cancel{\gamma \delta_{s,t}}$

$$\gamma(t+\tau, t) = \mathrm{Cov}(x_{t+\tau}, x_t) = \min(t, t+\tau)\,\sigma_w^2$$

$$\Rightarrow \delta \text{ stuff} = \gamma_{t+1,1} + \gamma_{t,0} - \gamma_{t+1,0} - \gamma_{t,1}$$

$$\propto 0.$$

$$\langle \dot{r}_n^\mu(\tau) \dot{r}_n^\nu(0) \rangle = \langle \dot{r}_n^\mu(0) \dot{r}_n^\nu(0) \rangle e^{-\gamma t}$$

$$= \delta^{\mu\nu} \langle |\dot{\underline{r}}_n|^2 \rangle \frac{1}{d} e^{-\gamma t}$$

$|\dot{\underline{r}}_n| = |\underline{S}_n|$

$\underline{S}_n = \sum_{m=1}^{t} \hat{\underline{e}}_m$

Where $\hat{\underline{e}}_m$ drawn from random ~~dist~~ uniform distributn.

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{2k} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_2 & 0 \end{pmatrix}$$

$$iY\Lambda = \begin{pmatrix} 0 & \lambda_2 & \\ -\lambda_1 & 0 & \\ & & \ddots \end{pmatrix} \qquad \Lambda iY = \begin{pmatrix} 0 & \lambda_1 & \\ -\lambda_2 & 0 & \\ & & \ddots \end{pmatrix}$$

$x \ln x - x \rightarrow \ln x$

$$\Rightarrow \{iY, \Lambda\} = \begin{pmatrix} 0 & \lambda_1+\lambda_2 & \\ -(\,'') & 0 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} (\lambda_1+\lambda_2) i\sigma_y & \\ & \end{pmatrix}$$

$$i\Lambda = -U^+ A U$$

$$\underbrace{iY\Lambda + \Lambda iY}_{\substack{\text{~~unitary~~} \\ \therefore\ \cancel{\lambda_i \in \mathbb{R}} \\ \text{real \& antisymm.}}} = -YU^+AU - U^+AUY$$

$$\cancel{= -U^+A^+UY^+} - Y^+U^+A^+U$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

U

$\frac{1}{2}(U+U^*)$

$\frac{1}{2}(U+U^+)$

$A^+ = -A$

$\therefore A^T = -A$

$A \in \mathbb{R}$

$Y^+ = Y$

$$Y = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & \ddots \end{pmatrix}$$

$$LHS^+ = -LHS$$

$$\cancel{= + U^T A U^* Y^T}$$

$$= -U^+A^+UY^+ - Y^+U^+A^+U$$

$$= +U^+AUY + Y^* U^+AU$$

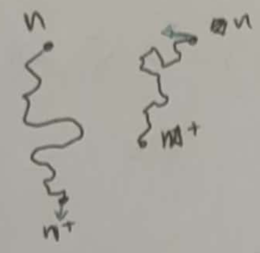
$$(U+U^+)^+ A (U+U^+)$$

$$= U^+AU + UAU + U^+AU^+ + UAU^+$$

Let's try & make some progress for small τ

$\Rightarrow \langle J^\mu_\tau J^\nu_0 \rangle \approx (4\lambda^2) \frac{\Delta_0}{2} e^{-4\lambda\beta} \left(\sum_n \delta^\mu_{n^+} \delta^\nu_n \right) (N_v - \cancel{2})$

Other terms assumed ~ 0 $\because$ dilute $\Rightarrow$ for a given $n \neq m$, $1 \neq 2$



$\langle \rangle_{t_0} = 0$

Only way we get correlat$^{\text{ns}}$ is when they get close together

$\because \sim$ hard-core random walks

For $l=1$, we can do something!

Fix prtcl ~~n~~ m & then n diffuses @ $2x$ rate in this frame of reference.

$\Rightarrow$ Problem reduces to diffusion on T^2 torus $\setminus \{\underline{0}\}$ (pos$^{\text{n}}$ of "2")

& we want to calculate

start pts. $\sum_{n_e} \delta^\mu_{n_e^+} \delta^\nu_{n_e}$ noting we have $\underline{8}$ hops / δt. now.

$\because$ other terms zero

$\because$ $T^2 \setminus \{0\}$ periodic (excl $\{\underline{0}\}$) we have $\sum_n \delta^\mu_{n^+} \delta^\nu_n \sim (N_v - 1) \delta_{\underline{r}^+} \delta_{\underline{r}}$

$\underline{r}$ completely arb. $\Rightarrow$ pick $(L_x/2, L_y/2)$

Ω : Generate 8 steps in a random walk.

$\langle \delta^\mu_{\underline{r}^+} \delta^\nu_{\underline{r}} \rangle \sim \underbrace{\langle \delta^\mu_{\underline{r}} \delta^\nu_{\underline{r}} \rangle}\, e^{-\gamma t}$ $\quad \gamma \sim 0 \because$ no potential / friction / etc.

$\sim \delta^{\mu\nu} \langle |\underline{\delta}_r|^2 \rangle \frac{1}{d}$

$\cancel{\delta}(2 \cancel{d} D \underline{\delta t}) \frac{1}{d}$ ($= 8 \because$ one set of 8 hops)

$\sim \delta^{\mu\nu} \, \underline{16 D}^{\,\approx 1}$

D

~~$\ddot{\underline{r}} = -\gamma \dot{\underline{r}} + \sqrt{2c}\,\underline{\eta}(t)$~~

$\frac{c}{\lambda m^2}$ $\frac{c}{m^2} = 2\gamma$

$$\dot{\underline{v}} = \underline{a}(t, \underline{v}) + \underline{\xi}(t) \qquad \text{s.t. } \langle \xi_i(t)\, \xi_j(t') \rangle = c\, \delta_{ij}\, \delta(t-t')$$

F-D : $c =$

$$\frac{\partial \rho}{\partial t} = \left(- \underline{\nabla} \cdot (\underline{a} \ldots) + D \nabla^2 \right) \rho = \left(-\gamma \underline{\nabla} \cdot (\underline{v} \ldots) + D \nabla^2 \right) \rho$$

$$= -\gamma \underline{\nabla} \cdot (\rho \underline{v}) + D \nabla^2 \rho = D \nabla^2 \rho - \alpha \underline{\nabla} \cdot \underline{J}.$$

We have $\underline{a} = \gamma \underline{v}$, but expect no such term for a free random walk

$\Rightarrow \gamma_{\text{eff}} = 0$? , $D_{\text{eff}} = 1$ as discussed.

$\delta t = 4$

$\dot{\underline{r}}_n = \underline{\delta}_n$ = hopping direc

= vector of 4x random hops

$$\Rightarrow \langle \dot{r}^{\mu}_n(t_0)\, \dot{r}^{\nu}_m(t_0+\tau) \rangle_{t_0} = \langle \dot{r}^{\mu}_n(t_0)\, \dot{r}^{\nu}_m(t_0) \rangle_{t_0}$$

$\frac{2dD\Delta t}{d}$ $\Delta t = 4$, $d = 2$

$$\Rightarrow \langle \dot{r}^{i\,2}_n \rangle = a^2$$

$$\langle \dot{r}^i_n \dot{r}^i_m \rangle = 0 \quad \text{if dilute}$$

avg: ~~steps~~

$\frac{2dD\Delta t}{d/2}$ (2, 2, 1, 4)

$= 8.$

$\Rightarrow$ only surviving term in random walk dynamics has no time-dependence ?!

$$\langle J^{\mu}_{\tau} J^{\nu}_{0} \rangle \propto \sum_{\ell} e^{-4\beta\lambda\ell} \binom{N_v - 1}{2\ell - 1} 2\ell$$

Actually still gives a decent $K(\tau)$ profile, albeit one that peaks way off where it should

	6-Vertex	8-Vertex
$K_{max} \sim$	(B): 0.25 – 0.3 (K): 0.15 – 0.25 (closer to 0.15 excl. noise @ peak?)	0.40 0.25 – 0.30
$C_{max} \sim$	0.25	0.25

Seem more correct.
UNLESS $N_V \to N_E$ & correct factor of 2 in δE or sth?

I expect $D \tilde{\propto} \dfrac{K_{max}}{C_{max}}$

$$\cancel{D^{(6)} = \frac{7}{6} \frac{a_{Rau}^2}{\delta t} = \frac{7}{6} \frac{1/2}{1/4} = \frac{7}{3}} \qquad \cancel{D^{(8)} = 1}$$

$$d_{Rau} \overset{!}{=} d_{hop}/\sqrt{2}$$

$$\cancel{\delta t_{Rau} = \delta t_{hop} = \delta t/4}$$

$$D^{(8)} = 1 \qquad D^{(6)}_{Rau} = \frac{7}{6} \frac{a_{Rau}^2}{\delta t} = \frac{7}{12}.$$

$$a_{Rau} = a/\sqrt{2}$$

<u>So</u> demon both is <u>universally</u> a factor of 2 too big. (not just 8-vertex)

$$K \propto \frac{1}{V} = \frac{1}{a^2 N_V}$$

$\Rightarrow$ We <u>expect</u>

$$K_{max} \approx 0.5 \, K_{max}^{(Rau)} \sim 0.15 \text{ ish!}$$

<u>Expect</u>

	(6)	(8)
$K_{max} \sim$	0.15 ~ 0.30 / 0.15	0.25 ~ 0.45 / 0.25
$C_{max} \sim$	0.25	0.25
$\Rightarrow D_{max} \sim$	$\frac{7}{12}$	1

Simulating the μcan dynamics directly $\forall\, \ell, \tau$ is prohibitive BUT we can sidestep this as follows:

We know that in each timestep we randomly pick edges & only those adjacent to e prtcls are accepted.

If $\exists\ 2\ell$ prtcls & we propose $N_E = 2N_v$ edges then on average we hit each prtcl 4 times on ~~each edg~~ adjacent edges

$\Rightarrow$ $\forall$ timesteps, iterate over prtcls $\underline{4}$ times & propose jump $\in \mathrm{Unif}(\pm\delta x, \pm\delta y)$

BUT exclude jumps to nearest neighbours.

On <u>average</u> these dynamics are the same!

<u>BUT</u> a pain to simulate still !!. (only a factor $\frac{2\ell}{2N_v}\cdot 4$ faster $= \cancel{4}\ell/N_v$)

$\ell \in \{0, N_v/2\}$

$\Rightarrow$ factor

on avg. each δ comprises $\underline{4}$ jumps per timestep.

$\Rightarrow$ <u>need</u> an approx$^{\underline{n}}$ for $\delta^{\mu}_{\Omega t(n)}\, \delta^{\nu}_{m} = \underline{\underline{\dot{r}^{\mu}_{n}(t)\, \dot{r}^{\nu}_{m}(0)}}$

i.e. the velocity autocorrelation.

$\because\ \underline{r}(t) = \underline{r}(0) + \int_0^t dt'\, \dot{\underline{r}}(t')$

General result: $D = \lim_{t\to\infty} \frac{1}{2t} \langle |\underline{r}(t) - \underline{r}(0)|^2 \rangle$

$$\Rightarrow \mu = \frac{1}{m\gamma} = \frac{D}{k_B T} = \frac{1}{k_B T}\int_0^\infty dt\, \langle \dot{\underline{r}}(t_0+t)\cdot \underline{r}(t_0)\rangle_{t_0}$$

$$\Rightarrow D = \frac{1}{2}\int_0^\infty dt\, \langle \dot{r}'(t_0+\tau)\, \dot{r}'(t_0)\rangle_{t_0}.$$

$\because$ isotropic

If Brownian motion, $m\ddot{\underline{r}} = -m\gamma\dot{\underline{r}} + \underline{\xi}(t)$

$$\langle \dot{r}^{\mu}(t_0)\,\dot{r}^{\nu}(t_0+\tau)\rangle_{t_0} = \langle \dot{r}^{\mu}(t_0)\,\dot{r}^{\nu}(t_0)\rangle_{t_0}\, e^{-\gamma t}$$

$$D = \frac{a_{hop}^2}{2d\,\delta t_{hop}} = \frac{2a^2}{d\,\delta t} = 2/d = 1$$

To get Langevin $\longleftrightarrow$ Fokker – Planck

you have to neglect the inertial term

$$\Rightarrow \quad m\gamma\,\dot{\underline{x}} = \underline{\xi}$$

We need to understand

$$r^{\mu}_{n^+} r^{\nu}_{m} = r^{\mu}_{\Omega^t(n)} r^{\nu}_{m} = r^{\mu}_{(n^+)}(t) \cdot r^{\nu}_{(m^+)}(0)$$

In <u>current</u> we end up with

$$r^{\mu}_{\Omega^{t+1}(n)} r^{\nu}_{\Omega^1(m)} + r^{\mu}_{\Omega^t(n)} r^{\nu}_{m} - r^{\mu}_{\Omega^{t+1}(n)} r^{\nu}_{m} - r^{\mu}_{\Omega^t(n)} r^{\nu}_{\Omega(m)}$$

$$= \left(r^{\mu}_{\Omega^{t+1}(n)} - r^{\mu}_{\Omega^t(n)}\right) \cdot \left(r^{\nu}_{\Omega^1(m)} - r^{\nu}_{m}\right)$$

$\delta t_{hop} = 1/4\, \delta t$ (i.e. 4 hops / timestep on avg)

$$= \delta^{\mu}_{n}(t) \cdot \delta^{\nu}_{m}(0) \qquad \text{where} \quad \underline{\delta} := \underline{r}(t + \delta t_{hop}) - \underline{r}(t)$$

i.e. $\propto$ correlation of quasiprtcl hopping direc·s. (as one might expect!)

autocorrelat^n

$$\langle J^{\mu}_{\tau} J^{\nu}_{0} \rangle = (2\lambda)^2 \frac{\Delta_0}{Z} \sum_{l=0}^{N_v/2} e^{-4\beta\lambda l} \left[\binom{N_v - 1}{2l-1} \sum_{n} \delta^{\mu}_{\Omega^t(n)} \delta^{\nu}_{n} + \binom{N_v - 2}{2l-2} \sum_{n \neq m} \delta^{\mu}_{\Omega^t(n)} \delta^{\nu}_{m} \right]$$

Should still be averaged over diff. start times.

cross-correlat^n.

Can't make more progress without specifying dynamics Ω beyond just "μcan $\Rightarrow$ l = const. on time evolution".

Decent model: for T small, l small

$\rightarrow$ ~free diffus^n on lattice.

Sum over possible start points

$$\langle J^{\mu}_{\tau} J^{\nu}_{0} \rangle \propto \sum_{l} e^{-4\beta\lambda l} \left[\binom{N_v - 2}{2l-1} \sum_{n} \delta^{\mu}_{n^+} \delta^{\nu}_{n} + \binom{N_v - 2}{2l-2} \sum_{nm} \delta^{\mu}_{n^+} \delta^{\nu}_{m} \right]$$

Start & end points fixed, $2l-1$ other prtcls free

Start & end points fixed, $2l-2$ other prtcls free.

ℓ

$$\langle J^\mu_\tau J^\nu_0 \rangle = \mathscr{L}(\tau+1, 1) + \mathscr{L}(\tau, 0) - \mathscr{L}(\tau+1, 0) - \mathscr{L}(\tau, 1)$$

$n^+ := \Omega^t(n)$

$\Omega^t =$ bijection

$\because$ μcan dynamics

$$\mathscr{L}(t,0) = \sum_{ij} r^\mu_i r^\nu_j \langle \epsilon_i(t) \epsilon_j(0) \rangle$$

$$= \sum_{ij} r^\mu_i r^\nu_j \; (2\lambda)^2 \frac{\Delta_0}{Z} \sum_{\ell=0}^{N\nu/2} \frac{1}{(2\ell)!} \underbrace{\sum_{n_1 \neq \cdots \neq n_{2\ell}}}_{\langle \underline{n} \rangle} e^{-4\beta\lambda\ell} \left(\sum_{\alpha=1}^{2\ell} \sum_{\beta=1}^{2\ell} \delta_{i=n^+_\alpha} \delta_{j=n_\beta} \right)$$

$$= (2\lambda)^2 \frac{\Delta_0}{Z} \sum_{\ell=0}^{N\nu/2} \frac{e^{-4\beta\lambda\ell}}{(2\ell)!} \sum_{\langle \underline{n} \rangle} \sum_{\alpha\beta} \sum_{ij} \delta_{i=n^+_\alpha} \delta_{j=n_\beta} \; r^\mu_i r^\nu_j$$

$$= (2\lambda)^2 \frac{\Delta_0}{Z} \sum_{\ell} \frac{e^{-4\beta\lambda\ell}}{(2\ell)!} \underbrace{\sum_{\langle \underline{n} \rangle} \sum_{\alpha\beta} r^\mu_{n^+_\alpha} r^\nu_{n_\beta}}_{?}$$

$$? = \sum_{n_1 \neq \cdots \neq n_{2\ell}} \sum_{\alpha\beta=1}^{2\ell} r^\mu_{n^+_\alpha} r^\nu_{n_\beta}$$

$$= \sum_{\alpha} \sum_{\substack{\underline{n} \setminus \{n_\alpha\} \\ \underline{n} \neq n_\alpha}} \sum_{n_\alpha} r^\mu_{n^+_\alpha} r^\nu_{n_\alpha} + \sum_{\alpha \neq \beta} \sum_{\substack{\underline{n} \setminus \{n_\alpha n_\beta\} \\ \underline{n} \neq n_\alpha, n_\beta}} \sum_{n_\alpha \neq n_\beta} r^\mu_{n^+_\alpha} r^\nu_{n_\beta}$$

$$= 2\ell \cdot \left\{ \begin{matrix} N\nu-1 \\ 2\ell-1 \end{matrix} \right\} \sum_n r^\mu_{n^+} r^\nu_n + 2\ell(2\ell-1) \cdot \left\{ \begin{matrix} N\nu-2 \\ 2\ell-2 \end{matrix} \right\} \sum_{n \neq m} r^\mu_{n^+} r^\nu_m$$

$$\Rightarrow \frac{?}{(2\ell)!} = \binom{N\nu-1}{2\ell-1} \sum_n r^\mu_{n^+} r^\nu_n + \binom{N\nu-2}{2\ell-2} \sum_{n \neq m} r^\mu_{n^+} r^\nu_m$$

$$\Rightarrow \mathscr{L}(t,0) = (2\lambda)^2 \frac{\Delta_0}{Z} \sum_\ell e^{-4\beta\lambda\ell} \left[\binom{N\nu-1}{2\ell-1} \sum_n r^\mu_{n^+} r^\nu_n + \binom{N\nu-2}{2\ell-2} \sum_{n \neq m} r^\mu_{n^+} r^\nu_m \right]$$

$$\sum_{\underline{\sigma}} e^{-\beta E(\underline{\sigma})} \quad O(\underline{\sigma})$$

$\underline{A}$ determines $\underline{\sigma}$ up to plaquette flips

$$= \sum_{\underline{A}} e^{-\beta E(\underline{A})} \quad O(\underline{A})\, \Delta_{\text{flips}}$$

$2\ell = \#\ A\text{'s} = -1$

~~requires that O depends only on number of $A = +$~~

$$= \cancel{2\sum_{\ell=0}^{N_v/2} e^{-4\beta\lambda\ell}\, O(\ell)}\ \underbrace{\Delta_{\text{flips}}\,\Delta_{\text{topo}}}_{\Delta_0}$$

$$= \Delta_0 \underbrace{\sum_{\ell=0}^{N_v/2}}_{\#A=-1} e^{-4\beta\lambda\ell} \sum_{n_1\neq\ldots\neq n_{2\ell}} \frac{1}{(2\ell)!}\, O_{n_1\ldots n_{2\ell}}$$

If $O(\sigma) = \epsilon_i(\sigma)$,

$$O_{n_1\ldots n_{2\ell}} = 2\lambda\ \underset{\delta_{i=n_1}+\delta_{i=n_2}+\ldots+\delta_{i=n_{2\ell}}}{\overset{\shortparallel}{\delta_{i\in\{n_1,\ldots,n_{2\ell}\}}}}$$

$$\Rightarrow \langle \epsilon_i \rangle_{eq} = \frac{\Delta_0}{Z}\sum_\ell e^{-4\beta\lambda\ell}\ \underbrace{\frac{1}{(2\ell)!}\cdot(2\ell)\cdot 2\lambda\cdot \begin{Bmatrix} N_v-1 \\ 2\ell-1 \end{Bmatrix}}_{2\lambda\,(2\ell/N_v)\binom{N_v}{2\ell}}$$

If $O(\sigma) = 1$,

$$O_{n_1\ldots n_{2\ell}} = 1$$

$$\Rightarrow Z = \Delta_0 \sum_\ell e^{-4\beta\lambda\ell} \binom{N_v}{2\ell}$$

If $O(\sigma) = \epsilon_i(\Omega^t(\sigma))\,\epsilon_j(\sigma)$,

$\Omega^t(n_i^-) = n_i$

$\Omega^t(n_i) := n_i^+ \quad \leadsto n_1\ \&\ n_1'$ are in bijection $\because$ μcan dynamics (no annihilat^n / creat^n).

$$O_{n_1\ldots n_{2\ell}} = (2\lambda)^2\ \delta_{i\in\{\Omega^t(n_1),\ldots,\Omega^t(n_{2\ell})\}}\ \delta_{j\in\{n_1,\ldots,n_{2\ell}\}}.$$

$$= (2\lambda)^2\left(\delta_{i=n_1'}+\ldots+\delta_{i=n_{2\ell}'}\right)\left(\delta_{j=n_1}+\ldots+\delta_{j=n_{2\ell}}\right)$$

$$\Rightarrow \sum_{n_1\neq\ldots\neq n_{2\ell}} O_{n_1\ldots n_{2\ell}} = \sum(2\lambda)^2\left(\delta_{i^-=n_1}+\ldots+\delta_{i^-=n_{2\ell}}\right)\left(\delta_{j=n_1}+\ldots+\delta_{j=n_{2\ell}}\right)$$

$$= \cancel{(2\lambda)^2\left(\ \right)}$$

$$\sum_{n_1 \neq \dots \neq n_{2l}} \sigma_{n_1 \dots n_{2l}} = \sum_{\underline{n}} (2\lambda)^2 \left(\sum_{\alpha=1}^{2l} \delta_{i^- = n_\alpha} \right) \left(\sum_{\beta=1}^{2l} \delta_{j = n_\beta} \right)$$

$$= \sum_{\underline{n}} (2\lambda)^2 \sum_{\alpha=1}^{2l} \sum_{\beta=1}^{2l} \delta_{i^- = n_\alpha} \delta_{j = n_\beta}$$

$$\cancel{= (2\lambda)^2 \sum_{\alpha\beta} \sum_{n_1 \neq \dots \neq n_{2l}} \delta_{i^- = n_\alpha} \delta_{j = n_\beta}}$$

$$\cancel{= (2\lambda)^2 \sum_{\alpha\beta} \sum_{n_1 \dots \cancel{n_\alpha} \dots \cancel{n_\beta} \dots n_{2l}}}$$

$$= (2\lambda)^2 \Bigg[\sum_{\alpha \neq \beta} \sum_{\underline{n}} \delta_{i^- = n_\alpha} \delta_{j = n_\beta}$$

$$+ \sum_{\alpha} \sum_{\underline{n}} \delta_{i^- = n_\alpha} \delta_{j = n_\alpha} \Bigg]$$

$$= (2\lambda)^2 \left[\sum_{\substack{\alpha \neq \beta}}^{\substack{j = n_\beta \\ i^- = n_\alpha}} \begin{Bmatrix} N_v - 2 \\ 2l - 2 \end{Bmatrix} \delta_{i \neq j} + \sum_{\alpha}^{i^- = j^-} \begin{Bmatrix} N_v - 1 \\ 2l - 1 \end{Bmatrix} \delta_{i = j} \right]$$

$$\qquad\qquad i^- \neq j \qquad\qquad\qquad i^- = j$$

$$\cancel{\frac{1}{(2l)!} (\,''\,) = (2\lambda)^2 \left[2l(2l-1) \begin{Bmatrix} N_v - 2 \\ 2l - 2 \end{Bmatrix} + 2l \begin{Bmatrix} N_v - 1 \\ 2l - 1 \end{Bmatrix} \right] \frac{1}{(2l)!}}$$

$$\cancel{= (2\lambda)^2 \left[2l(2l-1) \frac{(N_v-2)!}{(N_v-2l)!\,(2l)!} + 2l \frac{(N_v-1)!}{(N_v-2l)!\,(2l)!} \right]}$$

$$\cancel{= (2\lambda)^2 \left[\frac{2l(2l-1)}{N_v(N_v-1)} \binom{N_v}{2l} + \frac{2l}{N_v} \binom{N_v}{2l} \right]}$$

$$\cancel{= (2\lambda)^2 \binom{N_v}{2l} \left[\frac{2l}{N_v} \left(\frac{2l-1}{N_v-1} \right) + 1 \right]}$$

$$\frac{1}{(2l)!} \sum_{\underline{n}} \sigma_{\underline{n}} = (2\lambda)^2 \binom{N_v}{2l} \frac{2l}{N_v} \left[\left(\frac{2l-1}{N_v-1} \right) \delta_{i^- \neq j} + \delta_{i^- = j} \right].$$

$$= (2\lambda)^2 \binom{N_v}{2l} \frac{2l}{N_v} \Bigg[\underbrace{\left(\frac{2l-1}{N_v-1} \right)}_{A_l} + \underbrace{\left(1 - \frac{2l-1}{N_v-1} \right)}_{1 - A_l} \delta_{\substack{i^- = j \\ i^+ = j^+}} \Bigg]$$

$$\sum_{ij} \alpha_{ij} \langle \epsilon_i(t) \epsilon_j(0) \rangle_{eq}$$

$$= 4\lambda^2 \frac{\Delta_0}{Z} \sum_\ell e^{-4\beta\lambda\ell} \binom{N_v}{2\ell} \frac{2\ell}{N_v} \sum_{i j^+} \alpha_{i\Omega^{-t}(j^+)} \left(A_\ell + (1-A_\ell)\,\delta_{i=j^+} \right)$$

$j \sim j^+$ in bisection

$$= 4\lambda^2 \frac{\Delta_0}{Z} \sum_\ell e^{-4\beta\lambda\ell} \binom{N_v}{2\ell} \frac{2\ell}{N_v} \Big[A_\ell \sum_{ij} \alpha_{ij} + \underbrace{(1-A_\ell)}_{B_\ell} \sum_i \underbrace{\alpha_{i\Omega^{-t}(i)}}_{\alpha_{i i^-}} $$

$$B_\ell = \frac{N_v}{N_v-1}\binom{N_v-1}{2\ell-2}, \qquad \alpha_{ii^-} = \alpha_{j^+ j}$$

$$= \mathcal{X}(\tau, 0)$$

As before,

$$\langle J^\mu_\tau J^\nu_0 \rangle_{eq} = \cancel{2}\,\mathcal{X}(\tau+1, 1) + \mathcal{X}(\tau, 0) - \mathcal{X}(\tau+1, 0) - \mathcal{X}(\tau, 1)$$

Peq trans-inv-

$$= 2\mathcal{X}(\tau, 0) - \mathcal{X}(\tau+1, 0) - \mathcal{X}(\tau-1, 0)$$

$$= \underbrace{4\lambda^2 \frac{\Delta_0}{Z} \sum_\ell e^{-4\beta\lambda\ell} \binom{N_v}{2\ell} \frac{2\ell}{N_v} (1-B_\ell)}_{\mathcal{L}_\ell} \sum_j$$

$\alpha_{\tau+1,1} + \alpha_{\tau,0} - \alpha_{\tau+1,0} - \alpha_{\tau,1}$

finite difference.

$$= \mathcal{L}_\ell \sum_{j=1}^{2\ell} \dot r^\mu_j(\tau)\, \dot r^\nu_j(0)$$

exact.

$$= \mathcal{L}_\ell \sum_{j=1}^{2\ell} r^\mu_j(\tau+1)\left[r^\nu_j(1) - r^\nu_j(0) \right] - r^\mu_j(\tau)\left[r^\nu_j(1) - r^\nu_j(0) \right]$$

I need a model of Ω for our dynamics to get a good average.

$$= \mathcal{L}_\ell \sum_{j=1}^{2\ell} \delta^\mu_j(\tau)\, \delta^\nu_j(0)$$

BUT the random $\underline{\delta}$ we propose @ each timestep $\delta t/4$ are random! ⇒ uncorrelated

Only part which <u>isn't</u> random is when prtcls are neighbours – perhaps the diluteness assumption was too much?

Problem: If I neglect e prtcl "interactⁿs" then I get 0 ∵ motion uncorrelated @ diff. times.

Does $\ell/N\nu$ ~ macroscopic fix this?

Toric code has

$$Z = \Delta_0 \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\beta\lambda l} \qquad \Delta_0 = 2^{N_p - 1} \cdot 2^{b_1''^{\,2g}}$$

$$C_v \approx \left(\frac{4\lambda}{T}\right)^2 \frac{\binom{N_v}{2} e^{4\lambda/T}}{\left(\binom{N_v}{2} + e^{4\lambda/T}\right)^2}$$

for the $l=0,1$ approx$^{\underline{n}}$
($\beta\lambda \gg 1 \Rightarrow$ low temp.)

$\Rightarrow$ per spin, $C_v^{(1)} \approx \frac{1}{N_E} C_v = \frac{1}{2N_v} C_v$

does ~~peak @ $T \approx \lambda$ ... $C_v \approx 2.3$~~

$$Z/\Delta_0 = \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\beta\lambda l}$$

$k_B = 1$

$\beta = \frac{1}{\cancel{k_B} T}$

$$F = -\frac{1}{\beta} \ln Z$$

$$dF = -S\,dT - \cancel{p\,dV}^{\to 0} + \dots$$

$$S = -\frac{\partial}{\partial T}\left(\frac{1}{\beta}\ln Z\right) =$$

$$\frac{d}{dT} = \frac{d\beta}{dT}\frac{d}{d\beta} = -\frac{1}{T^2}\frac{d}{d\beta} = -\beta^2\, d/d\beta$$

$$= \beta^2 \frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\ln Z\right)$$

$$\cancel{= \beta \ln Z \mp}$$

$$= \beta \frac{1}{Z}\frac{\partial Z}{\partial\beta} - \ln Z$$

$$C = T\frac{\partial S}{\partial T} = -\beta\frac{\partial S}{\partial\beta} = +\beta \cancel{\frac{1}{Z}} \frac{\partial Z}{\partial\beta} - \beta\cancel{\frac{1}{Z}}\frac{\partial Z}{\partial\beta}$$

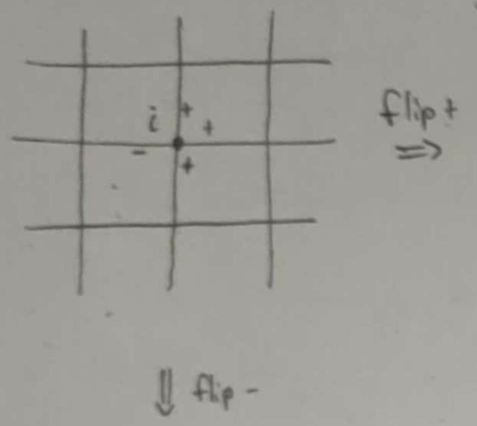
$$-\beta^2\frac{1}{Z^2}\left(\frac{\partial Z}{\partial\beta}\right)^2 - \beta\frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2}$$

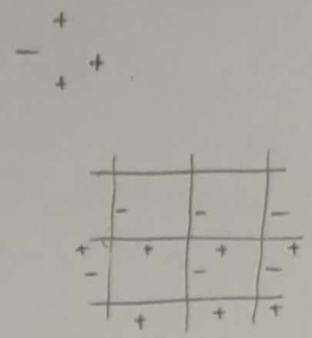
$$\frac{\partial Z}{\partial\beta} = \sum_{l=0}^{N_v/2} \binom{N_v}{2l}\cdot(-4\lambda l)\, e^{-4\beta\lambda l}$$

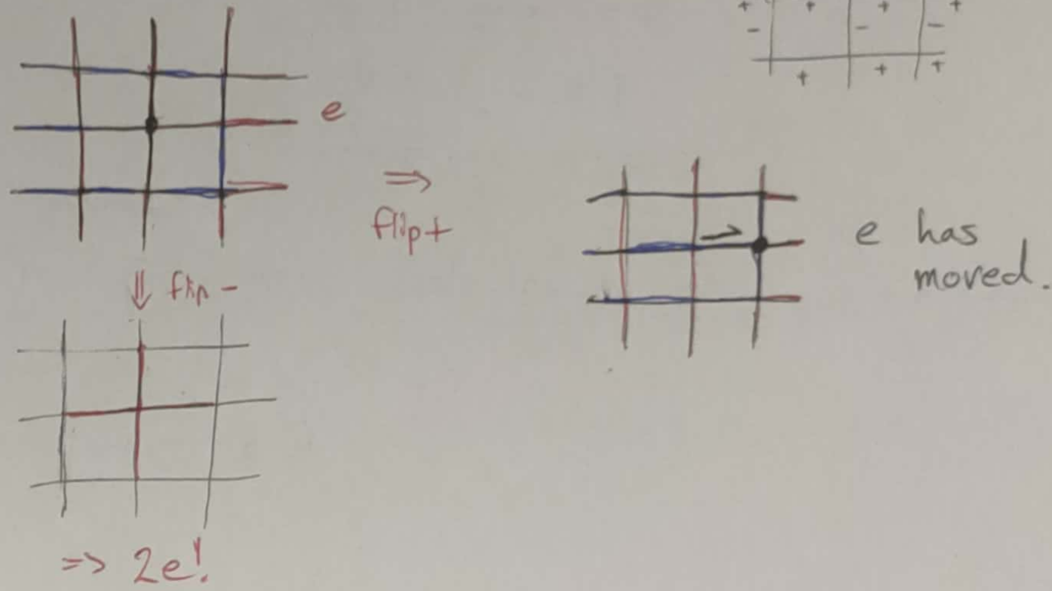
Consider the following:

$$\tilde{A}_j = \Big(\sum_{\alpha \in \partial^+ j} \sigma_\alpha\Big)^2 \in \left\{ \begin{matrix} 0 \\ 4 \\ 16 \end{matrix} \right.$$

$$\tilde{A}_i = 4$$

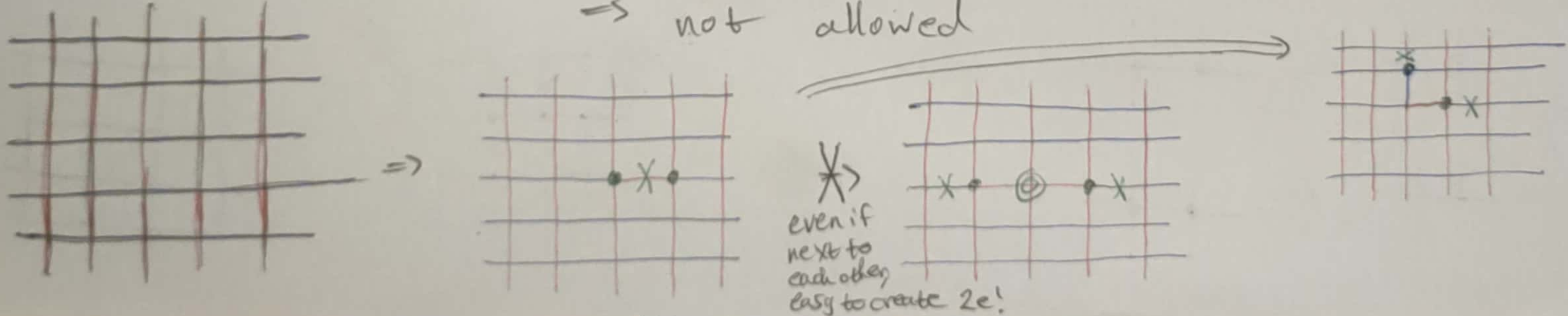


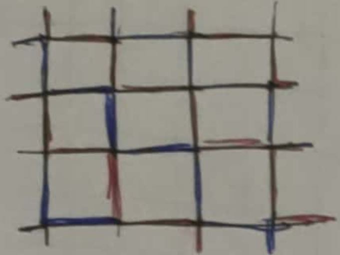


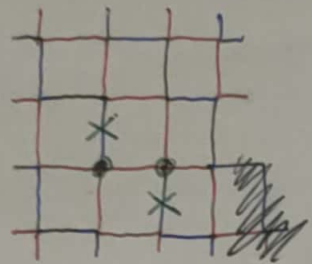


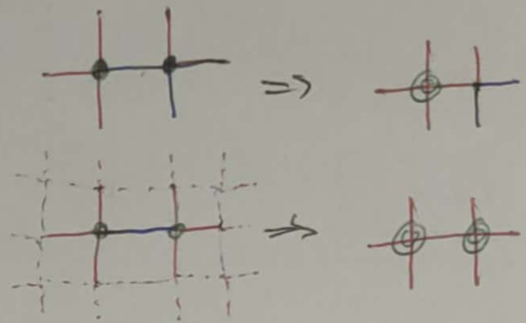
Key point: For an e excitat$^{\bar{n}}$ $\exists$ always a spin for which flipping it would create a $2e$ excitat$^{\bar{n}}$

$\Rightarrow$ not allowed









So the 6-vertex dynamics are diffusive e monopoles with one disallowed direction (depends <u>entirely</u> on bckgd GS config).

No consistent fusion rules.

I called it 2e but we can have e.g.

$$e+e \xrightarrow[]{\overset{\mathrm{sgn}(\Delta E)}{+}} e+e+2e$$

$$e+e \xrightarrow{-} 1$$

$$e+e \xrightarrow{+} 2e$$

$$e+e \xrightarrow{+} 2e+2e$$

$$\left.\begin{array}{l} 2e \not\longrightarrow 1 \\ 1 \not\longrightarrow 2e \end{array}\right\} \text{have to go via e+e states}$$

$$1 \xrightarrow{+} e+e$$

$$e+e \xrightarrow{0} e+e$$

$\Rightarrow$ a <u>lot</u> more complicated than 8-vertex, where:

$$e+e \xrightarrow{0} e+e$$

$$e+e \xrightarrow{-} 1$$

$$1 \xrightarrow{+} e+e$$

6-vertex

$$1 \underset{-}{\overset{+}{\rightleftharpoons}} e \cdot e$$

$$1 \not\rightleftharpoons 2e$$

$$e \circ e \underset{-}{\overset{+}{\rightleftharpoons}} e \cdot e \circ 2e$$

$$\underset{-}{\overset{+}{\rightleftharpoons}} 2e$$

$$\underset{-}{\overset{+}{\rightleftharpoons}} 2e \circ 2e$$

$$\cancel{e \underset{0}{\overset{0}{\rightleftharpoons}} e}$$

$$\cancel{2e \circ e \underset{0}{\overset{0}{\rightleftharpoons}} 2e \circ e}$$

e's can move @ 0 cost
~~except~~ in 1 dir where $e \rightarrow e \ne 2e$

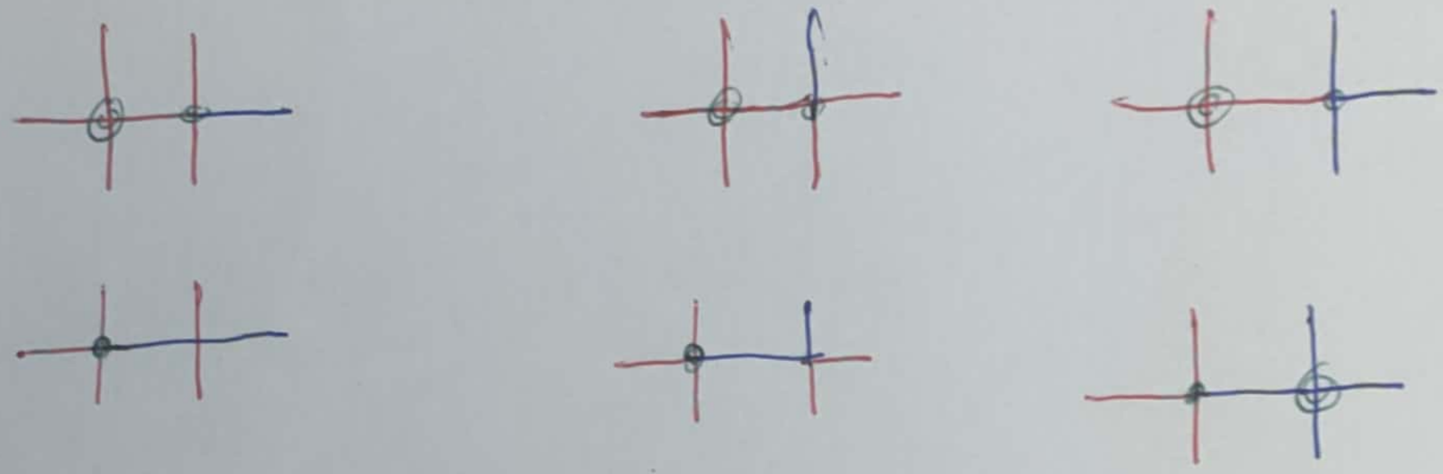
2e's require a neighbouring e to swap with and in a certain unique config.

8-Vertex

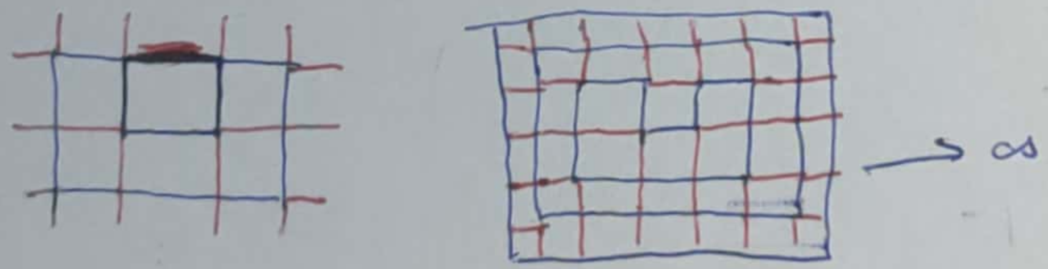
$$1 \underset{-}{\overset{+}{\rightleftharpoons}} e \cdot e$$

$$\cancel{e \underset{0}{\overset{0}{\rightleftharpoons}} e}$$

e's can move @ 0 cost



e/2e – facilitated hopping.



Toric code → transverse Ising mapping?

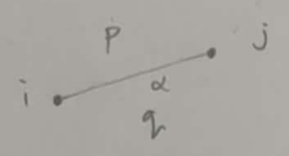
$$-\beta\hat{H} = \Lambda \sum_i \hat{A}_i + H \sum_\alpha \hat{Z}_\alpha \qquad A_i = \prod_{\alpha \in \partial^+ i} \hat{Z}_\alpha.$$

Define ref. state $\sigma_\alpha = +1 \ \forall \alpha$.

Then define $\hat{W}_p = (-1)^{n_p}$ → eigenvalues. w/. n_p = # times plaquette flipped.

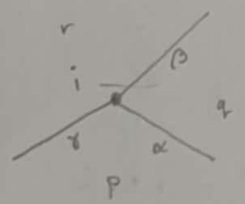
→ Within each topo sector, any config. can be written in terms of n_p by:

$\sigma_\alpha = +1$ initially



$$W_p W_q = (-1)^{n_p + n_q} = (-1)^{n_\alpha} = Z_\alpha$$

X Y Z
U V W



$$Z_\alpha Z_\beta Z_\gamma = W_p W_q W_q W_r W_r W_p$$

$$v_i = (-1)^{n_i}$$

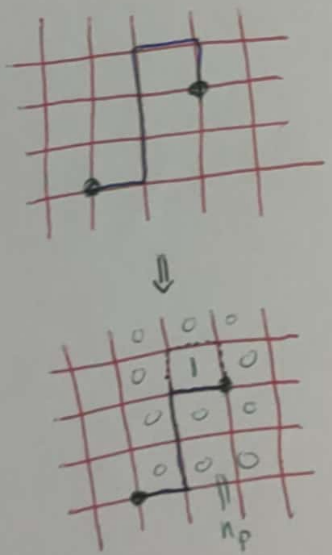
$$Z = \sum_{\{\sigma_i = \pm 1\}} e^{A \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta + B \sum_\alpha \sigma_\alpha}$$

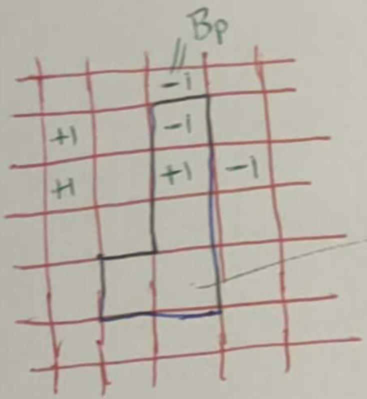
Define $G_{ij} = A$ for α, β sharing a vertex.
& zero otherwise.

Then $$Z \propto \int \left(\prod_i d\varphi_i\right) e^{-\frac{1}{2}\sum_{\alpha\beta} \varphi_\alpha G^{-1}_{\alpha\beta} \varphi_\beta} \, e^{\sum_\alpha \ln\cosh(\varphi_\alpha + H)}$$

$\Rightarrow$ Landau-Ginzburg Hamiltonian on dual lattice.
defined à la Rau et al.

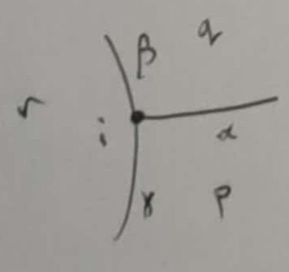
$\Rightarrow$ Same critical behaviour.





all $n_p = 1$ here

$$\sigma_\alpha{}^{\not{t}} = \prod_{p \in \partial^+\alpha} \mu_p \, \sigma_\alpha{}^{ref}$$



$$A_i = \prod_{\alpha \in \partial^+ i} \sigma_\alpha$$

$$= \prod_{\alpha \in \partial^+ i} \prod_{p \in \partial^+\alpha} \underbrace{\mu_p}_{(-1)^{n_p}} \sigma_\alpha{}^{ref}$$

$$= \underbrace{\mu_p \mu_q \mu_q \mu_r \mu_r \mu_p}_{1} \; \sigma_\alpha{}^{ref\,2} \, \sigma_\beta{}^{ref\,2} \, \sigma_\gamma{}^{ref\,2} \to 1$$

$$= 1$$

Okay. $-\beta H = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha + \underbrace{J \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta}_{?}$ $\quad J = \beta j$

$$A_i = \prod_{\alpha \in \partial^+ i} \sigma_\alpha$$

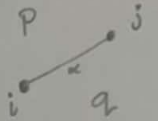
Key point: In GS, all $\sigma_\alpha = +1$ ($\Rightarrow$ all $A_i = +1$)

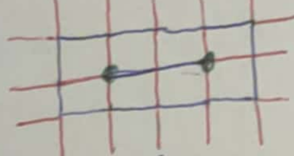
We can explore the $A_i = +1\ \forall i$ states by applying plaquette flips only (excl. other topo sectors).

$\Rightarrow$ Valid to define B_p variables $= (-1)^{n_p}$

$\to$ plaquette flip operator.

$\Rightarrow \sigma_\alpha = B_p B_q$



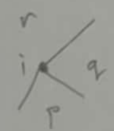


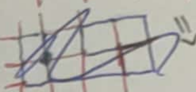
$A_i = (-1)^{n_i}$?

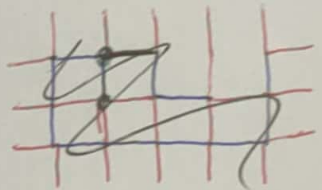
$\sigma_\alpha = A_i A_j$.

$$A_i = \prod_{\alpha \in \partial^+ i} \sigma_\alpha$$

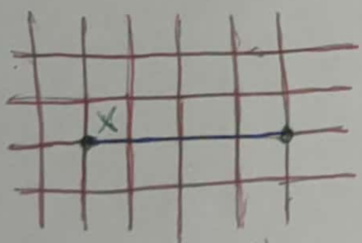
$$= B_p B_q B_q B_r B_r B_p = 1$$





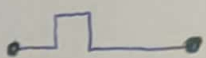


For each ℓ $\exists$ $\binom{N_v}{2\ell}$ deg. for pos^{n}s of prtcls



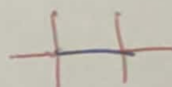
2h

$\Rightarrow$



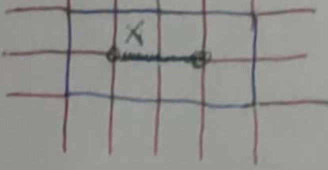
$\Delta E_H = \pm 4h$

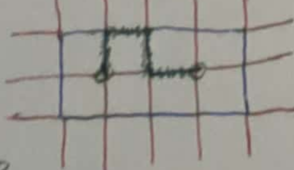
$-2h$



In each ℓ state we can move the strings with plaquette flips.

Two key observat^{n}s: * Plaquette flips <u>do</u> change energy now, but <u>don't</u> change <u>or</u> move e prtcls!





$\Delta E_H = 0$ ✓: $2 \times r$, $2 \times b$.

not

Perturbed Toric Code.

$X Y Z \to U V W$

* x - field $\quad \underline{h} = (g, 0, 0)$

We have $[\hat{B}_p, \hat{H}] = 0 \quad \Rightarrow$ look @ GS w/. $b_p = +1 \ \forall p$

$$\Rightarrow \hat{H} = -\lambda \sum_i \hat{A}_i - g \sum_\alpha \hat{X}_\alpha$$

Let $\hat{V}_i = \hat{A}_i \quad \Rightarrow \quad \hat{X}_\alpha = \prod_{i \in \partial\alpha} \hat{V}_i := \tilde{B}_{\tilde{p}}$

$$\Rightarrow \hat{H} = -\lambda \sum_i \hat{V}_i - g \sum_\alpha \prod_{i \in \partial\alpha} \hat{V}_i$$

$\Rightarrow$ || field Ising model

* z - field $\quad \underline{h} = (0, 0, h)$

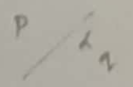
We have $[\hat{A}_i, \hat{H}] = 0 \quad \Rightarrow$ GS has $a_i = +1 \ \forall i$

$$\Rightarrow \hat{H} = -\lambda \sum_p B_p - h \sum_\alpha \hat{Z}_\alpha .$$

Let $\hat{W}_p = \hat{B}_p \quad \Rightarrow \quad \hat{Z}_\alpha = \prod_{p \in \partial^+\alpha} \hat{W}_p$

$$\Rightarrow \hat{H} = -\lambda \sum_p \hat{W}_p - h \sum_\alpha \prod_{p \in \partial^+\alpha} \hat{W}_p .$$

In 2D, $\quad \hat{X}_\alpha = \hat{V}_i \hat{V}_j \qquad \hat{Z}_\alpha = \hat{W}_p \hat{W}_q .$



Okay. $-\beta H = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha + \underbrace{J \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta}_{?}$ $\quad J = \beta j$

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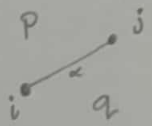
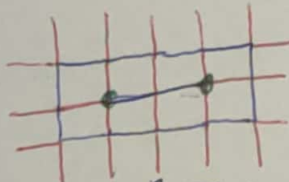
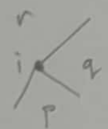


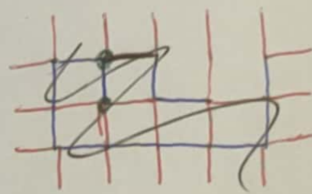


$A_i = (-1)^{n_i}$?

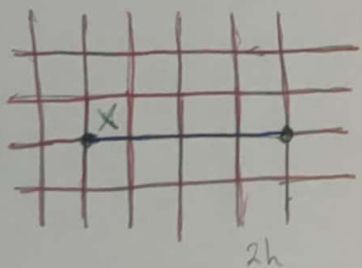
$\sigma_\alpha = A_i A_j$.



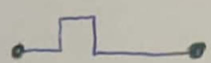
$$A_i = \prod_{\alpha \in \partial^+ i} \sigma_\alpha = B_p B_q B_q B_r B_r B_p = 1$$



For each ℓ $\exists$ $\binom{N_v}{2\ell}$ deg. for pos$^{\text{ns}}$ of prtcls

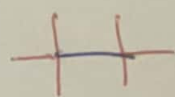


$\Rightarrow$



$\Delta E_H = \pm 4h$

$-2h$



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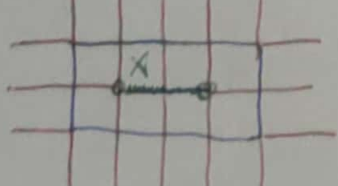
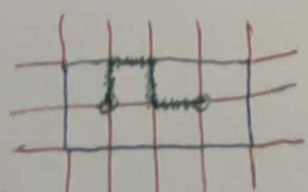




$\Delta E_H = 0 \vee$; $2\times r$, $2\times b$.

$$\Rightarrow -\beta H = \Big[2\Lambda\sigma^{z-1}(1-\overbrace{(z-1)}^{\mathcal{O}(\delta\sigma^2)\text{ bit.}}) + H\Big]\sum_\alpha \sigma_\alpha$$

$$+ \Big[\tfrac{1}{2}\Lambda\sigma^{z-2}\Big]\sum_i\sum_{\langle\alpha\beta\rangle}\sigma_\alpha\sigma_\beta \quad + \text{const.} + \mathcal{O}(\delta\sigma^3)$$

$\because \sigma_\alpha^2 = 1 \quad \Rightarrow \quad \sum_{\alpha\neq\beta}\sigma_\alpha\sigma_\beta = \sum_{\alpha\beta}\sigma_\alpha\sigma_\beta - z$

Where we used that:

$$\sum_i\sum_{\alpha\in\partial^+ i}\sigma_\alpha = 2\sum_\alpha\sigma_\alpha$$

$$\sum_i\sum_{\substack{\alpha\in\partial^+ i\\ \beta\in\partial^+ i}}\sigma_\alpha\sigma_\beta = \sum_{\langle\alpha\beta\rangle}\sigma_\alpha\sigma_\beta$$

n.n. means α & β share a site.

In MFT we just keep $\mathcal{O}(\delta\sigma)$

$\Rightarrow H_{eff} = [2\Lambda\sigma^{z-1} + H]$

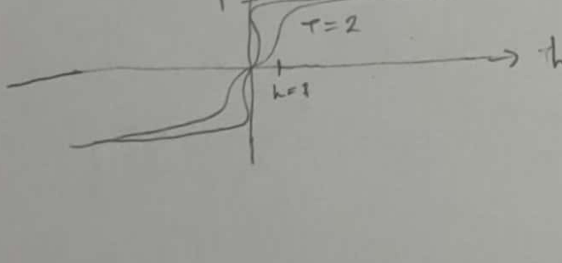
$\Downarrow \ldots$ $\qquad\qquad m = \sum_\alpha \sigma_\alpha / N_E$

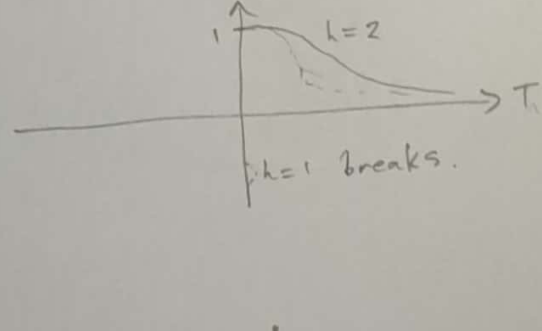
$\qquad\qquad\qquad\qquad = \sigma$

$m = \tanh(H_{eff})$

On square lattice, $z = 4$

$\Rightarrow \quad m = \tanh(2\Lambda m^3 + H) \qquad \lambda = 1$





$$Z = \sum_{\underline{\sigma}} e^{+\Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha}.$$

$\beta\lambda = \Lambda$
$\beta h = H$

Hypothesis: $\sigma_\alpha = \bar{\sigma} + \delta\sigma_\alpha$

$$\Rightarrow \quad A_i = \prod_{\alpha\in\partial^+ i} \sigma_\alpha = \prod_{\alpha\in\partial^+ i} (\bar{\sigma} + \delta\sigma_\alpha)$$

$$= \bar{\sigma}^z + \bar{\sigma}^{z-1} \sum_{\alpha\in\partial^+ i} \delta\sigma_\alpha + \bar{\sigma}^{z-2} \sum_{\alpha\neq\beta} \delta\sigma_\alpha \delta\sigma_\beta + \dots$$

$$\sum_i \sum_{\alpha\in\partial^+ i} = 2\sum_\alpha$$

$$\Rightarrow \quad Z \approx \sum e^{(2\Lambda\bar{\sigma}^{z-1} + H)\sum_\alpha \delta\sigma_\alpha}$$

$$\sigma_\alpha = \sigma + \delta\sigma_\alpha \qquad (\sigma + \delta\sigma_\alpha)^4$$

$$\Rightarrow \quad A_i = \sum_{n=0}^{z} \sigma^{z-n} \sum_{\alpha_1\neq\dots\neq\alpha_n} \frac{1}{n!} \delta\sigma_{\alpha_1}\dots\delta\sigma_{\alpha_n}$$ (If $\delta\sigma_\alpha = y$, reduces to $\binom{z}{n} y^n$)

$$= \sum_{n=0}^{z} \sigma^{z-n} \left(\sum_\alpha \delta\sigma_\alpha + \frac{1}{2} \sum_{\alpha\neq\beta} \delta\sigma_\alpha \delta\sigma_\beta + \dots \right)$$

$$\approx \sigma^{z-1} \sum_\alpha (\sigma_\alpha - \sigma) + \frac{\sigma^{z-2}}{2} \sum_{\alpha\neq\beta} (\sigma_\alpha - \sigma)(\dots)_\beta + \dots)$$

$$\approx \sigma^{z-1}\left(\sum_\alpha \sigma_\alpha - z\sigma\right) + \frac{\sigma^{z-2}}{2} \sum_{\alpha\neq\beta} \left(\sigma_\alpha\sigma_\beta - \sigma(\sigma_\alpha + \sigma_\beta)\right) + \binom{z}{2}\sigma^2$$

$$\approx \underbrace{\sigma^z\left(-\binom{z}{1} + \binom{z}{2}\right)}_{\substack{\text{if } n\to\infty \text{ this} \\ \text{is } 0 \Rightarrow \text{self-} \\ \text{consistent}}} \xrightarrow{} \text{const.} + \sigma^{z-1}\sum_\alpha \sigma_\alpha + \frac{\sigma^{z-2}}{2} \sum_{\alpha\neq\beta} \sigma_\alpha\sigma_\beta - \frac{\sigma^{z-1}}{2}(z-1)2\sum_\alpha \sigma_\alpha$$

$$\approx \text{const.} + \sigma^{z-1}\left(1 - (z-1)\right)\sum_\alpha \sigma_\alpha + \frac{1}{2}\sigma^{z-2} \sum_{\alpha\neq\beta} \sigma_\alpha\sigma_\beta$$

$\Rightarrow$ deg. of each m, n is $|\mathbb{Z}|$ as one might guess
$\therefore$ overcomplete basis. $\{H, |\Lambda + H|, |\Lambda - H|\}$.

$$\Rightarrow \quad Z = \sum_{m,n} \cancel{\Delta_{mn}}^{\,|\mathbb{Z}|} e^{-\beta(m\Lambda + nH)}$$

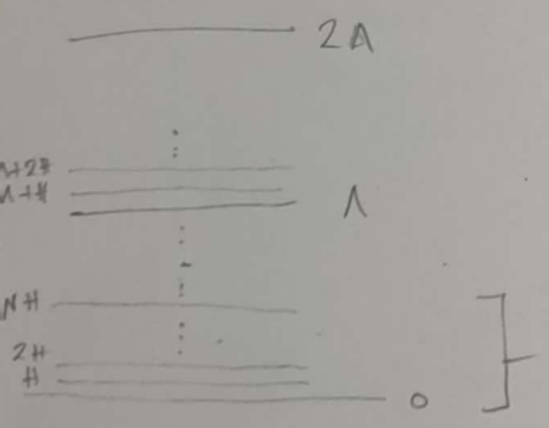
$$\propto \sum_{m,n} e^{-\beta(m\Lambda + nH)}$$

Good!

Worth checking: Using $\langle \Lambda, H \rangle$ as generators is okay!

$\Rightarrow$ Previous result is correct.

<u>Now</u> the problem reduces to:



When taking limit, I want all levels here to collapse onto 0 level.

For that to be true, I demand that $\underline{\underline{NH \longrightarrow 0}}$.

$\Rightarrow$ $H \longrightarrow 0$ faster than $N \longrightarrow \infty$.

$$D = \begin{cases} \langle 0, \Lambda \rangle \\ \langle H, \Lambda + H, |\Lambda - H| \rangle \end{cases}$$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & \\ 2 & 2 & & \end{array}$$

$$Z(H=0) = \sum_{\ell m n} e^{-\beta(\ell H + m(L+H) + n|L-H|)}$$

$$\downarrow p = n+m$$

$$\to (N+1) \sum_{p} e^{-\beta p L} (p+1)$$

Implies $(p+1)$ - deg of each level for infinitesimal H.

We're clearly counting wrong - can we not form a basis using just two of the generators? (over counting)

$n,m \geq 0$ $\qquad$ $p,q \not\geq 0$ → perfectly fine to take energy out as long as $D \geq 0$ in total.

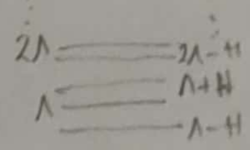
$$nH + m\Lambda = pH + q(H+\Lambda) \quad \checkmark \text{ works}$$

e.g. $n=0$, $m=1$

$$= rH + s|H-\Lambda| \quad \checkmark \text{ works}$$

$$= t(H+\Lambda) + s|H-\Lambda| \quad \checkmark \text{ works.}$$

Yep! The problem is as $H \to 0$ get continuum of states when we actually want only Λ levels.



Fix: set $H = \Lambda$?

Would fix level spacing but would need to correct

Our problem is that the basis

$\{\pm H, \pm(\Lambda+H), \pm|\Lambda-H|\}$ is overcomplete

<u>and</u> we need to allow −ve coeffs.

Let's try to work out the degeneracy of $m\Lambda + nH$

$$m\Lambda + nH = \begin{cases} m(H+\Lambda) + (n-m)H \\ \mp m|\Lambda-H| + (n+m)H \\ \frac{n+m}{2}(H+\Lambda) \pm \frac{n-m}{2}|H-\Lambda| \end{cases}$$

Key point: $\forall\, m,n$ $\exists$ a unique decomposition into any two basis elements.

Problem is, we have three!

$$m\Lambda + nH = pH + q(\Lambda+H) \pm r(\Lambda-H)$$
$$= (p+q\mp r)H + (q\pm r)\Lambda$$

$$\Rightarrow \begin{array}{l} m = q \pm r \\ n = q + p \mp r \end{array} \quad \Rightarrow \quad \begin{array}{l} m+n = 2q+p \\ (\pm r = m-q) \end{array}$$

Remember $m,n \in \mathbb{Z}_{\geq 0}$ & $p,q \in \mathbb{Z}$

$\Rightarrow$ countably ∞ soln.s!

$$m\Lambda + nH = (m+n-2q)H + q(\Lambda+H) \pm (m-q)|\Lambda-H|$$

e.g. $(m,n) = (0,0) = -2qH + q(\Lambda+H) \mp q(\Lambda-H)$
$= 0 \;\; \forall\, q \in \mathbb{Z}$

$(m,n) = (0,1) = (1-2q)H + q(\Lambda+H) - q(\Lambda-H) = H \;\; \forall\, q \in \mathbb{Z}$.

Demons

$$\Delta E^{(\beta)} = \Delta E^{(\beta)}_{h=0} + 2h\,\sigma_\beta$$

$$\Rightarrow \begin{cases} 8 \text{ vertex}: & \{0, \pm 4\lambda\} \pm 2h \\ 6 \text{ vertex}: & \{0, \pm \underbrace{8\lambda}_{A}\} \pm \underbrace{2h}_{B} \end{cases}$$

$$\Rightarrow \Delta E \in \{\pm B, \pm A \pm B\}.$$

$$= \{\pm B, \pm(A+B), \pm(A-B)\}$$

$$\Rightarrow D = \langle |B|, |A+B|, |A-B| \rangle \to \langle A, B, C \rangle$$

$$A = \delta E_h$$
$$B = |\delta E_h + \delta E_\lambda|$$
$$C = |\delta E_h - \delta E_\lambda|$$

Actually,

$$Z = \sum_{lmn} e^{-\beta(lA + mB + nC)}$$

So we have $Z = Z_{\Lambda+H} \cdot Z_{|\Lambda-H|} \cdot Z_H$

$$\langle D \rangle = \langle D_{\delta E_h} \rangle + \langle D_{\delta E_h + \delta E_\lambda} \rangle + \langle D_{|\delta E_h - \delta E_\lambda|} \rangle$$

$$= \text{1 terms} + \text{2 terms}.$$

$$\delta E_h = H$$
$$\delta E_\lambda = \Lambda$$

① terms $= \dfrac{H}{e^{\beta H}-1} + \dfrac{\Lambda + H}{e^{\beta(\Lambda+H)}-1} + \dfrac{|\Lambda - H|}{e^{\beta|\Lambda-H|}-1}$

$\Lambda > H$
$\Lambda < H$

$$Z = \sum_{\ell m n} e^{-\beta(\ell H + m(\Lambda+H) \pm n(\Lambda - H))}$$

$$= \sum_{\ell m n}{}' \, e^{-\beta((\ell + m \mp n)H + (m \pm n)\Lambda)}$$

$H \quad \Lambda + H \quad \Lambda - H / H - \Lambda$

We were correct, it's quantised in H, Λ

$\Rightarrow \quad Z = \sum_{mn} e^{-\beta(mH + n\Lambda)}$

$\Lambda > H$

① $H, \quad \Lambda = (\Lambda + H) - H$
$\qquad\qquad = (\Lambda - H) + H$

$\Lambda < H$

② $H, \quad \Lambda = (\Lambda + H) - H$

$\Rightarrow$ in either case, can write

$$\alpha H + \beta\Lambda = \gamma H + \delta(\Lambda + H) + \varepsilon|H - \Lambda|$$

Do we care about degeneracy though? Yes!

$\Rightarrow$ better to do things directly as above...

Generalising the lattice

* PBCs?
* Defining dimensions $\underline{L}$
* PBCs in relative displacement (divide by abs if $\underline{> 1 ?}$)

bad ∵ disallows $\alpha \neq 1$ ⇒ maybe $> L$?

~~Okay. If $\frac{h}{\lambda} \in \mathbb{Q}$, we can write~~

$$Z = \sum_{n=0}^{\infty} e^{-\beta n \underbrace{\min(\delta E_\lambda, \delta E_h)}_{\gamma}}$$

$$\Rightarrow \langle D \rangle = \frac{\gamma}{e^{\beta\gamma} - 1}$$

$$Z \langle D \rangle = \sum_{n\,m}^{N\,N} (n \underbrace{\delta E_\lambda}_{A} + m \underbrace{\delta E_h}_{B}) e^{-\beta(n\delta E_\lambda + m\delta E_h)}$$

$$= -\frac{\partial}{\partial\beta} Z$$

BUT $Z_B = \frac{1 - e^{-\beta B(N+1)}}{1 - e^{-\beta B}}$ iff $\beta B > 0$ i.e. sum converges.

I $\underline{\text{know}}$ $\beta A > 0$

$\Rightarrow$ can write $Z = Z_A \sum_m^N e^{-\beta m A}$

$$\rightarrow \langle D \rangle = -\frac{\partial}{\partial\beta} \ln(Z) = \underbrace{-\frac{\partial}{\partial\beta} \ln(Z_A)}_{\langle D \rangle_A} - \frac{\partial}{\partial\beta} \ln\left(\sum_m^N e^{-\beta m B}\right)$$

$$\Rightarrow \langle D \rangle = \langle D \rangle_A + \langle D \rangle_B$$

$$\langle D_A \rangle = \overset{0 \text{ as } M\to\infty}{\cancel{\frac{-MA}{e^{\beta MA}-1}}} + \frac{A}{e^{\beta A}-1}$$

$$\langle D_B \rangle = \begin{cases} \overset{\to 0\,?}{\cancel{\frac{-MB}{e^{\beta MB}-1}}} + \underset{\downarrow \lim B\to 0}{\frac{B}{e^{\beta B}-1}} & B>0 \\ 0 & B=0 \end{cases}$$

Key point: When we take $\lim_{M\to\infty}$ we've assumed

$A>0 \Rightarrow$ fine

BUT $B=0$?

Important limit is in $\underline{\underline{BM}}$

Let's try $B\to 0$ & $M \to \infty$ BUT with $BM \to 0$ also. Works better than $BM \to \infty$ according to Desmos.

In our code, $B \in [0, 2h \sim 10)$

$N \to \infty$?

$\rightarrow$ $B \ll M$ always BUT $BM = 0$ @ $B=0$

& $BM = \infty$ elsewhere

$$\Rightarrow \langle D_B \rangle = \begin{cases} \frac{B}{e^{\beta B}-1} - MB\overset{\to 0}{\cancel{e^{-\beta MB}}} & B>0 \\ \overset{\to \frac{1}{\beta}}{\cancel{\frac{B}{e^{\beta B}-1}}} - \frac{1}{\beta} = 0 & B=0 \end{cases}$$

$M = N+1$

$$Z = \begin{cases} M Z_A \\ Z_B Z_A \end{cases}$$

$$Z_A = \frac{1 - e^{-\beta M A}}{1 - e^{-\beta A}}$$

$$Z' = \begin{cases} M Z_A' \\ Z_A' Z_B + Z_A Z_B' \end{cases}$$

$$\Rightarrow \frac{Z'}{Z} = \begin{cases} Z_A'/Z_A \\ Z_A'/Z_A + Z_B'/Z_B . \end{cases}$$

$$e^{-\beta A} = \alpha$$

$$-\frac{Z_A'}{Z_A} = \frac{\cancel{(1-\alpha)}}{(1-\alpha^M)} \frac{MA\alpha^M(1-\alpha) + A\alpha(1-\alpha^M)}{(1-\alpha)^{\cancel{2}}}$$

$$= AM \frac{\alpha^M}{1-\alpha^M} + A \frac{\alpha}{1-\alpha}$$

$$= A\left(\frac{M\alpha^M}{1-\alpha^M} + \frac{\alpha}{1-\alpha}\right) = \langle D \rangle$$

$$\Rightarrow \langle D \rangle = A\left(\frac{M e^{-\beta M A}}{1 - e^{-\beta M A}} + \frac{e^{-\beta A}}{1 - e^{-\beta A}}\right)$$

$$+ B(\ldots)_B$$

$$= A\left(\frac{M}{e^{\beta M A} - 1} + \frac{1}{e^{\beta A} - 1}\right)$$

$$+ B(\ldots)_B$$

It seems like the order of $\lim_{B \to 0}$ and $\lim_{M \to \infty}$ matters?

$$B(\dots)_B \longrightarrow B\left(\frac{M}{1+\beta MB-1} + \frac{1}{1+\beta B-1}\right)$$

$$= B\left(\frac{2}{\beta B}\right) = \frac{2}{\beta} \; ?$$

It seems like for $B \to 0^+$ we get

$$\lim_{M\to\infty}\lim_{B\to 0}\langle D\rangle = A\left(\cancelto{M\to\infty}{\frac{M}{e^{\beta MA}-1}} + \frac{1}{e^{\beta A}-1}\right) + 2T$$

$$= \frac{A}{e^{\beta A}-1} + 2T$$

$$\lim_{B\to 0}\lim_{M\to\infty}\langle D\rangle = \frac{A}{e^{\beta A}-1} + T$$

Either way, we get this offset $\propto T$???

I'd have expected continuity but maybe that was silly...

It's the latter that Desmos prefers...

Very strange ∴ it's extensive → good

BUT $\lim_{B\to\infty}\langle D_{A,B}\rangle = \langle D_A\rangle$

I.e. you have to take the field to infinity, i.e. no excitations in $Z_B = \sum_m e^{-\beta B m}$ to recover the desired form.

I.e. take $\delta E_h = 2h \to \infty$???

Time to add an h-field to our sim!

3 key changes:

* ΔE_flip ⇝ trivial

* Demon energies ⇝ now quantised in units ~~of δE and~~

of δE_λ and δE_h

$\delta E_\lambda \sim \cancel{2\lambda(A_i + A_8)} \sim \begin{cases} 8\lambda & \text{6-vertex} \\ 4\lambda & \text{6-vertex} \\ 2\lambda & \text{Ferro} \end{cases}$

$\therefore \Delta E_H = 2\lambda \sum_{\text{next}} \hat{z}_i A$

actually depends on z_i

e.g. if $z=2$, $\Delta E_H = 0, 4\lambda$

$\delta E_h \sim 2h\sigma \sim 2h$

↳ Anyway this means D should be a float and also we need to modify $\langle D(T) \rangle$.

$\frac{1}{1-e^{-\beta\varepsilon}}$

β

For $h=0$, $\langle D \rangle = ?$ $\quad Z_D = \sum_{n=0}^{\infty} e^{-n\delta E_\lambda / T}$

$\langle D \rangle Z_D = -\frac{1}{\beta}\frac{\partial Z_D}{\partial \beta}$.

$$\text{Now, } Z_D = \sum_{n,m} e^{-\beta(n\delta E_\lambda + m\delta E_h)}$$

$$= \frac{1}{1-e^{-\beta\delta E_\lambda}} \frac{1}{1-e^{-\beta\delta E_h}}$$

$$\Rightarrow \beta Z_D \langle D \rangle = -\frac{\beta\delta E_\lambda e^{-\beta\delta E_\lambda}}{(\ldots)_\lambda^2}\frac{1}{(\ldots)_h} - \frac{\beta\delta E_h e^{-\beta E_h}}{(\ldots)_h^2}\frac{1}{(\ldots)_\lambda}$$

$$\Rightarrow \langle D \rangle = \frac{\delta E_\lambda}{e^{\beta\delta E_\lambda}-1} + \frac{\delta E_h}{e^{\beta\delta E_h}-1}$$

$\delta E_\lambda = A$

$\delta E_h = B$

$$= \frac{A(e^{\beta B}-1) + B(e^{\beta A}-1)}{(e^{\beta B}-1)(e^{\beta A}-1)}$$

$$De^{\beta(A+B)} + D - De^{\beta A} - De^{\beta B} = Ae^{\beta B} - A + Be^{\beta A} - B$$

$$\Rightarrow \quad D e^{\beta(A+B)} - (D+B)e^{\beta A} - (D+A)e^{\beta B} + (D+A+B) = 0$$

Take

$$a e^{(A+B)x} - b e^{Ax} - c e^{Bx} + d = 0$$

If $h=0$, $B=0$ $\Rightarrow$ $\cancel{\Delta B}$ ~~$D+A = DPA$~~

Just going to have to solve numerically "n".
for ~~ΔPX~~.
$T(\langle D \rangle)$

* Change temp. estimat^{n} in Demon Kubo also.

$$a \alpha^x \beta^x - b \alpha^x - c \beta^x + d = 0$$

$$A = \delta E_x \qquad B = \delta E_h$$

Mean-Field Approach

$$H = -\lambda \sum_i A_i - h \sum_\alpha \sigma_\alpha$$

Let $\sigma_\alpha = \sigma + \delta_\alpha$

$$\Rightarrow \sum_i A_i = \sum_i \prod_{\alpha \in \partial^+ i} (\sigma + \delta_\alpha)$$

$$(\sigma + \delta_1)(\sigma + \delta_2)\dots(\sigma + \delta_4) = \sigma^3(\delta_1 + \delta_2 + \delta_3 + \delta_4)$$

$$\approx \sum_i \left(\sigma^{z_i - 1} \Big(\sum_{\alpha \in \partial^+ i} \delta_\alpha \Big) + O(\delta^2) + \overset{\sigma^4}{\text{const.}} \right)$$

$$\approx N_v \sigma^4 + \sum_i \sigma^{z_i - 1} \Big(\sum_{\alpha \in \partial^+ i} (\sigma_\alpha - \sigma) \Big)$$

$$= N_v \sigma^4 + \sum_i \sigma^{z_i - 1} \Big(\sum_{\alpha \in \partial^+ i} \sigma_\alpha - z_i \sigma \Big)$$

$$= N_v \sigma^4 - \sum_i z_i \sigma^{z_i} + \sum_i \sigma^{z_i - 1} \sum_{\alpha \in \partial^+ i} \sigma_\alpha$$

Suppose $z_i = z \ \forall i$ (makes sense for mean-field).

$$\sum_i A_i = N_v \sigma^4 - \cancel{z} N_v \sigma^z + \dots$$

$$= N_v (\sigma^4 - z\sigma^z) + \sigma^{z-1} \underbrace{\sum_i \sum_{\alpha \in \partial^+ i} \sigma_\alpha}_{2 \sum_\alpha \sigma_\alpha}$$

$$\Rightarrow H = -\lambda N_v (\sigma^4 - z\sigma^z) - 2\lambda \sigma^{z-1} \sum_\alpha \sigma_\alpha - h \sum_\alpha \sigma_\alpha$$

$$h_{eff} = h + 2\lambda \sigma^{z-1}$$ ~~if z...~~

~~If $\lambda \gg h > 0$, $\sigma \to 1 \Rightarrow h_{eff} =$~~

$$h_{eff} = h + 2\lambda \sigma^{z-1}$$

$$\Rightarrow \quad H_{eff} = -h_{eff} \sum_{\alpha} \sigma_\alpha$$

$$\Rightarrow \quad Z = \prod_{\beta=1}^{N_E} \left(\sum_{\sigma_\beta = \pm 1} e^{\cancel{+}\beta h_{eff} \sum_\alpha \sigma_\alpha} \right)$$

$$= \left(2\cosh(\beta h_{eff})\right)^{N_E}$$

$\therefore$ ignoring constant.
(fine for observable σ)

$$\sigma = \cancel{} \langle \sigma_i \rangle = \frac{1}{N_E \beta} \frac{\partial \ln Z}{\partial h_{eff}}$$

$$= \tanh(\beta h_{eff})$$

$$= \tanh(\beta h + 2\sigma^{z-1}\beta\lambda)$$

$\tanh(x) \approx x - \dfrac{x^3}{3}$ for x small

$$\Rightarrow \quad \sigma \approx \beta h + 2\beta\lambda\sigma^{z-1} \cancel{+} - \cancel{\tfrac{1}{3}(\beta h)^3} - \cancel{\tfrac{1}{3}(\dots)^3}$$

Not needed unlike Ising case where $h_{eff} = \beta h + \beta z J \sigma$

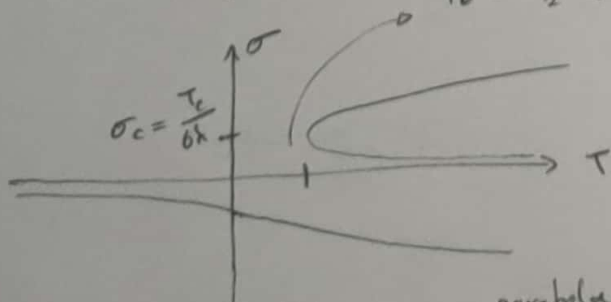
$\Downarrow$

Take $z = 4$

$$\Rightarrow \quad \sigma \approx (\beta h) + 2(\beta\lambda)\sigma^3$$

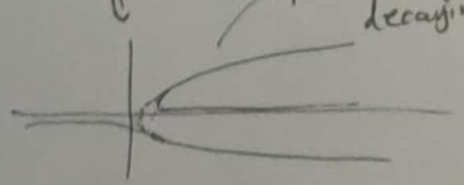
for λh = const,

$T_c = \frac{3h}{2}\sqrt[3]{\frac{4\lambda}{h}}$

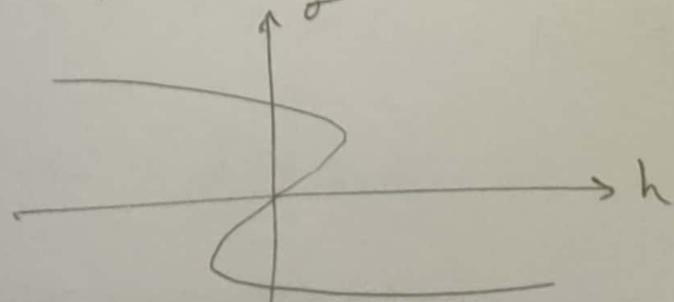


$\Downarrow$

parabola + decaying soln.



for T = const,



Maxwell constructⁿ required?

We further need to take $\sigma < 1$ & $\beta\lambda, \beta h < 1$ for our expans^{n}

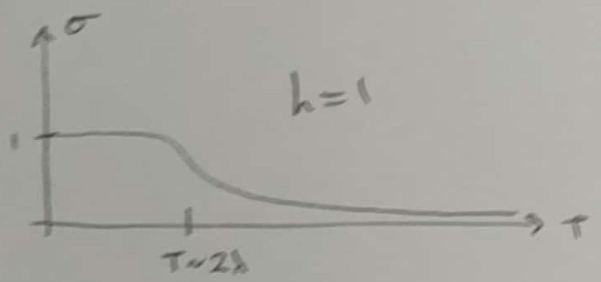
=> <u>Most</u> realistic is to throw away soln.s for large σ

=> take decaying soln?

Recall total energy $= -h_{eff} N_E \sigma$

<u>x small invalid</u>

Numerically plot soln: $(z=4)$



$\because N_E = 2N_V$

We still require $2h \gtrsim \lambda$ for MFT to work (i.e. h term wins) $N_E h >> N_V \lambda$.

More generally to get T_c find inflect^{n} point.

$\Rightarrow \sigma''(T) = 0 \Rightarrow \cancel{\sigma''(\beta) = 0}$

$\beta = \frac{1}{T} \Rightarrow \cancel{\frac{\partial \beta}{\partial T} = -\beta^2}$

$$\cancel{\sigma''(\beta) = \frac{\partial\beta}{\partial T}\left(\frac{\partial}{\partial\beta}\left(\frac{\partial\beta}{\partial T}\right)\frac{\partial}{\partial\beta}\sigma\right)}$$

$$\cancel{= \beta^2 \frac{\partial}{\partial\beta}\left(\beta^2 \frac{\partial}{\partial\beta}\sigma\right)}$$

$$\cancel{\frac{\partial\sigma}{\partial\beta} = h_{eff}\,\mathrm{sech}^2(\beta h_{eff})}$$

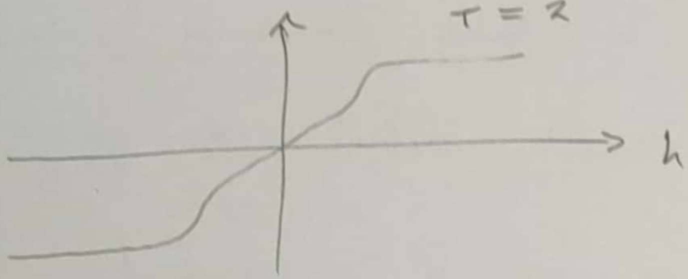
$$\cancel{\frac{\partial}{\partial\beta}(\beta^2\,'') = 2\beta h_{eff}\,\mathrm{sech}^2(\ldots) - 2\beta^2 h_{eff}^2 \tanh(\ldots)\,\mathrm{sech}^2(\ldots) = 0}$$

$$\cancel{\text{iff. } (1 - \beta h_{eff} \tanh(\beta h_{eff})) = 0.}$$

Wolfram! :)

$$\sigma_0'' = s^2\left(2h\beta^3 + \frac{6\sigma^2}{\beta}\sigma_0'' + 12\sigma^2\sigma' + \frac{12}{\beta}\sigma\sigma'^2\right) \quad t = \tanh$$
$$- 2\left(-h\beta^2 + \frac{6\sigma^2}{\beta}\sigma' + 2\sigma^3\right) t s .$$

<u>Nope</u>.



Better approach.

$\tanh(\beta h + 2\sigma^{z-1}\beta\lambda)$

I.e. in bulk, $\sigma^{ref} = +1$

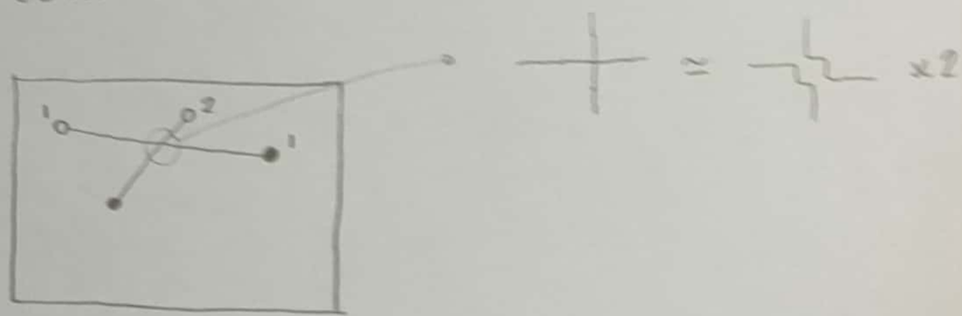
$\Rightarrow$ $\mu_p \mu_q$ <u>ferro</u> magnetic $\Rightarrow$ no loops or v. large loops to $\downarrow$ energy cost.

Near a string in ref state, $\sigma^{ref} = -1$

$\Rightarrow$ $\mu_p \mu_q$ <u>anti</u>ferro $\Rightarrow$ flip only on one side to move string

So effectively we have a ferromagnet w/. fixed domain wall defects.

First problem: Define the ref config. given $2l$ prtcls (labelled with which should be connected).



In continuum limit, if sufficiently local graph $\rightarrow \mathbb{R}^d$ with non-flat metric.

$\Downarrow$ all cells id. $\begin{cases} A_p = A \ \forall A \\ a_\alpha = a \ \forall \alpha \end{cases}$

Hypothesis: if $z_i = z \ \forall i$, the graph metric is <u>flat</u> i.e. Euclidean.

$\Downarrow$

$|\cdot|_2$ norm makes sense?

(On a generic graph the minimal paths are non-unique and non-trivial (e.g. need Dijkstra)

On a <u>cubic</u> graph,

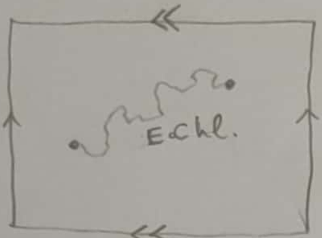
same # -ve spins
$\Rightarrow$ same energy.

We need $N_{steps} < |R|_1 / a$ $\Rightarrow$ L_1 norm.

On e.g. another local graph, it's $L_2 < l < L_1$

I want to define auxiliary spins à la $T=0$ case such that I can track the σ_α for a given l-state
(also remaining in one topo sector $\Rightarrow$ additional 2^{β_1}).

For each l, what is the reference state?
Pick all $\sigma_\alpha = +1$ except for $2l$ sites where $A_i = -1$



$$\sigma_\alpha = \Big(\prod_{p\in\partial^+\alpha} \mu_p\Big)\, \sigma_\alpha^{ref}$$

w/. $\mu_p = (-1)^{n_p}$.

& $A_i = \prod_{\alpha\in\partial^+ i} \sigma_\alpha = A_i^{ref}$. $\because \mu_p^2 = 1$

I'm really struggling to see how I can define anything that won't depend on the prtcl positions

$\Rightarrow$ Within each l, the Hamiltonian is $\qquad Z = \sum_{\underline{\sigma}} e^{-\beta F[\underline{\sigma}]}$

$$\begin{aligned} -\beta F &= \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha \\ &= \Lambda \sum_i A_i^{ref} + H \sum_\alpha \sigma_\alpha^{ref} \prod_{p\in\partial^+\alpha} \mu_p \\ &= \Lambda (N_v - 4l) + H \sum_{\langle pq\rangle} \mu_p \mu_q \, \sigma^{ref}_{\langle pq\rangle} \end{aligned}$$

$\Downarrow$
ferro/anti-" depending on σ ref

<u>Now</u> we can do mean-field, i.e. take ref state as straight line strings & so $\mu_p \approx \mu + \delta\mu_p$ so not many isolated loops / wiggly strings.
I.e. bias towards nearby plaquettes flipping to create v. large loops (minimise perim/area ratio)

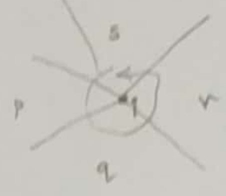
$$-\beta F = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha$$

Now let $\mu_p = (-1)^{n_p}$

$$\Rightarrow \quad \sigma_\alpha = \Big(\prod_{p\in\partial^+\alpha}\mu_p\Big)\sigma_\alpha^{ref}$$

$$A_i = \prod_{\alpha\in\partial^+ i}\sigma_\alpha = \prod_{\alpha\in\partial^+ i}\Big(\Big(\prod_{p\in\partial^+\alpha}\mu_p\Big)\sigma_\alpha^{ref}\Big)$$

$$= \prod_{\alpha\in\partial^+ i}\sigma_\alpha^{ref} = \underline{A_i^{ref}}$$



$$\Rightarrow -\beta F = \Lambda(N_v - 4l) + H\sum_{\langle pq\rangle}\sigma^{ref}_{\langle pq\rangle}\mu_p\mu_q$$

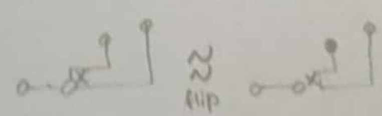
for a ref. state w/. $2l$ e prtcls.

On square lattice, define ref. state for any two prtcls:

$\alpha\ \beta\ \gamma$
$\nu\ \mu\ \eta$

 = 

Uh oh...



So instead use:

 $\not\simeq$ 

which has the same issue

Need a way to pick ℓ non-edge sharing strings given the endpoints. Choice of pairing doesn't matter!

$\Rightarrow$ fine to use shortest (linear) path, understanding that we're interested in the 1-norm.

Can do MFT in μ_p to bias towards aligned neighbouring plaquettes i.e. large or no loops within each ℓ-sector.

MFT in σ is a bit more direct & demands $H \gg \Lambda$

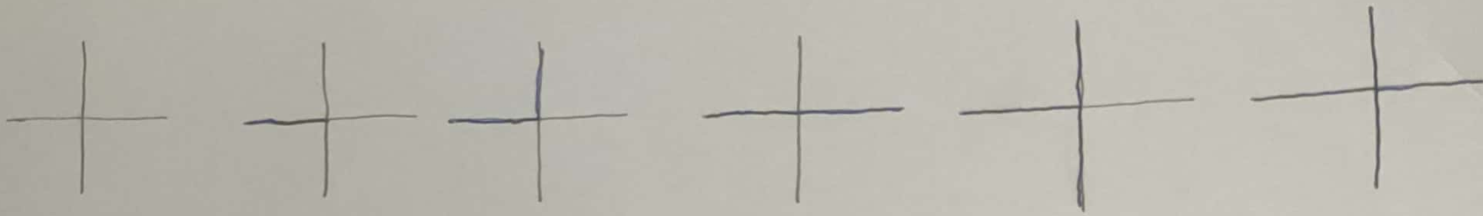
Let's do MFT in μ: $\mu_p = \bar{\mu} + \delta\mu_p$

$$\Rightarrow \mu_p \mu_q \approx \bar{\mu}^2 + \bar{\mu}(\delta\mu_p + \delta\mu_q) + O(\delta\mu^2)$$
$$= -\bar{\mu}^2 + \bar{\mu}(\mu_p + \mu_q) + O(\delta\mu^2)$$

$$\Rightarrow -\beta F = \Lambda(N_v - 4\ell) + H\bar{\mu} \sum_{\langle pq \rangle} \sigma^{ref}_{\langle pq \rangle}(\mu_p + \mu_q)$$
$$= \Lambda(N_v - 4\ell) + H\bar{\mu}\frac{1}{2}\sum_p \sum_{q \in nn(p)} \sigma^{ref}_{\langle pq \rangle}(\mu_p + \mu_q)$$
$$= \Lambda(N_v - 4\ell) + H\bar{\mu}\sum_p \mu_p \sum_{\alpha \in \partial p} \sigma^{ref}_\alpha$$

$$H_{eff}(p) = \bar{\mu} H \underbrace{\sum_{\alpha \in \partial p} \sigma^{ref}_\alpha}_{\sim |\partial p|}$$

Vertices :



Δ	1	4	4	2	4	1
2Λ	0	1	0	0	1	0
2H	0	1	2	2	3	4

BUT this is silly!

We know a spin can either be flipped / unflipped

$\Rightarrow D_\lambda \in \{0, \underset{\delta E_h}{\overset{\shortparallel}{2h}}\}$

Whereas the λ part depends on configs of neighbouring sites $\rightarrow$ in principle

D_λ ~~...~~ $= n\,\delta E_\lambda$.

$$\Rightarrow Z = \sum_{n=0}^{\infty} \sum_{m=0}^{1} e^{-\beta \delta E_\lambda n} e^{-\beta \delta E_h m}$$

$$= Z_\lambda \left(1 + e^{-\beta \delta E_h}\right)$$

$$\Rightarrow \langle D \rangle = \langle D_\lambda \rangle - \frac{1}{1+e^{-\beta H}} \frac{\partial}{\partial \beta} \left(1 + e^{-\beta H}\right)$$

$$= \langle D_\lambda \rangle + \frac{H}{e^{\beta H} + 1}$$

$$\underline{\underline{C}} = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & \searrow & \searrow & & \searrow \\ \vdots & & & & \\ c_{n-2} & & \searrow & \searrow & \\ c_{n-1} & \searrow & \cdots & \searrow & \searrow \end{pmatrix}$$ is diagonalised by: $\underline{v}_j = (1, \omega^j, \omega^{2j}, \ldots \omega^{(n-1)j})$, $j \in \{0, \ldots, n-1\}$.

& $\lambda_j = c_0 + c_{n-1}\omega^j + c_{n-2}\omega^{2j} + \ldots + c_1 \omega^{(n-1)j}$

For us, $\underline{c_i = c_{n-i}}$ $\quad c$ $\quad$ $\omega^m = e^{2\pi i m/N} = e^{i\eta_m}$

$\Rightarrow$ $\underline{v}_j = (1, \omega^j, \ldots)$ & $\lambda_j = c_0 + c_1\omega^j + c_2\omega^{2j} + \ldots + c_2\omega^{(n-2)j} + c_1\omega^{(n-1)j}$

$= c_0 + 2c_1 \omega^{nj/2} \cos(\omega^j) + \ldots$

$= c_0 + 2\sum_{m=1}^{(n-1)/2} c_m \cos(\eta_m j)$

(Take n odd so don't need $c_{n/2}\,\omega^{j(n/2)}$ term).

λ_j^L = still annoying.

Write $c_1 = 1$, others 0 $\Rightarrow$ $\lambda_j^L = 2^L c_1^L \cos^L(\eta_1 j)$

$$\Rightarrow \underline{\underline{\Lambda}}^L = \begin{pmatrix} \lambda_0^L & & \\ & \lambda_1^L & \\ & & \ddots \end{pmatrix}$$

& $\underline{\underline{C}}^L = \underline{\underline{U}}\,\underline{\underline{\Lambda}}^L\,\underline{\underline{U}}^+$

$$= \frac{1}{n}\begin{pmatrix} \underline{v}_0 & \underline{v}_1 & \cdots & \underline{v}_{n-1} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}\begin{pmatrix} \lambda_0^L & & \\ & \lambda_1^L & \\ & & \ddots \\ & & & \lambda_{n-1}^L \end{pmatrix}\begin{pmatrix} \underline{v}_0^* \longrightarrow \\ \underline{v}_1^* \longrightarrow \\ \vdots \\ \underline{v}_{n-1}^* \longrightarrow \end{pmatrix}$$

$$= \frac{1}{n}\begin{pmatrix} & '' & \end{pmatrix}\begin{pmatrix} \lambda_0^L \underline{v}_0^* \longrightarrow \\ \lambda_1^L \underline{v}_1^* \longrightarrow \\ \vdots \\ \lambda_{n-1}^L \underline{v}_{n-1}^* \longrightarrow \end{pmatrix} = \frac{1}{n}\begin{pmatrix} 1 & 1 & & \omega^{n-1} \\ 1 & \omega & & \\ & & \ddots & \\ 1 & \omega^{n-1} & & \omega^{(n-1)^2} \end{pmatrix}\begin{pmatrix} \lambda_0^L(1 & 1 & \cdots & 1) \\ \lambda_1^L(1 & \bar{\omega} & \cdots & \bar{\omega}) \\ \lambda_{n-1}^L(1 & \bar{\omega}^{n-1} & \cdots &) \end{pmatrix}$$

$$= \frac{1}{n}\begin{pmatrix} & ew \end{pmatrix}$$

$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & & 1 \\ 1 & \omega & & \omega^{n-1} \\ & & & \\ 1 & \omega^{n-1} & & \omega^{(n-1)^2} \end{pmatrix} \begin{pmatrix} \lambda_0^L (1 & 1 & & 1) \\ \lambda_1^L (1 & \bar\omega & & \bar\omega^{n-1}) \\ & & & \\ \lambda_{n-1}^L (1 & \bar\omega^{n-1} & & \bar\omega^{(n-1)^2}) \end{pmatrix}$$

$$= \frac{1}{n} \begin{pmatrix} \sum \lambda_m^L & \sum \lambda_m^L \bar\omega^m & \sum \lambda_m^L \bar\omega^{2m} & \cdots \\ \sum \lambda_m^L \omega^m & \sum \lambda_m^L \omega\bar\omega^{m} & \sum \lambda_m^L & \\ & & \ddots & \end{pmatrix}$$

$0 \to n-1$ indexing

$$(\underline{\underline{T}}^L)_{jk} = \frac{1}{n} \sum_m \lambda_m^L (\bar\omega^j \omega^k)^m \qquad m = 0 \ldots n-1$$

$$\omega = e^{2\pi i/n}$$

$$= \frac{1}{n} \sum_{m} e^{\frac{2\pi i m}{n}(k-j)} \left(c_0 + 2 \sum_{m} c_m \cos\left(\frac{2\pi m}{n} j\right)\right)$$

Take only $c_1 = 1$

$$\Rightarrow (\text{''}) = \frac{2^L}{n} c_1^L \sum_{\ell} e^{\frac{2\pi i m}{n}(k-j)} \cos^L\left(\frac{2\pi m}{n} j\right)$$

$$= (2c_1)^L \frac{1}{2\pi} \int d\eta \; e^{i\eta(k-j)} \cos^L(\eta j)$$

We want to calculate:

$$\sum_{\tau=0}^{\tau_{max}} \langle A_{t+\tau} A_t \rangle_t$$

If $\tau_{max} = t_{max} - 1$,
$LHS = \sum_{t=1}^{t_{max}} \left(\sum_{\tau=0}^{t_{max}-1} \cdots \right)$
i.e. sums then commute!
(sort of)

$$= \sum_{\tau=0}^{\tau_{max}} \frac{1}{t_{max} - \tau} \sum_{t=1}^{t_{max}-\tau} A_{t+\tau} A_t$$

$$= \sum_{t=1}^{t_{max}} \sum_{\tau=0}^{\min\{\tau_{max}, t_{max}-t\}} \frac{1}{t_{max} - \tau} A_{t+\tau} A_t \neq t_{max} \left\langle \sum_{\tau=0}^{\min\{\tau_{max}, t_{max}-t\}} \frac{A_{t+\tau} A_t}{t_{max} - \tau} \right\rangle_t$$

Okay... but this isn't true when bootstrapping!

For bootstrap,

$$\langle A_{t+\tau} A_t \rangle_t = ?$$

Fix N_{bps} & W

$m \in \{1, \ldots, M = \lfloor N/W \rfloor\}$

We split $\underline{A}$ up into blocks of size W ($\underline{A}_m$)

$\Rightarrow$ calculate $f(\underline{A}_m) = f_m \quad \forall m \in \{1, \ldots, M\}$

$\Rightarrow$ Calculate $f_{\langle n \rangle} = \frac{1}{M} \sum_{m}^{M} f_{rand(m)} \quad \forall n \in \{1, \ldots, N_{bps}\}$

i.e. avg. over M random f_m (w/. replacement) N_{bps} times.
(should these be reversed here?)

Then $\bar{f} = \frac{1}{N_{bps}} \sum_{n}^{N_{bps}} f_{\langle n \rangle}$

$$\sigma_{\bar{f}}^2 = \frac{1}{N_{bps} - 1} \sum_{n}^{N_{bps}} (f_{\langle n \rangle} - \bar{f})^2$$

Mean-Field:

$$-\beta F = \Lambda \sum_i A_i + \Gamma \sum_i \tilde{A}_i + H \sum_\alpha \sigma_\alpha$$

Let $\sigma_\alpha = \sigma + \delta\sigma_\alpha$

$$\Rightarrow \quad A_i = \prod_{\alpha \in \partial^+ i} (\sigma + \delta\sigma_\alpha) = \sigma^{z_i} + \sigma^{z_i - 1} \sum_{\alpha \in \partial^+ i} \delta\sigma_\alpha + \sigma^{z_i-2} \sum_{\substack{\alpha \neq \beta \\ \alpha\beta \in \partial^+ i}} \delta\sigma_\alpha \delta\sigma_\beta + O(\delta\sigma^3)$$

$$\approx \sigma^{z_i}(1 - z_i + z_i(z_i - 1)) + \sigma^{z_i-1}(1-(z_i-1)) \cdot \sum_\alpha \sigma_\alpha + \sigma^{z_i-2} \sum_{\alpha \neq \beta} \sigma_\alpha \sigma_\beta$$

$$\tilde{A}_i = z_i + \sum_{\substack{\alpha \neq \beta \\ \alpha\beta \in \partial^+ i}} \sigma_\alpha \sigma_\beta \qquad \text{exactly to } O(\delta\sigma^2)$$

$$\Rightarrow -\beta F^{(2)} = \text{const.} + \left(\sigma^{z-1}(1-(z-1))\Lambda\right) \overset{2\sum_\alpha}{\cancel{\sum_i \sum_{\alpha \in \partial^+ i}}} \sigma_\alpha + H \sum_\alpha \sigma_\alpha + \left(\sigma^{z-2}\Lambda + \Gamma\right) \sum_i \sum_{\substack{\alpha \neq \beta \\ \alpha\beta \in \partial^+ i}} \sigma_\alpha \sigma_\beta + O(\delta\sigma^3)$$

$$= O(1) + O(\delta\sigma^3) + \left[2\Lambda\sigma^{z-1}(1-(z-1)) + H\right] \sum_\alpha \sigma_\alpha + \left[\Lambda\sigma^{z-2} + \Gamma\right] \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta$$

This is to <u>second</u> order.

To <u>first</u> order, $-\beta F^{(1)} = O(1) + O(\delta\sigma^2) + \left[2\Lambda\sigma^{z-1} + 2\Gamma\sigma(z-1) + H\right] \sum_\alpha \sigma_\alpha$

$$\Rightarrow -\beta F^{(2)} = -\beta F^{(1)} + \left[\Lambda\sigma^{z-2} + \Gamma\right]\left[\sum_{\alpha \neq \beta} \sigma_\alpha \sigma_\beta - 2\sigma(z-1)\sum_\alpha \sigma_\alpha\right.$$

Let's minimise $F^{(1)}$ $\quad z = 4$ for 2d square lattice

$$\beta F^{(1)} \propto 2\Lambda\sigma^{z} + 2\Gamma\sigma^2(z-1) + H\sigma$$

$$\Rightarrow \frac{\partial F^{(1)}}{\partial \sigma} = 0 \Rightarrow 0 = H + 8\Lambda\sigma^3 + 4\Gamma\sigma(z-1)$$

$g = \Gamma/\beta$

gives $\bar{\sigma}(\lambda, g, h)$

Just a quick consideration, what is

$$H = -\lambda \sum_i A_i - J \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta$$ as on Rau's lattice!

$$= -\lambda \sum_i A_i - \frac{J}{2} \sum_i \tilde{A}_i .$$

$\Gamma = \beta J/2$

Very easy! For ϵ_i & j_{ij} will just have superposⁿ of two cases!.

$\tilde{A}_i = (\sum_{\alpha\in\partial^+ i} \sigma_\alpha)^2$

Proof on general graph:

$$\sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta \doteq \sum_i \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\in\partial^+ i}} \sigma_\alpha \sigma_\beta = \sum_i (\tilde{A}_i - z_i)$$
$$= \sum_i \tilde{A}_i - 2N_E$$

$$\tilde{A}_i = \Big(\sum_{\alpha\in\partial^+ i} \sigma_\alpha\Big)^2 = \sum_{\alpha\in\partial^+ i} \sum_{\beta\in\partial^+ i} \sigma_\alpha \sigma_\beta$$
$$= \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\in\partial^+ i}} \sigma_\alpha \sigma_\beta + z_i$$

const. $\alpha=\beta$ term

$\Rightarrow$ 6-vertex actually quite general & useful ∵ adds magnetic Ising term to 8-vertex!

(In μ_p language, $\sigma_\alpha = \prod_{p\in\partial^+\alpha} \mu_p \sigma_\alpha^{ref}$

$\Rightarrow \sigma_{\langle\alpha} \sigma_{\beta\rangle} = \sigma_\alpha^{ref} \sigma_\beta^{ref} \cdot (\mu_p \mu_q \mu_s \mu_t$ $\Rightarrow$ bit icky)

Suppose we've fixed σ^{ref}.

Then $$Z = 2^{\beta_1} \sum_{l=0}^{N_V/2} \frac{e^{-4\Lambda l}}{(2l)!} \sum_{\underbrace{x_1 \neq y_1 \neq \dots \neq x_l \neq y_l}_{\{x,y\}}} \mathcal{Z}(\{x,y\})$$

Now $$\mathcal{Z}(\{x,y\}) = \sum_{\mu} e^{H \sum_{\langle pq \rangle} \sigma^{ref}_{\langle pq \rangle}(\{x,y\}) \mu_p \mu_q}$$

So write $$H_{pq} = H \sigma^{ref}_{\langle pq \rangle}(\{x,y\})$$

ref. config. for a given $\{x,y\}$.

Avg. Spin energy.

$$e^{\beta(-\lambda A + h\Sigma)} \qquad e^{-\beta\lambda A + \dots}$$

μ	$\tilde{A}$	$\Sigma\sigma$
++++	16	+4
+++−	4	+2
++−+	4	+2
++−−	0	0
+−++	4	+2
+−+−	0	0
+−−+	0	0
+−−−	4	−2
−+++	4	+2
−++−	0	0
−+−+	0	0
−+−−	4	−2
−−++	0	0
−−+−	4	−2
−−−+	4	−2
−−−−	16	−4

$$Z = \sum_\mu e^{-\beta(\lambda(\tilde{A})_\mu - h(\Sigma\sigma)_\mu)}$$

$$= e^{-16\beta\lambda + 4\beta h} + e^{-4\beta\lambda + 2\beta h} + e^{-4\beta\lambda+2\beta h} + 1 + e^{-4\beta\lambda+2\beta h} + 1 + 1 + e^{-4\beta\lambda - 2\beta h} + e^{-4\beta\lambda+2\beta h} + 2 + e^{-4\beta\lambda-2\beta h} + 1 + e^{-4\beta\lambda-2\beta h} + '' + e^{-16\beta\lambda+\dots}$$

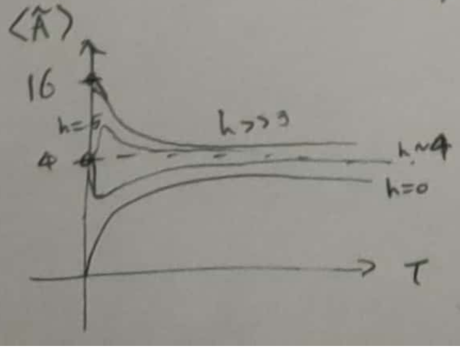
$$= 6 + 2e^{-16\beta\lambda}\cosh(4\beta h) + 4e^{-4\beta\lambda+2\beta h} + 4e^{-4\beta\lambda-2\beta h}$$

$$= 6 + 2e^{-16\beta\lambda} C_{4\beta h} + 8e^{-4\beta\lambda} C_{2\beta h}$$

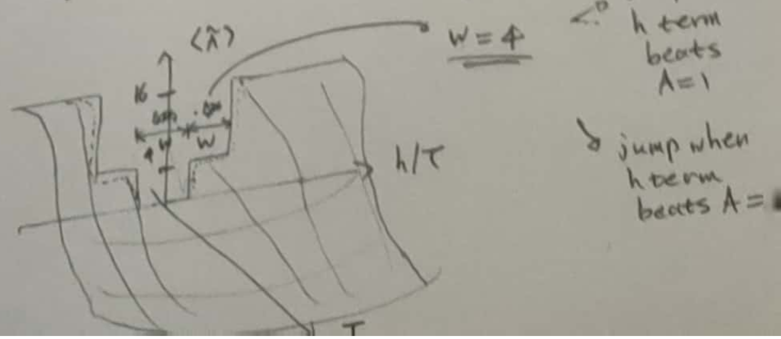
Once again, for $h=0$ we recover the previous result.

$$\langle\tilde{A}\rangle Z = -\frac{1}{\beta}\frac{\partial Z}{\partial\lambda} = 0 + 32e^{-16\beta\lambda}C_{4\beta h} + 32e^{-4\beta\lambda}C_{2\beta h}$$

$$\Rightarrow \langle\tilde{A}\rangle = \frac{0 + 16e^{-16\beta\lambda}C_{4\beta h} + 16e^{-4\beta\lambda}C_{2\beta h}}{3 + e^{-16\beta\lambda}C_{4\beta h} + 4e^{-4\beta\lambda}C_{2\beta h}}$$



$\Rightarrow$



And finally the 6-vertex case.

<u>Same</u> as 8-vertex except

$$\lambda \to -\lambda \qquad A_i \longrightarrow \tilde{A}_i = \Big(\sum_{\alpha \in \partial^+ i} \sigma_\alpha\Big)^2$$

$(\sum_\alpha \sigma_\alpha - \sigma_\beta)^2$

$A - 2\sigma_\beta \sum_\alpha \sigma_\alpha + 1$

$\sigma_\beta(\sigma_\beta - 2\sum_\alpha \sigma_\alpha$

$$\Rightarrow \Delta A_i^{(\beta)} = -4\sigma_\beta B_i \, \delta_{i \in \partial\beta} \, \Theta \qquad B_i = \sum_{\substack{\alpha \in \partial^+ i \\ \alpha \neq \beta}} \sigma_\alpha$$

$$\Rightarrow \Delta \epsilon_i^{(\beta)} = \lambda \Delta \tilde{A}_i^{(\beta)} + \frac{1}{2} \sum_{\alpha \in \partial^+ i} (\Delta D_\alpha - h \Delta \sigma_\alpha)$$

$$\vdots \qquad i \overset{\beta}{\text{———}} j$$

$$= 2\lambda \sigma_\beta (B_j - B_i) \, \delta_{i \in \partial\beta} \, \Theta$$

$$\Rightarrow j_{ij}^{(\beta)} = -2\lambda \sigma_\beta (B_j - B_i) \, \delta_{i,j \in \partial\beta} \, \Theta .$$

Precisely as for the others (kind of obvious in retrospect...)

Ferromagnet case

$$E = \underbrace{-\lambda \sum_\alpha A_\alpha - h \sum_i \sigma_i}_{E_H} + \underbrace{\sum_i D_i}_{E_P}$$

$$A_\alpha = \sigma_{\alpha_1} \sigma_{\alpha_2} = \prod_{i \in \partial\alpha} \sigma_i$$

$$\Delta E_H^{(i)} = -\lambda \sum_\alpha \Delta A_\alpha^{(i)} - h \sum_i \Delta\sigma_i^{(i)}$$

$$= +2\lambda \sum_\alpha A_\alpha \delta_{\alpha\in\partial^+ i} + 2h\sigma_j$$

$$\sum_\alpha A_\alpha \delta_{\alpha\in\partial^+ i} = \sum_{\alpha\in\partial^+ i}$$

$$\Delta A_\alpha^{(i)} = -2A_\alpha \delta_{i\in\partial\alpha}$$

$$\Delta\sigma_j^{(i)} = -2\sigma_j \delta_{ij}$$

$$\Delta D_j^{(i)} = -\Delta E_H^{(i)} \delta_{ij}$$

First we write:

$$E = \sum_i \epsilon_i$$

$$= -\frac{\lambda}{2} \sum_i \sum_{\alpha\in\partial^+ i} A_\alpha - h\sum_i \sigma_i + \sum_i D_i$$

$$\Rightarrow \epsilon_i = D_i - h\sigma_i - \frac{\lambda}{2} \sum_{\alpha\in\partial^+ i} A_\alpha$$

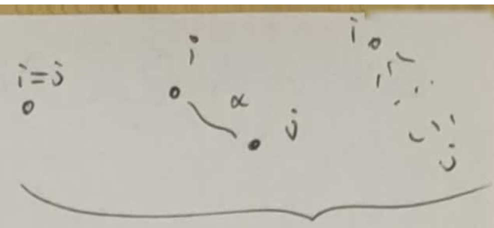
$$\Rightarrow \Delta\epsilon_i^{(j)} = -\Theta\delta_{ij}\Delta E_H^{(j)} - h\Delta\sigma_i^{(j)} - \frac{\lambda}{2}\sum_{\alpha\in\partial^+ i} \Delta A_\alpha^{(j)}$$

$$= -\Theta\delta_{ij}\left(\cancel{2h\sigma_j} + 2\lambda \sum_{\alpha\in\partial^+ j} A_\alpha\right) + \cancel{2h\sigma_j\delta_{ij}\Theta}$$

$$+ \frac{\lambda}{\cancel{2}} \cancel{2} \sum_{\alpha\in\partial^+ i} A_\alpha \delta_{j\in\partial\alpha}$$

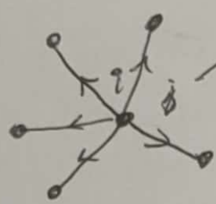
$$= \lambda\Theta \sum_{\alpha\in\partial^+ i} A_\alpha \left(\delta_{j\in\partial\alpha} - 2\delta_{ij}\right)$$

$$= \begin{cases} i \not\sim j & 0 \\ i \neq j,\ i - j & \Theta \lambda A_{(i-j)} \\ i = j & -\Theta \lambda \sum_{\alpha \in \partial^+ j} A_\alpha \end{cases}$$



3 cases

$\Downarrow$ i.e.



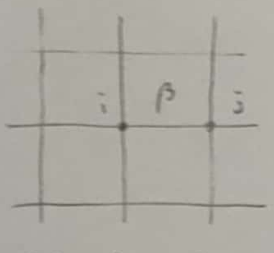
flipping i sends λA_β <u>out</u> along each bond $\beta \in \partial^+ i$

& $\quad j^{(k)}_{ij} = \Theta \lambda A_{(i-j)} \left(\delta_{ik} - \delta_{jk} \right)$ as in Rau's paper.

<u>All</u> indep of h!.

In both 8-vertex & Ising models.

Check: $\epsilon_i = -\lambda A_i + \frac{1}{2} \sum_{\alpha \in \partial^* i} (D_\alpha - h\sigma_\alpha)$



flip β $\Longrightarrow$

All $A = 1$

$\sigma = 1$

$\Downarrow$

$\epsilon_i = -\lambda - \frac{z}{2}h + \epsilon_D$

Pick all $D = \infty$

~~wher~~ so ~~item~~ flip accepted.

$A_i = A_j = -1$

$\sigma_\beta = -1$

$\epsilon_i = +\lambda - \frac{z}{2}h + h + \epsilon_D - \frac{1}{2}\Delta E^{(\beta)}$

$D_\beta \to D_\beta - \Delta E^{(\beta)}$

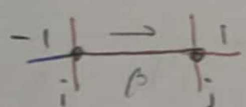
$\Rightarrow \Delta\epsilon_i = \lambda + h - \lambda - h$

$= 0.$

$\Delta E^{(\beta)} = +2h$

$+ 2\lambda(A_i + A_j)$

$= 2h + 2\lambda$

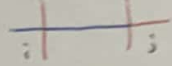


~~$\Delta E^{(\beta)} = 2(h+\lambda)$~~

$\Delta E^{(\beta)} = 2\lambda(1-1)$

$+2h$

$= 2h$



$\epsilon_i = +\lambda - \frac{z}{2}h + E_D$

$\epsilon_j = \cancel{\pm}\lambda - \frac{z}{2}h + E_D$

$\epsilon_j = +\lambda - 2h + h + (E_D \mp h)$

$\epsilon_i = -\lambda + (E_D \mp h)$

$\epsilon_j = -\lambda - 2h + E_D$

$\epsilon_i = +\lambda - 2h + \frac{z}{2}h + E_D$

$\Rightarrow \quad \epsilon_i = \lambda - h + E_D \to -\lambda - h + E_D$

$\epsilon_j = -\lambda - 2h + E_D \to +\lambda - 2h + E_D$

$\Delta\epsilon_i = -2\lambda \quad \Delta\epsilon_j = 2\lambda \quad \Rightarrow$ h does <u>not</u> make a difference!

h <u>only</u> affects the dynamics through the MC condition.

$\cancel{Z = 2e^{\beta\lambda}\cosh(4h)}$

$$Z = e^{\beta(\lambda+4h)} + 3e^{\beta(-\lambda+2h)} + e^{\beta(-\lambda+h)} + 3e^{\beta(\lambda)}$$
$$+ e^{\beta(\lambda-4h)} + 3e^{\beta(-\lambda-2h)} + e^{\beta(-\lambda-h)} + 3e^{\beta(\lambda)}$$

$$= 6e^{\beta\lambda} + 2\cosh(\beta h)e^{-\beta\lambda} + 6\cosh(2\beta h)e^{-\beta\lambda}$$
$$+ 2\cosh(4\beta h)e^{\beta\lambda}$$

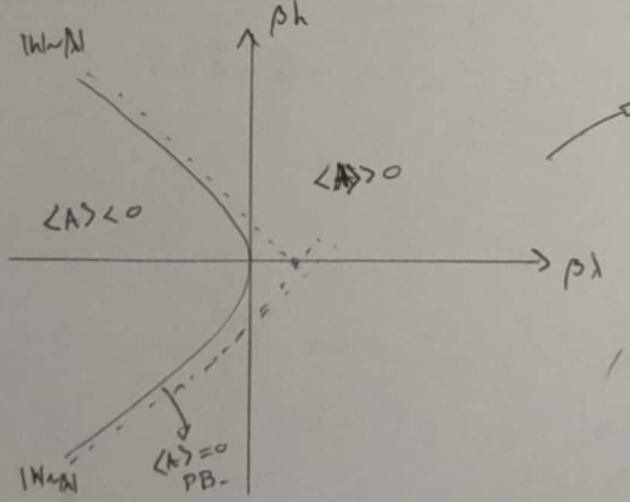
$\cosh\theta = C_\theta$

If $h=0$, $Z = 8(e^{\beta\lambda}+e^{-\beta\lambda}) = 16\cosh(\beta\lambda)$ as before

Now we can find:

$$Z\langle A\rangle = \cancel{8}\left(6e^{\beta\lambda} + 2C_{4\beta h}e^{\beta\lambda} - 2C_{\beta h}e^{-\beta\lambda} - 6C_{2\beta h}e^{-\beta\lambda}\right)$$

$$\Rightarrow \langle A\rangle = \frac{3(e^{\beta\lambda} - C_{2\beta h}e^{-\beta\lambda}) + (C_{4\beta h}e^{\beta\lambda} - C_{\beta h}e^{-\beta\lambda})}{3(e^{\beta\lambda} + C_{2\beta h}e^{-\beta\lambda}) + (C_{4\beta h}e^{\beta\lambda} + C_{\beta h}e^{-\beta\lambda})}$$



We get $\langle A\rangle$ varying from -1 to $+1$ depending on h/λ.

$$\Delta \epsilon_i^{(\beta)} = \frac{1}{2} \sum_{\alpha \in \partial^+ i} (\Delta D_\alpha - h \Delta \sigma_\alpha) - \lambda \cancel{\sum_{\alpha \in \partial^+ i} \Delta \sigma_\alpha} \Delta A_i$$

$$= \frac{\Theta}{2} \sum_{\alpha \in \partial^+ i} \left(-\delta_{\alpha\beta} \Delta E_H^{(\beta)} + 2h \overset{\delta_{\alpha\beta}}{\cancel{\quad}} \sigma_\alpha\right) + 2\lambda \Theta A_i \, \delta_{\beta \in \partial^+ i}$$

$$= \Theta\left(-\Delta E_H^{(\beta)} \frac{1}{2} + \cancel{2} h \delta \sigma_\beta + 2\lambda A_i \right) \Theta_{\beta \in \partial^+ i}$$

$$= \Theta_{D_\beta \geq \Delta E_H^{(\beta)}} \Theta_{\beta \in \partial^+ i} \left(2\lambda A_i + \cancel{h\sigma_\beta} - \lambda(A_i + A_j) - \cancel{h\sigma_\beta}\right)$$

$$i \overset{\beta}{\text{—}} j$$

$$= \underline{\Theta\, \Theta \left[\lambda (A_i - A_j)\right]}$$

So energy density & current <u>unchanged</u> by h-field!

Only change is in $\Theta_{D_\beta \geq \Delta E_H^{(\beta)}}$ part!

Avg. spin energy.

For a single vertex,

μ	A	$\Sigma \sigma$
1111	$+1$	4
$11\,1\bar{1}$	-1	2
$11\,\bar{1}1$	-1	2
$11\,\bar{1}\bar{1}$	$+1$	0
$1\bar{1}11$	-1	1
$1\bar{1}1\bar{1}$	$+1$	0
$1\bar{1}\bar{1}1$	$+1$	0
$1\bar{1}\bar{1}\bar{1}$	-1	-2
$\bar{1}111$	-1	2
$\bar{1}11\bar{1}$	$+1$	0
$\bar{1}1\bar{1}1$	$+1$	0
$\bar{1}1\bar{1}\bar{1}$	-1	-1
$\bar{1}\bar{1}11$	$+1$	0
$\bar{1}\bar{1}1\bar{1}$	-1	-2
$\bar{1}\bar{1}\bar{1}1$	-1	-2
$\bar{1}\bar{1}\bar{1}\bar{1}$	$+1$	-4

$$E = -\lambda A - h \Sigma \sigma$$

$$Z = \sum_\mu e^{\beta \lambda (A)_\mu + \beta h (\Sigma \sigma)_\mu}$$

$$\langle A \rangle = \frac{1}{Z} \frac{1}{\beta} \frac{\partial Z}{\partial \lambda}$$

$$Z = e^{\beta(\lambda + 4h)} + e^{\beta(-\lambda + 2h)}$$
$$+ e^{\beta(-\lambda + 2h)} + e^{\beta(\lambda)}$$
$$+ e^{\beta(-\lambda + h)} + e^{\beta(\lambda)} + e^{\beta(\lambda)}$$
$$+ e^{\beta(-\lambda - 2h)}$$
$$+ e^{\beta(\lambda - 4h)} + e^{\beta(-\lambda - 2h)} + e^{\beta(-\lambda - 2h)}$$
$$+ e^{\beta(\lambda)} + e^{\beta(-\lambda - h)} + e^{\beta(\lambda)} + e^{\beta(\lambda)}$$
$$+ e^{\beta(-\lambda + 2h)}$$

8 - Vertex Model w/. h-field

$$\cancel{E} = \underbrace{-\lambda \sum_i A_i - h \sum_\alpha \sigma_\alpha}_{E_H} + \underbrace{\sum_\alpha D_\alpha}_{E_D}$$

$$= \sum_i \epsilon_i$$

where $\quad \epsilon_i = \frac{1}{2} \sum_{\alpha \in \partial^+ i} (D_\alpha - h\sigma_\alpha) - \lambda \overbrace{\prod_{\alpha \in \partial^+ i} \sigma_\alpha}^{A_i}$

Now consider flipping a spin: $\quad i \overset{\beta}{\text{———}} j$

$$\sigma_\beta \to -\sigma_\beta$$
$$A_{i,j} \to -A_{i,j} \quad \Rightarrow \quad \underline{\Delta E_H^{(\beta)} = 2\lambda(A_i + A_j) + 2h\sigma_\beta}$$

Now how do <u>all</u> the variables change?

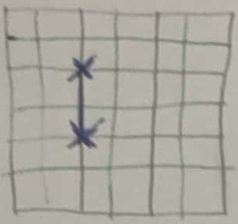
$\Theta = \Theta_{D_\beta > \Delta E_H^{(\beta)}}$

$$\sigma'_\alpha = \sigma_\alpha(1 - 2\,\delta_{\alpha\beta}\,\Theta)$$
$$A_i' = A_i(1 - 2\,\delta_{i\in\partial\beta}\,\Theta)$$
$$D_\alpha' = D_\alpha - \delta_{\alpha\beta}\,\Theta\,\Delta E_H^{(\beta)}$$

$1_\beta \overset{\beta}{\text{———}} 2_\beta$

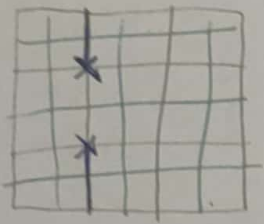
$$\Rightarrow \quad \Delta E_{\cancel{H}}^{(\beta)} = -\lambda \sum_i \Delta A_i^{(\beta)} - h \sum_\alpha \Delta\sigma_\alpha^{(\beta)} + \sum_\alpha \Delta D_\alpha^{(\beta)}$$
$$= +2\lambda\Theta \sum_i A_i\,\delta_{i\in\partial\beta} + 2h\Theta \sum_\alpha \sigma_\alpha\,\delta_{\alpha\beta} - \Delta E_H^{(\beta)}\,\Theta \sum_\alpha \delta_{\alpha\beta}$$
$$= 2\lambda\Theta(A_{1_\beta} + A_{2_\beta}) + 2h\Theta\sigma_\beta - 2\lambda\Theta(A_{1_\beta} + A_{2_\beta}) - 2h\Theta\sigma_\beta$$
$$= 0$$

Worth checking! $\Rightarrow$ μcan overall!

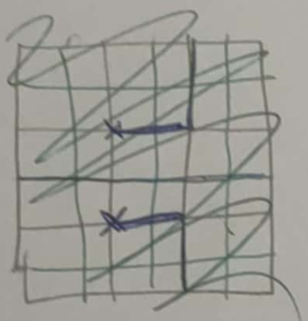


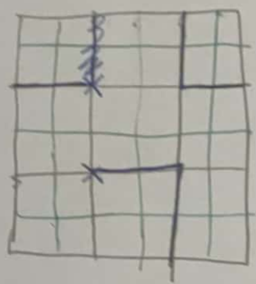
$$t^3 \cdot (6-2) = 4t^3$$

$$\neq$$



$$t^3(6-4) = 2t^3$$

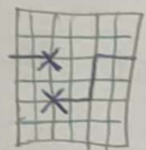




$$t^3(6-10) = -4t^3$$

??? t^3 dependent on σ-ref?

How??



$= -2t^3$

$$\sum_{\underline{\mu}} e^{H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q}$$

$$= \sum_{\underline{\mu}} \prod_{\langle pq \rangle} e^{H \eta_{pq} \mu_p \mu_q}$$

$$= \sum_{\underline{\mu}} \prod_{\langle pq \rangle} C_H (1 + t_H \eta_{pq} \mu_p \mu_q)$$

$$= \cancel{\sum_{\underline{\mu}}} \cancel{\prod_{\langle pq \rangle}} C_H^{N_E} \sum_{\underline{\mu}} \underbrace{\prod_{\langle pq \rangle} (1 + t_H \eta_{pq} \mu_p \mu_q)}_{\neq 0 \text{ iff } \mu_p\text{'s repeated an even no. of times.}}$$

For $H \ll 1$, cut off below ~~topologically~~ non-contractible F^* $\Rightarrow$ no issues with counting crossings wrt topo sector strings.

Then $$\sum_{\{x\}} (-1)^{|F^* \cap S|} = \underbrace{\Delta_F}_{\#\text{ F configs}} \sum_{m=0}^{V_F} \underbrace{\binom{N-V_F}{n-m}}_{\text{ways of placing prtcls}} (-1)^m$$

For $H \gg 1$, ~~cut off~~ need to estimate n° shared edges. That's quite hard...

~~for diff topo sectors we get~~

~~$|F \cup S|$ $|F \cup S|$ $|F| + |S|$ $|F \cap S|$~~

$|F| + 2d$

$\Delta\ \beta\ \gamma\ \delta\ \varepsilon\ \nu\ \eta\ \Omega$

$\pi\ \tau\ \alpha\ \rho\ \mu\ \omega$

$\alpha\ \beta\ \gamma$ G Q

ω ν $\delta\ \varepsilon\ \mu$ G $\eta\ \nu\ \rho$

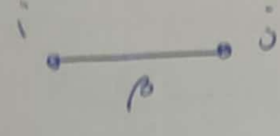
α

β

Two Cases:

① Allow double flips of β.

We sample N_v vertices.



$\Rightarrow$ β get $\underline{2}$ chances w/. i & j on average

$\mathbb{P}(\beta_1 = \beta_2 = \beta)$ $\mathbb{P}(\beta_1 = \beta, \beta_2 \neq \beta)$ + vice-versa

@ each site, $\cancel{\mathbb{P}(!\beta_2)} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$\mathbb{P}(\beta) = \left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\,\frac{3}{4}\right) = \cancel{\frac{1}{4}}\ \frac{7}{16} \sim \frac{1}{2}$

$\Rightarrow$ each edge hit an average of

$\cancel{n = 2 \cdot \frac{1}{4} = \frac{1}{2}}$

$$n = 2 \cdot \frac{7}{16} = \frac{14}{16} \sim \frac{7}{8} \sim 1$$

② Don't!

Also 2 chances, but $\cancel{\mathbb{P}(!\beta) =}$

$\mathbb{P}(\beta) = \mathbb{P}(\beta_1 = \beta \wedge \beta_2 \neq \beta)$ + vice-versa

$= \frac{1}{4}\,\frac{3}{4} \cdot 2 = \frac{3}{8}$

$\frac{1}{16}$ $\frac{6}{16}$

$\Rightarrow n = 2 \cdot \frac{3}{8} \sim \frac{6}{8} \sim \frac{3}{4}.$

<u>SO</u> ratio is

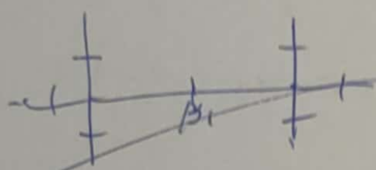
$$\frac{n_{①}}{n_{②}} \sim \frac{7}{8}\,\frac{4}{3} = \frac{7}{6}$$

$\sim \cancel{1.33}$

~ 1.17 hmmm . . .

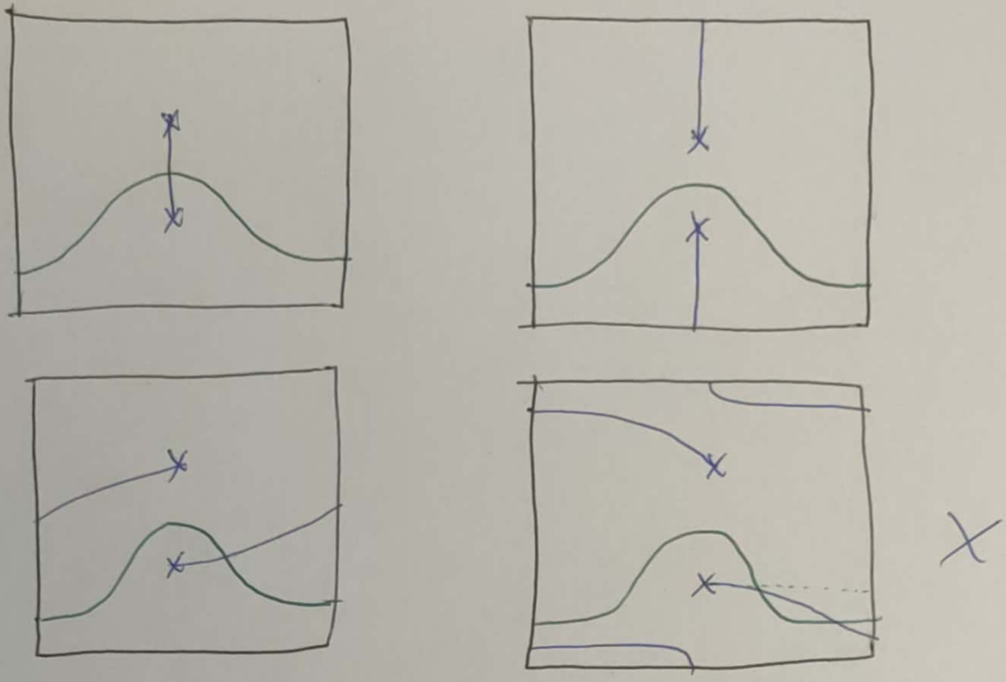
If <u>instead</u> we sample <u>edges</u> (β_1 then $\beta_2 \in nn(\beta_1)$).

①

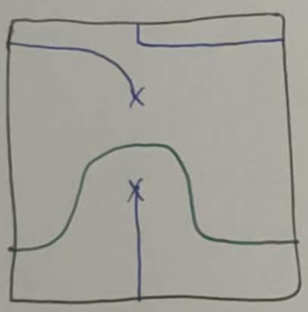


$$\mathbb{P}(\beta) = \mathbb{P}(\beta = \beta_1) + \mathbb{P}(\beta \in nn(\beta_1))\,\mathbb{P}(\beta \text{ chosen})$$

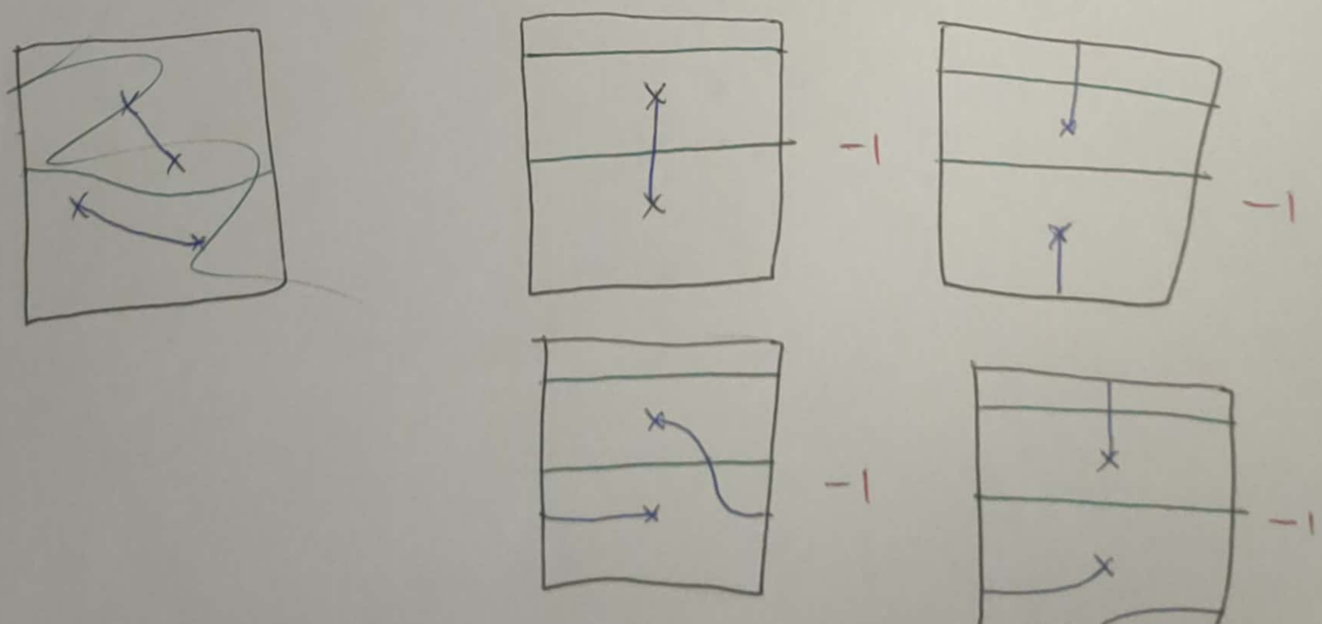
$$n = \underset{\substack{\uparrow \\ \beta \text{ chosen}}}{\frac{1}{2}} + \underset{\substack{\uparrow \\ nn(\beta)}}{6} \cdot \frac{1}{6} = \frac{3}{4}\ ?$$



This is rubbish.

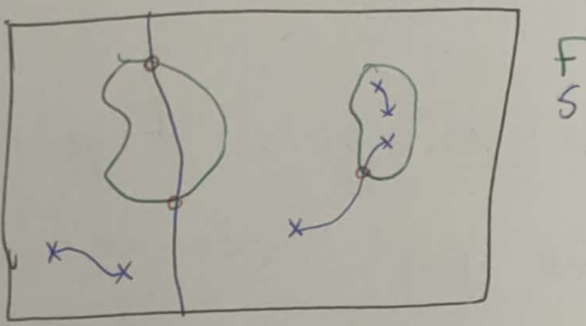


We <u>can't</u> just sum over σ-ref in diff. sectors?



$$Z = \sum_{F \subset G} \underbrace{\left(C_H^{N_E} \cdot t_H^{|F|}\right)}_{A_F} \underbrace{(-1)^{|F \cap S|}}_{P_F}$$

For F a closed loop, this is simple.



Here,

$P_F = (-1)^{\#\text{ enclosed prtcls in each constituent loop.}}$

$\hookrightarrow$ For F cont, it defines an inside & outside

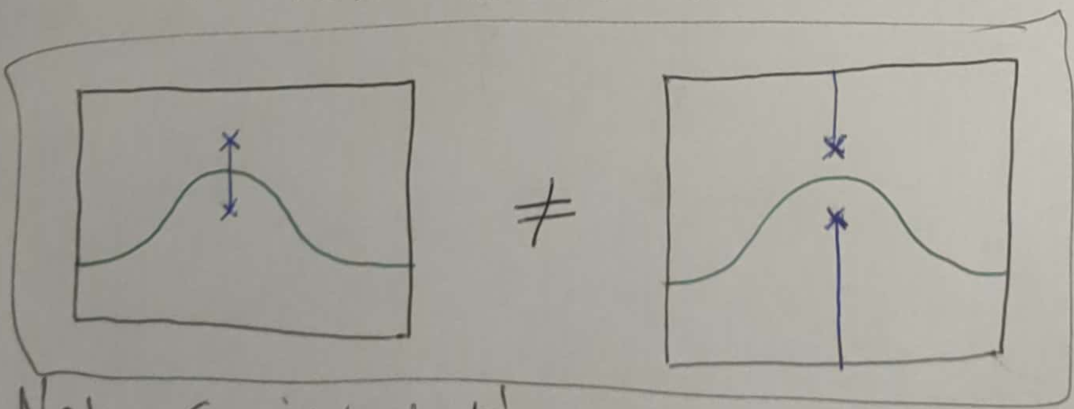
More precisely, $T^2 \simeq \mathbb{R}^2$

$\Rightarrow$ can define

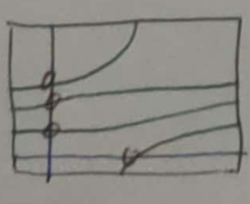
$P_F = \prod_{\text{prtcls}} (-1)^{\text{winding n}^\circ \text{ of } F \sim \text{prtcl.}}$

$\hookrightarrow$ Indep. of $\underline{w}$ lines.

For F non-cont. we have issues.



Not S-invariant!



We can count these fine $\sim$ Just depends which $\underline{w}$ S is in & which sector F is in.

We <u>must</u> have that Z is σref_ indep

So in summing these F, we <u>must</u> be able to kill off this dependence.

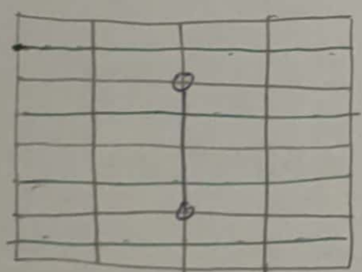
Can we?

Propose: Summing over configs of F good enough?

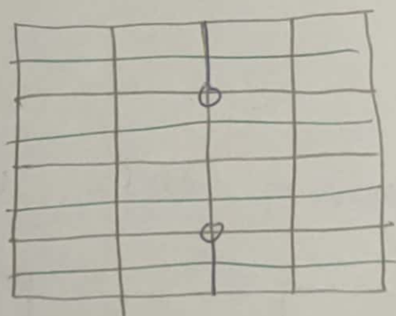
↳ À priori, no good reason – in general, require only that ~~#~~ $\boxed{\sum_{F \mid |F| = \ell}} (\ldots)$ is σref_ inv.

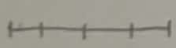
Test:

(1)



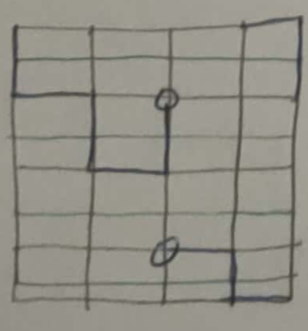
(2)



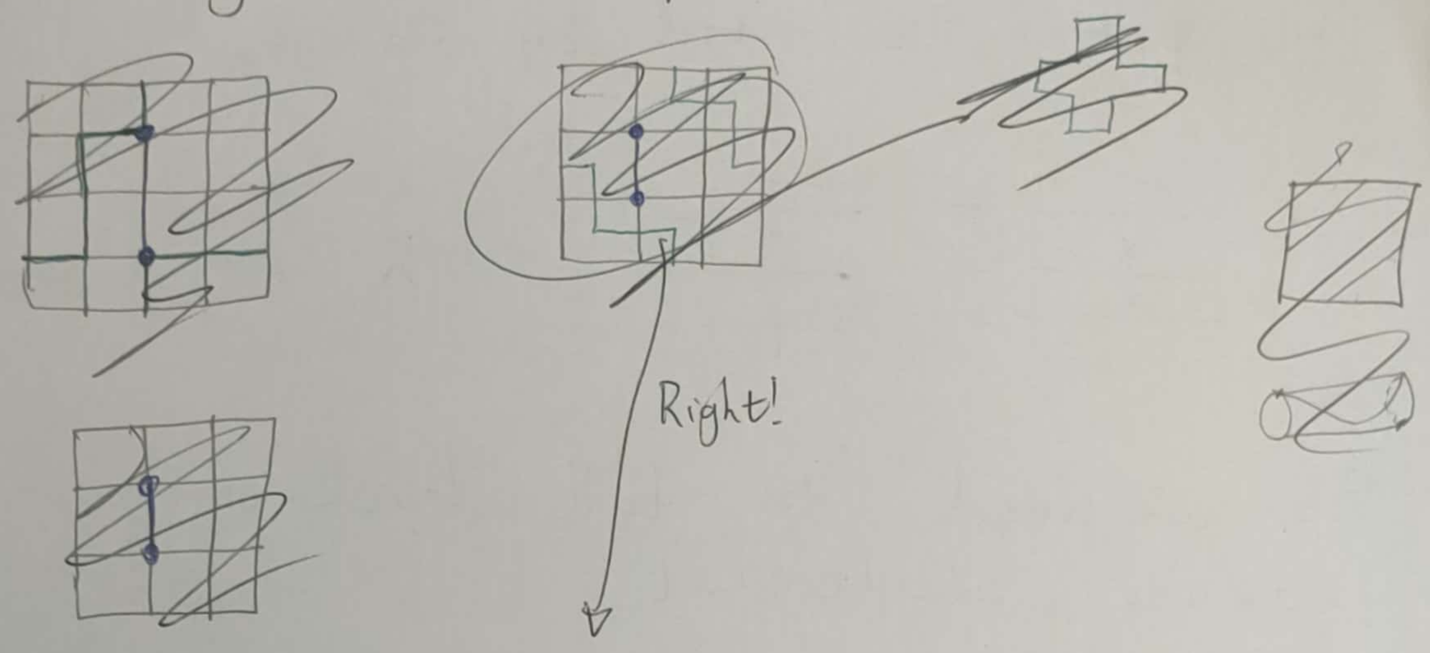
$F =$ 

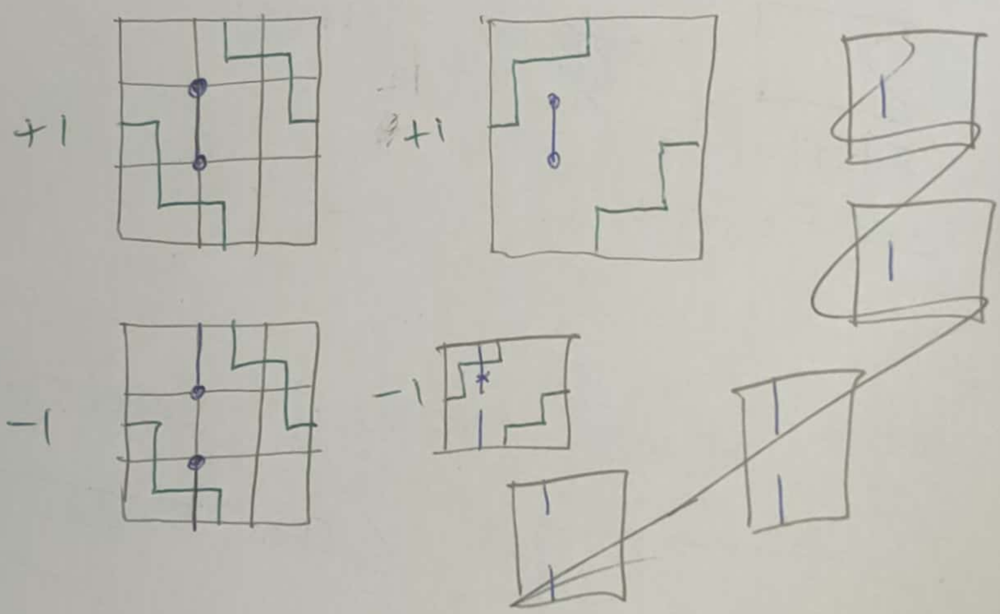
For this F, it <u>is</u> sufficient.
∃ only 4 possible pos^ns!

(3)

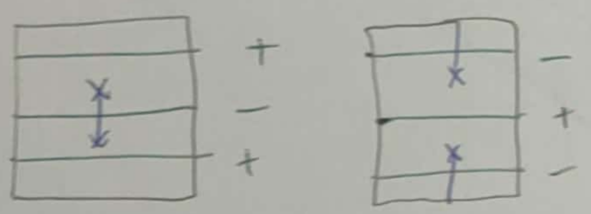


Let's try a more complicated F.





I want to count the # intersect^{ns} of F w/. prtcls connected by strings.

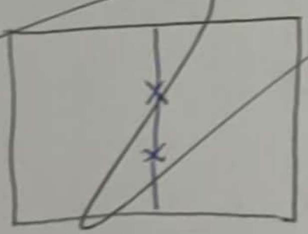


$\overset{=}{\wedge}$. $\cancel{\ell}$ ~~#2~~

So we need to fall back on our earlier Statement.

$$\sum_{\substack{F \,|\, |F|=\ell \\ \text{and } F \text{ non-cont.}}} (-1)^{|F \wedge S|} \; ?$$

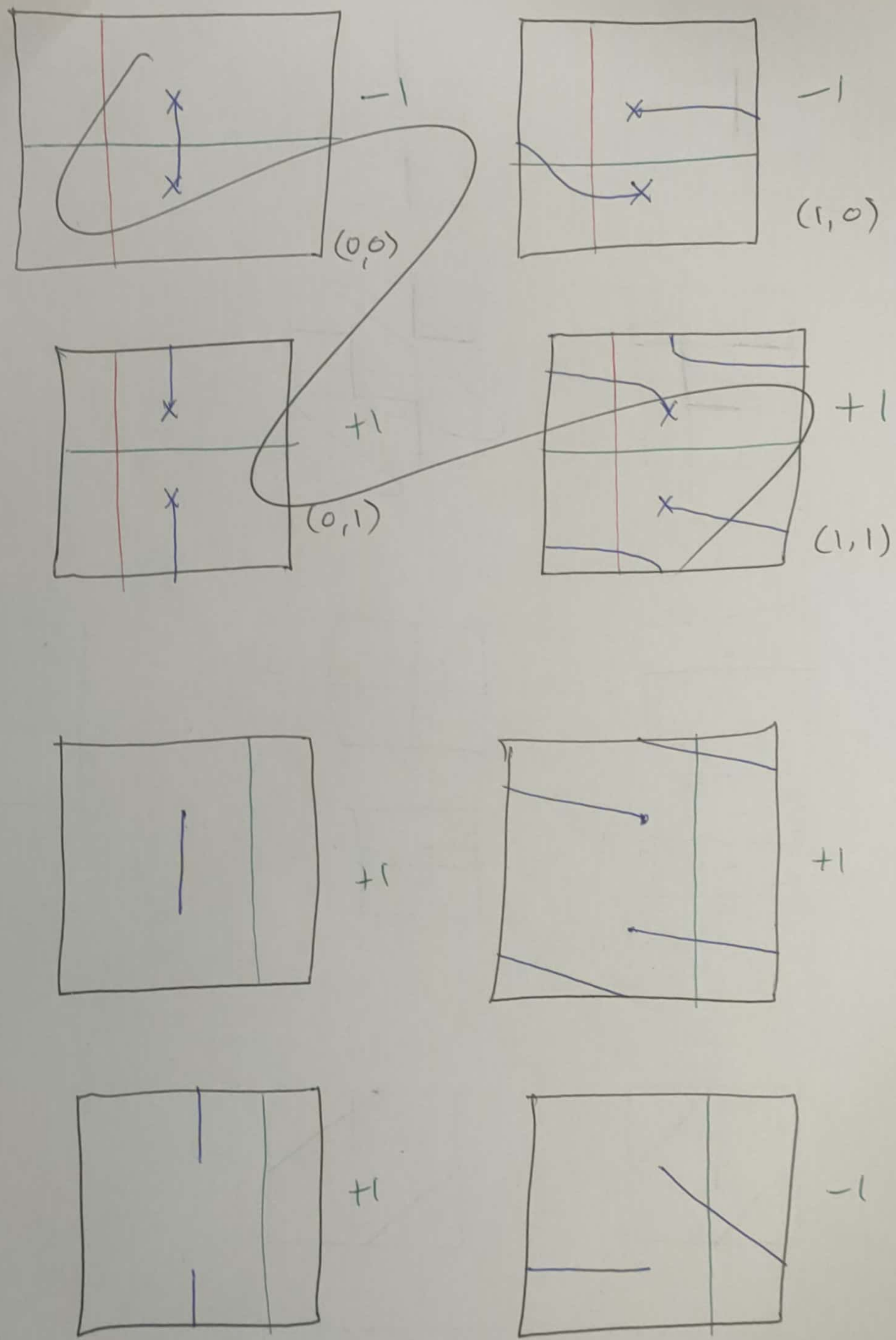
~~What about joining prtcls like this?~~



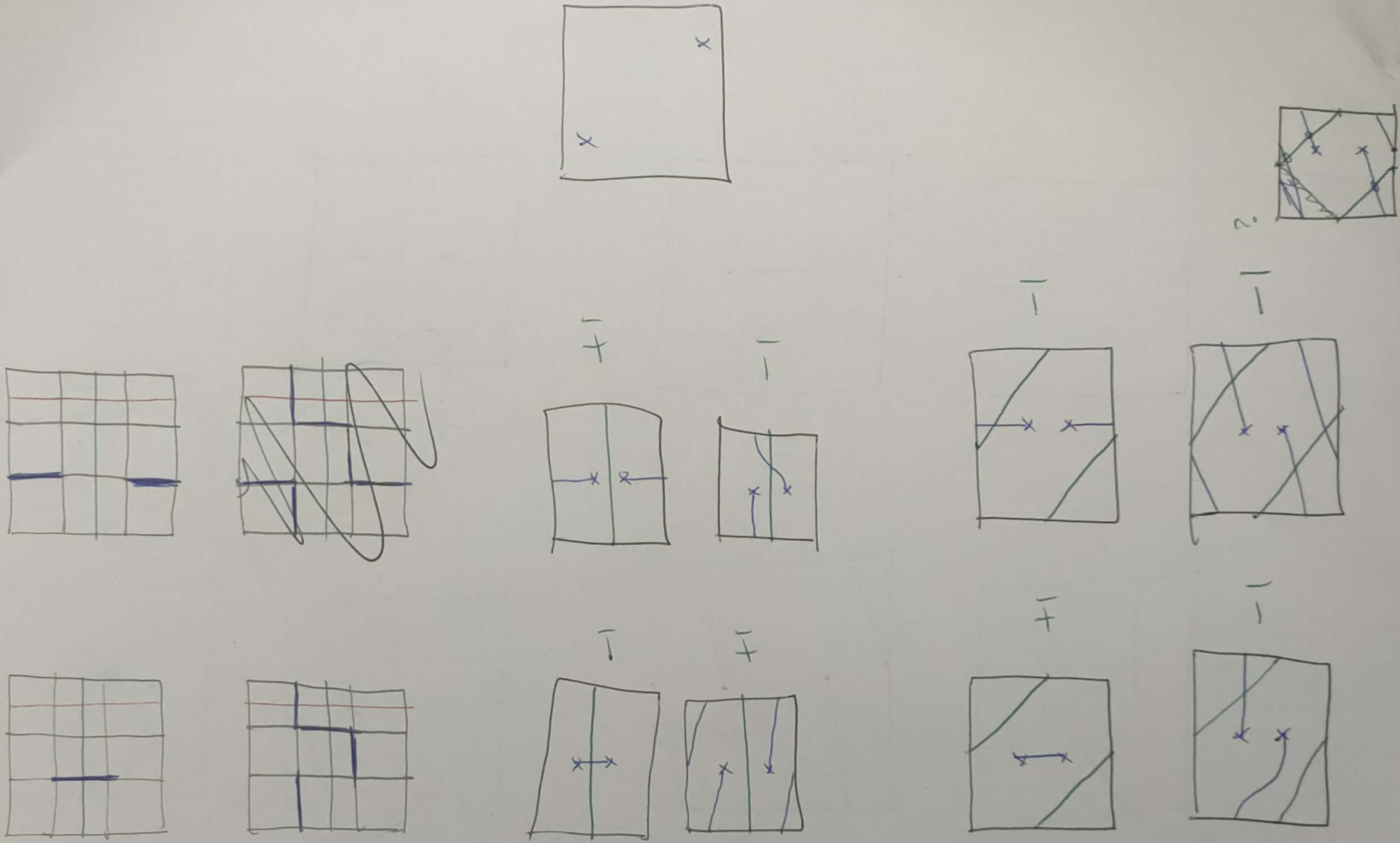
~~Then divide by 2?~~

The only way I can see to progress is to sum over different $\underbrace{\sigma\text{-ref}}_{\eta}$.

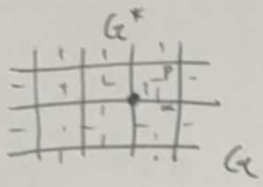
Define $\eta_{(++)}$, $\eta_{(+-)}$, ... etc. as the following: ref states w/. prtcls joined... trivially, then in various sectors.



−1
(0,0)
−1
(1,0)
+1
(0,1)
+1
(1,1)
+1
+1
+1
−1



$$\tilde{Z} = \begin{cases} 2^{N_P} C_H^{N_E} \displaystyle\sum_{F^* \subset G^*} t_H^{|F^*|} (-1)^{|F^* \cap S|} \\ 2 e^{H N_E} \displaystyle\sum_{F \subset G} e^{-2H|F \cup S|} \end{cases}$$



$$Z = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} Y_n$$

$$Z = \sum_{\sigma} e^{\Lambda \sum_i A_i + H \sum}$$

$$Y_n = \sum_{S_{v,\Sigma}} \tilde{Z}$$

$$\sum_{S_{v,\Sigma}} = \sum_{W} \frac{1}{n!} \sum_{\{\Sigma\}}.$$

$$= \begin{cases} \displaystyle\sum_{F^* \subset G^*} \left(2^{N_P} C_H^{N_E} t_H^{|F^*|}\right) \sum_{S_n} (-1)^{|F^* \cap S|} & H \ll 1 \\ \displaystyle\sum_{F \subset G} \left(2 e^{H N_E}\right) \sum_{S_n} e^{-2H|F \cup S|} & H \gg 1 \end{cases}$$

$$\langle n \rangle = -\frac{1}{N} \frac{\partial \ln Z}{\partial (2\Lambda)} \qquad (K \propto \langle n \rangle)$$

$$\left\langle \frac{1}{N_E} \sum_{\alpha} \sigma_\alpha \right\rangle = + \frac{1}{N_E} \frac{\partial \ln Z}{\partial H} = \langle M \rangle$$

$$C = \frac{\beta^2}{N_E} \frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$= \frac{1}{N_E} \left[(2\Lambda)^2 \frac{\partial^2 \ln Z}{\partial (2\Lambda)^2} + H^2 \frac{\partial^2 \ln Z}{\partial H^2} \right]$$

$$-\beta F = \begin{cases} O(\delta\sigma) : \frac{1}{2}\sigma^{z-1}\Lambda\sum_\alpha \sigma_\alpha + H\sum_\alpha \sigma_\alpha \\ O(\delta\sigma^2) : \frac{1}{2}\sigma^{z-1}(1-2z)\Lambda\sum_\alpha \sigma_\alpha + H\sum_\alpha \sigma_\alpha \\ \qquad + \sigma^{z-2}\sum_{\langle\alpha\beta\rangle}\sigma_\alpha\sigma_\beta \end{cases}$$

$$z=4 \Rightarrow \begin{cases} H_{eff} = H + \frac{1}{2}\Lambda\sigma^3 & (1) \\ H_{eff} = H - \frac{7}{2}\Lambda\sigma^3, \quad J_{eff} = \sigma^2 & (2) \end{cases}$$

First, (1)

$$Z = \sum_{\{\sigma\}} e^{+H_{eff}\sum_\alpha \sigma_\alpha}$$

$$= \prod_\alpha 2\cosh(H_{eff}) = \cancel{2^N}^{\text{const.}}\cosh^N(H_{eff})$$

$N = N_E$

$$C = \frac{H^2}{N_E}\frac{\partial^2}{\partial H^2}\ln Z$$

$$\sigma := \langle \tfrac{1}{N}\sum_\alpha \sigma_\alpha \rangle \quad \forall \alpha$$

$$= +\frac{1}{N}\frac{\partial \ln Z}{\partial H_{eff}}$$

$$= \frac{1}{\cosh(H_{eff})}\frac{\partial}{\partial H_{eff}}\cosh(H_{eff}) = \tanh(H_{eff})$$

$$\Rightarrow \sigma = \tanh(H + \tfrac{1}{2}\Lambda\sigma^3)$$

$$C = \cancel{N}\frac{H^2}{\cancel{N}}\frac{\partial}{\partial H}\tanh(H_{eff}) \neq H^2\operatorname{sech}^2(H_{eff})$$

$$Z = \sum_{n=0}^{z} \binom{z}{n} e^{-\Xi (z-2n)^2}$$

$$Z\, n_{min} = \sum (\text{''}) \, \delta_{|z-2n| = q_{min}}$$

$$Z\, n_{all} = \sum (\text{''}) \, \delta_{|z-2n| \neq q_{GS}} = Z - \sum (\text{''}) \, \delta_{|z-2n| = q_{GS}}$$

$z = 4 \rightsquigarrow q_{GS} = 0$

$q_{min} = 2$

$T \to \infty$: $Z n_{min} \to 2 \binom{z}{\frac{z - q_{min}}{2}} = 2 \cdot \binom{4}{2} = 12$

$Z n_{all} \to 1 - 2 \binom{z}{\frac{z - q_{GS}}{2}} = Z - 2 \cdot \binom{4}{1} = Z - 8$

$Z \to \sum \binom{z}{n} = 2^z = 2^4 = 16$

$\Rightarrow n_{min} \to \frac{12}{16} =$

$\cancel{n_{all} \to 1 - \frac{8}{16} = \frac{1}{2}}$

$\Rightarrow n_{min} \to 2 \cdot \binom{4}{1} / 16 = 1/2$

$n_{all} \to 1 - \frac{2 \cdot \binom{4}{2}}{16} = \frac{5}{8}.$

$z = 3 \rightsquigarrow q_{GS} = 1$

$q_{min} = 3$

$T \to \infty$: $Z \to 2^3$

$Z n_{min} \to 2 \binom{3}{\frac{3-3}{2}} = 2$

$Z n_{all} \to Z - 2 \binom{3}{\frac{3-1}{2}} = Z - 6$

$\Rightarrow n_{min} \to 2/8 = 1/4$

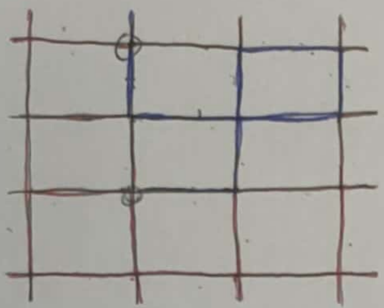
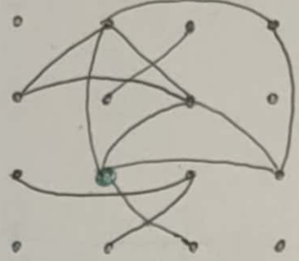
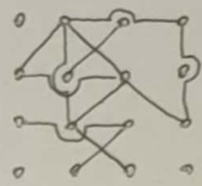
$n_{all} \to 1 - 6/8 = 1 - 3/4 = 1/4$

Good!

<u>How</u> do I relate M & $\mathbb{P}$?

Percolation of +− paths?

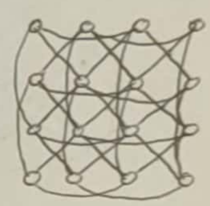
$E = 13$

 $\simeq$  ? $\simeq$ 

Problem: spinon motion alters the bckgd
BUT it leaves M const.

Let's estimate $\mathbb{P}$ for the above graph

Out of ...



$$M = (24 - 2\cdot 8)/24$$
$$= (24 - 16)/24$$
$$= \cancel{4}\ 8/24$$
$$= \cancel{1/2}\ 1/3$$

$E = 18 + 16$
$= 34.$

$\Rightarrow M = \cancel{1/2}\ 1/3,\quad \mathbb{P} \approx 0.4 = \frac{13}{34}$

$$\frac{13}{34} \sim \frac{14}{35} \sim \cancel{\frac{2}{5}}\ \frac{2}{5} \sim 0.4$$

I proposed $\mathbb{P} = \frac{1-M}{2}$

$$= \frac{1 - 1/12}{2} = \frac{11}{24} \sim 0.5 \text{ ish}$$

$\Rightarrow$ not <u>far</u> off...

~~PZ~~ @ each vertex i, add a bond to E for each combo of +− bonds $\in \partial^+ i$

Define $z_i^+ + z_i^- = z_i$ $(= 4\ \forall i)$ @ each vertex

$\Rightarrow i$ contributes $\frac{1}{2}\binom{z_i^+}{1}\binom{z_i^-}{1} = z_i^+ z_i^- / 2$

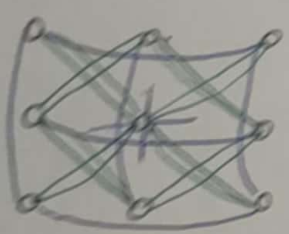
$\hookrightarrow$ ∵ order doesn't matter

$$\Rightarrow |E| = \sum_i \frac{1}{2} z_i^+ z_i^-$$

$$= \sum_i \frac{1}{2} z_i^+ (z_i - z_i^+)$$

~~We define P to be~~

We're really considering hops on a <u>next</u>-nearest neighbour graph ($\sim$ bipartite graph)



P is defined relative to the graph as left here, i.e. each vertex contributes

w/ $\cancel{z \neq 4}$

$= \underbrace{4}_{\text{linear}} + \underbrace{12}_{\text{diagonals (2 routes)}}$ $\Rightarrow \binom{12}{2} = 33$

$z=4$ $\rightsquigarrow$ pick any 2

$\Rightarrow \binom{z}{2}$

no extra 1/2 ∵ no +/− to worry about the order of!

$z = 4 \Rightarrow \binom{z}{2} = 6$

$$\Rightarrow P = \frac{1}{N_E \binom{z}{2}} \sum_i z_i^+ z_i^-$$

$$= \frac{1}{6 N_E} \sum_i z_i^+ z_i^- .$$

$$M = \frac{1}{N_E} \sum_{\alpha} \sigma_\alpha$$

$$= \frac{1}{2N_E} \sum_i \sum_{\alpha \in \partial^+ i} \sigma_\alpha .$$

$$= \frac{1}{2N_E} \sum_i \left(z_i^+ - z_i^- \right)$$

$$\boxed{\begin{aligned} M &= \frac{1}{2N_E} \sum_i \left(z_i^+ - z_i^- \right) \\ \mathbb{R} &= \frac{1}{6N_E} \sum_i z_i^+ z_i^- \end{aligned}}$$

assumes PBCs.

Recall that

$$z_i^+ + z_i^- = z_i$$

$$\Rightarrow \quad M = \frac{1}{2N_E} \sum_i \left(2z_i^+ - z_i \right) = \left(\frac{1}{N_E} \sum_i z_i^+ \right) - 1$$

$$\mathbb{P} = \frac{1}{6N_E} \sum_i z_i^+ \left(z_i - z_i^+ \right)$$

$$\overset{!!}{\approx} \quad = \frac{1}{6N_E} \sum_i \left(z_i^+ z_i - (z_i^+)^2 \right) \qquad z_i = 4 \; \forall i$$

$$\overset{!!}{\approx} \quad = \frac{2}{3} M - \frac{1}{6N_E} \sum_i (z_i^+)^2$$

Let's try mean-field! $\quad M = \langle z_i^+ \rangle$

$$\Rightarrow \quad (z_i^+)^2 \approx M^2 + 2M\delta z_i^+ = 2M z_i^+ - M^2 \quad \text{to } \mathcal{O}(\delta z$$

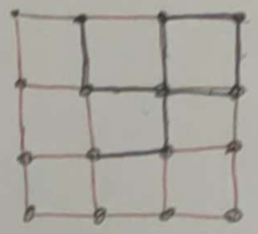
$$\Rightarrow \quad \mathbb{P} \approx \frac{2}{3} M - \frac{M}{3} \underbrace{\frac{1}{N_E} \sum_i z_i^+}_{M}$$

$$+ \frac{M^2}{6} \frac{1}{N_E} \underbrace{\sum_i (1)}_{N_V}$$

$$\frac{1}{12} - \frac{4}{12} = -\frac{3}{12}$$

$$= \frac{2}{3} M - \frac{1}{3} M^2 + \frac{1}{12} M^2 \approx M \left(\frac{2}{3} - \frac{1}{4} M \right)$$

Check:



$\sum_i z_i^+ z_i^-$

$= 0 + 2 + 2 + 0 + 0 + 4 + 0 + 2$
$+ 0 + 3 + 4 + 0 + 0 + 0 + 0 + 0$

$= 2 + 2 + 4 + 2 + 3 + 4$

$= 17$

$\mathbb{P} \overset{?}{=} \dfrac{17}{6 \cdot 24} \sim \dfrac{3}{24} \sim \dfrac{1}{8}$???

~~Wrong!?~~

Problem: I assumed periodicity!
Also I undercounted diagonals the first time around!

and... $M = \dfrac{24 - 2 \cdot 8}{24}$

$= 1/3$

Formula: $\mathbb{P} \approx \dfrac{2}{9} - \dfrac{1}{36} = \dfrac{7}{36} \sim \dfrac{1}{5}$

Properly, $\mathbb{P} = \dfrac{1}{2} \dfrac{\sum_i z_i^+ z_i^-}{\sum_i \binom{z_i}{2}}$ $\longrightarrow = \dfrac{z(z-1)}{2}$

$= \dfrac{17}{\sum_i z_i(z_i - 1)}$

$= \dfrac{17}{4 \cdot (4(3)) + 4(2(1)) + 8(3(2))}$

$= \dfrac{17}{48 + 8 + 48} = \dfrac{17}{104}$

~ 0.16

Okay... Disregard BCs So reasonable

$$\cancel{\mathbb{P} = \frac{1}{3}\overline{y} - \frac{1}{12}\overline{y^2}}$$

$$\mathbb{P} = \frac{1}{3}\langle y \rangle - \frac{1}{12}\langle y^2 \rangle$$

$$N_v \mathbb{P} = \frac{1}{3}\left\langle \sum_i y_i \right\rangle - \frac{1}{12}\left\langle \sum_i y_i^2 \right\rangle$$

$$= \frac{1}{3}\left\langle 4\cancel{N_v} - \sum_i w_i \right\rangle - \frac{1}{12}\left\langle 16\cancel{N_v} - 8\sum_i w_i + \sum_i w_i^2 \right\rangle$$

$$= \frac{1}{3}\left\langle \sum_i w_i \right\rangle - \frac{1}{12}\left\langle \sum_i w_i^2 \right\rangle$$

$$\sum_i \left(y_i^2 + w_i^2\right) = \sum_i y_i^2 + (z_i - y_i)^2$$

$$= \sum_i \left(2y_i^2 - 8y_i + 16\right)$$

$$\sum_i (-1)^{w_i} = \sum_i (-1)^{w}(-1)^{\delta w_i}$$

irrelevant

$$\cancel{\sum_{\underline{w}} \left(\sum_i w_i^2\right) e^{+\Lambda\sum_i (-1)^{w_i} + H\left(\sum_i y_i^2 - N_E\right)}}$$

$$\sum_{\underline{y}} \left(\sum_i y_i^2\right) e^{\Lambda\sum_i (-1)^{y_i} + H\sum_i y_i}$$

$y_i \in \{0,1,2,3,4\}$

$y_i^2 \in \{0,1,4,9,16\}$.

@ MF level,

$$\mathbb{P} \sim \frac{1}{3}(1 - M^2)$$

@ large T though, $\bar{y} \ll \delta y \Rightarrow$ MF fails.
We <u>should</u> get $\mathbb{P} \rightsquigarrow 0.25 - \varepsilon$
small offset.

$$\mathbb{P} = \frac{1}{4}(1 - M^2)$$

$$= \begin{cases} T=0: & M=1 \Rightarrow \mathbb{P}=0 \\ T=\infty: & M=0 \Rightarrow \mathbb{P}=1/4 \end{cases}$$

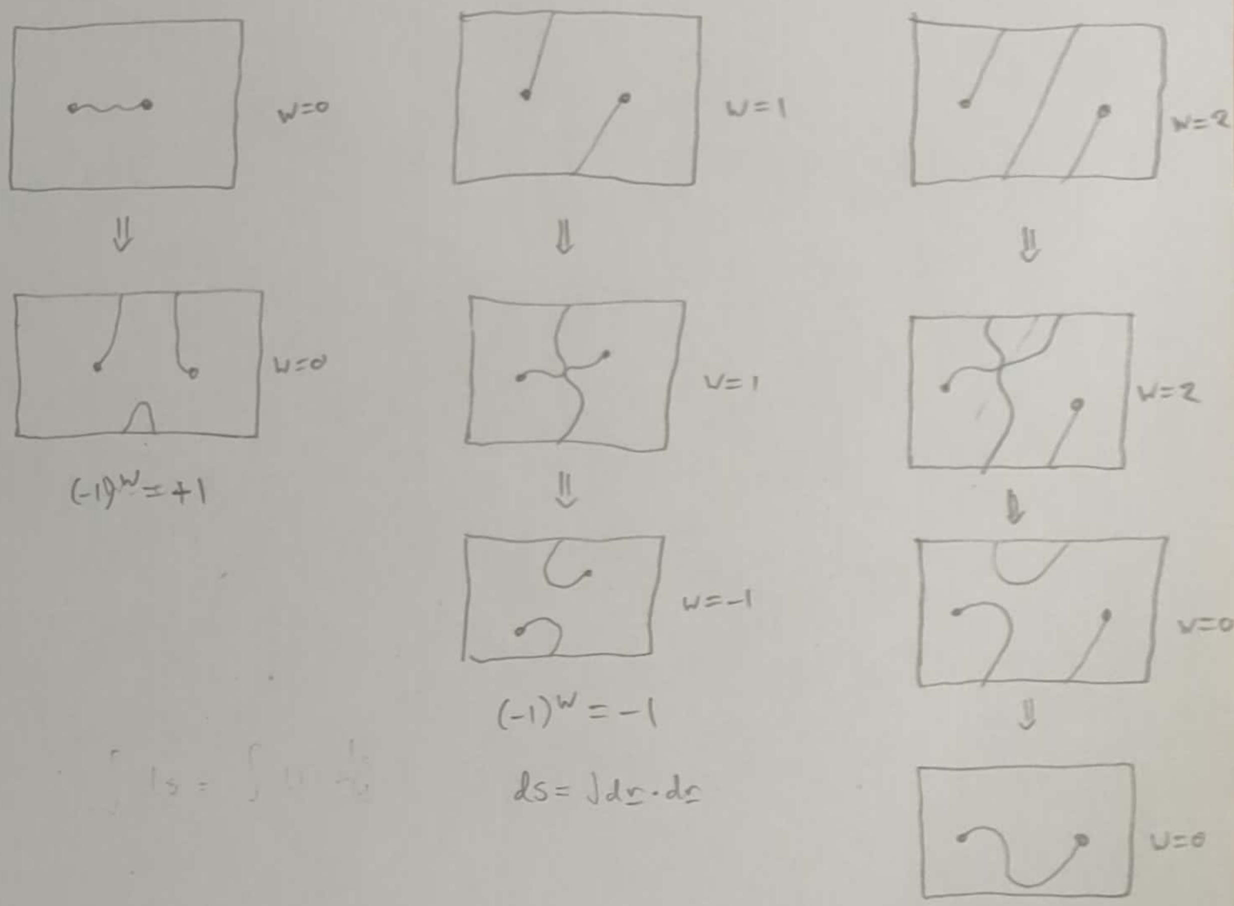
$\mathbb{Q} = 1 - \mathbb{P} =$ "percolation probability"

Take the continuum Limit:

Now, $Z = \sum_{l} \cdot \int_{\substack{\text{start} \\ \text{endpoints}}} \cdot$ energy exp. factor $\propto l_{string}$.

(don't care ~ overlaps ∵ indistinguishable)

Note: One might think we need to do this on the torus BUT only the parity of winding matters ⇒ can just add a 2^{β_1} factor to Z:



$(-1)^w = +1$

$(-1)^w = -1$

$ds = \int d\underline{r} \cdot d\underline{r}$

$(-1)^w = +1$

$G \propto H = ?$

$$Z = 2^{\beta_1} \sum_{l=0}^{N/2} e^{-4\Lambda l} \frac{1}{(2l)!} \prod_{m=1}^{l} \underbrace{\left(\int d^d \underline{x}_m \int d^d \underline{y}_m \int_{\underline{x}}^{\underline{y}} \mathcal{D}\underline{r}(t)\, e^{-G\int dt\, |\dot{\underline{r}}|} \right)}$$

pick 2 ptcls arb.

Need another comb. factor?

Nope, already included in $\frac{1}{(2l)!}$

I also need to account for (contractible) loops generated by plaquette flips...

$$\int_{\underline{x}}^{\underline{y}} \mathcal{D}\underline{r}(t)\, e^{-G\int|d\underline{r}|} = \int_{\underline{x}}^{\underline{y}} \mathcal{D}\underline{r}(t)\, e^{-G\sum_{m=0}^{N-1}|\underline{r}_{m+1}-\underline{r}_m|}$$

$$= \prod_{\substack{m=0 \\ \underline{r}_0=\underline{x} \\ \underline{r}_N=\underline{y}}}^{N-1}$$

Okay, maybe we can think of this as a random walk with variable length for each step?

As a first approx^n, $\int_{\underline{x}}^{\underline{y}} \mathcal{D}\underline{r}\, e^{-G\int|d\underline{r}|} \approx e^{-G|\underline{x}-\underline{y}|}$

$$\Rightarrow Z = 2^{\beta_1} \sum_{\ell} e^{-4\Lambda\ell} \frac{1}{(2\ell)!} V \underbrace{\int d^d\underline{x}\, e^{-G|\underline{x}|}}_{= S_d \int_0^\infty dr\, r^{d-1} e^{-Gr}}$$ • isolated loops

$$= 2^{\beta_1} \sum_{\ell} e^{-4\Lambda\ell} \frac{1}{(2\ell)!} \Big(\frac{V}{G^d} \underbrace{S_d}_{= \frac{(2\pi)^{d/2}}{\Gamma(d/2)}} \Gamma(d) \Big)^{\ell}$$ $G = 2\beta H$?

$$= 2^{\beta_1} \sum_{\ell} e^{-4\Lambda\ell} \frac{1}{(2\ell)!} \left(\frac{1}{\beta H}\right)^{d\ell}$$ • loop factor ⁚⁀

$$Z = 2^{\beta_1} Z_{loops} \sum_{l=0}^{N_V/2} e^{-4\Lambda l} \frac{1}{(2l)!} \left(Z_{string} \right)^{l!}$$

$$Z_{string} = \int d^d \underline{x} \int d^d \underline{y} \; \mathcal{Z}(\underline{y} - \underline{x})$$

$$Z_{loops} = \sum_{n=0}^{\sim N_P} \frac{1}{n!} \left(\int d^d \underline{x} \; \mathcal{Z}(\underline{x} - \underline{0}) \right)^n$$

Where...

$$\mathcal{Z}(\underline{y}) = \int_{\underline{0}}^{\underline{y}} D\underline{r}(t) \, e^{-\alpha \int |d\underline{r}|_n}$$

$\underline{r}_0 = \underline{0}$
$\underline{r}_N = \underline{y}$

$$= \prod_{m=0}^{N-1} \left(\int d^d \underline{r}_m \, e^{-\alpha |\underline{r}_{m+1} - \underline{r}_m|_n} \right)$$

$$= \prod_{m=0}^{N-1} \left(\int d^d \underline{w}_m \, e^{-\alpha |\underline{w}_m|_n} \right) \delta\left(\sum_{k=0}^{N-1} \underline{w}_k - \underline{y} \right)$$

Then use $\quad \delta(\underline{x}) = \int đ^d \underline{k} \, e^{i \underline{k} \cdot \underline{x}}$

$$\Rightarrow \mathcal{Z}(\underline{y}) = \int đ^d \underline{k} \, e^{-i \underline{k} \cdot \underline{y}} \prod_{m=0}^{N-1} \underbrace{\left(\int d^d \underline{w}_m e^{i \underline{k} \cdot \underline{w}_m} \overbrace{e^{-\alpha |\underline{w}_m|_n}}^{g(w)} \right)}_{F_m(k)}$$

For the 2-norm...

$$f_m(k) = \int d^d \underline{w} \; e^{i \underline{k} \cdot \underline{w}} \; g(w)$$

$$= \dots$$

$$= k \left(\frac{2\pi}{k} \right)^{d/2} \int_0^\infty dw \; w^{d/2} \; J_{d/2-1}(kw) \, \overbrace{e^{-\alpha w}}^{\shortparallel \; g(w)}$$

$$\Rightarrow \mathcal{Z}(\underline{y}) = \int đ^d \underline{k} \; e^{-i \underline{k} \cdot \underline{y}} \; F_m(k)$$

$$= y \left(\frac{2\pi}{y} \right)^{d/2} \int_0^\infty dk \; k^{d/2} \; J_{d/2-1}(-ky) \; f_m(k)$$

$\mathcal{N}(y) = $ take $(2\pi)^{d/2} \; y^{1-d/2} \int_0^\infty dk\, k^{d/2} J$

$$\Rightarrow \mathcal{Z}(y) = (2\pi)^{d/2} \, y^{1-d/2} \int_0^\infty dk\, k^{d/2} \, J_{d/2-1}(ky)$$

$$\cdot \prod_{m=0}^{N-1} \left((2\pi)^{d/2} \, k^{1-d/2} \int_0^\infty dw\, w^{d/2} \, J_{d/2-1}(kw) \, e^{-\alpha w} \right)$$

$$= (2\pi)^{\frac{d}{2}(N+1)} \, y^{1-\frac{d}{2}} \int_0^\infty dk \; k^{N-\frac{d}{2}(N-1)} \prod_{m=0}^{N-1} \left(\int_0^\infty dw\, w^{d/2} \, J_{d/2-1}(kw) \, e^{-\alpha w} \right)$$

$$= (2\pi)^{\frac{d}{2}(N+1)} \, y^{1-\frac{d}{2}} \prod_{m=0}^{N-1} \left(\int_0^\infty dw_m \, w_m^{d/2} \, e^{-\alpha w_m} \right) \int_0^\infty dk\, k^{N-\frac{d}{2}(N-1)}$$

$$\cdot J_{\frac{d}{2}-1}(ky) \prod_{m=0}^{N-1} J_{\frac{d}{2}-1}(k w_m)$$

Take $d = 2$: (J_0 even)

$$\mathcal{Z}(y) = \cancel{y^{1-\frac{d}{2}}} \, (2\pi)^{N+1} \prod_{m=0}^{N-1} (\dots) \int_0^\infty dk\, k \; J_0(ky) \, J_0(kw_0) \dots J_0(kw_{N-1})$$

Icky... maybe try and find

$f_1 = \int_0^\infty dw \; w^1 \, J_0(kw) \, e^{-\alpha w}$ first? $\qquad J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{4^n \, n!^2}$

We know $f_0 = \frac{1}{\sqrt{k^2+\alpha^2}}$

$$-\frac{\partial f_0}{\partial \alpha} = \int_0^\infty dw\, w\, J_0(kw)\, e^{-\alpha w} = f_1$$

$$\Rightarrow f_1 = -\frac{\partial}{\partial \alpha} (k^2+\alpha^2)^{-1/2} = \cancel{\tfrac{1}{2}} \cdot \cancel{2}\alpha \, (k^2+\alpha^2)^{-3/2}$$

$$\Rightarrow \mathcal{Z}(y) = (2\pi\alpha)^{N+1} \cdot \frac{1}{2\pi} \int_0^\infty dk\; k \, J_0(ky) \, \frac{\alpha^{-1}}{(k^2+\alpha^2)^{3N/2}} .$$

$$\mathcal{Z}(0) = (2\pi\alpha)^{N+1} \, \frac{1}{2\pi} \int_0^\infty \frac{dk\, k}{(k^2+\alpha^2)^{3N/2}} \qquad x = k^2 \quad (x+\alpha^2)^{-3N/2}$$

$$= (2\pi\alpha)^{N+1} \, \frac{1}{2\pi} \left[\frac{2}{3N} \, \frac{1}{(k^2+\alpha^2)^{3N/2-1}} \right]_\infty^0$$

$(n=1)$-norm Case

$$f_m(\underline{k}) = \prod_{i=1}^{d} \left(\int_{\mathbb{R}} dw \, e^{ik_i w} \, e^{-\alpha |w|} \right)$$

$$= \prod_{i=1}^{d} \left(\frac{2\alpha}{\alpha^2 + k_i^2} \right)$$

$$\Rightarrow \; Z(y) = \int_{\mathbb{R}} \text{đ}^d \underline{k} \; e^{-i\underline{k}\cdot \underline{y}} \cancel{\prod_{m=0}^{N-1}} \left(\prod_{i=1}^{d} \frac{2\alpha}{\alpha^2 + k_i^2} \right)^N$$

$$= \prod_{i=1}^{d} \int_{\mathbb{R}} \text{đ}k \; e^{-ik_i y_i} \left(\frac{2\alpha}{\alpha^2 + k_i^2} \right)^N$$

$$= \prod_{i=1}^{d} (2\alpha)^N \int_{\mathbb{R}} \frac{\text{đ}k \, e^{-ik_i y_i}}{(\alpha^2 + k_i^2)^N}$$

$N = ?$

$$\Rightarrow \quad \mathcal{Z}(0) = (2\pi\alpha)^{N+1} \frac{1}{2\pi} \frac{2}{3N} \frac{1}{\alpha^{3N-2}}$$

$$= (2\pi)^N \frac{2}{3N} \alpha^{N+1+2-3N}$$

$$= \left(\frac{2\pi}{\alpha^{2}}\right)^N \frac{2}{3N} \alpha^3 .$$

$$\propto \alpha^{3-2N}$$

$\alpha \propto \beta H$

~~$\Rightarrow$ is important?~~

~~No! Just a prefactor!~~

divergent prefactor...

$$\Rightarrow \quad Z_{\text{loops}} = \sum_n \frac{1}{n!}\left(A\, \alpha^{3-2N}\right)^n$$

$$\approx \frac{1}{1 - A\alpha^{3-2N}} \quad \underline{\underline{\text{if}}} \text{ convergent...}$$

$$Z_{\text{string}} = \int d^2y\, \mathcal{Z}(y)$$

$$= 2\pi \int_0^\infty dy\, y\, \mathcal{Z}(y).$$

$$= (2\pi\alpha)^{N+1} \int_0^\infty \frac{dk\, k}{(k^2+\alpha^2)^{3N/2}} \underbrace{\int_0^\infty dy\, y\, J_0(ky)}_{G}$$

$$G = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2 4^{2n}} \int_0^\infty dy\, y^{1\pm 2n} + \ldots \quad ??$$

$$J = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2 4^n} \frac{1}{2(n+1)} [\, y$$

$$\mathcal{Z}(y) = \int_0^y \mathcal{D}\underline{r}(t)\, e^{-\alpha \int |d\underline{r}_m|_n}$$

$$= \lim_{N\to\infty} \prod_{m=0}^{N-1} \left(\int d^d \underline{r}_m\, e^{-\alpha |\underline{r}_{m+1} - \underline{r}_m|_n} \right)$$

$$= \lim_{N\to\infty} \prod_{m=0}^{N-1} \left(\int d^d \underline{w}_m\, e^{-\alpha |\underline{w}_m|_n} \right) \delta\left(\sum_{k=0}^{N-1} \underline{w}_k - y \right)$$

But this is (a) divergent & (b) doesn't capture the physics for generic graphs.

A discrete version would be better formulated as

ℓ = length of path

$$\mathcal{Z}(y) = \sum_{\ell=\ell_{min}}^{\ell_{max}} e^{-2H\ell}\, \Delta_{\#\text{ length-}\ell\text{ paths}}.$$

Let's start w/. Z_{loop} for a single loop.

For length l, there must be...

n_x x_+ moves

n_x x_- moves

$n_y = k - n_x$ y_+ moves

$n_y = k - n_x$ y_- moves

w/. $2(n_x + n_y) = l = 2k$

Must have l even obviously.

$$\Rightarrow \Delta(n_x|l) = \underbrace{\binom{l}{n_x}}_{x_+ \text{ moves}} \underbrace{\binom{l-n_x}{n_x}}_{x_- \text{ moves}} \underbrace{\binom{l-2n_x}{l/2 - n_x}}_{y_+ \text{ moves}}$$

& y_- are left over.

$$\Delta(l) = \sum_{n_x=0}^{l/2} \Delta(n_x|l)$$

$$= \sum_{m=0}^{k} \binom{2k}{2m}\binom{2k-2m}{2m}\binom{2k-4m}{k-2m} \quad ?$$

$$\Delta(2p) = \sum_{s=0}^{p} \binom{2p}{2s}\binom{2s}{s}\binom{2p-2s}{p-s}$$

$$= \ldots$$

$$= \binom{2p}{p}^2 = \binom{l}{l/2}^2$$

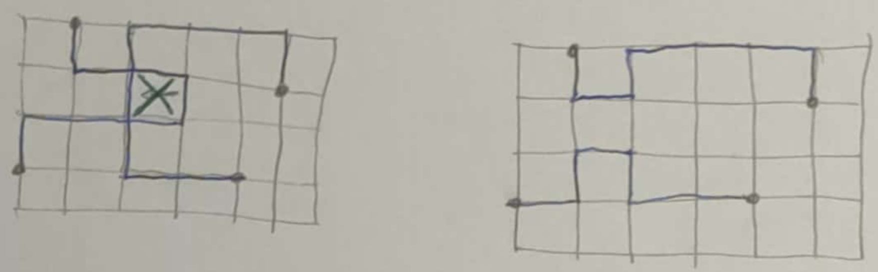
$l = 0, 2, \ldots, N_E$

Starting site

$$\Rightarrow Z_{loop} = N_V \cdot \sum_{p=0}^{N_E/2} \binom{2p}{p}^2 e^{-2H(2p)}$$

$$= N_V \sum_{p=0}^{N_E/2} \binom{2p}{p}^2 e^{4Hp}.$$

M



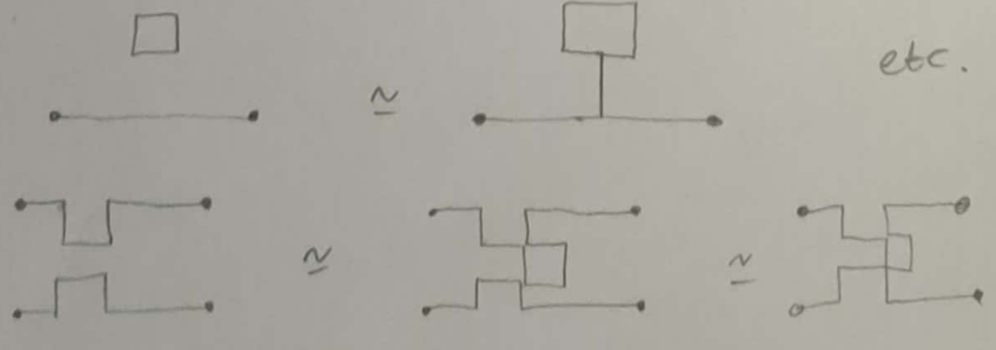
→ Can generate <u>all</u> pairings of vertices using plaquette flips.

<u>SO</u> all configs of given $\sum_i A_i$ are generated by starting w/. the vertices linked up & then applying plaquette flips. ($\cdot\, 2^{\beta_1}$ for topo sectors)

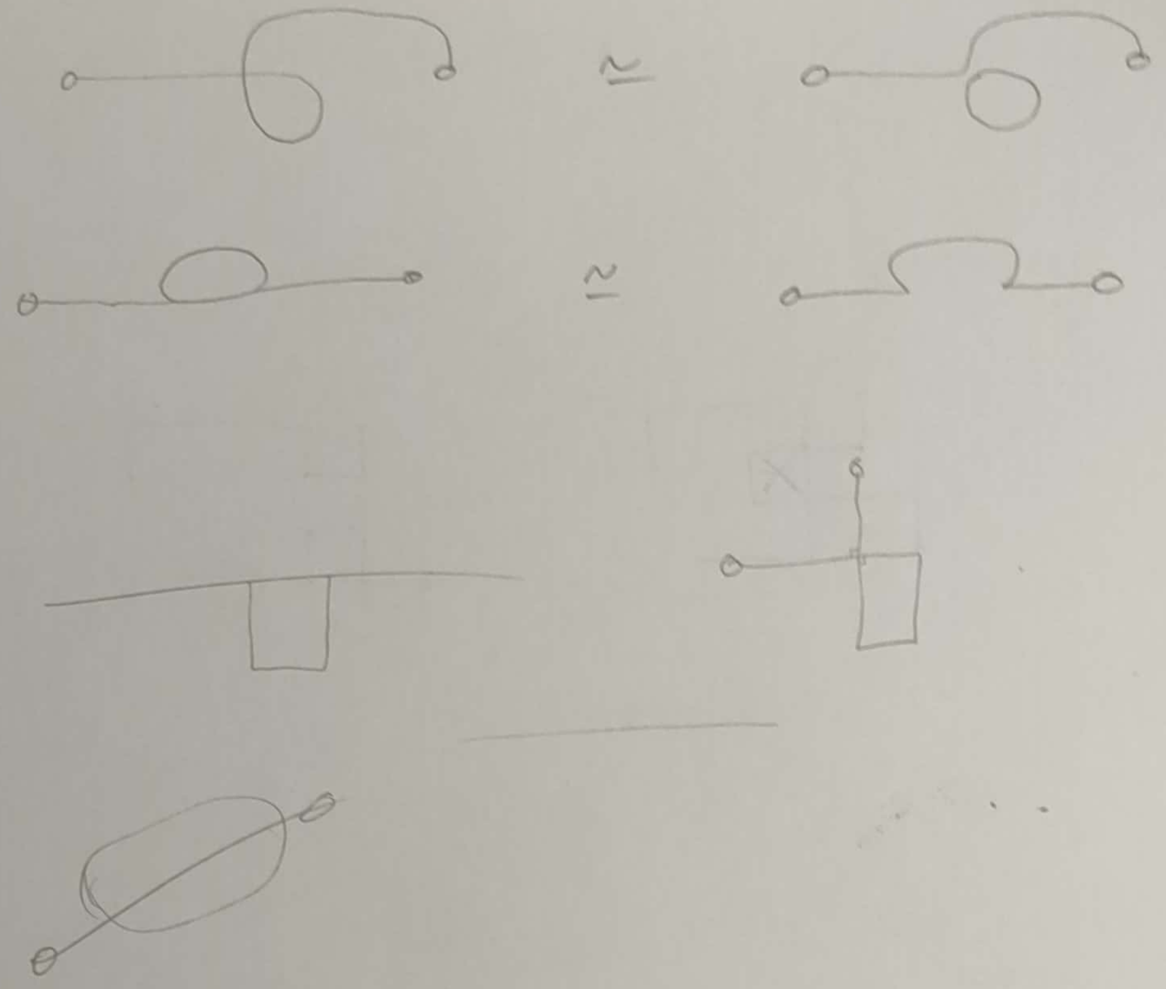
<u>BUT</u> this is bad ∵ $\sum_\alpha \sigma_\alpha$ changes...

→ I want to reorder sum to be over n°s of blue edges.

Suppose I allow <u>any</u> path. Is this sufficient to generate <u>every</u> config?



etc.



* Square lattice
* Pick 2ℓ vertices

I want the number of configurations

Alternative: Enforce δ-constraint w/. Lagrange multiplier $\underline{\eta}$.

$$\bar{Z}(\underline{\eta}) = \int d^d y \; Z(y)\, e^{-\underline{\eta}\cdot\underline{y}} \qquad \text{Laplace tf.}$$

$$= \prod_{m=0}^{N-1} \Big(\underbrace{\int d^d \underline{w}_m \, e^{-\underline{\eta}\cdot\underline{w}_m} \, e^{-\alpha |\underline{w}_m|_n}}_{f_m(\underline{\eta})} \Big)$$

For the 1-norm...

$$f_m(\underline{\eta}) = \prod_{i=1}^{d} \Big(\int_{\mathbb{R}} dw \, e^{-\eta_i w} \, e^{-\alpha |w|} \Big)$$

diverges unless $\eta_i \in i\mathbb{R}$. $\Rightarrow$ write $\underline{\eta} = i\underline{s}$

$$f_m(\underline{s}) = \prod_{i=1}^{d} \Big(\int_{\mathbb{R}} dw \; e^{-\alpha|w|} \, e^{-i s_i w} \Big)$$

$$= \prod_{i=1}^{d} \Big(\frac{2\alpha}{\alpha^2 + s_i^2} \Big)$$

$$\Rightarrow \bar{Z}(\underline{s}) = \Big(\prod_{i=1}^{d} \frac{2\alpha}{\alpha^2 + s_i^2} \Big)^N$$

$$\bar{F} = -\frac{1}{\beta} \ln \bar{Z} = -\frac{N}{\beta} \Big(\sum_{i=1}^{d} \ln\Big(\frac{2\alpha}{\alpha^2 + s_i^2}\Big)\Big)$$

$$\langle \underline{y} \rangle = \underline{\nabla}_s \bar{F} \Rightarrow -\frac{N}{\beta} \cdot \frac{2 s_i}{(\alpha^2+s_i^2)^2} \cdot \frac{\alpha^2+s_i^2}{2\alpha} = -\frac{2N}{\beta} \frac{s_i}{(\alpha^2+s_i^2)} = \langle y_i \rangle$$

$$F(\langle \underline{y} \rangle) = \bar{F}(\underline{s}) - \underline{s}\cdot\langle \underline{y} \rangle \qquad \langle \underline{y} \rangle = \underline{z}$$

$$\Rightarrow 0 = s_i^2 + \frac{2N}{\beta z_i} s_i + \alpha^2$$

$$\Rightarrow \Big(s_i + \frac{N}{\beta z_i}\Big)^2 - \Big(\frac{N}{\beta z_i}\Big)^2 + \alpha^2 = 0 \quad \leadsto \quad s_i = -\frac{N}{\beta z_i} \pm \sqrt{\Big(\frac{N}{\beta z_i}\Big)^2 - \alpha^2}$$

For the ℓ-norm...

$\bar{Z}(y) = \prod_{m=0}^{N-1} f_m(y) = (f_m(y))^N \qquad y = i\underline{s}$ for convergence.

$$f_m(\underline{s}) = \int d^d \underline{w}\, e^{-\alpha |\underline{w}|}\, e^{-i \underline{s}\cdot\underline{w}}$$

$$= s \left(\frac{2\pi}{s}\right)^{d/2} \int_0^\infty dw\, w^{d/2}\, J_{d/2-1}(-sw)\, e^{-\alpha w}$$

$$\Rightarrow \bar{F}(y) = -\frac{1}{\beta} \ln(\bar{Z}(y))$$

$$= -\frac{N}{\beta} \ln f_m(y)$$

$\underline{z} = \langle y \rangle_{\bar{Z}}$

$$F(\underline{z}) = \bar{F}(y) - y\cdot\underline{z}$$

$$= \bar{F}(\underline{s}) - i\underline{s}\cdot\underline{z}$$

$$\underline{z} = \nabla_y \bar{F}(y) = -i \nabla_s \bar{F}(\underline{s}) = +i \frac{N}{\beta} \frac{1}{f_m} \nabla_{\underline{s}} f_m$$

$$\frac{\partial f_m}{\partial \underline{s}} = \frac{\partial s_i}{\partial s} \frac{\partial f_m}{\partial s_i}$$

$s^2 = s_i s_i$

$$\Rightarrow \frac{\partial s}{\partial s_i} = \frac{s_i}{s}$$

$$\Rightarrow s \nabla_{\underline{s}} f_m = \underline{s} \frac{\partial f_m}{\partial s}.$$

$$\Rightarrow \underline{z} = i \frac{N}{\beta} \frac{1}{f_m} \frac{1}{s} \underline{s} \frac{\partial f_m}{\partial s}$$

$$\Rightarrow F(\underline{z}) = \bar{F}(\underline{s}) + \frac{N}{\beta} \frac{s}{f_m} \frac{\partial f}{\partial s}$$

So... Can I consider non-self avoiding walks (paths) on $G \setminus \{$other paths' edges?$\}$.

No. Walks are no good...

Okay. Gotta be careful

String 1: Path on G.

⋮

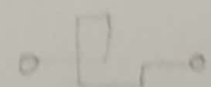
String 2: Path on $G \setminus \{$loop 1 edges$\}$.

⋮

Loops also need to be on $G \setminus \{$other things$\}$ (paths o → o)

"Path" = walk but disallowing repeated edges (repeated vertices fine?)

↓



Sure, but better to not allow these states & put them in with loops later.

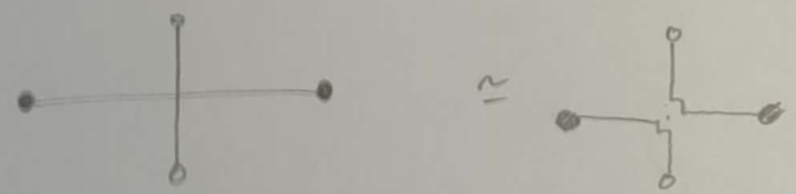
⇒ "Path" = walk w/o repeated vertices (⇒ edges)

This sounds nice <u>but</u> doesn't work ∵ diff. n° blue edges...

Alt: Pick unique paths between prtcls & use plaquette flips to move them / add loops.

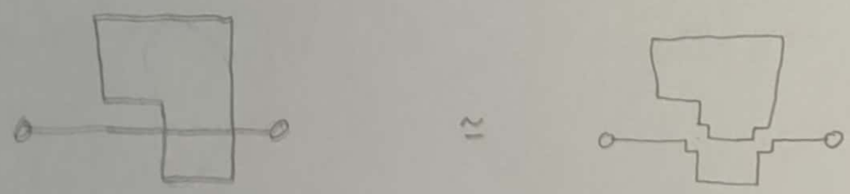
But ~~this~~ this is <u>also</u> a problem. ∵ plaquette flips also change #.

Okay. I have a hypothesis. I can treat the ~~strings~~ strings as indep. as well as the loops!



BUT let's not do that yet?

~~Loops~~ Loops indep ✓.



ah. Shit.

Ideas:

Replica trick: $\lim_{n\to 0} \frac{Z^n - 1}{n} = \ln Z$.

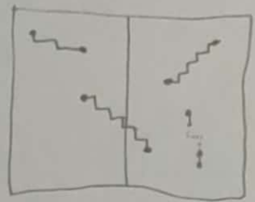
① Replace η w/. some sum of η's that smooths out the variations.

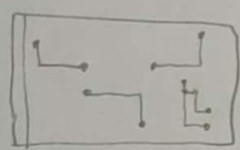
② Define unique ref. state.

* Pick loops to be straight lines (not overlapping prtcl strings, ~~ideally~~)

* Pick strings to be diagonals ?

$\Rightarrow$ if we have:  is okay.

$\Rightarrow$ $\underline{\sigma}^{ref}_{v}$ =  $W = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

= 

BUT

 $\}$ Given pts in between each other, can $\underline{\text{always}}$ accommodate min-dist. pairing

$\Rightarrow$ In σ^{ref}, # blue edges =

$x_i \leftrightarrow x_{2i}$ arb.

$$\sum_{p=1}^{\ell} |x_p - x_{2p}|_1 \to \text{1-norm}$$

$$+ \sum_{q=1}^{b_1} \begin{Bmatrix} W_q = 1 : L_q \\ W_q = 0 : 0 \end{Bmatrix}$$

minimal length of q^{th} non-cont. loop.

element-wise product

$$= \sum_{p=1}^{\ell} |x_p - x_{2p}|_1 + |W \odot L|_1 = |\mathcal{L}|$$

Can we use this?

Approximation: $\sum_{\langle pq\rangle} \eta_{pq} \mu_p \mu_q \simeq \sum_{\langle pq\rangle} \mu_p \mu_q - 2 \sum_{\alpha \in \mathcal{L}} \mu_{p_\alpha} \mu_{q_\alpha}$

Assume $\underline{\text{bulk wins}}$ $\Longrightarrow$ $\approx \sum_{\langle pq\rangle} \mu_p \mu_q - 2|\mathcal{L}|$

$$\Rightarrow \quad \hat{Z}_w(\underline{x}) \approx e^{-2H|\ell_w(\underline{x})|} \underbrace{\sum_{\underline{\mu}} e^{H\sum_{\langle pq\rangle}\mu_p\mu_q}}_{Z^{(H)}_{Ising}}$$

$$\Rightarrow Z^{(\,)} \approx \sum_{\ell=0}^{N_v/2} e^{-4\Lambda\ell} \underbrace{\frac{1}{(2\ell)!}\sum_{\underline{x}_1\neq\cdots\neq\underline{x}_{2\ell}} e^{-2H\sum_{p=1}^{\ell}|\underline{x}_p-\underline{x}_{2p}|_1}}_{I} \underbrace{\sum_{W=\underline{0}}^{1} e^{-2H|W\cdot L|_1}}_{J} Z^{(H)}_{Ising}$$

$$J = \prod_{n=1}^{b_1}\left(\sum_{W=0}^{1} e^{-2HWL_n}\right) = \prod_{n=1}^{b_1}\left(1+e^{-2HL_n}\right)$$

$$I = \frac{1}{(2\ell)!}\cdot\left(\sum_{\underline{x}_1\neq\cdots\neq\underline{x}_{2\ell}} e^{-2H\sum_{p=1}^{\ell}|\underline{x}_p-\underline{x}_{2p}|_1}\right) \quad \text{ew.}$$

$$\approx \frac{1}{(2\ell)!}\cdot\left(\int d^2\underline{x}\int d^2\underline{y}\; e^{-2H|\underline{x}-\underline{y}|_1}\right)^{\ell}$$

continuum limit & ignore collis^ns

$\because$ 1-norm

$$= \frac{1}{(2\ell)!}\left(\int_0^{L_1}dx\int_0^{L_1}dy\; e^{-2H|x-y|}\cdot(\text{''})_{L_2}\right)^{\ell}$$

$\because$ 2D

$x \to x/L_1$

$dx \to dx/L_1$

$$= \frac{1}{(2\ell)!}\left(L_1^2\,\frac{2-2HL_1-2e^{-HL_1}}{H^2L_1}\cdot(\text{''})_{L_2}\right)^{\ell}$$

$$= \frac{1}{(2\ell)!}\left(\frac{2}{H^2}\right)^{2\ell}N_v^{\ell}\left(1-HL_1-e^{-HL_1}\right)^{\ell}\left(1-HL_2-e^{-HL_2}\right)^{\ell} \quad N_v = L_1L_2$$

H small

$$\approx \frac{1}{(2\ell)!}N_v^{2\ell}\left(1-\tfrac{1}{3}HL_1\right)^{\ell}\left(1-\tfrac{1}{3}HL_2\right)^{\ell}$$

$$\sim \binom{N_v}{2\ell} + O(\ell HL)$$

$$\left(\int d^2\underline{x} \int d^2\underline{y}\; e^{-2H|\underline{x}-\underline{y}|_1}\right)^{\ell} = I$$

$$= \left(\int_{-L_1/2}^{L_1/2} dx \int_{-L_1/2}^{L_1/2} dy\; e^{-2H|x-y|} \cdot (\,'' \,)_{L_2}\right)^{\ell}$$

$$= \left(\int_{-L_1/2}^{L_1/2} dx \underbrace{\int_{\underset{[-L_1,0]}{\overset{\wedge}{x-L_1/2}}}^{\overset{\in[0,L_1]}{x+L_1/2}} dz\; e^{-2H|z|}}_{} \cdot (\,''\,)_{L_2}\right)^{\ell} \quad \Big\downarrow \quad \begin{array}{l} z = x-y \\ dy = -dz \end{array}$$

$$\underbrace{\begin{aligned} &= \int_0^{x+L/2} dz\, e^{-2Hz} - \int_0^{x-L/2} dz\, e^{2Hz} \\ &= \frac{1}{2H}\left(\left[e^{-2Hz}\right]_{x+L/2}^{0} + \left[e^{2Hz}\right]_{x-L/2}^{0}\right) \\ &= \frac{1}{2H}\left(1 - e^{-2Hx-HL} + 1 + e^{2Hx-HL}\right) \\ &= \frac{1}{H}\left(1 - e^{-HL}\cos(Hx)\right) \end{aligned}}_{}$$

$$= \frac{1}{H}\left(L_1 - e^{-HL}\int_{-L/2}^{L/2} dx \cos(Hx)\right)$$

$$e^{-HL}\frac{2}{H}\sin(HL/2) \cancel{\sin}$$

$$= \frac{L_1}{H} + \frac{2e^{-HL}}{H^2}\sin(HL_1/2)$$

$$I = \left(\frac{L_1}{H}\left(1 + e^{-HL_1}\mathrm{sinc}(HL_1/2)\right)\frac{L_2}{H}\left(1 + e^{-HL_2}\mathrm{sinc}(HL_2/2)\right)\right)^{\ell}$$

$$= \frac{N_v^{\ell}}{H^{2\ell}}\left(\ldots\right)^{\ell}\left(\ldots\right)^{\ell} \qquad N_v = L_1 L_2$$

$$\Big\downarrow \; H \ll L$$

$$= \frac{N_v^{\ell}}{H^{2\ell}}\left(1 + (1-HL_1)\right)^{\ell}\left(1 + (1-HL_2)\right)^{\ell}$$

$$\sim \frac{N_v^{2\ell}}{H^{\ell}}.$$

$$J \sim \prod_{n=1}^{b_1}(2 - 2HL_n) = 2^{b_1}\prod(1-HL_n) = 2^{b_1}\prod e^{-HL_n}$$

$$= 2^{b_1} e^{-H\sum L_n}$$

$$\int \cancel{d^2 x_1 \delta_1} \int d^2 y_1 \delta_1 e^{-2H|x_1-y_1|} \; \dots \int d^2 x_\ell \delta_\ell \int d^2 y_\ell \, e^{-2H|x_\ell-y_\ell|} \left(1-\delta(x_1 \dots\right.$$

$$\Rightarrow Z \approx \sum_{\ell=0}^{N_v/2} e^{-4\Lambda \ell} \underbrace{\frac{N_v^{2\ell}}{(2\ell)!}}_{\approx \binom{N_v}{2\ell} \text{ for } 2\ell \ll N_v} \frac{1}{H^\ell} \overset{\text{const.}}{\cancel{2^{b_1}}} \prod_{n=1}^{b_1} e^{-2HL_n} \; Z_{ising}^{(H)}$$

$$\tilde{\propto} \; Z_{ising}^{(H)} \, e^{-2H \overset{\sim -4HL}{\cancel{\sum_n L_n}}} \sum_{\ell=0}^{N_v/2} e^{-4\Lambda\ell} \binom{N_v}{2\ell} \left(\frac{1}{H}\right)^\ell$$

$$Z \propto \underbrace{Z_{Ising} \sum_{W=0}^{1/2} e^{-2H|W \cdot L|_1}}_{\underline{\text{Cancels}} \text{ in } K.} \cdot \underbrace{\sum_{l=0}^{Nv/2} \frac{e^{-4\Lambda l}}{(2l)!} \sum_{\{\underline{x}_p\}} e^{-2H \sum_{q=1}^{l} |x_q - x_{2q}|_1}}_{Z_1}$$

$$\sum_{\underline{\sigma}} O(\underline{\sigma}) e^{-\beta E[\underline{\sigma}]} = \frac{1}{Z_1} \sum_{l=0}^{Nv/2} \frac{e^{-4\Lambda l}}{(2l)!} \sum_{\{\underline{x}_p\}} O_{\{\underline{x}_p\}} e^{-2H \sum_{q=1}^{l} \underbrace{|x_q - x_{2q}|_1}_{d(n_q, n_{2q})}}$$

We consider first $O(\underline{\sigma}) = \epsilon_i(\underline{\sigma})$

$\{\underline{x}_p\} \to \{n\} = \underline{n}$

$\delta_{i \in \underline{n}} = \sum_j \delta_{i = n_j}$

$$\Rightarrow O_{\{\underline{x}_p\}} = (2\lambda)\, \delta_{i \in \{\underline{x}_p\}}$$

$$\Rightarrow \langle \epsilon_i \rangle_{eq} = \frac{1}{Z_1} \sum_{l=0}^{Nv/2} \frac{e^{-4\Lambda l}}{(2l)!} (2\lambda) \sum_{\underline{n}} \delta_{i \in \underline{n}}\, e^{-2H \sum_{q=1}^{l} d(n_q, n_{2q})}$$

$$= (2\lambda) \frac{1}{Z_1} \sum_{l=0}^{Nv/2} \frac{e^{-4\Lambda l}}{(2l)!} \sum_{p=1}^{2l} \sum_{\underline{n}} \delta_{i = n_j}\, e^{-2H \sum_{q=1}^{l} d(n_q, n_{2q})}$$

$$= (\ldots) \sum_{l} (\ldots) \underbrace{\cancel{\sum_{j=1}^{2l}}}_{2l} \underbrace{\sum_{m} e^{-2H\, d(i, m)}}_{D} \underbrace{\sum_{\underline{n} \setminus \{n_j, n_{2j}\}} e^{-2H \sum_{q \neq j} d(\ldots)}}_{C_l}$$

$m = n_{2j}$

D: $\underline{\text{indep}}$ of i $\because$ periodic lattice.

C_l: Sum over $2l-2$ sites (no collisions) s.t. none @ i or m.

B_l: Sum over $2l-1$ sites (no collisions) s.t. none @ i

Likewise,

$$\langle \epsilon_i(\Omega^{t}(\sigma))\, \epsilon_j(\sigma) \rangle = (2\lambda)^2 \frac{1}{Z_1} \sum_{\substack{\ell \\ \underline{n}}} \frac{e^{-4\Lambda \ell}}{(2\ell)!} I_\ell^{ij}$$

$$I_\ell^{ij} = \sum_{\underline{n}} \sum_{p,q=1}^{2\ell} \delta_{i=n_p^{+}}\, \delta_{j=n_q}\, e^{-2H \sum_{r=1}^{\ell} d(n_r,\, n_{2r})}$$

Question: for $2l$ large, can we switch to a hole model?

In μcan dynamics, yes! $\exists$ prtcl-hole symm.

Suppose $2l = N_v$: $\sum_{z_1 \neq \dots \neq z_{N_v}} = (2l)! = N_v!$ (worth checking)

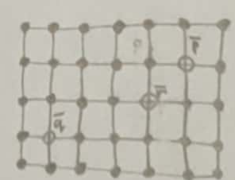
Suppose $\overset{z}{2h} = N_v - \overset{y}{2l}$ = nº holes

$$\cancel{\sum_{z_1 \neq \dots \neq z_{N_v}} = \sum_{z_1 \neq \dots \neq z_{2h}} \sum_{y_1 \neq \dots \neq y_{2l}}}$$

$$\frac{N!}{2l!\,(N-2l)!} = \frac{N!}{2h!\,(N-2h)!}$$

$$\sum_{pq} \sum_{z_1 \neq \dots \neq z_{N-2h}} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{q}(0)$$

$$= \sum_{y_1 \neq \dots \neq y_{2h}} \sum_{pq} \dot{x}^{\mu}_{p}(t)\, \dot{x}^{\nu}_{q}(0)$$



All $\dot{x} = 0$ except next to holes.

holes undergo random walk

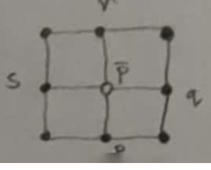
↓

how do we write the $\dot{x}$ in terms of $\dot{y}$?

$$= x^{\mu}_{p}(t+1)\, x^{\nu}_{q}(1) - x^{\mu}_{p}(t)\, x^{\nu}_{q}(1)$$
$$- x^{\mu}_{p}(t+1)\, x^{\nu}_{q}(0) + x^{\mu}_{p}(t)\, x^{\nu}_{q}(0)$$

For our hole RW approx^{n}, just look @ 1 ~~prtcl~~ hole:

$t=0$



$$\dot{x}_{\bar{p}} = \begin{cases} -\dot{x}_q \\ \vdots \\ s \\ p \end{cases}$$

$n = N_v - \ldots$

$$\left(\frac{(2\lambda)^2}{Z_1} \sum_{h=0}^{N N_v^2} \frac{e^{2\Lambda h}}{h!} \underbrace{\sum_{n_1 \neq \ldots \neq n_h} \delta_{i \notin \underline{n}^+} \delta_{j \notin \underline{n}}}_{\mathcal{M}} \right) \cdot e^{-2N_v \Lambda}$$

h can be odd!

$$\delta_{i \in \underline{n}} = \sum_j \delta_{i = n_j}$$

$$\mathcal{M} = \sum_{\langle \underline{n} \rangle} \left(1 - \sum_p \delta_{i = n_p^+}\right)\left(1 - \sum_q \delta_{j = n_q}\right)$$

$$= \sum_{\langle \underline{n} \rangle} \left(1 + \sum_{pq} \delta_{i = n_p^+} \delta_{j = n_q} - \sum_p \delta_{i = n_p^+} - \sum_q \delta_{j = n_q}\right)$$

$$I_{t0} = \sum_{ij} r_i^\mu r_j^\nu \mathcal{M}$$

$$= \sum_{\langle \underline{n} \rangle} \left(\sum_{ij} r_i^\mu r_j^\nu + \sum_{pq} r_{n_p^+}^\mu r_{n_q}^\nu + \sum_{pj} r_{n_p^+}^\mu r_j^\nu - \sum_{iq} r_i^\mu r_{n_q}^\nu \right)$$

Take eg. $\sum_{iq} r^\mu_i r^\nu_{n_q}$ in $J_{t,0}$

$$= \sum_{iq} r^\mu_i \left(r^\nu_{n_q} + r^\nu_{n_q^+} - r^\nu_{n_q} - r^\nu_{n_q^+} \right) = 0$$

& likewise for const. term

$\Rightarrow$ get exactly the same sum except

except

$$\underbrace{\frac{1}{(2h)!} \sum_{y_1 \neq \ldots \neq y_{2h}}}_{\sim \binom{N_v}{2l} = \binom{N_v}{2l}} r^\mu_p(t) \, r^\nu_q(0)$$

$\sum_{pq} \sim 2h$ not $2l$ & $\Lambda \to -\Lambda$

$\Rightarrow$ for avg. dynamics it's the same!

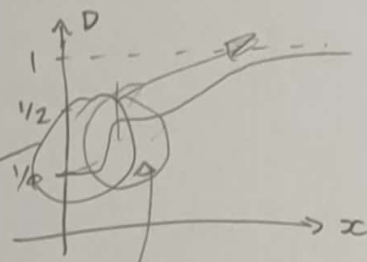
$(1+e^{-2\Lambda})^N + (1-e^{-2\Lambda})^N$

$=$

In the limit, $K \sim$ const.

$C \sim$ subs. peak $\longrightarrow$ $\tau = 0^+$

$\Rightarrow$ $D \rightarrow$



Can't see for N large

(no factor of 4 for timestep)

Take x large $\Rightarrow$ D goes very slowly to 1

as $\tau \rightarrow \infty$

$\Rightarrow$ we see this bit $\frac{1}{2}$ – ~ flat.

$\Rightarrow$ flat @ $D = \frac{1}{2}$ in limit. $N \rightarrow \infty$ $\Rightarrow$

indep. random walks.

Why out?. Would expect

$$D \sim \frac{a^2}{2d\,\delta t} \overset{d=2}{\sim} \frac{1}{4\,\delta t} \qquad \delta t = \frac{1}{4}$$

$\Rightarrow$ $D \sim 1$

I don't get where the factor of 2 comes from — N_V vs N_E? Or sth else?

$$K = \frac{(2\lambda)^2}{Z_1} \frac{1}{N_E T^2} \cdot \underbrace{\frac{1}{2} \sum_{l=0}^{N\nu/2} \binom{N\nu}{2l} e^{-4\Lambda l} \, 2l}_{J}$$

$$C = \frac{1}{N_E} \frac{\partial}{\partial T}\left(T^2 \frac{\partial}{\partial T} \ln Z_1\right) =$$

$$= \frac{1}{N_E} \beta^2 \frac{\partial^2}{\partial \beta^2} \ln Z_1$$

$$Z_1 = \sum_{l=0}^{N\nu/2} \binom{N\nu}{2l} e^{-4\Lambda l}$$

$$= \frac{1}{2} \sum_{p=0}^{N\nu} (1+(-1)^p) \binom{N\nu}{p} (e^{-2\Lambda})^p$$

$$J = \frac{1}{2} \sum_{p=0}^{N\nu} \underbrace{(1+(-1)^p)}_{\alpha_p} \binom{N\nu}{p} (e^{-2\Lambda})^p \; p$$

$$\underbrace{\binom{N}{p} p}\, x^p = N \binom{N-1}{p-1} x^p \quad \text{for } p > 0$$

$$\Rightarrow J = \cancel{p=0}\, 0 + \frac{N}{2} \sum_{p=1}^{N} \binom{N-1}{p-1} x^p \alpha_p \qquad q = p-1$$

$$= \frac{Nx}{2} \sum_{q=0}^{N-1} \binom{N-1}{q} x^q \alpha_q$$

$$= \frac{Nx}{2} \sum_{q=0}^{N-1} (1-(-1)^q) \binom{N-1}{q} x^q$$

$$= \frac{Nx}{2} \left[(1+x)^{N-1} + (1-x)^{N-1} \right]$$

$\Rightarrow$ For holes,

$$A^{\mu\nu}_{J}(t,0) \sim (2\lambda) \cdot N\nu \left(\frac{Z_1}{Z_1} - \langle \epsilon_i \rangle_{+\Lambda} \right) \delta_{t0} \delta^{\mu\nu} \quad (.4?)$$

$$= (2\lambda) \cdot N\nu \tanh(+\Lambda)\, \delta_{t0} \delta^{\mu\nu} \quad (.4?)$$

& $Z_1 \sim \cancel{\alpha} \cosh^N(+\Lambda) = \cosh^N(\Lambda)$

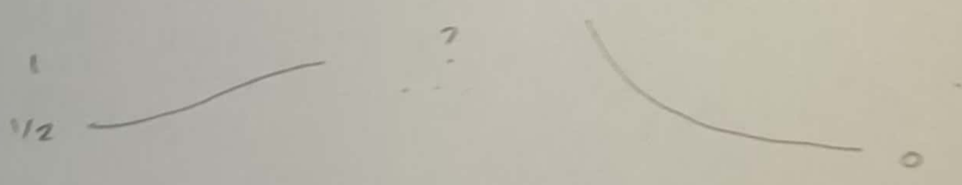
$\Rightarrow$ C unchanged (expected – RW irrelevant to C)

$$\Rightarrow \quad K^{\mu\nu} = +(2\lambda) N\nu \tanh\Lambda \cdot \frac{1}{2} \cdot \frac{1}{2N\tau^2}$$

$$= + \frac{\Lambda^2 \tanh(\Lambda)}{2\lambda}$$

$$\Rightarrow \quad D(\Lambda) = + \frac{1}{\lambda} \frac{\tanh\Lambda}{\operatorname{sech}^2\Lambda}$$

Key point: $D \to 0$ as $T \to \infty$ as we'd maybe expect?



But shouldn't we expect prtcl-hole symmetry?

No! High $T \not\Rightarrow$ max excited state. It rather implies random spin state

So to interpolate, we should say:

$$A_{s}^{\mu\nu}(t,0) \sim \frac{(2\lambda)^2}{Z_1} \sum_{p=0}^{N/2} \frac{(1+(-1)^p)}{2} \binom{N}{p} e^{-2\Lambda p} \cdot p \, \delta_{t0} \, \delta^{\mu\nu} \quad (.4?$$

$$+ e^{-2N\Lambda} \frac{(2\lambda)^2}{Z_1} \sum_{h=0}^{N/2} \frac{(1+(-1)^{h+N})}{2} \binom{N}{h} e^{+2\Lambda p} \cdot h \, \delta_{t0} \, \delta^{\mu\nu} \quad (.4?$$

$$\Rightarrow \text{get} \sim (2\lambda)^2 \left[\frac{N_v}{2} \langle \epsilon_i \rangle + e^{-2N\Lambda} \frac{N_v}{2} \left(1 - \langle \epsilon_i \rangle\right) \right] \ldots \quad (.4?$$

$$\sim (2\lambda) \frac{N_v}{2} \left[1 - \tanh(\Lambda) + e^{-2N\Lambda} \tanh(\Lambda) \right]$$

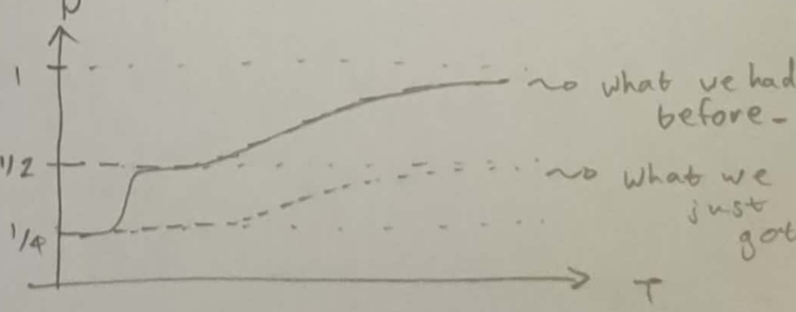
$$= (2\lambda) \frac{N_v}{2} \left[1 - 2\tanh(\Lambda) e^{-N\Lambda} \sinh(N\Lambda) \right]$$

$$\text{Then } K^{\mu\nu} \sim \frac{1}{2N_v T^2} (\,''\,) \cdot \frac{1}{2} (\text{from } \delta \text{ to sum})$$

$$= \frac{1}{4\lambda} \Lambda^2 \left(1 - 2e^{-N\Lambda} \sinh(N\Lambda) \tanh(\Lambda)\right)$$

$$\Rightarrow D \sim \frac{1}{2\lambda} \frac{1 - 2e^{-2N\Lambda} \sinh(N\Lambda) \tanh(\Lambda)}{\operatorname{sech}^2(\Lambda)}$$

<u>**Yes!**</u> We get.



As

Firstly, we have:

$$Z_\Lambda = \sum_{l=0}^{\lfloor N_v/2 \rfloor} \binom{N_v}{2l} e^{-4\Lambda l}$$

$$= \sum_{p=0}^{N_v} \binom{N_v}{2l} \frac{(1+(-1)^p)}{2} e^{-2\Lambda p}$$

} Binomial thm.

$$= \frac{1}{2}\left[(1+e^{-2\Lambda})^N + (1-e^{-2\Lambda})^N \right]$$ Nice!

Importantly, we need to check analyticity as $N \to \infty$.

$$-\beta F_\Lambda = \ln Z_\Lambda = -\ln 2 + \ln\left((1+e^{-2\Lambda})^N + (1-e^{-2\Lambda})^N \right)$$

$$\to -\ln 2 + N \ln(1+e^{-2\Lambda})$$

$\Rightarrow$ Manifestly analytic

$\Rightarrow$ PTs only occur in Z_H (or Z_5?)

For conductivity, we want

$$y = \sum_{l=0}^{\lfloor N_v/2 \rfloor} \binom{N_v - 2}{2l-1} e^{-4\Lambda l} \cdot (\dots) \quad \Rightarrow \text{ too complicated.}$$

So for the singular bits we only need Z_H (& ~~Z~~ Z_J)

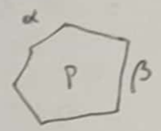
$Z_H = Z_{Ising}$ => looks just like Ising model w/o field.

$Z_J \sim Z_{Ising-next\,nn.}$ => bit more complicated, either next-nn or diag + 4-~~spin~~ interact$^{\bar{n}}$ depending on nn-def$^{\bar{n}}$ for spins.

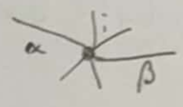
$$J \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta = J \sum_{\langle\alpha\beta\rangle} \overset{\sigma_\alpha^{ref}\sigma_\beta^{ref}}{\overset{\parallel}{\tilde{\eta}_{\alpha\beta}}} \prod_{\substack{p\in\partial^+\alpha \\ q\in\partial^+\beta}} \mu_p \mu_q$$

Three ~~Two~~ natural def$^{\bar{n}}$s of $\langle\alpha\beta\rangle$:

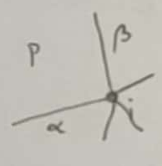
① $\beta \in nn(\alpha)$ iff $\exists p \mid \alpha, \beta \in \partial p$



② $\beta \in nn(\alpha)$ iff $\exists i \mid \alpha, \beta \in \partial^+ i$

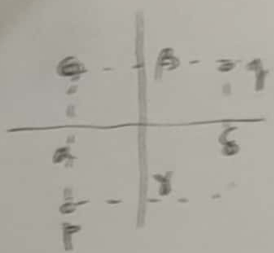


③ $\beta \in nn(\alpha)$ iff both!



Blindly using the 6-vertex term assumes ② which makes the most sense given our data structure.

Implementing the "more natural" ③ would require adding plaquettes explicitly to our code.
<u>However</u> on the square lattice it has an appealing form:



etc. $\Rightarrow J \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta$

$$= J \sum_{\langle\langle pq \rangle\rangle} N_{pq} \mu_p \mu_q$$

$\Rightarrow$ looks like next-nn. BUT note the N_{pq}.

$N_{pq} = \tilde{\eta}_{\alpha\beta} + \tilde{\eta}_{\gamma\delta}$ $\because$ $\exists$ 2 contributing terms

$\Rightarrow N_{pq} \in \{0, \pm 2\}$ $\Rightarrow$ <u>no longer</u> sth we can ignore σ^{ref} for...

Would need to genuinely deal w/. the $\underline{\sigma}^{ref}$ config $\forall$ prtcl configs.

Another current issue:

Demon heat baths

$$\Delta_\lambda = \cancel{}\,\delta E_\lambda = \begin{cases} 4\lambda & 8 \\ 8\lambda & 6 \end{cases}$$

For $h=0$ it makes sense to approx

$$Z_\lambda = \sum_{n=0}^{\infty} e^{-n \Delta_\lambda \beta}$$

$$\Rightarrow \langle D_\lambda \rangle = \frac{\Delta_\lambda \beta}{-1 + e^{+\Delta_\lambda \beta}}$$

But for $h \neq 0$ the demon energies now come in units of Δ_h & Δ_λ

$$\Rightarrow Z_* \overset{?}{=} Z_\lambda Z_h$$

$$= \sum_{n=0}^{\infty} \sum_{m=-1}^{1} e^{-(n\Delta_\lambda + m\Delta_h)\beta}$$

Each σ_α can only benefit/lose from this once $\because$ $E_h = -h\sum_\alpha \sigma_\alpha$ $(\Delta_h = 2h)$

had $\sum_0^1$ before but actually this makes more sense & converges better

$$\cancel{\Rightarrow \langle D_\lambda \rangle = \frac{\Delta_\lambda}{-1 + e^{+\Delta_\lambda}}}$$

$$\Rightarrow Z_h = 1 + 2\cosh(\Delta_h)$$

$$\Rightarrow \langle D_h \rangle = -\frac{1}{Z_h}\frac{\partial Z_h}{\partial \beta} = -\frac{\Delta_h}{Z_h} 2\sinh(\Delta_h)$$

$$= -\Delta_h \frac{2\sinh(\Delta_h)}{1 + 2\cosh(\Delta_h)} < 0 \quad \ldots \quad \text{Ono.}$$

β

$\Rightarrow \exists\, \beta$ for which $\langle D_\lambda \rangle + \langle D_h \rangle < 0$

$\Rightarrow$ PT is @ $H=0$! where ferro/antiferro preference ($\mathbb{Z}_2$ plaq- flip all symm). breaks.

Recall that $Z = \sum_{\mu} e^{H\sum_{\langle pq\rangle}\mu_p\mu_q + \cancel{\alpha}\sum_p \mu_p}$ ($\alpha \to 0$)

$\Big\}$ H-S decoupling

$$\beta H[\psi(x)] = \int d^d x \, \frac{1}{a^d}\left(\frac{t'}{2}|\psi|^2 + \frac{a^2}{8Hd^2}|\nabla\psi|^2 + \frac{1}{12}|\psi|^4 - \cancel{\alpha}\psi\right)$$

$$t = \frac{1}{2Hd} - 1$$

~~$\Rightarrow Z\beta H$~~

We already must have $d=2$ for " to be quadratic
$|\partial^+\alpha| = 2$ in $d=2$.

$(a=1)$

$$\Rightarrow \beta H = \int d^d x \left(\frac{t}{2}|\psi|^2 + \frac{K}{2}|\nabla\psi|^2 + \frac{\lambda}{4!}|\psi|^4\right)$$

w/. $t = a^{-d}\left(\frac{1}{4H} - 1\right) = \cancel{a^{-d}}\left(\frac{T}{4h} - 1\right)$ ($a^{-d} \to 1$)

$\Rightarrow T_c = 4h$

& $K = \cancel{a^{2-d}}\,\frac{1}{4H}$ ($a^{2-d} \to 1$)

$$\lambda = 4!/12 = \frac{4\cdot 3\cdot 2\cdot 1}{12} = 2$$

$$\Rightarrow \beta H = \int d^d x \left(\frac{\left(\frac{1}{4H}-1\right)}{2}|\psi|^2 + \frac{1}{8H}|\nabla\psi|^2 + 2|\psi|^4\right)$$

$\Rightarrow$ non-perturbative $|\psi|^4$ theory.

@ mean-field, $T_c = 4h$ & $|\psi|^2 = \begin{cases} 0 \\ \frac{1}{4}\left(1 - \frac{1}{4H}\right) \end{cases}$

Hmmm...

Still better than applying H-S directly to

$$\cancel{H = \sum\sum} \quad \Lambda \sum_i A_i + H \sum_\sigma \sigma_\alpha + J \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta$$

$\because$ would be $|\phi|^4$ theory $\propto \Lambda$!

$\Rightarrow$ very bad.

Attempt to proceed:

$$\hat{Z}_w(x) = \sum_{\mu} e^{+H\sum_{\langle pq\rangle} \eta_{pq}\mu_p\mu_q} = \hat{Z}_\eta$$

$$= 2^{N_P} C_H^{N_E} \left(1 + \square + \square + \dots \right)$$

$$= 2^{N_P} C_H^{N_E} \sum_{F\subseteq G} t_H^{L(G)} \left(\sum_{\ell\in F} \left(\prod_{\alpha\in\ell} \eta_\alpha\right)\right)$$

I.e. each term comes w/. a product of η_α along the loop.

H-S transformat^{n} ?

$$G_{pq} = \begin{cases} H\eta_{pq} & \langle pq\rangle \\ 0 & \text{otherwise} \end{cases}$$

const. factor

$$\hat{Z}_\eta = \cancel{\det(2\pi G_{pq}^{-1})^{-1/2}} \sum_{\mu} \int \left(\prod_p d\varphi_p\right) e^{\left(-\frac{1}{2}\sum_{pq}\varphi_p G_{pq}^{-1}\varphi_q + \sum_p \varphi_p\mu_p\right)}$$

$$\propto \int D\varphi_p \, e^{-\frac{1}{2}\sum_{pq}\varphi_p G_{pq}^{-1}\varphi_q + \sum_p (\ln(\cosh\varphi_p))}$$

Assume we're near a critical point

$\Rightarrow$ φ_p small

Problem: G_{pq} is nn., symm. & circulant BUT change in sign problematic

$$H\sum_{pq}\phi_p\eta_{pq}\phi_q = \frac{H}{N}\sum_{kk'}\phi_k\phi_{k'}\sum_{pq}\eta_{pq}e^{i(pk+qk')}$$

$$\frac{(N-2\ell)(N-2\ell-1)}{2}$$

+ +	$\frac{1}{2}(N-2\ell)(N-2\ell-1)$	
+ −	$(N-2\ell)\,2\ell$	} same graph ⇒ $\cdot\frac{1}{2}$
− +	$(N-2\ell)\,2\ell$	
− −	$2\ell(2\ell-1)$	

$\Rightarrow \quad N-2 \qquad \frac{(N-p)(N-p-1)}{2}$

$$-\beta F = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha$$

Take $\sigma_\alpha = s + \delta\sigma_\alpha$

$\Rightarrow$ to $O(\delta\sigma_\alpha^2)$,

$$\sum_i A_i \approx \sum_i \left(s^4 + \cancel{4} s^3 \sum_{\alpha \in \partial^+ i} \delta\sigma_\alpha + \cancel{6} s^2 \sum_{\alpha \neq \beta \in \partial^+ i} \delta\sigma_\alpha \delta\sigma_\beta \right)$$

$$= \text{const.} + \cancel{4} s^3 \underbrace{\sum_i \sum_{\alpha \in \partial^+ i}}_{= 2\sum_\alpha} \sigma_\alpha$$

$$- 2 s^3 \sum_i (z_i - 1) \sum_{\alpha \in \partial^+ i} \sigma_\alpha$$

$$+ s^2 \underbrace{\sum_i \sum_{\alpha \neq \beta}}_{= \sum_{\langle\alpha\beta\rangle}} \sigma_\alpha \sigma_\beta$$

$$= O(1) - 2 s^3 (\Lambda + 2H(z-1)) \sum_\alpha \sigma_\alpha$$

$$+ s^2 H \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta .$$

$\Rightarrow$ end up w/. a quadratic theory in σ's.

We have:

$$Z = \sum_{\underline{\sigma}} e^{-\beta F[\underline{\sigma}]}$$

and

$$-\beta F = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha + \cancel{\gamma} \sum_{\langle\alpha\beta\rangle} \sigma_\alpha \sigma_\beta$$

0 for now.

The first step is to pull out the Λ-term which only depends on the nº prtcls: (2ℓ = # prtcls)

$$Z = \sum_{\ell=0}^{N_V/2} e^{-4\Lambda \ell} \sum_{\underline{\sigma}_\ell} e^{H \sum_\alpha \sigma_\alpha}$$

$\underline{\sigma}_\ell$ = configs w. 2ℓ prtcls

We then expand

sum over all length-b_1 binary nº.s i.e. topo sectors

$$\sum_{\underline{\sigma}_\ell} = \frac{1}{(2\ell)!} \sum_{x_1 \neq \dots \neq x_{2\ell}} \sum_{w=\underline{0}}^{\underline{1}} \hat{Z}_w(\underline{x})$$

I.e. pulling out the pos$^{\text{ns}}$ of prtcls & topo sector info BUT we keep that info ~~inside~~ by defining new plaquette variables

$\mu_p = (-1)^{n_p}$ $\quad$ n_p = nº plaq. flips from some reference config $\underline{\sigma}^{ref}$

Useful property: * $\sigma_\alpha = \left(\prod_{p \ni \partial\alpha} \mu_p\right) \sigma_\alpha^{ref} \overset{2D}{=} \mu_p \mu_q$

i.e. becomes quadratic

* Plaquette flips are all that are left in $\hat{Z}_w(\underline{x})$ $\because$ they deform loops/strings & create new loops, as long as we define a $\underline{\sigma}^{ref}$ with the right prtcl config etc (unchanged by flips)

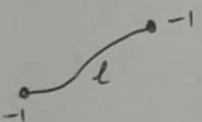
$$\Rightarrow \hat{Z}_w(\underline{x}) = \sum_{\underline{\mu}} e^{H \sum_{\langle pq \rangle} \eta_{pq}\, \mu_p \mu_q}$$

where $\eta_{pq} = \eta_{\underbrace{pq}_{\alpha \,\because\, nn.}}(\underline{x}, w) = \sigma_\alpha^{ref}(\underline{x}, w) \in \pm 1$

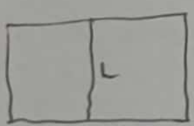
$\Rightarrow$ need a ref. state which is constructed as:

1. Start w/. GS
2. Create $2l$ prtcls @ desired posns x_i
3. Join them up w/. strings ~~or~~
4. Apply loop operators according to w.
5. Apply any plaquette flips you like.

This seems like a big mess but we can see how it works: $\hat{Z}_w$ is an Ising model w/. fixed domain wall defects.

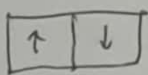


$\sim$ domain wall length l in GS of $\hat{Z}_w$



non-cont. loop

$\sim$ in GS we have



$\Rightarrow$ energy $\cancel{\gtrsim}$ $2hL$

$\Rightarrow$ exp. suppressed in L as expected

Can we somehow tackle this complicated beast? We could try to come up with a canonical choice for ~~η_r~~ $\underline{\sigma}^{ref}$ but that's hard.

Instead, let's follow the method of Kramers & Wannier...

For H small, use $e^{Hs} = (\cosh H) + s(\sinh H)$

$\Rightarrow \hat{Z}_w(\underline{x}) = \sum_{\mu} \prod_{\langle pq \rangle} C_H(1 + \eta_{pq}\mu_p\mu_q t_H)$

Now note $\sum_{\mu_p} \mu_p = 0 \Rightarrow$ only terms even in μ_p survive

$\Rightarrow$ only terms even in η_{pq} survive on average if we sum over all possible ref. states ⊛ — not true by itself

$\Rightarrow \hat{Z}_w(\underline{x}) = 2^{N_p} C_H^{N_E} f(t_H)$

$$f(x) = \sum_{L=0}^{L_{max}} \#_L x^L = \sum_{F \subseteq G} x^{L(F)}$$

F = ~~closed~~ subgraphs of G=latti

$\Rightarrow \hat{Z}_w(\underline{x})$ indep. of η_{pq}

$$\Rightarrow Z = 2^{b_1} 2^{N_p} \sum_{\ell=0}^{N_v/2} e^{-4\Lambda\ell} \binom{N_v}{2\ell} C_H^{N_E} \sum_{F \subseteq G} t_H^{L(G)}$$

Which, in the limit $H \to 0$ recovers

$$Z_0 = 2^{b_1} 2^{N_p} \sum_{\ell=0}^{N_v/2} e^{-4\Lambda\ell} \binom{N_v}{2\ell}$$

Where the extra 2^* factor is ∵ R-W assumes OBCs $\Rightarrow$ flipping all plaq.-s $\neq$ identity operat^{n}.

⊛ If we don't make this approx^{n} we get terms like

$\underbrace{\eta_{pq}\eta_{qr}\cdots\eta_{sp}}_{2n} t^{2n}$ i.e. sth a lot more complicated that depends on the ref. State.

Idk if this approx^{n} is even valid...

For the high H we need to be a bit cleverer.
First, flip our measurement axis for μ_p such that all $\eta_{pq} = +1$ ⇝ only possible with OBCs. ?

Then switch to auxiliary variables on the sites etc. to get:

$$\hat{Z} = 2e^{N_E H^*} f(e^{-2H^*})$$

on square lattice ∵ self-dual
doesn't hold in general
(see paper on hyperbolic graphs K-W).

Definitely possible for non-cont. loops or closed loops ⟶ NOT possible for open strings ☹

To be honest it's not a surprise this didn't work, it predicts a very strange heat capacity & gives $\partial K / \partial h = 0$ which is very unexpected (prtcl transport indep. of line tension??

To linear order in H,

$$\hat{Z}_\eta = \left(1 + \left(\sum_p \left({}_p\square\right)\right) t_H^4\right) 2^{N_p} C_H^{N_E}.$$

ew.

We know $\hat{Z}_\eta$ indep of η as long as they're consistent w/. $\{w, \underline{x}\}$.

Can we sum up multiple η in a clever way to deal w/. this?

Remember all terms in $\hat{Z}_\eta$ ~ even n°s products of η_{pq}'s.

$\Rightarrow$ $\eta \to -\eta$ does nothing.

All I can do are plaquette flips around $\eta = \underline{\sigma}$ ref

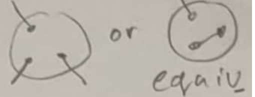
$\Rightarrow$ only cares about loops w/. endpoints!

Note: Can only get $\prod_{\alpha \in C} \eta_\alpha = -1$ if the loop of plaquettes encloses an endpoint! (or is non-cont. & crosses a non-cont. loop $\because w \neq \underline{0}$)

important! $\Rightarrow$ The $\eta_\alpha = 1 \ \forall\ \alpha$ approx$^{\bar{n}}$ is valid in the limit of $l \to 0$ i.e. no quasiprtcl endpoints for which C can enclose & give a -ve term! ~~Also~~

$\hookrightarrow$ Ising approx$^{\bar{n}}$ isn't useless – works if density of quasiprtcls small

Let's work to low order in H,



$$\hat{Z}_p \approx 2^{N_p} C_H^{N_E} \sum_{F \subseteq G} t_H^{\sum_{l \in F} L(l)} \left(\sum_{l \in F} \left(\prod_{\alpha \in l} \eta_\alpha \right) \right)$$

def.

$$= 2^{N_p} C_H^{N_E} \sum_{F \subseteq G} t_H^{L(F)} \left(\sum_{l \in F} \Theta_l \right) \qquad \Theta_l = \begin{cases} -1 & l \text{ encl} \\ & \text{odd \# precl} \\ +1 & \text{otherwis} \end{cases}$$

$$\approx 2^{N_p} C_H^{N_E} \left(1 + t_H^4 (N_p - 4l) + t_H^6 \left(2(N_p - 8l) \right) + t_H^8 (N_p - 2l) \ ? \quad) + \dots \right)$$

Note that to low order $\hat{Z}_w(x)$ indep. of w *
& indep. of x, it's indep. of x

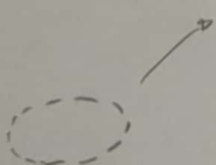
=> at very least we can kill off $\sum_w \rightarrow 2^{b_1}$ term

* ∵ The loops $l \in F \subseteq G$ are by defn non-contractible ?

=> can't have e.g.:

Is this true? Could we not have e.g. a term on un. all the way around?

Yep!



Why does w drop out @ low H? Means there's little cost to breaking domain wall preferences.

① is satisfied but ② isn't ∵ Ω becomes a random walk w/. no double occupancy

This is where the above steps past ↯ break down – the autocorrelatⁿ of a given start point is not indep. of where the other prtcls are in the higher temperature limit.

→ above approxⁿ of taking:

remember @ each timestep, z^{μ} moves are proposed in random dir.s

$$\langle \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{q}(0) \sim \langle \text{''} \rangle_{\text{random walk}} \sim \delta_{pq}\, \delta^{\mu\nu}\, \delta_{t0} \cdot 4$$

(for the 6-vertex case it in fact decays algebraically) $\left(\sim t^{\beta/2-2},\ \beta = 1+\alpha,\ \langle |r(t)|^2 \rangle \sim t^{\alpha}\right)$

fails for l large!

→ fails for $\Lambda \lesssim 1$ → $T \gtrsim \lambda$

Fortunately, $J_{t,0}$ is sth we can actually measure! We can then semi-analytically compute the Kubo conductivity from observing the dynamics alone!

$$J_{t,0} \approx \begin{Bmatrix} N_v \\ 2l \end{Bmatrix} \sum_{pq} \langle \underline{\dot{r}^{\mu}_{p}(t)}\, \underline{\dot{r}^{\nu}_{q}(0)} \rangle_{\text{all } 2l \text{ prtcls' initial posⁿs}}$$

Where the $\approx$ is exact but reflects making Ω random.

$$\rightarrow \langle J^{\mu}_{t} J^{\nu}_{0} \rangle_{eq} \approx \frac{(2\lambda)^2}{Z_1} \underbrace{\sum_{l=0}^{N_v/2} \left\{ \binom{N}{2l} e^{-4\Lambda l}\right.}_{= Z_1} \underbrace{\left.\sum_{pq} \langle \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{q}(0) \rangle_{x_s\, \forall s \in \{1,\dots,2l\}} \right\}}_{= f(l) \text{ so can't cancel}}$$

In random walk approxⁿ we recover

$$\langle J^{\mu}_{t} J^{\nu}_{0} \rangle_{eq} \approx \frac{(2\lambda)^2}{Z_1} \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\Lambda l}\ 2l\, \delta^{\mu\nu}\, \delta_{t0} \cdot 4$$

also note by switching to holes $\binom{N}{2l} = \binom{N}{N-2l}$ → same result @ l large

Which matches what I obtained before $\left(\binom{N_v - 1}{2l-1} N_v = \binom{N_v}{2l} 2l \right)$

If we introduce a field, recall that in our dodgy approx^{n} Z_{Ising} (& the W parts) drop out & we're left with an extra $e^{-2H\sum_{p=1}^{\ell}|x_p - x_{2p}|_1}$

→ fine to pick p & $2p$ WLOG ∵ indistinguishable.

This term carries through all the way to:

$$J_{t,0}^{(H)} = \sum_{pq} \sum_{x\ldots} \dot{r}_p^{\mu}(t)\, \dot{r}_q^{\nu}(0)\, e^{-2H\sum_{s=1}^{\ell}|x_s - x_{2s}|_1}$$

$$\approx \begin{Bmatrix} N_v \\ 2\ell \end{Bmatrix} \sum_{pq} \left\langle \dot{r}_p^{\mu}(t)\, \dot{r}_q^{\nu}(0)\, e^{-(\ldots)} \right\rangle_{x_s\ \forall s}$$

Note we'll also have this in the denominator $Z_1^{(H)}$ but without the $\dot{r}$ parts.

Unlike this part, however, the denominator will be deterministic.

Note: the requirement ① is a lot more restrictive ~~than~~ nowms for line tension

⇒ W/o demons, the only allowed moves are those where the string length is unchanged.

⇒ no equivalent random walk approx^{n} ...

⇒ are the dynamics in a field even interesting? Seems like for μcan we'll only get confinement if demons are present.

It will look very different in the two cases I should think...

Note that in the random walk approx^{n},

$$N_v \langle \epsilon_i \rangle = \frac{2\lambda}{Z_1} \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\Lambda l} \, 2l$$

$$A_J^{\mu\nu}(t,0) \approx \frac{(2\lambda)^2}{Z_1} \sum_{l=0}^{N_v/2} \binom{N_v}{2l} e^{-4\Lambda l} \quad 2l \cdot \langle 1 \cancel{\dot{x}}^{2} 1 \rangle^{4?} \delta_{t0} \delta^{\mu\nu}$$

$$= \Big((2\lambda) \cdot N_v \underbrace{\langle \epsilon_i \rangle_\Lambda}_{\sim \lambda (1 - \tanh(\Lambda))} \Big) \delta_{t0} \delta^{\mu\nu} (\cdot 4?)$$

$$\Rightarrow A_J^{\mu\nu}(t,0) \sim 2\lambda^2 N_v (1 - \tanh(\Lambda)) \delta_{t0} \delta^{\mu\nu}$$

Can we also approximate C?

Well recall that $Z_1 \sim (16 \cosh(\Lambda))^{N_v} \cancel{\left(\frac{1}{N_v}\right)}?$: N vertices

irrelevant for heat capacity

$$\cancel{t^2} \frac{\partial}{\partial T} \left(\frac{T^2}{Z} \frac{\partial Z}{\partial T} \right)$$

$$= \cancel{2} \frac{\partial}{\partial T}$$

$$= \cancel{2} \Lambda^2 \frac{\partial}{\partial \Lambda} \left(\frac{1}{Z} \frac{\partial Z}{\partial \Lambda} \right)$$

$$= \Lambda^2 \frac{\partial}{\partial \Lambda} (\tanh \Lambda)$$

$$= (\Lambda \operatorname{sech} \Lambda)^2$$

$$\Lambda = \frac{\lambda}{T}$$

$$\Rightarrow \frac{d\lambda}{dT} = - \cancel{\frac{\lambda}{T^2}} \frac{\lambda}{T^2}$$

$$= - \frac{\Lambda^2}{\lambda}$$

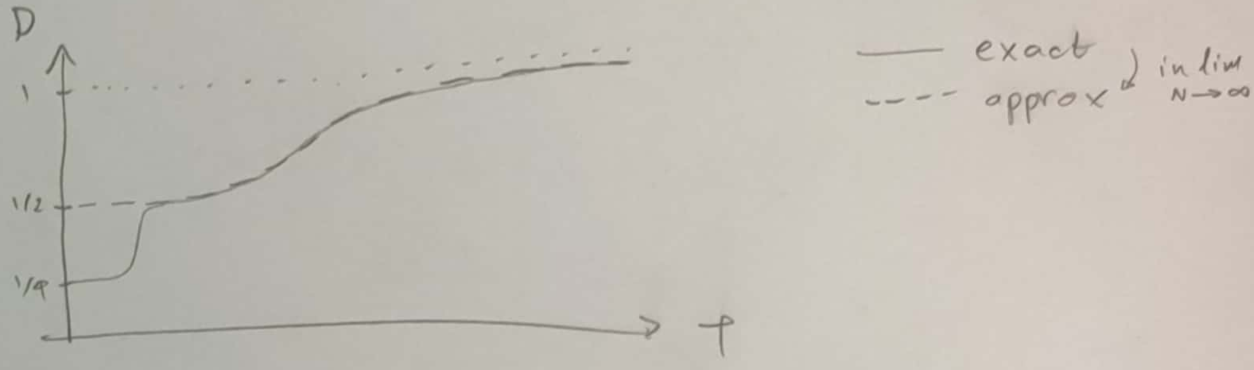
$$=$$

$$\left(\frac{1}{T}\right)^2 = \frac{\Lambda^2}{\lambda^2}$$

$$\Rightarrow \boxed{\begin{aligned} C &= \tfrac{1}{2} (\Lambda \operatorname{sech} \Lambda)^2 \\ K^{\mu\nu} &= \frac{1}{2N_v T^2} (2\lambda) N_v \langle \epsilon_i \rangle \, \delta^{\mu\nu} \underbrace{\sum_{\tau=0}^{\infty} \delta^{\tau 0} \left(1 - \frac{\delta^{\tau 0}}{2}\right)}_{1/2} \\ &= \frac{\Lambda^2}{\lambda} N_v \langle \epsilon_i \rangle = \frac{1}{2\lambda} \Lambda^2 (1 - \tanh \Lambda) \end{aligned}}$$

$$\Rightarrow D(\Lambda) = \frac{1}{\lambda} \frac{1-\tanh\Lambda}{\operatorname{sech}^2\Lambda} \quad \Big\} \quad \text{indep of } N!$$

which goes like:



$\Rightarrow$ should expect interpolation between two plateaus?

HOWEVER I'd expect that for $\Lambda \lesssim 1$ the RW approx$^{\bar{n}}$ becomes bad for "prtcls" & good for holes.

Dynamics " the same but instead, Λ?

$$Z_1 \rightarrow \sum_{h=0}^{N_v} \Big(\underbrace{\frac{(1+(-1)^{h+N_v})}{2}}_{\text{keep only odd or even depending on } N\text{'s parity.}} \Big) \binom{N_v}{h} e^{+2\Lambda(h-N_v)}$$

$N_v = h + \underbrace{2\ell}_{p}$

$$A_J^{\mu\nu}(t,0) \rightarrow \frac{(2\lambda)^2}{Z_1}\Big(\text{"} \quad \text{"} \Big) \sum_{\bar{p}\bar{q}} \langle \dot{r}_{\bar{p}}^{\mu}(t)\, \dot{r}_{\bar{q}}^{\nu}(0) \rangle_{\Sigma}$$

$\downarrow$ RW

$$\frac{(2\lambda)^2}{Z_1}\Big(\text{"} \quad \text{"} \Big)\, h \cdot \delta_{t0}\, \delta^{\mu\nu} (\cdot\, 4?)$$

If N even, this looks the same as for prtcls except $\Lambda \rightarrow -\Lambda$, $2\ell \rightarrow h'' \overset{2\ell}{=} N - 2\ell$ & extra factor of $e^{-2N_v\Lambda}$ which cancels?? -

Horizontal line @ $\sqrt{L^2/N}$ for μcanonical.

Be careful: need $\rho \gtrsim \frac{1}{V}$ prtcl density.

Averaging zero w/. multiple runs!

Use μcan ensemble
Fix n° prtcls $\Rightarrow Z$ becomes const.

Think ~ adding spin correlations.

$$Z = \sum_{l=0}^{\lfloor N_V/2 \rfloor} e^{-4\Lambda l} \sum_{W=\underline{0}}^{\infty} \frac{1}{(2l)!} \sum_{\underline{x}_1 \neq \dots \neq \underline{x}_{2l}} \sum_{\underline{\mu}} e^{-H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q}$$

$$= \sum_{l=0}^{\lfloor N_V/2 \rfloor} Z_{2l}$$

$$Z_l \xrightarrow{N \to \infty} Z \text{ but w/. } 2\Lambda \to 2\Lambda + U \qquad U = \beta\mu$$

$$\& \quad \frac{\partial F}{\partial U} = 2l \qquad F = \ln Z$$

$$\Rightarrow \quad 2l = \frac{1}{Z} \frac{\partial Z}{\partial U}$$

$$= \langle 2l \rangle$$

$n = 2l$ fixed

$$\langle \epsilon_i(\Omega^\tau(\sigma)) \epsilon_j(\sigma) \rangle_{eq}^{(n)} = \frac{1}{Z_n} \frac{e^{-2\Lambda n}}{n!} \sum_{W=\underline{0}}^{1} \sum_{\underline{x}_1 \neq \dots \neq \underline{x}_n} \sum_{\underline{\mu}} e^{-\dots}$$

$$\cdot (2\lambda) \delta_{i \in \underline{x}^\tau} \cdot (2\lambda) \delta_{j \in \underline{x}}$$

$$= \left(\frac{(2\lambda)^2}{Z_n} \frac{e^{-2\Lambda n}}{n!} \right)^{A} \sum_{\underline{x} \neq \dots \neq \underline{x}_n} \delta_{i \in \underline{x}^\tau} \delta_{j \in \underline{x}}$$

$$\left(\sum_W \sum_\mu e^{-H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q} \right)^{B_{\underline{x}}}$$

$$= A \sum_{\langle \underline{x} \rangle = 1}^{N_V} \sum_{pq=1}^{n} \delta_{i \in x_p^\tau} \delta_{j \in x_q} \; B_{\underline{x}}$$

$$= A \sum_{pq} \sum_{\langle \underline{x} \rangle} \delta_{i \in x_p^\tau} \delta_{j \in x_q} \; B_{\underline{x}}$$

$$\sum_{ij} x_i^\mu x_j^\nu (\text{''}) = A \sum_{pq} \sum_{\langle \underline{x} \rangle} x_p^\mu(\tau) \, x_q^\nu(0) \; B_{\underline{x}}$$

$$\langle J^\mu_\tau J^\nu_0 \rangle^{(n)}_{eq} = \frac{(2\lambda)^2}{Z_n} \frac{e^{-2\Lambda n}}{n!} \sum_{pq} \underbrace{\sum_{\langle x(0)\rangle} \dot{x}^\mu_p(\tau) x^\nu_q(0)\, B_{x}}$$

$$\approx \frac{(2\lambda)^2}{Z_n} \binom{N_V}{n} e^{-2\Lambda n} \sum_{pq} \langle \dot{x}^\mu_p(\tau) \dot{x}^\nu_q(0)\, B_{x(0)} \rangle$$

$\{x_p(0) \mid x_p(0) \neq x_q \ \forall pq\}$.

First let $H=0 \Rightarrow$ all the stuff in $B \to$ const.

$$\langle J^\mu_\tau J^\nu_0 \rangle^{(n)}_{eq} \xrightarrow[\text{RWA}]{H=0} (2\lambda)^2 \sum_{pq} C\, \delta_{pq} \delta^{\mu\nu} \delta_{t0}$$

$$= (2\lambda)^2\, n\, C\, \delta^{\mu\nu} \delta_{t0}$$

$$\& \ Z_n \to \binom{N_V}{n} e^{-2\Lambda n} \sum_{w} \sum_{\mu_p} (1)$$

$$\Rightarrow K_n = \frac{1}{N_E T^2} \sum_{t=0}^{\infty} (2\lambda)^2 n C \delta^{\mu\nu} \underbrace{\delta_{t0} \left(1 - \frac{\delta_{t0}}{2}\right)}_{\frac{1}{2}\delta_{t0}}$$

$$= \frac{1}{2N_E T^2} (2\lambda)^2 C\, n\, \delta^{\mu\nu} . = \frac{(2\lambda)^2 n}{N_E T^2} \frac{C \delta^{\mu\nu}}{2}$$

Replace w/. $\langle n \rangle_T \approx N_V (1 - \tanh(\Lambda))/2$

$$C_n = \frac{1}{N_E} \frac{\partial}{\partial T} \left(T^2 \frac{\partial \ln Z_n}{\partial T} \right)$$

$$= \frac{1}{N_E T^2} \frac{\partial^2 \ln Z_n}{\partial \beta^2} = \frac{\Lambda^2}{N_E} \frac{\partial^2 \ln Z_n}{\partial \Lambda^2}$$

$$= \frac{\Lambda^2}{N_E} \frac{\partial^2}{\partial \Lambda^2} (-2\Lambda n) = 0$$

$$\langle \dot{x}(t)\,\dot{x}(0)\rangle = \langle x(t+1)\,x(1)\rangle + \langle x(t)\,x(0)\rangle - \langle x(t)\,x(1)\rangle - \langle x(t+1)\,x(0)\rangle$$

$$= \delta_{t,0} + \delta_{t,0} - \delta_{t,1} - \delta_{t+1,0}$$

$$= 2\delta_{t,0} - \delta_{t,1} - \delta_{t,-1}$$

$$T = \frac{dE}{dS}$$

$$T\frac{dS}{dT} = ?$$

$$\frac{\partial S}{\partial \tau}\frac{\partial T}{\partial S}\frac{\partial E}{\partial S} = -1$$

$$\frac{\partial S}{\partial \tau} T = -\frac{\partial E}{\partial T}$$

$E = E(S)$ $\quad T = \frac{dE}{dS}$

$E = E(S, T)$ $\quad C = \frac{\partial E}{\partial T}$

$Z = \text{const.}$

$Z \propto e^{-\cancel{\Delta A} 2\beta \lambda n}$

$$F = -\frac{1}{\beta}\ln Z$$

$$= \underline{2\lambda n}$$

$$\Rightarrow 2 D_n \overset{?}{=} \frac{2n_2}{c\delta^{\mu\nu}}$$

We know $C(\tau) = \dfrac{\Lambda^2 \operatorname{sech}^2 \Lambda}{N_E}$

$$K^{\mu\nu}(\tau) = K^{\mu\nu}_{\langle n(\tau)\rangle} = \Lambda^2 \left(\frac{n/2}{N_E}\right) c\,\delta^{\mu\nu} \qquad n = 2\ell$$

$$\Rightarrow D(\tau) = \frac{1}{2} \frac{\langle n(\tau)\rangle}{\operatorname{sech}^2 \Lambda} \qquad \langle n(\tau)\rangle = \tfrac{1}{2}(1-\tanh\Lambda) N_v \qquad c = 1?$$

$$\propto \frac{1-\tanh\Lambda}{\operatorname{sech}^2\Lambda}$$

$\dfrac{N!}{}$

$$Z \langle n(\tau)\rangle = \sum_{n=0}^{N_v} \frac{(1+(-1)^n)}{2} e^{-2\Lambda n} \binom{N_v}{n} n$$

$m = n-1$

$$= N_v \sum_{m=0}^{N_v-1} \frac{(1-(-1)^m)}{2} e^{-2\Lambda(m+1)} \binom{N_v-1}{m}$$

$$= N_v \frac{e^{-2\Lambda}}{2} \left[(1+e^{-2\Lambda})^{N_v-1} - (1-e^{-2\Lambda})^{N_v-1} \right]$$

$$\Rightarrow \langle n(\tau)\rangle = N_v e^{-2\Lambda} \frac{(+)^{N_v-1} - (-)^{N_v-1}}{(+)^{N_v} + (-)^{N_v}}$$

But I'd expect $\langle n(\tau)\rangle \sim \frac{N_v}{2}(1-\tanh\Lambda)$?

Need either $\Lambda \to N\Lambda$ or Λ/N preferred

$C = \frac{1}{2}\Lambda^2 \operatorname{sech}^2\Lambda$

$K = \frac{1}{2}\Lambda^2 (1 - \tanh\Lambda)$

My calc$^{\underline{n}}$s diverge @ $\Lambda \to \infty$ i.e. where there's an inflection in D

~~$D = \cosh^2\Lambda - \sinh\Lambda\cosh\Lambda$~~

$-\Lambda \frac{\partial D}{\partial \Lambda} = T\frac{\partial D}{\partial T}$

$\Lambda/T = \Lambda^2/\lambda$

$\frac{\partial^2 D}{\partial T^2} = 0 = \frac{\Lambda^2}{\lambda}\frac{\partial}{\partial\Lambda}\left(\frac{\Lambda^2}{\lambda}\frac{\partial D}{\partial\Lambda}\right)$

Wrong D!

$= \frac{\Lambda^2}{\lambda^2}\frac{\partial}{\partial\Lambda}\left(\Lambda^2(\sinh(2\Lambda) - \cosh(2\Lambda))\right)$

$= \frac{\Lambda^2}{\lambda^2}\left(2\Lambda(\ldots) + 2\Lambda^2(\cosh(2\Lambda) - \sinh(2\Lambda))\right)$

$\Rightarrow c + s = \Lambda(s - c)$

$e^{2\Lambda} = \Lambda e^{-2\Lambda} \Rightarrow \underline{\underline{e^{4\Lambda} = \Lambda}}$

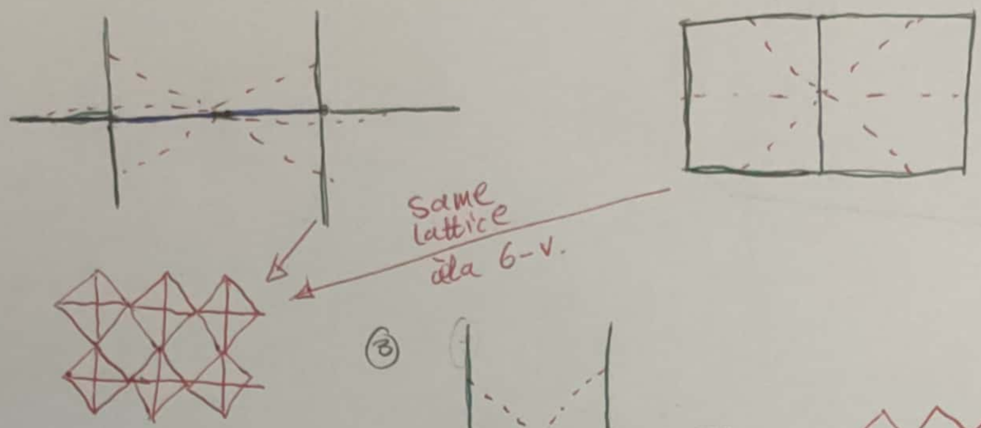
Inflection point occurs when $e^{4\Lambda} = \Lambda$, i.e.

I.e. @ $\Lambda \to \infty$ $\exists$ no excitat$^{\underline{n}}$s below a certain temp (both $C, K \sim 1/T^2$)

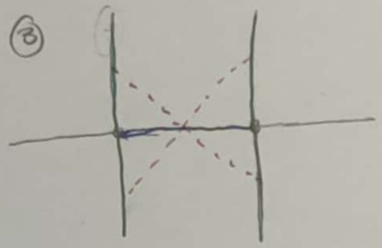
Inflect$^{\underline{n}}$ point for D_{approx} is $\Lambda \lesssim 1$ i.e. $\lambda < T$

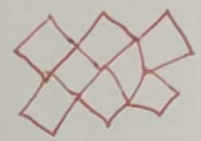
$\Rightarrow$ makes sense!

①



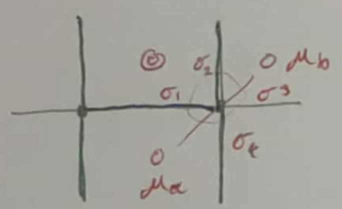
③





just a square lattice.

③

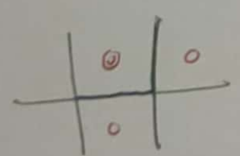


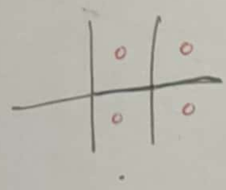
$\sim$ next-nn interact^n

$$\left(\sum_{n=1}^{4} \sigma_n\right)\mu_a\mu_b$$

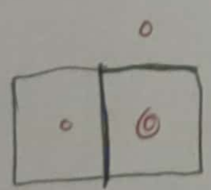
no longer simple...

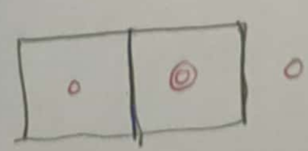
①

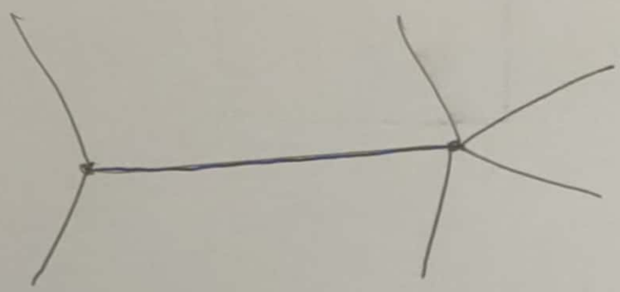


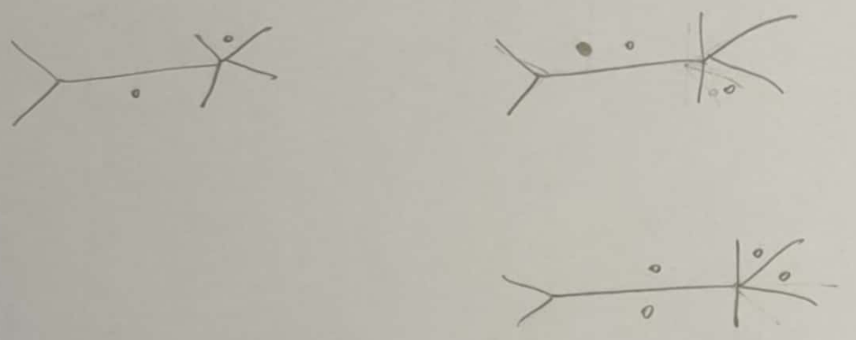


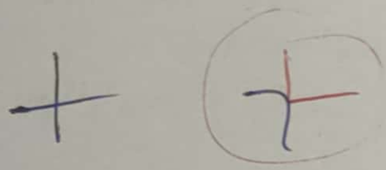
③





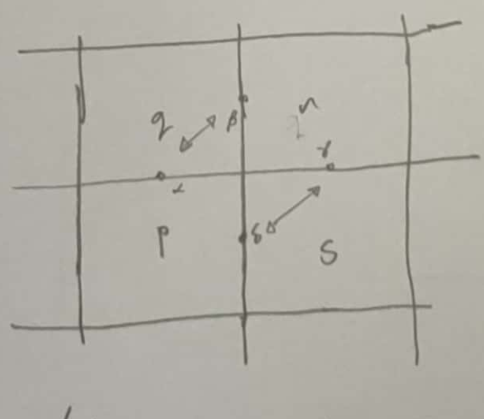




8-vertex 

6

Cost



$$((\sigma_\alpha \sigma_\beta) + (\sigma_\gamma \sigma_\delta))\, \mu_p \mu_s$$

$$+ ((\sigma_\alpha \sigma_\gamma) + (\sigma_\beta \sigma_\delta))\, \mu_p \mu_q \mu_r \mu_s.$$

$$Z' = -2\cancel{\Lambda}Ne^{-2\Lambda}\left[(+)^{N-1} - (\mp)^{N-1}\right]$$

$$\cancel{Z'' = -2\Lambda Ne^{-2A}}$$

$$Z'' = -2Ne^{-2\Lambda}\left[(+)^{N-1} - (-)^{N-1}\right]$$

$$+$$

$$C = \Lambda^2 \frac{\partial^2 \ln Z}{\partial \Lambda^2} = (2\Lambda)^2 \frac{\partial^2 \ln Z}{\partial (2\Lambda)^2}$$

$$Z' = -e^{-2\Lambda} N\left[(+)^{N-1} - (-)^{N-1}\right]$$

$$Z'' = +e^{-2\Lambda} N\left[(+)^{N-1} - (-)^{N-1}\right]$$

$$+e^{-4\Lambda} N(N-1)\left[(+)^{N-2} + (-)^{N-2}\right]$$

$$= -Z' + Z^{(2)}$$

$$\Rightarrow \cancel{C/\Lambda^3} \quad C/(2\Lambda)^2 = \frac{Z''}{Z} - \frac{Z'^2}{Z^2}$$

$$= \frac{Z^{(2)}}{Z} - \frac{Z'}{Z} - \left(\frac{Z'}{Z}\right)^2$$

$$H \sum_{\langle pq \rangle} \sigma^{ref}_{pq} \mu_p \mu_q + J \sum_i (\ldots) \cancel{\mu} \cancel{A} B_i$$

$$+ J \sum_{\substack{pq \\ diag}} (\ldots) \mu_p \mu_q$$

$$\mu_p \approx \mu + \delta \mu_p$$

to $\mathcal{O}(\delta \mu_p)$

$\Rightarrow (\ldots) =$

$$Z = \underbrace{\cancel{\tfrac{1}{2}}}_{\text{(irrelevant)}} \left[(1 + e^{-2\lambda})^N + (1 - e^{-2\lambda})^N \right]$$

~~ln Z =~~

$$\frac{\partial Z}{\partial \lambda} = -2 \cancel{4} e^{-2\lambda} N \left[(1 + e^{-2\lambda})^{N-1} - (\ldots)^{N-1} \right]$$

$$\cancel{G =} \quad \frac{\partial^2 \ln Z}{\partial \lambda^2} = \frac{\partial}{\partial \lambda} \left(\frac{Z'}{Z} \right)$$

$$= \frac{Z''}{Z} - \left(\frac{Z'}{Z} \right)^2$$

Take $N \to \infty \Rightarrow$ for $\Lambda \geq 0$,

$$Z \to \frac{n}{2}(1+e^{-2\Lambda})^N$$

$$\Rightarrow \quad C = N \frac{\partial^2}{\partial(2\Lambda)^2}(1+e^{-2\Lambda})\ (2\Lambda)^2$$

$$= N\ (2\Lambda)^2 e^{-2\Lambda}$$

Which would be our phenomenological guess.

$$\Rightarrow \quad n \sim \frac{e^{-2\Lambda}}{(1+e^{-2\Lambda})} \equiv \frac{1}{2}(1-\tanh\Lambda)$$

$T > \lambda/2$

$\lambda/T < 2$

$\Lambda < 2$

$2\Lambda n \sim 1$

$\Lambda \sim 2$

$n \sim 1/4$

Vertices

Cost					
Λ	~~3Λ~~ 0	2Λ	0	2Λ	0
J	$8J$	$2J$	0	$2J$	$8J$
H	0	H	$2H$	$3H$	$4H$

$$J\sum_{\langle\alpha\beta\rangle}\sigma_\alpha\sigma_\beta = \frac{J}{2}\sum_j \tilde{A}_j$$

$$= \underbrace{\frac{J}{2}}_{\tilde{\lambda}}\sum_j\Big(\sum_{\alpha\in\partial^+ j}\sigma_\alpha\Big)^2$$

$4\tilde{\lambda}$ $\quad$ $16\tilde{\lambda}$

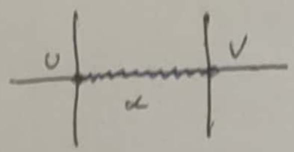
$2J$ & $8J$

Λ has the advantage that A is unchanged under plaquette flips. What leaves $\tilde{A}$ invariant? Only rotations/reflections, & these <u>don't</u> leave neighbours inv.

$\Rightarrow$ $\exists$ no local move which preserves the no prtcls.

$$-\beta F = \Lambda \sum$$

Exactly what values can D take?



$\cancel{\Delta} D_\alpha \in \underline{\ell}(4\lambda) + \underline{m}(8j) + n(2h)$

both $\ell, m \in \{0, 1, \dots, \infty\}$ ∵ no energy cost in principle to bring a prtcl to vertices u or v

$n \in \{0, \pm 1\}$ depending on initial spin state.

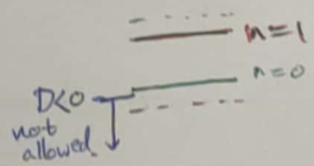
Suppose that we <u>choose</u> all spins $\parallel$ initially i.e. GS of H & Λ terms (not of J term)

$\Rightarrow$ $n \in \{0, \cancel{\mp} 1\}$ <u>only</u>. $\leadsto$ E_α can only $\uparrow$
D_α " " $\downarrow$

~~BUT could have~~

For $2h < 4\lambda, 8j$ this is fine ∵ can't remove the 2h energy to use for anything else.

Gets messy @ crossover.



Fix: treat D as 3 <u>independent</u> resevoirs.

This removes the issue of λ, h, j incommensura[bility]
& allows us to interpolate.

<u>However</u> it means that for a flip to take place, $\Delta E_G \leq D_G \quad \forall G \in \{\Lambda, H, J\}$.

Is this a problem to have 3 separate demons?

Means no sharing of the energy

$\Rightarrow$ 3 demons working in concert $\gg$ 1 demon trying to keep track of different quantisat$^{\overline{ns}}$ ~ should all be in eq$^{\overline{m}}$ w/. system @ the end...

In our case it would give

$$Z_{D_\alpha} = \underset{\frac{1}{1+e^{-4\lambda}}}{Z_\Lambda}\ \underset{1+e^{-2H}}{Z_H}\ \underset{\frac{1}{1+e^{-8J}}}{Z_J}$$

$4\lambda = \frac{4\lambda}{\beta}$

$\frac{\partial}{\partial \beta} = \frac{\partial 4\lambda}{\partial \beta}\frac{\partial}{\partial 4\lambda}$

$-\beta\frac{\partial}{\partial\beta} = 4\lambda\frac{\partial}{\partial 4\lambda}$

$$\Rightarrow \langle E^{(\pm)}_{D_\alpha}\rangle = -\frac{1}{Z}\frac{\partial}{\partial\beta}Z = \langle E_\Lambda\rangle + \cdots$$

$$= \frac{4\lambda}{e^{4\lambda}+1} + 2he^{-2H} + \frac{8J}{e^{8J}+1}$$

In principle can solve the above for $T(\langle E_{D_\alpha}\rangle)$ $\because$ will be monotonic $\uparrow$ng if $\lambda, j, h \geq 0$

We've assumed all ≥ 0

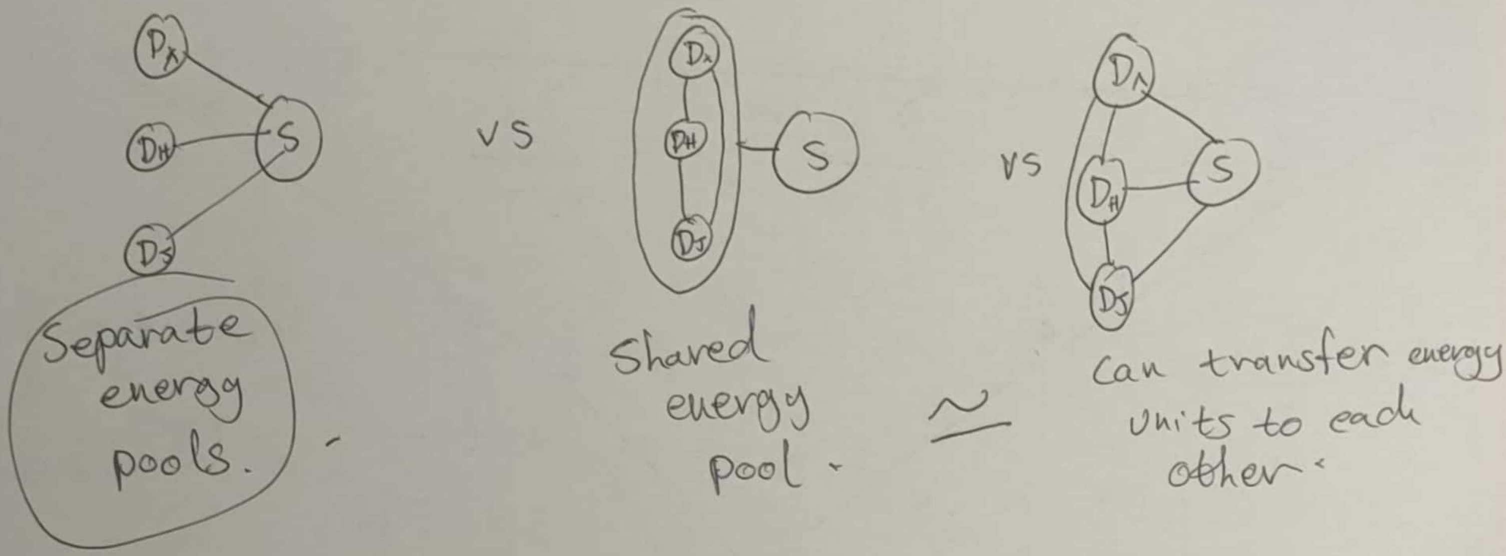
~~If untrue, replace w/. |value|~~

If untrue, need to alter partition func$^{\overline{ns}}$ $\because$ demon only allow +ve changes for each.

BUT J, Λ has $\mathbb{Z}_2$ symm. $\Rightarrow$ sgn(h), sgn(J) not interesting?.

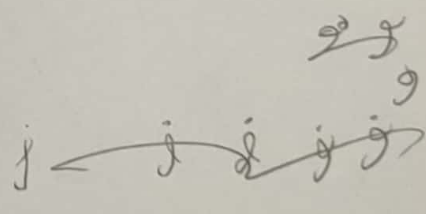
Can we allow energy transfer between the 3 demons?

I would argue no. This restricts possible moves* $\Rightarrow \uparrow$s thermalisat$^{\overline{n}}$ time but in principle it should still thermalise eventually.

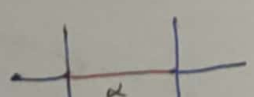


$\Rightarrow$ Replace D w/. 3-tuple.

& Randomly initialise D w/. quantised values, starting in a GS for H & Λ?



* e.g.



$D_H = 0(2h)$ $D_\Lambda = 10(4\lambda)$ $D_J = 8(8$

Can't flip α $\because$ no energy available in H-resevoir. Even if $\Lambda >> H, J$!

Not ideal!

Resolution: Allow $D_G < 0$ as long as

$\sum D_G \geq 0$? $\longrightarrow$ new flip requirement

This would give $2(hn + 2\lambda\ell + 4jm) \geq 0$ in partit$^{\overline{n}}$ func$^{\overline{n}}$.

$n \in \{0, \pm 1\}$ $\quad \ell, m \in \mathbb{Z}$.

$$Z = \sum_{\ell=-\infty}^{\infty} e^{-4\lambda\ell} \sum_{n=-1}^{1} e^{-2hn} \sum_{m=-\infty}^{\infty} e^{-8jm} \; \Theta_{(\ldots) \geq 0}$$

$$= \sum_n \begin{cases} n=0: & \sum_\ell e^{-4\lambda\ell} \sum_m e^{-8jm} \; \Theta_{2\lambda\ell \geq -4jm} \\ n=\pm 1: & e^{\mp 2h} \sum_\ell e^{-4\lambda\ell} \sum_m e^{-8jm} \; \Theta_{2\lambda\ell \geq -4jm \mp h} \end{cases}$$

$$\sum_{\ell=-\infty}^{\infty} e^{-4\lambda\ell} \sum_{m=-\infty}^{\infty} e^{-8jm} \; \Theta_{2\lambda\ell \geq 4jm}$$

$$\approx \int_{-\infty}^{\infty} dm\, e^{-8jm} \int_{-\frac{4jm}{2\lambda} + \left\{ {0 \atop \mp h/2\lambda} \right.}^{\infty} d\ell\, e^{-4\lambda\ell}$$

$$= \int_{-\infty}^{\infty} dm\, e^{-8jm} \, \frac{1}{4\lambda} \, \underbrace{e^{-4\lambda \cdot \frac{4jm}{2\lambda}}}_{e^{+8jm}} \cdot \left\{ {1 \atop e^{\pm 4\lambda \cdot h/2\lambda}} \right.$$

$$= \frac{e^{\pm 2h}}{4\lambda} \cdot \infty \quad \ldots \text{ erm.}$$

$$= \frac{1}{4\lambda} \left\{ {1 \atop e^{\pm 2h}} \right\} \cdot \Big(\int_{\mathbb{R}} dm\Big) \; ?L = \infty$$

Take $J = 0$

$$\Rightarrow \quad Z = \sum_{n=-1}^{1} \sum_{l \in \mathbb{Z}} e^{-(4\Lambda l + 2Hn)} \quad \Theta_{4\Lambda l + 2Hn \geq 0}$$

$$= \sum_{n=-1}^{1} \sum_{l=l_{min}}^{\infty} e^{-(\dots)} \qquad \text{w/. } l_{min} = -\left\lfloor \frac{h}{2\lambda} n \right\rfloor$$

$$\approx \frac{1}{4\Lambda} \sum_{n=-1}^{1} e^{-2Hn} \cdot e^{+4\Lambda \left\lfloor \frac{H}{2\Lambda} n \right\rfloor}$$

$\underline{J = 0}$

$$Z = \sum_{l=-\infty}^{\infty} e^{-4\Lambda l} \sum_{n=-1}^{1} e^{-2Hn} \quad \Theta_{4\Lambda l + 2Hn \geq 0}$$

$$= \sum_{l=0}^{\infty} e^{-4\Lambda l} + e^{-2H} \sum_{\substack{l=\lceil -H/2\Lambda \rceil \\ =-\lfloor H/2\Lambda \rfloor}}^{\infty} e^{-4\Lambda l}$$

$$+ e^{+2H} \sum_{l=\lceil H/2\Lambda \rceil}^{\infty} e^{-4\Lambda l}$$

$$= Z_\Lambda + e^{-2H}\left(Z_\Lambda + \sum_{l=-\lfloor H/2\Lambda \rfloor}^{-1} e^{-4\Lambda l} \right)$$

$$+ e^{+2H}\left(Z_\Lambda - \sum_{l=0}^{\lfloor H/2\Lambda \rfloor} e^{-4\Lambda l} \right)$$

$p = -l$ reversal

$$= Z_\Lambda(1 + 2\cosh(2H)) + e^{-2H} \sum_{p=+1}^{\lfloor H/2\Lambda \rfloor} e^{4\Lambda p} - e^{2H} \sum_{p=0}^{\lfloor H/2\Lambda \rfloor} e^{-4\Lambda p}$$

$$= \cancel{e^{-2H-4\Lambda}} - 1$$

$$+ 2 \sum_{p=0}^{\lfloor H/2\Lambda \rfloor} \cosh(4\Lambda p - 2H)$$

$$\boxed{= Z_\Lambda \underset{(Z_\Lambda Z_H)}{\underset{\|}{(1 + 2\cosh(2H))}} + e^{-2H}\left(\sum_{l=1}^{\lfloor H/2\Lambda \rfloor} e^{+4\Lambda l} \right) + e^{+2H}\left(\sum_{l=1}^{\lfloor H/2\Lambda \rfloor} e^{-4\Lambda l} + 1 \right)}$$

$$Z = \sum_{l} \sum_{m} e^{-4\Lambda l} e^{-8Jm} \sum_{n=-1}^{1} e^{-2Hn} \Theta(\ldots)$$

$$2Hn \geq -(4\Lambda l + 8Jm)$$

$$n \geq -\frac{2\Lambda}{H} l - \frac{4J}{H} m$$

$$\cancel{l \geq \frac{H}{2\Lambda} n - \frac{4J}{H} m}$$

$$= \sum_{m} \left(\sum_{l=-\lfloor \frac{2J}{\Lambda} m \rfloor}^{\infty} (\ldots) + \sum_{l=-\lfloor \frac{2J}{m} - \frac{H}{2\Lambda} \rfloor}^{\infty} (\ldots) e^{+2H} \quad l \geq -\frac{H}{2\Lambda} n - \frac{2J}{\Lambda} m \right.$$

$$\left. + \sum_{l=-\lfloor \frac{2J}{m} + \frac{H}{2\Lambda} \rfloor}^{\infty} (\ldots) e^{-2H} \right)$$

$$= \sum_{m \in \mathbb{Z}} e^{-8Jm} \left(Z_\Lambda (1 + 2\cosh(2H)) + \sum_{l=-\lfloor \ldots \rfloor}^{-1} e^{-4\Lambda l} + ('') + ('') \right)$$

$$\cancel{\frac{Z_\Lambda(1 + 2\cosh(2H))}{1 - e^{-8J}}}$$

$$\parallel$$

$$= (Z_\Lambda Z_H Z_J) + \sum_{m} e^{-8Jm} (\Sigma + \Sigma + \Sigma).$$

$$\langle \epsilon_i(\Omega^t(\sigma))\, \epsilon_j(\sigma)\rangle_{eq} = \frac{(2\lambda)^2}{Z_1} \sum_{l=0}^{N\nu/2} \frac{e^{-4\Lambda l}}{(2l)!} \sum_{\langle n\rangle} \sum_{\langle p,q} \delta_{i=n_p^t}\, \delta_{j=n_q}$$

$$\Rightarrow \sum_{ij} r_i^\mu r_j^\nu (\,\text{''}\,) = \frac{(2\lambda)^2}{Z_1} \sum_{l=0}^{N\nu/2} \frac{e^{-4\Lambda l}}{(2l)!} \sum_{\langle n\rangle} \sum_{pq} \sum_{ij} \delta_{i=n_p^t}\, \delta_{j=n_q}\, r_i^\mu r_j^\nu$$

$$= \frac{(2\lambda)^2}{Z_1} \sum_{l=0}^{N\nu/2} \frac{e^{-4\Lambda l}}{(2l)!} \underbrace{\sum_{\langle n\rangle} \sum_{pq} r^\mu_{n_p^t}\, r^\nu_{n_q}}_{I_{t,0}}$$

r_{n^t} = position of a prtcl starting @ site r_n after t timesteps.

n = just a site index

Let's drop the indices ∵ confusing.

Define $r_p(t)$ to be the pos$^{\underline{n}}$ of a prtcl starting @ $r_p(0) = x_p$ after t timesteps

$$I_{t,0} = \sum_{x_1\neq\ldots\neq x_{2l}} \sum_{pq} r_p^\mu(t)\, \underbrace{r_q^\nu(0)}_{= x_q^\nu}$$

$$= \sum_{pq} \sum_{x_1\neq\ldots\neq x_p\neq\ldots\neq x_q\neq\ldots x_{2l}} r_p^\mu(t)\, r_q^\nu(0)$$

$$\langle J_t^\mu J_0^\nu\rangle_{eq} = \frac{(2\lambda)^2}{Z_1} \sum_l \frac{e^{-4\Lambda l}}{(2l)!} \Big(\underbrace{I_{t+1,1} + I_{t,0} - I_{t+1,0} - I_{t,1}}_{J_{t0}} \Big)$$

$$J_{t,0} = \sum_{pq} \sum_{\ldots} \Big(r_p^\mu(t+1)\big(r_q^\nu(1) - r_q^\nu(0)\big) + r_p^\mu(t)\big(r_q^\nu(0) - r_q^\nu(1)\big)\Big)$$

$$= \sum_{pq} \sum \Big(\big(r_p^\mu(t+1) - r_p^\mu(t)\big)\, \dot{r}_q^\nu(0)\Big)\, \delta t$$

$$J_{t,0} = \sum_{pq} \underbrace{\sum_{\cdots} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{q}(0)\, \delta t^2}_{\text{Invariant under changing } p,q \;\because\; \text{summed over all } x_p, x_q.} \to 1 = \delta t = 1 \text{ timestep}$$

$= 4$ proposed jumps on avg

$= 1$

$$= \sum_{p}^{2l} K_{pp} + \sum_{p \neq q}^{2l(2l-1)} L_{pq}$$

$$K_{pp} = \sum_{\substack{x_1 \neq \cdots \neq x_p = \cdots \neq x_{2l} \\ \text{s.t. } x_i \neq x_p}} \sum_{x_p} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{p}(0)$$

$$= \begin{Bmatrix} N_v - 1 \\ 2l - 1 \end{Bmatrix} \sum_{x_p} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{p}(0)$$

$$L_{pq} = \begin{Bmatrix} N_v - 2 \\ 2l - 2 \end{Bmatrix} \sum_{x_p \neq x_q} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{q}(0)$$

This seems great! We get

$$(2l)!\, J_{t,0} = \binom{N_v}{2l} \frac{2l}{N_v} \sum_{x_p} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{p}(0) + \binom{N_v}{2l} \frac{2l(2l-1)}{N_v(N_v-1)} \sum_{x_p \neq x_q} \dot{r}^{\mu}_{p}(t)\, \dot{r}^{\nu}_{q}(0)$$

where we've assumed the prtcl dynamics are:

① μcanonical $\Rightarrow$ prtcls conserved

② deterministic.

① & ② imply Ω is at <u>worst</u> injective i.e. $\exists$ sites which aren't reached in t timesteps.

BUT we're <u>not</u> dealing with deterministic dynamics

Low H

$$Z_{\underline{w}}(\{x\}) = 2^{N_P} \cosh^{N_E}(H) \sum_{FCG} \tanh^{|E_F|}(H) \underline{\left(\prod_{\alpha \in E_F} \sigma_\alpha^{ref} \right)}$$

FCG → closed polygon subgraphs. {even no. lines per site}

$= (-1)^{\#\text{ intersecti[ons] w/ string}}$

High H

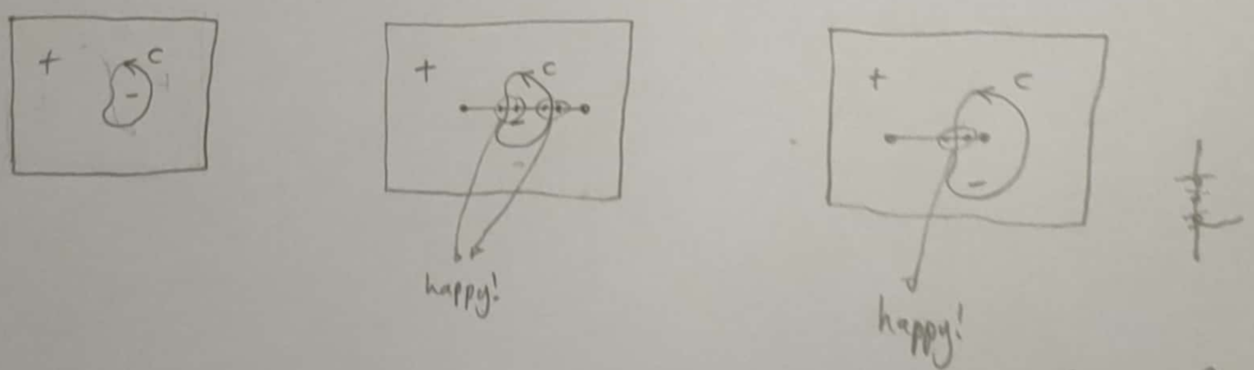
For $\sigma_\alpha^{ref} = +1 \ \forall \alpha$,

$$Z = 2e^{N_E H} \sum_{\substack{\text{islands} \\ \text{of } \downarrow \mu\text{'s}}} e^{-2H \cdot \text{perimeter of island}}$$

$$Z = \sum_{\mu} e^{H(N_E - 2 \cdot \# \underbrace{\text{unhappy neighbours}}_{\text{i.e. } \sigma_{pq}^{ref} \mu_p \mu_q = -1})}$$

assumes σ ref only has non-cont[ractible] strings/loo[ps] & they're all minimal

$\#$ unhappy neighbours $= \#\{\sigma\text{ ref} = -1\} +$



$\Rightarrow$ # unhappy neighbours $= \#\ \sigma_{ref} = -1$ → indep. of $\underline{\mu}$

+ perim.s of islands

− # intersect[io]ns of strings & islands

$$\Rightarrow \quad Z = e^{H\left(N_E - 2|\{\alpha \mid \sigma_\alpha^{ref} = -1\}|\right)}$$

$$\cdot \sum_{\text{islands}} e^{-2H(\text{Perim. of islands} - \#\text{intersect}^{\text{ns}} \text{ w/. strings})}$$

islands ↓ of ↓ spins (μ) in ↑ balk

$$= e^{H(\ldots)} \sum_{\text{islands}} e^{-2H(\text{Perim})} \; e^{+2H \# \text{intersect}^{\text{ns}}}$$

strings.

$$Z_{low} = 2^{N_P} C_H^{N_E} \sum_{F \subset G} t_H^{|E_F|} (-1)^{|F \cap S|}$$

$$Z_{high} = 2 e^{N_E(H - 2|\{\alpha \mid \sigma_\alpha^{ref} = -1\}|)} \sum_{F^* \subset G^*} e^{-2H|E_{F^*}|} \; e^{+2H|F^* \cap S|}$$

$G \simeq G^*$

$F \longleftrightarrow F^*$

$t_H \longleftrightarrow e^{-2H}$

$(-1)^{|F \cap S|} \longleftrightarrow e^{+2H|F^* \cap S|}$

$e^{2H} \neq -1$ ...

Hmmm.

$2H_c \sim \ln(p-p_c)$?

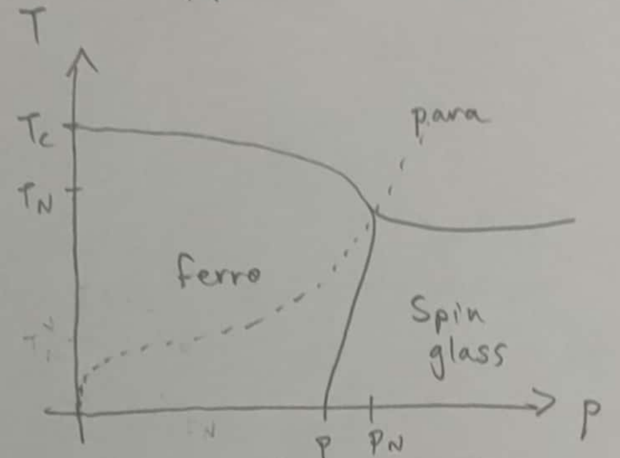
$\underline{p_c = 0.17}$ for $\pm J$-model.

$p_c \sim 0.15$

$T_c \sim 0.2$

$p_c \sim$ n° quasiparticles in random bond approx$^{\underline{n}}$

Our approx$^{\underline{n}}$ will be to use $p_c(T)$:



phase diagram for $[Z]$ where $[\cdot] =$ disorder avg.

Not quite what we want!

..... = Nishmori line w/. $e^{2H} = \frac{p}{1-p}$

BUT $p_c = p_c(T)$

$\rightsquigarrow$ Conjecture by Takeda & Nishimori that $H(p_N) + H(p_N^*) = 1$

Kramers - Wannier as FT

Define $C_m = \{ \sum_{v \in \Delta_m} a_v v \mid a_v \in G = \mathbb{Z}_2 \}$ to be the m-cells of the cell complex.

$\Delta_0 = V$
$\Delta_1 = E$

Consider scalar fields $\phi : C_1 \to C_0$

$\exists$ also operators ∂ & δ

We observe that

$$\sum_{\langle ij \rangle} \sigma_i \sigma_j = |E| - 2|\delta\phi|$$

$|\cdot|$ = Hamming weight $= ?$

$\Rightarrow Z = \sum$ over all gradients of $\mathbb{Z}_2$ fields

$$= \sum_{\phi \in C_0} f(\delta\phi) \overset{DFT}{=} \sum_{\phi \in C_0} \frac{1}{\sqrt{|C_1|}} \sum_{\xi \in C_1} (-1)^{\langle \delta\phi, \xi \rangle} \hat{f}(\xi)$$

Characters of C_m are $\chi_d(c) = (-1)^{\langle c, d \rangle}$

w/. $f(x) \doteq e^{KN - 2|x|}$

Use that δ is adjoint of ∂

Sum over characters zero unless trivial.

$$\Rightarrow Z = \frac{|C_0|}{\sqrt{|C_1|}} \sum_{\gamma \in Z_1} \hat{f}(\gamma)$$

$$= 2^{N - |E|/2} \sum_{''} ''$$

$Z_1 = \ker(\partial)$

$$\hat{f}(\gamma) = \frac{1}{\sqrt{|C_1|}} \sum_{\phi \in C_1} f(\phi)\,(-1)^{\langle \gamma | \phi \rangle}$$

$$= 2^{-|E|/2} \prod_{e \in E} \left(e^{K} + (-1)^{\gamma_e} e^{-K}\right)$$

$$= 2^{-|E|/2}\, C_K^{|E|} \prod_{e \in E} \left((1 + (-1)^{\gamma_e}) + (1 - (-1)^{\gamma_e}) \tanh K\right)$$

$$= 2^{|E|/2}\, C_K^{|E|}\, t_H^{|\gamma|}$$

The K-W duality is ~~r~~ ~~just~~ usually the FT:

$$U_\pm(H) = e^{\pm H} \quad \rightsquigarrow \quad U^*_\pm(H) = \frac{1}{\sqrt{2}}\left(e^{+H} \pm e^{-H}\right)$$

$$\Rightarrow \quad + H \sum_{e\in\Delta_n} \prod_{v\in\partial e} \sigma_v \longrightarrow + J \sum_{e^*\in\Delta_n^*} \prod_{p\in\partial e^*} \sigma_p$$

for us, $n=1$

$\Rightarrow \Delta_n^* = \Delta_1$

but $\partial\Delta_1^* = \delta\Delta_1$

$\subset \Delta_2$

Generally, we'll get

$$Z(U_\pm(H)) = 2^a Z^*(U^*_\pm(H))$$

w/. a dependent on Δ

For self-dual cases, 2^a is negligible in the thermodynamic limit

---

To introduce randomness, write:

$$-\beta F = H \sum_{e\in\Delta_n} \eta_e \prod_{v\in\partial e} \sigma_v \qquad \text{w/.} \quad \eta_e = \begin{cases} 1 & \text{prob. } p \\ -1 & \text{" } 1-p \end{cases}$$

One can make progress by performing the replica trick n times & conjecturing sth but I'll leave that for the paper.

Following the paper,

$$Z = 2^{|\Delta_0|} C_H^{|\Delta_1|} \sum_{\substack{\gamma \in \ker(\partial_1) \\ \cap \\ \Delta_1}} t_H^{|\gamma|}$$

$$= \sum_{\mu} e^{+H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q}$$

$$Z^* = \sum_{v} \sum_{\mu} e^{-H^* \sum_{\substack{\langle pq \rangle \\ p \overset{x}{-} q}} \eta_{pq} \underbrace{\left(\prod_{l | x \in l} v_l \right) \mu_p \mu_q}}$$

v_l are additional Ising spins for each non-contractible cocycles

↳ domain walls. ~ diff. BCs

↳ b_1 of them.

For Δ a torus, $Z - Z^* \sim L e^{-2H^* L} \to 0$

so we needn't worry about v ∵ only 2 dof.s

For Δ more general, $Z - Z^* \not\to 0$

e.g. on hyperbolic spaces $\partial b_1 / \partial N \neq 0$.

Note: $$Z(U_\pm(H)) = 2^a Z^*(U_\pm^*(H))$$

$$\Downarrow$$

$$Z(U_\pm(H_1))\, Z^*(U_\pm(H_2)) = Z(U_\pm^*(H_2))\, Z^*(U_\pm^*(H_1))$$

$$\Downarrow \quad @ \; H_{c_1}$$

$$U_\pm(H_{1c})\, U_\pm(H_{2c}) = U_\pm^*(H_{1c})\, U_\pm^*(H_{2c})$$

$$\Updownarrow$$

$$e^{-2H_{2c}} = \tanh(H_{1c})$$

To define the random model we note that for $e \in \Delta_1$, the Boltzmann avg. factor is:

$$x_k(p, H) = p e^{(n-2k)H} + (1-p) e^{-(n-2k)H}$$

for n replicas where k of them are $-$ve

Then

$$[Z^n]_{av} = Z_n(x_0, x_1, \dots, x_n)$$

Likewise,

$$x_{2k}^* = 2^{-n/2}\left((e^H + e^{-H})^{n-2k}(e^H - e^{-H})^{2k}\right)$$
$$x_{2k+1}^* = 2^{-n/2}\left((e^H + e^{-H})^{n-2k-1}(e^H - e^{-H})^{2k+1}\right) \cdot \underline{(2p-1)}$$

$$\Rightarrow \quad Z_n(\underline{x}) = 2^{\tilde{a}} Z_n^*(\underline{x}^*)$$

We can't just proceed as before :

$\underbrace{x_n = x_n^* \text{ can't be satisfied } \forall n}_{\text{condit}^{\bar{n}} \text{ for self-duality.}}$

Conjecture : only $x_0 = x_0^*$ matters.

↳ gives crit point only on Nishmori line $e^{-2H} = \frac{1-p}{p}$

Generalising to H_1, H_2 we get:

$$x_0(p_{1c}, H_{1c})\, x_0(p_{2c}, H_{2c}) = x_0^*(p_{1c}, H_{1c})\, x_0^*(p_{2c}, H_{2c})$$

and $e^{-2H_n} = \frac{1-p_n}{p_n} \quad \forall n \in \{1, 2\}$.

In $n \to 0$ limit of quenched randomness

i.e. a fixed "random" config,

$H(p_{1c}) + H(p_{2c}) = 1$

where $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$

↳ mutual info.

For $n \to \infty$, the lowest-energy random config dominates the replica average

$\Rightarrow$ get non-random Ising.

What we want to do is perform the replica trick but only with replicas w/. σ^{ref}'s satisfying the necessary condit^{ns}.

This will inform the value of p.

Remember that the $\#\{\sigma^{ref}_i = -1\}$ is bounded from below by:

$$\underline{W} \odot \underline{L} + \min_f \left\{ \sum_{m=1}^{n} |\underline{x}_m - \underline{x}_{f(m)}|_1 \right\} \frac{1}{2}$$

$\because$ 2x counting.

$f(m) \neq m$ $\forall m$

$f: \mathbb{Z}_n \to \mathbb{Z}_n$

$\Rightarrow$ taking every edge $\sim$ equiv. (clearly not true)

$$p \geqslant \frac{''}{N_E} = \frac{\underline{W} \odot \underline{L}}{2\Pi_d L d} + \dots$$

We should have that $\left.\begin{matrix} p_1 = p_2 \\ H_1 = H_2 \end{matrix}\right\}$?

$$\Rightarrow \quad x_0(p_c, H_c) = x_0^*(p_c, H_c) \quad \& \quad e^{-2H} = \frac{1-p}{p}$$

$$x_0(p, H) = p e^{nH} + (1-p) e^{-nH}$$

$$2^{-n/2} x_0^*(p, H) = C_H^n$$

Let's try a mean-field approach:

$$-\beta V f = H \sum_{\langle pq\rangle} \eta_{pq}\, \mu_p \mu_q + J \sum \dots$$

Let $J = 0$ for now & $\mu_p \approx \mu + \delta\mu_p$

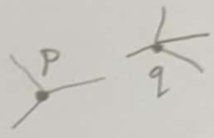
$\Rightarrow \mu_p \mu_q \approx \mu^2 + \mu(\delta\mu_p + \delta\mu_q)$ to $\mathcal{O}(\delta\mu_p)$

$= \mu(\mu_p + \mu_q) - \mu^2$ " "

" $\sum_p b_p^{eff}$

$$\Rightarrow -\beta V f = -H\mu^2 \sum_{\langle pq\rangle} \eta_{pq}$$

$$+ H\mu \sum_p \mu_p \Big(\sum_{q \in nn(p)} \eta_{pq}\Big)$$



$\Rightarrow b_p^{eff} = H\mu \sum_{q \in nn(p)} \eta_{pq}$ $\quad b^{eff} = \sum_p b_p^{eff}$

$\rightarrow Vf = h\mu^2 b^{eff} - h\mu \sum_p \mu_p b_p^{eff}$

By H-S decoupling, we'll get

$G_{pq} = H\eta_{pq}$ for nearest neighbours & 0 otherwise

$F = \tau \ln Z$ $\quad C = \frac{\partial U}{\partial \tau}$

$C = \tau \frac{\partial S}{\partial \tau}$

$S = \frac{\partial F}{\partial \tau}$

$$\Rightarrow C = \tau \frac{\partial^2}{\partial \tau^2}(\tau \ln Z)$$

$$= \tau \frac{\partial}{\partial \tau}\left(\ln Z + \tau \frac{\partial}{\partial \tau} \ln Z\right)$$

$$= 2\tau \frac{\partial \ln Z}{\partial \tau} + \tau^2 \frac{\partial^2 \ln Z}{\partial \tau^2}$$

$\langle Z_n \rangle \approx 4\left(1 - \frac{\langle n \rangle}{N_V}\right)$ very approximate!

$p_{occ} = \frac{\langle n \rangle}{N_V}$ $\forall$ sites

$$\langle \dot{x}_p^{\mu}(t)\, \dot{x}_q^{\nu}(0) \rangle$$

$$\langle n(\Lambda) \rangle = \frac{1}{2}(1 - \tanh(\Lambda))$$

$\gamma_n = \frac{1}{2} Z_n$

$D_n = \frac{K_n}{C}$ $\quad \cancel{(\partial C / \partial n = 0)}$

$$D = \left(\gamma_n \frac{n}{N_V}\right) \cosh^2 \Lambda$$

$D = \frac{1}{2}(1 - t_\Lambda^2)\, c_\Lambda^2$

$\langle \gamma_n \rangle = \frac{4}{2} \cdot \left(1 - \frac{1}{2}(1 - t_\Lambda)\right)$

$= (1 + t_\Lambda)$

$$\langle n(\Lambda)\rangle = \frac{1}{2}\left(1 - \tanh(\Lambda)\right)$$

$$\overset{T \ll \lambda}{\approx} e^{-2\lambda/T} = e^{-2\Lambda} \quad \text{(binomial)}$$

To find the average prtcl separat$^{\bar{n}}$ $\mathfrak{z}$, we note that

$$\rho := \frac{\langle n\rangle}{N_V} = \cancel{k}\, \mathfrak{z}^{-2}$$

$$\Rightarrow \quad \mathfrak{z}^2 = N_V e^{2\Lambda}$$

$$\mathfrak{z} = \sqrt{N_V}\, e^{\Lambda}$$

We expect issues when $\mathfrak{z} \sim \sqrt{N_V} \sim L$

$$\Rightarrow \text{ when } e^{\lambda/T} \sim 1 \quad ?$$

$$\Rightarrow \quad T/\lambda \sim 0$$

Reason: above n too coarse an approx$^{\bar{n}}$.

We have $\cancel{k}$ $\langle n(\Lambda)\rangle = e^{-2\Lambda}\left(\frac{(1+e^{-2\Lambda})^{N-1} + (\ldots - \ldots)^{N-1}}{(\ldots + \ldots)^{N} + (\ldots - \ldots)^{N}}\right)$

$$\mathfrak{z} = \sqrt{N_V}\, \frac{1}{\sqrt{\langle n\rangle}} = \sqrt{N_V}\, e^{\Lambda}\left(\ldots\right)^{-1/2}$$

$\langle \dot{x}^\mu(t)\, \dot{x}^\nu(0) \rangle = A\, \delta^{\mu\nu}\, \delta_{t0}$

$\dot{\underline{x}} = \frac{1}{\delta t}\, \underline{d} = \frac{\underline{x}(t+\delta t) - \underline{x}(t)}{\delta t}$

To find A, ~~setr~~ ~~make~~ set $t=0$

$A\, \delta^{\mu\nu} = \langle \dot{x}^\mu(0)\, \dot{x}^\nu(0) \rangle$

$\delta t = 1 \qquad \cancel{\&}\; a = 1$

$\delta t_M = \delta t / 4 \qquad a_M = a$

$A = \langle \dot{x}^\mu(0)\, \dot{x}^\mu(0) \rangle$ (no μ sum)

~~We expect that $\dot{x}(0) \sim$~~ 4

We expect $|\underline{x}(t+\delta t) - \underline{x}(t)|^2 \sim 4D\, \delta t$ in 2D

$D := \frac{a^2}{\delta t}$

$\Rightarrow \sim \underline{\underline{4a^2}}$

$2a$

---

no sum

In 1D, $|d^\mu(t)|^2 \sim 2D\, \delta t \sim 2a^2$

$\Rightarrow$ ~~$\cancel{\&}$~~ $\underline{\underline{A = 2a^2}}$?

$$|d^{\mu}(t)|^2 \sim \cancel{2D\delta t}\ \underbrace{\frac{z}{2}D}_{\gamma}\delta t$$

$$\cancel{\langle z_n \rangle_n = 4 \cdot (1-n)^4 + 3 \cdot (1-n)^3 n + \cdots}$$

$$\langle z_n \rangle_n = \frac{4\ {''} + 3\ {''} + 2\ {''} + 1\ {''}}{\underbrace{1\,(1-n)^4 + 4(1-n)^3 n + 6(1-n)^2 n^2 + \cdots}_{(p+q)^n}}$$

Simplest option: 2x spins flipped per move.

$\Rightarrow$ would expect

$\delta t_{\mu} \rightarrow \delta t_{\mu}/2$

but not much else new.

Don't really trust the demons...

$\rightarrow$ Key is to compare w/. $H=0$ case for original dynamics...

Problems: * starting state = ?

Need to initialise w/. can. dist$^{\underline{n}}$ $\Rightarrow$ will be a mess...

~~For T = 0, above will break~~

Could always explicitly disallow this... * Can now annihilate prtcls & create them elsewhere.

@ each order we want to calculate.

$$f(x) := \sum_{\lambda \subset G} x^{|E_\lambda|} \underbrace{\left(\prod_{\alpha \in E_\lambda} \sigma_\alpha^{\text{ref}} \right)}_{(-1)^{\#\text{int.s}}}$$

$$= \forall \text{ loops} \begin{cases} (-1)^{\#\text{crossings of non-cont. loops}} \\ \cdot (-1)^{\#\text{winding no}} \\ \cdot (-1)^{\#\text{prtcls enclosed}} \end{cases}$$

$$\to f(x) = \sum_{\lambda \subset G} x^{\overbrace{\sum_{l \in \lambda} \cancel{\ } |l|}^{|\lambda|}} \cdot \prod_{l \in \lambda} \left((-1)^{\#\text{cross } w} (-1)^{w_l} (-1)^{\#\text{enclosed prtcls}} \right)$$

λ = ~~set of~~ admissible graphs i.e.
each vertex has even no. bonds
= product of a bunch of loops $l \in \lambda$

$\lvert\lambda\rvert$	$f_{\lvert\lambda\rvert}$
0	1
4	$\square = \overset{\square}{N_v} - 2\overset{\boxdot\cdot\cdot}{(n)}$
6	$\square = \overset{\square}{2N_v} - 2\overset{\boxdot}{(4n)} + \cancel{2}\ \overset{\boxdot\,\boxdot}{\#}$ such pairs ew...

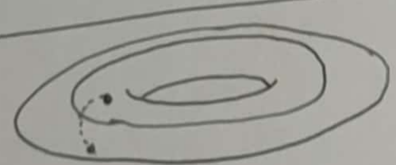
BUT remember we want to find:

$$S = \sum_{\underline{w}} \sum_{\underline{x}_1 \neq \dots \neq \underline{x}_n} Z_n$$

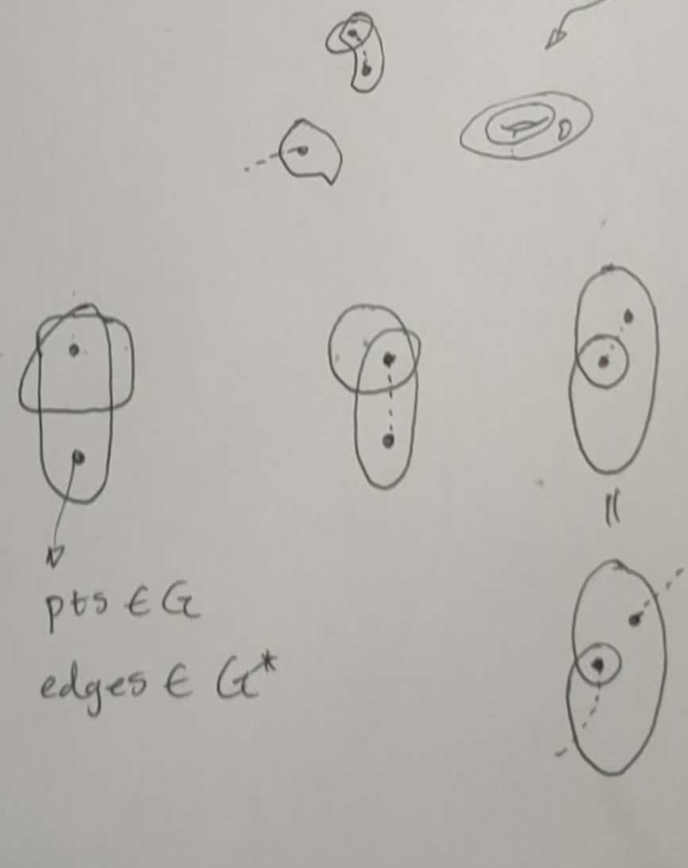
So we're interested in

$$F(x) = \sum_{\lambda \subset G^*} x^{\lvert\lambda\rvert} \sum_{\underline{w}} \sum_{\{\underline{x}\}} \left(\prod_{\ell \in \lambda} (-1)^{\cdots} (-1)^{\cdots} (-1)^{\cdots} \right)$$

$\cancel{=}$



For non-cont. loops
we're kinda screwed
$\because$ depends on location of
string!
$\Rightarrow$ Restrict to lower
orders



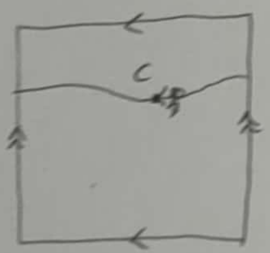
1) Calculate all components of subgraphs of a given order.

2) Calculate whether loop non-cont. in a given direc.

3) Calculate winding $n^{\underline{o}}$ of a loop ~ any point ~~(ideally only enclosed points)~~

= 0 for non-enclosed points

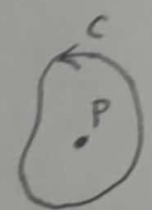
(2)



Sum up the displacement vectors along C. If it's non-zero, C is non-cont. We can determine the sector by which components are non-cont!

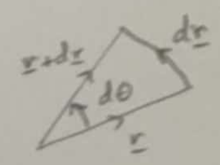
$$\oint_C d\underline{r} \neq \underline{0} \implies C \text{ non-cont.}$$

(3)



$$w = \oint_C d\theta \,.$$

~~$\theta = 2\pi$~~

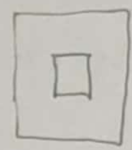


θ should be signed angle ⇒ be careful w/. atan...

How can I find all possible closed-loop graphs naïvely? Just do ~~over~~ parity of square loops $\Rightarrow \# = 2^{N_P}$ states total!

$\Downarrow$

length-N_p binary numbers.



Defects also live on plaquettes.

$$\frac{1}{n!}\sum_{\{x\}} \left(\sum_{\lambda \text{ of order } \ell} \left(\prod_{c \in \lambda} (-1)^{\cdots} \right) \right) =$$

$\lvert\lambda\rvert$	
0	$1 \cdot \binom{N_v}{n}$
4	$\square = (\overset{\square}{N_v} - \overset{\boxdot}{2n}) \cdot \binom{N_v}{n}$
6	$\square = \frac{1}{n!}\sum_{\{x\}} (\overset{\square\ \square}{2 N_v} - \overset{\square}{2 \cdot 4n} + 2 \cdot \# \boxed{\cdot\ \cdot}\ \text{pairs})$

$$= \binom{N_v}{n}(2N_v - 8n) + \binom{N_v}{n} \cdot 2 \cdot\ ?$$

$$-\beta F = H \sum_\alpha \sigma_\alpha + \Lambda \sum_i \prod_{\alpha \in \partial^+ i} \sigma_\alpha$$

$$\sigma_\alpha = \sigma + \delta\sigma_\alpha$$

$$\sum_i (\sigma_{i_0} + \delta\sigma_{i_1})(\dots)_{i_2}(\dots)$$

$$= \sum_i \left(\sigma^{z_i} + \sigma^{z_i - 1}(\delta\sigma_{i_1} + \delta\sigma_{i_2} + \dots) + O(\delta\sigma^2)\right)$$

$$= \sum_i \left(-\sigma^4 + \sigma^3 \sum_{\alpha \in \partial^+ i} \sigma_\alpha\right) \qquad z_i = 4$$

$$-\beta F = -N_v \sigma^4 + H \sum_\alpha \sigma_\alpha$$

$$+ \Lambda \sigma^3 \sum_i \sum_{\alpha \in \partial^+ i} \sigma_\alpha$$

$$2\sum_\alpha$$

$$-\beta F = -N_v \sigma^4 + \underbrace{(H + 2\Lambda\sigma^{z-1})}_{H_{eff}.} \sum_\alpha \sigma_\alpha .$$

$$H_{eff} = H + 2\Lambda\sigma^{z-1} \qquad \sigma := \langle \sigma_\alpha \rangle$$

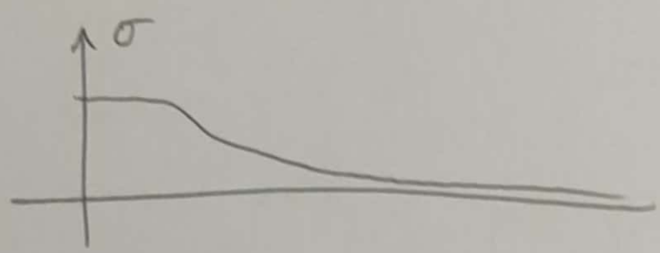
$$= \sum_{\text{states}} P_\alpha \sigma$$

$$\cancel{M = \tanh(H_{eff} M)}$$

$$M = \tanh(\beta(h + 2\lambda$$

$$\underline{\sigma = \tanh(H_{eff})}$$

$$\sigma = \tanh(\beta(h + 2\lambda\sigma^{z-1}))$$



For $\sigma = \frac{1}{2}$,

$$\frac{1}{2} = \tanh(H + 2\Lambda(\tfrac{1}{2})^3)$$

$$\frac{1}{2} = \tanh(H + \cancel{2}\tfrac{1}{4}\Lambda)$$

$$\approx H + \frac{1}{4}\Lambda - \cancel{\tfrac{1}{3!}(\dots)^3 + \dots} \quad \text{for } T \text{ large } (\gg \Lambda)$$

$$\underline{H \approx \frac{1}{2} - \frac{1}{4}\Lambda}$$

$h \gg \lambda$

$$h \approx \frac{T}{2} - \frac{\lambda}{4} \approx \frac{1}{2}T.$$

@ percolation threshold,

$$H \gg \Lambda \Rightarrow \frac{1}{2} \approx \tanh(H)$$

$$H \approx \operatorname{arctanh}(\tfrac{1}{2})$$

$h \approx \frac{1}{2}T$. gives $\sigma \approx \frac{1}{2}$.

$\Rightarrow p_\uparrow \sim p_\downarrow \sim \frac{1}{2}$?

---

p $\underline{\underline{T \sim 2h}}$

Solution: Relate K and D

via $\langle \dot{x}^\mu_p(t)\, \dot{x}^\nu_q(0) \rangle$

We'll have $\sim \sum_{pq} \sum_{\{x\}} Z_H(\{x\}) \langle \dot{x}^\mu_p(t)\, \dot{x}^\nu_q(0) \rangle$

$\cancel{\sum_{\{x\}} Z_H (}$ $\approx \sum_{pq} \underline{\underline{\langle \dot{x}^\mu_p(t)\, \dot{x}^\nu_q(0) \rangle}} \sum_{\{x\}} Z_H(\{x\})$

Can be obtained from percolation sims?

First thing: Relate diffusion to percolation

Relate perc. to magnetisation

Relate mag. to temperature / field.

$\underline{\text{indep prtcl}}$
$\underline{\text{approx}^{n}}$

But first: We can find $\langle x(t)\, x(0) \rangle \sim t^\alpha$

but what about $\langle \dot{x}(t)\, \dot{x}(0) \rangle$?

$\dot{x}(t) = x(t+1) - x(t)$

$\Rightarrow \langle \dot{x}(t)\, \dot{x}(0) \rangle \sim x(t+1)\, x(1) - \ldots$

$\sim t^\alpha$ also? No!

w/. $\beta = 1 + \alpha$ $\qquad \sim t^{\beta/2 - 2}$

Comes from Ansatz $\chi(t) \propto (\tau/t)^{1-\beta/2}\, e^{-\gamma_0 t}$ $\quad$ $\gamma_0 \sim 0$ on our timescales

Let's assume sth similar is happening here

So $\langle \dot{x}(t)\, x(0) \rangle \sim t^{\beta/2 - 2} \sim t^{\alpha/2 - 3/2} \sim t^{\frac{\alpha-3}{2}}$

When $\exists$ no correlat$^{\underline{n}}$s, we get instead $x(t) \sim \delta(t)$

$\Rightarrow \langle x(t)\, x(0) \rangle \sim t$ & $\langle \dot{x}(t)\, \dot{x}(0) \rangle \sim \delta_{t0}$.

More precisely, we get a stochastic force

$$\underline{f} = \underset{\substack{\downarrow \\ \text{(white noise)}}}{\eta} - (1 - \beta/2) \underbrace{\int_{0}^{t} d\tau\, \eta(\tau)\, (t-\tau)^{\beta/2 - 1}}_{\underline{x}(t) \text{ (for long times)}}$$

$$\Rightarrow \dot{\underline{x}}(t) = \left(\frac{\beta}{2} - 1\right) \int_{-\infty}^{t} d\tau\, \eta(\tau)\, (t-\tau)^{\beta/2 - 2}$$

$$\cancel{\Rightarrow \langle \dot{\underline{x}}(t) \cdot \dot{\underline{x}}(0) \rangle = (\beta/2 - 1)^2 \int_{-\infty}^{t} d\tau\, \underbrace{\langle \eta(\tau) \cdot \eta(0) \rangle}_{\overline{\eta^2}\, \delta(\tau - 0)}\, (t-\tau)^{\beta/2 - 2}}$$

$$\cancel{= (\beta/2 - 1)^2\, \overline{\eta^2}\, t^{\beta/2 - 2}.}$$

$\cancel{\beta = 1 + \alpha}$

$$\cancel{\langle \dot{\underline{x}}(t) \cdot \underline{x}(0) \rangle = \overline{\eta^2}\, t^{\beta/2 - 1}}$$

$$\cancel{\langle \dot{\underline{x}}(t) \cdot \underline{x}(0) \rangle = \int_{-\infty}^{t} d\tau \int_{-\infty}^{0} d\tau'\, \underbrace{\langle \eta(\tau) \cdot \eta(\tau') \rangle}_{\overline{\eta^2}\, \delta(\tau - \tau')}\, (t-\tau)^{\beta/2 - 1} (-\tau')^{\beta/2 - 1}}$$

$$\cancel{= \overline{\eta^2} \int_{-\infty}^{t} d\tau\, (-\tau)^{\beta/2 - 1} (t - \tau)^{\beta/2 - 1}}$$

$$\langle \dot{\underline{x}}(t) \cdot \dot{\underline{x}}(0) \rangle = (\beta/2 - 1)^2\, \overline{\eta^2}\, t^{\beta/2 - 2}$$

$$\langle \underline{x}(t) \cdot \underline{x}(0) \rangle = \overline{\eta^2}\, t^{\beta/2 - 1} \;?\quad \text{No!}$$

$$Z\, K^{\mu\nu} = \frac{1}{N_E} \sum_{t=0}^{t_{max}-1} \left(1 - \frac{\delta_{t0}}{2}\right) \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\underline{w}} \frac{1}{n!} \sum_{ij} x_i^\mu x_j^\nu$$

$$\cdot \sum_{\{\underline{x}\}} \sum_{\underline{\mu}} e^{+H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q} \cdot \cancel{\Big[(2\Lambda)^2 \delta_{i \in \omega^t(\underline{x})} \delta_{j \in \underline{x}} + (2\Lambda) H \big(\delta_{i \in \omega^t(\underline{x})} m_j(t) + m_i \delta_{j \in \underline{x}} \big)}$$

$$\cdot \Big[\underset{①}{(2\Lambda)^2 \dot{\Theta}_i(t) \dot{\Theta}_j(0)} + (2\Lambda)(H)\big(\underset{②a}{\dot{\Theta}_i(t) \dot{m}_j(0)} + \underset{②b}{\dot{\Theta}_j(0) \dot{m}_i(t)} \big) + \underset{③}{(H)^2 \big(\dot{m}_i(t) \dot{m}_j(0) \big)} \Big].$$

$$\beta \epsilon_i = \underset{0,\,2\Lambda}{\Lambda(1 - A_i)} - H\underset{0,\,\dots,\,2Hz}{\Big(z_i - \sum_{\alpha \in \partial^+ i} \sigma_\alpha\Big)}$$

$$\Theta_i = \sum_p \delta_{i = \omega^t(x_p)}$$

① is easy(ish!)

$$① = \frac{(2\Lambda)^2}{N_E} \sum_t \left(1 - \frac{\delta_{t0}}{2}\right) \sum_n \frac{1+(-1)^n}{2} \frac{e^{-2\Lambda n}}{n!} \sum_{\underline{w}} \sum_{pq} \sum_{\{\underline{x}\}} \dot{x}_p^\mu(t) \dot{x}_q^\nu(0) \sum_{\underline{\mu}} (\dots)$$

$$②a = \frac{(2\Lambda H)}{N_E} \Big(\text{''} \qquad \text{''} \Big) \sum_p \sum_{\{\underline{x}\}} \dot{x}_p^\mu(t) \sum_{\underline{\mu}} e^{\dots}$$

$$\sum_j x_j^\nu \dot{m}_j(0)$$

$$m_i := \cancel{z}_i - \overbrace{\sum_{\alpha \in \partial^+ i} \sigma_\alpha}^{S_i}$$

arb. const. vanishes in $\dot{m}_j \propto m_j(1) - m_j(0)$

$$\dot{m}_j = S_j[\sigma] - S_j[\Omega(\sigma)].$$

Key point: m only changes by a particle moving to/from the site!
No plaquette flips (in dynamics Ω)!

Either $\dot{m}_i = 0$ OR
$\dot{m}_i = \pm 2$ from a prtcl starting or ending there.

OR can pass over it! But that will give $\dot{m} = 0$ $\because$ m must be conserved by our moves!

Unfortunately the sign isn't knowable from where the spinons are $\therefore$.

$\Rightarrow \quad \dot{m}_j = \pm 2\,\delta_{j\in \underline{x}} \;\pm\; 2\,\delta_{j \in \partial(\underline{x})}$

$\sum_{\mu}$ etc in there now too...

$\Rightarrow \quad (2a) = \dfrac{(2\Lambda)(2H)}{N_E}\Big(\text{''} \quad \text{''}\Big) \sum_{pq} \sum_{\{\underline{x}\}} \dot{x}^{\mu}_{p}(t) \overset{\text{determines signs}}{\Big(} \pm \dot{x}^{\nu}_{q}(0) \pm \dot{x}^{\nu}_{q}(1)\Big)$

$(2b) = \dfrac{(2\Lambda)(2H)}{N_E}\Big(\text{''} \quad \text{''}\Big) \sum_{pq} \sum_{\{\underline{x}\}} \Big(\pm \dot{x}^{\mu}_{p}(t) \pm \dot{x}^{\mu}_{p}(t+1)\Big)\, \dot{x}^{\nu}_{q}(0).$

$(3) = \dfrac{(2H)^2}{N_E}\Big(\text{''} \quad \text{''}\Big) \sum_{pq} \sum_{\{\underline{x}\}} \Big(\pm \dot{x}^{\mu}_{p}(t) \pm \dot{x}^{\mu}_{p}(t+1)\Big) \Big(\ldots \Big)$

$\Rightarrow$ Total $\propto (2\Lambda)^2 \Big(\langle \dot{x}^{\mu}_{p}(t)\, \dot{x}^{\nu}_{q}(0) \rangle \Big)$

$+ (2\Lambda)(2H) \Big(\pm_{p_1} \langle \dot{x}^{\mu}_{p}(t+1)\, \dot{x}^{\nu}_{q}(0) \rangle \pm_{p_1} \langle \dot{x}^{\mu}_{p}(t)\, \dot{x}^{\nu}_{q}(0) \rangle$

$\pm_{q_2} \langle \dot{x}^{\mu}_{p}(t)\, \dot{x}^{\nu}_{q}(0) \rangle \pm_{q_2} \langle \dot{x}^{\mu}_{p}(t)\, \dot{x}^{\nu}_{q}(1) \rangle \Big)$

$+ (2H)^2 \Big(\pm_{p_1} \pm_{q_1} \langle \ldots \rangle + \ldots \Big)$

The signs are again...

Is there a saving grace? Yes! We sum over <u>all</u> $\underline{\mu}$ amenable to prtcl pos$^{\underline{n}}$s $\Rightarrow$ all signs will be represented, though perhaps not equally.

We can work on <u>avg.</u> wrt $\sum_{\underline{\mu}}(\ldots)$ by saying ~~MPA~~ $M(H) \sim \tanh(H)$.

All I want is $z_i^+ \, z_i^-$

$$A_i = \cancel{(-1)^{z_i^+ + z_i^-}} \quad (-1)^{z_i^-}$$

$$-\beta F = \Lambda \sum_i (-1)^{z_i^-} + H \sum_\alpha \sigma_\alpha$$

$$= \Lambda \sum_i (-1)^{z_i - y_i} + \frac{H}{2} \sum_i (z_i^+ - z_i^-)$$

$z_i^+ + z_i^{-1} = z_i$

$w = z_i^-$ $\therefore$ easier! $\quad y + w = z_i = 4$

$$-\beta F = \Lambda \sum_i (-1)^{w_i} + \frac{H}{2} \underbrace{\sum_i (z_i}_{2N_E} - 2w)$$

$$= H N_E + \Lambda \sum_i (-1)^{w_i} \cancel{} - H \sum_i w_i$$

$$= H\left(N_E - \sum_i w_i\right) + \Lambda \sum_i (-1)^{w_i}$$

$$= H\left(\sum_i y_i^2 - N_E\right) + \Lambda \sum_i (-1)^{y_i} \quad \because z_i = 4$$

I want to find $\langle \sum_i w_i^2 \rangle$

I can easily get $\langle \sum_i w_i \rangle$ & $\langle \sum_i (-1)^{w_i} \rangle$

Does knowing this give enough? $\quad w_i \in \{0, 1, 2, 3, 4\}$.

$(-1)^{w_i} \in \{1, -1, 1, -1, 1\}$.

$$\sum_i w_i^2 = \sum_i (4 - y_i)^2 = 16 N_v - 8 \sum_i y_i + \sum_i y_i^2$$

$$\left\langle \sum_i (y_i^2 - w_i^2) \right\rangle = 8 \sum_i y_i - 16 N_v$$

$$\left\langle \sum_i (-1)^{y_i} \right\rangle = \left\langle \sum_i (-1)^{w_i} \right\rangle.$$

$$\frac{1}{N_v}\sum_i y_i^2 = \frac{1}{N_v}\sum_i (z_i^+)^2$$

$$\sum_i (z_i^+ - z_i^-)^2 = \sum_i {z_i^+}^2 + {z_i^-}^2 - 2z_i^+ z_i^-$$

$$= \sum_i 4y_i^2 - 4y_i z_i + z_i^2$$

$$\sum_i (z_i^+ + z_i^-)^2 = \sum_i (1) = N_v$$

$$\sum_i (z_i^+ - z_i^-)^2$$

$$-\beta F = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha$$

hmmm.

I want $\sum_i \left(\sum_{\alpha\in\partial^+ i} \sigma_\alpha\right)^2$

$$\sum_i A_i = \sum_i \prod_{\alpha\in\partial^+ i} \sigma_\alpha.$$

In mean-field it almost has the right form.

$$= \sum_i \sum_{\alpha\neq\beta} \sigma_\alpha \sigma_\beta.$$

$$\sum_i z_i = 2N_E = 4N$$

$$M = \frac{1}{4N}\sum_i (2y_i - z_i)$$

$$= \left(\frac{1}{2N}\sum_i y_i\right) - 1$$

$$\Rightarrow \frac{1}{N}\sum_i y_i = 2(M+1) = \bar{y}$$

$$2\sum_i \left(\frac{z_i}{2}\right) = 2N\left(\frac{z}{2}\right) = 2N\cdot\frac{4(3)}{2}$$

$$\mathbb{P} = \frac{1}{12N}\sum_i \overset{4}{z_i} y_i - y_i^2 = \frac{1}{3N}\sum_i y_i - \frac{1}{12N}\sum_i y_i^2$$

$$= \frac{2}{3}(M+1) - \frac{1}{12N}\sum_i y_i^2$$

$$y_i = y + \delta y_i$$

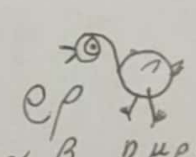
$$\frac{1}{N}\sum_i y_i^2 \approx \frac{1}{N}\sum_i \bar{y}^2 + 2\bar{y}\delta y_i$$

$$= \frac{1}{N}\sum_i 2\bar{y}y_i - \bar{y}^2$$

$$= \frac{2\bar{y}}{N}\sum_i y_i - \bar{y}^2$$

$$= 2\bar{y}^2 - \bar{y}^2 = \bar{y}^2$$

$$\Rightarrow \mathbb{P} = \frac{1}{3}\bar{y} - \frac{1}{12}\bar{y}^2$$

ℙ 𝔾 eρ 

♡ αβ ημρω

$$M = \frac{1}{2N_E} \sum_i (\underbrace{z_i^+}_{y_i} - z_i^-) = \frac{1}{4N} \sum_i (2y_i - z_i)$$

$$\mathbb{P} = \frac{1}{6N_E} \sum_i z_i^+ z_i^- = \frac{1}{12N} \sum_i y_i(z_i - y_i)$$

$$\Rightarrow M = \frac{1}{4N}\left(\sum_i 2y_i - \cancel{4}N\right) = \frac{1}{2N}\sum_i y_i - 1 = \frac{\sum_i y_i}{N_E} - 1$$

$$= \frac{2 \cdot \#+\text{bonds}}{N_E} - 1$$

$$= 2 \cdot \text{frac of } +\text{bonds} - 1$$

$$\in [-1, 1].$$

$$\& \quad \mathbb{P} = \frac{1}{12N} \sum_i z_i y_i - y_i^2$$

$$= \frac{\sum_i (z_i - y_i) y_i}{6N_E}$$

For square lattice, $\mathbb{P} = \frac{4\sum_i y_i}{6N_E} - \frac{\sum_i y_i^2}{6N_E}$

$$= \frac{2}{3}(M+1) - \frac{\sum_i y_i^2}{6N_E}$$

$$\sum_i y_i^2 \approx \sum_i y^2 + 2y y_i \quad \text{w/.} \quad y = 2(M+1)$$

$$:= \sum_i y_i / N_v$$

$$= N_v y^2 + 2y \underbrace{\sum_i y_i}_{N_E(M+1)}$$

$$\Rightarrow \mathbb{P} = \frac{2}{3}(M+1) - \frac{1}{6N_E}\left(N_v \cdot 4(M+1)^2 + 4(M+1) \cdot N_E(M+1)\right)$$

$$= \frac{2}{3}(M+1) - \frac{1}{6}\left(2(M+1)^2 + 4(M+1)^2\right)$$

$$= \frac{2}{3}(M+1) - (M+1)^2$$

P will be the same for each sublattice ∵ square

If all bonds possible, $E_\bullet = 6N_v$

$$= N_v \left(\frac{z}{2}\right) = \sum_i \left(\frac{z}{2}\right)$$

$$= E_o$$

When finding $E_\bullet$ <u>with</u> bonds removed,

$$E_o + E_\bullet = \sum_i \frac{1}{2} z_i^+ z_i^-$$

$$E_\bullet = \frac{1}{4} \sum_{i \in V_0} z_i^+ z_i^-$$

→ Sum over other sublattice.

We need $\frac{1}{N} \sum_i y_i =$

& $\frac{1}{N} \sum_i y_i^2$

Where $M = \left(\frac{1}{2N} \sum_i y_i\right) - 1$

$$= \sum_\alpha \sigma_\alpha$$

$$\Rightarrow \sum_\alpha \sigma_\alpha =$$

$$\sum_\alpha \sigma_\alpha^2 = \frac{1}{2} \sum_i (z_i^+ - z_i^-)^2$$

$$= \frac{1}{2} \sum_i (2z_i^+ - z_i)^2$$

$$= 2 \sum_i z_i^{+2} - 2 \sum z_i z_i^+ + \frac{1}{2} \sum_i z_i^2.$$

$$M = \frac{1}{2N_E} \sum_i (z_i^+ - z_i^-)$$

$$(z_i^+ + z_i^-) = z_i$$

$$\mathbb{P} = \frac{\sum_i \frac{1}{2} z_i^+ z_i^-}{\sum_i \binom{z_i}{2}} = \frac{\sum_i z_i^+ z_i^-}{\sum_i z_i (z_i - 1)}$$

If $z_i = 4 \ \forall i$,

$$\sum_i \binom{z_i}{2} \overset{z_i=4}{=} 6N_V = 3N_E$$

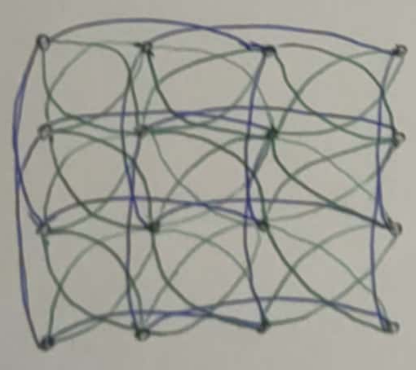
$$\left\{ \mathbb{P} = \frac{1}{6N_E} \sum_i z_i^+ z_i^- \right\}$$

In mean-field,

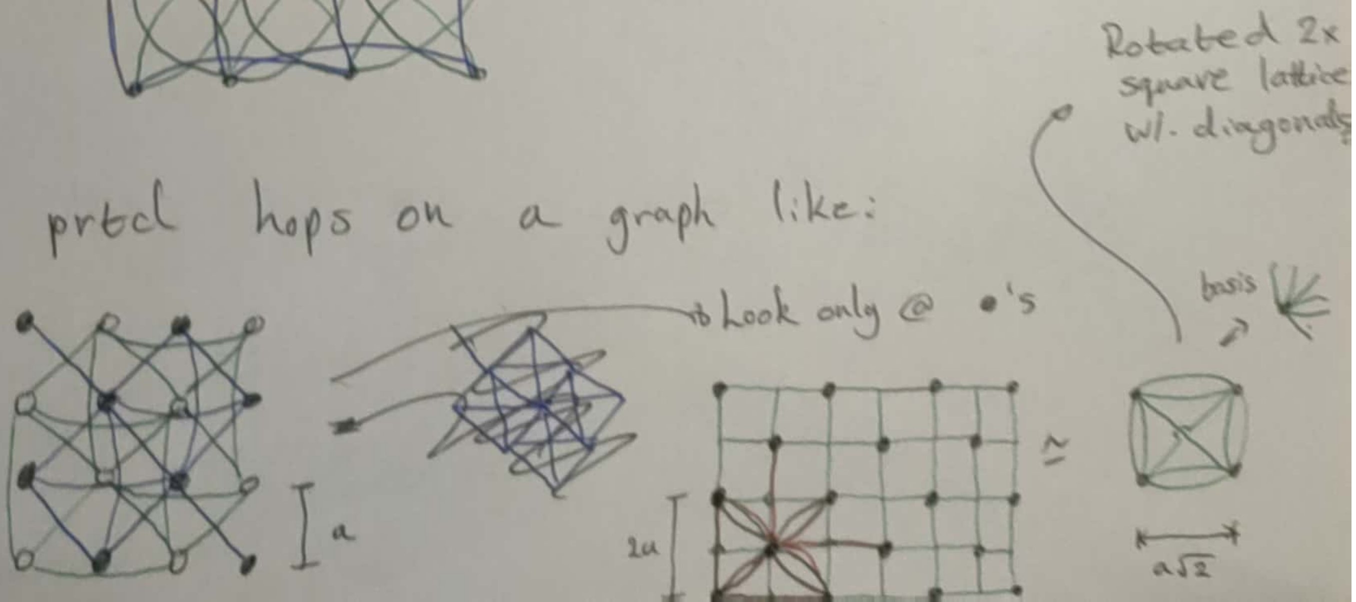
$$\boxed{\mathbb{P} \approx \frac{2}{3} M - \frac{1}{4} M^2 .}$$

<u>But</u> importantly on a next-nearest neighbour graph.

I.e. percolation refers to ~~a~~ possible +−/−+ 2-step "edges"



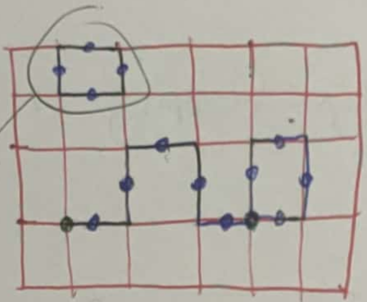
Each prtcl hops on a graph like:



Picture: Spinons • on bonds

Prtcls • on vertices.
(visons)

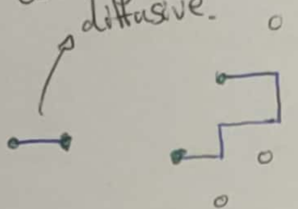
For H large,



Can only have spinons move if a vison collides w/. ~~...~~ / passes over it!

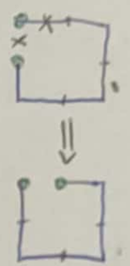
For H very large,

~~totally~~ diffusive.



→ endpoints can jiggle around but length fixed.

Prtcls can move iff neighbours



⇒ pairs diffusive

ends restricted to $\mathbb{P} \sim 100\%$

⊥ or ┘ etc. or — or ·

i.e. almost 100% percolated

⇒ highly subdiffusive!

Can only move around in a circle.

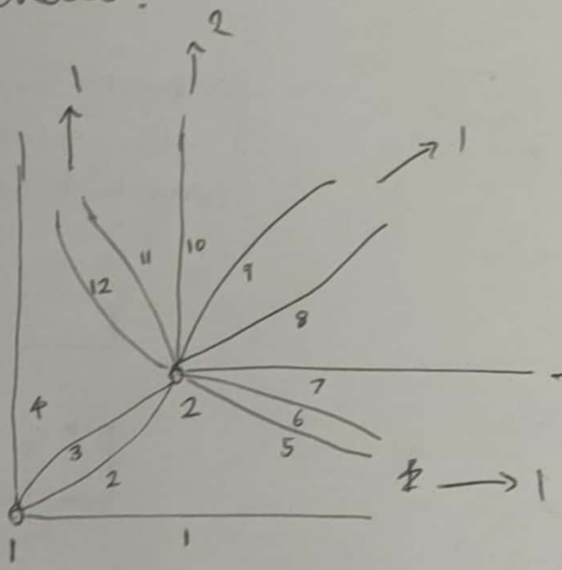
For H small, we have a random background.

I want to model the walks of:

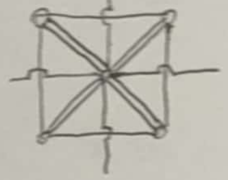
① An individual particle doing double hops on the square lattice.

② A vison pair flip-flopping on the square lattice

For ①, we should instead consider <u>single</u> hops on a rotated 2x square lattice with diagonals.



ρ α x γ ζ μ



μ ζ ρ d ρ p G d b Q Q

For ②, we need a different sim setup?

Maybe we could centralise this...
Instead of <u>removing</u> edges, just ~~only picks~~ for each vertex pick a spin, then pick another w/. opposite spin.
Allows prtcls to "hop over" each other
Key: Would need to ensure prtcls have vison config...

Vison picture: ① Isolated visons

T=0: Stuck, can only wiggle @ ends of spinon string

T=∞: Move on +− paths

② Vison pairs

T=0: Perfectly diffusive, RW

T=∞: Move on "opposite spin" paths to the spin between them.

o.e. they flip-flop.

Spinon picture: ① Long spinon strings

T=0 : Only the end visons can move

T=∞ : Spinons are pretty much free to move <u>if</u> a vison comes along!

② Isolated spinon

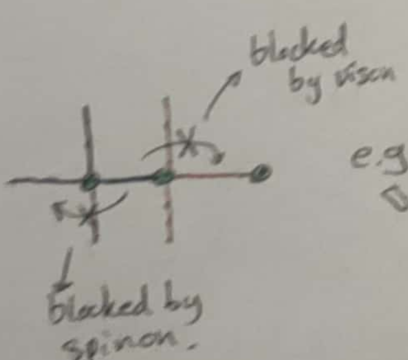
T=0 : RW on edges!

T=∞ : Can only move to where there <u>isn't</u> a spinon

⇒ ~ RW w/. collisions.

② No percolation at all: D ~ const & α ~ const.

∀ T



e.g.

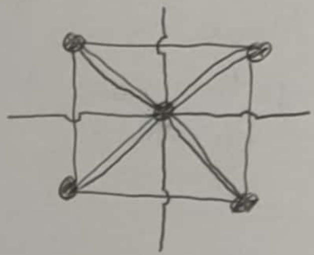
Well, will have a $\frac{1}{2}(1+M(T))$ factor for spinon collisions!

As well as $(1-n(T))$ for vison collis^ns

① Complicated! @ $T=\infty$, +−+− vison path picture easier & $T=0$ vison picture is okay.

BUT there is percolation. of the +−+− paths

↳ Isolated visons move in a RW on a graph like this:



I.e. one of the sublattices of the square lattice (call it A & the other B)

Which is then percolated & has bonds removed if they're not +−

Assuming this is done randomly, we have that the total possible nº bonds v is:

choose 2 edges "via" i (0% percolation)

$$|E_A| = \sum_{i\in V_B} \binom{z_i}{2} = \frac{1}{2}\sum_{i\in V_B} z_i(z_i-1)$$

To go from the original lattice to the A sublattice,

choose 2 opposite edges "via" i

$$|E_A'| = \frac{1}{2}\sum_{i\in V_B} z_i^+ z_i^-$$

⇒ Percolation probability of the A sublattice is:

$$\mathbb{P}_A = 1 - \frac{|E_A'|}{|E_A|} = 1 - \frac{\sum_{i\in V_B} z_i^+ z_i^-}{\sum_{i\in V_B} z_i(z_i-1)}$$

$z_i^+ + z_i^- = z_i$

$$= 1 - \frac{\cancel{\tfrac{1}{2}}\sum_{i\in V}(\ldots)}{\cancel{\tfrac{1}{2}}\sum_{i\in V}(\ldots)} = 1 - \frac{\sum_i z_i^+(z_i - z_i^+)}{\sum_i z_i(z_i-1)}$$

$$= 1 - \frac{\sum_i z_i^+(4-z_i^+)}{12 N_v}$$

We know $M = \sum_\alpha \sigma_\alpha / N_E$

$$= \frac{1}{4N_v} \sum_i \left(z_i^+ - z_i^- \right)$$

$$= \frac{1}{2N_v} \sum_i z_i^+ - \frac{1}{4N_v} \cancel{\sum_i z_i}^{\,4N_v}$$

$$= \frac{1}{N_E} \sum_i z_i^+ - 1!$$

$$\mathbb{P} = 1 - \frac{1}{6N_E} \sum_i \left(4 z_i^+ - z_i^{+2} \right)$$

$$= 1 - \frac{2}{3N_E} \sum_i z_i^+ + \frac{1}{6N_E} \sum_i z_i^{+2}$$

$\frac{1}{N_E} \sum z_i^+ = M + 1 \cancel{= \frac{1}{2} z}$ We know $M(H) \sim \tanh(H)$.

$\sum (z + \delta z_i)^2$

$\sum z^2 + 2z\delta z_i + \delta z_i^2$

$\Rightarrow z_i^+ \sim z^+ + \delta z_i^+$ w/. $z = 2(M+1) = \left(\frac{\sum z_i}{N_v} \right)$

$$\Rightarrow \sum_i z_i^{+2} \sim \overset{\mathcal{O}(1)}{N_v z^{+2}} + \overset{\mathcal{O}(\delta z_i^+)}{2 z^+ \sum_i (z_i^+ - z^+)} + \overset{\mathcal{O}(\delta z_i^{+2})}{\sum_i (z_i^+ - z^+)^2}$$

$\cancel{2z \sum_i \delta z_i}$

$\cancel{2z \sum_i z_i - \sum_i 2z^2}$

$\cancel{2z^2 - 2z^3}$

~~To $\mathcal{O}(1)$, $\mathbb{P} \approx 1 - \frac{2}{3} 2(M+1)$~~

To $\mathcal{O}(1)$, $\mathbb{P} \approx 1 - \frac{1}{3} z^+ + \frac{1}{12} z^{+2}$

$$= 1 - \frac{2}{3}(M+1) + \frac{4}{12}(M+1)^2$$

$$= 1 - \frac{2}{3} - \cancel{\frac{2}{3} M} + \frac{1}{3} M^2 + \cancel{\frac{2}{3} M} + \frac{1}{3}$$

$$= \frac{2}{3} + \frac{1}{3} M^2 = 1 - \frac{1}{3}\left(1 - M^2\right)$$

To $\mathcal{O}(\delta z_i)$, $\frac{1}{N_v} \sum_i z_i^{+2} \sim z^{+2} + 2z^{+2} - 2z^{+2} \sim z^{+2}$ (the same!)

To $\mathcal{O}(\delta z_i)$, ... need $\sum_i (\underset{y_i}{z_i^+} - \underset{\bar{y}}{z^+})^2 = \sum_i (y_i - \bar{y})^2$.

Small only @ ~~Large T~~ $\rightarrow$ low T $\Rightarrow \bar{y} \gg \delta y$

$\Rightarrow \quad \mathbb{P}_{MF} = 1 - \frac{1}{3}(1 - M^2)$ valid for low $T \Rightarrow M \lesssim 1$

$= \begin{cases} T=\infty : 2/3 \\ T=0 : 1 \end{cases}$

$\mathbb{P} = 1 - \frac{1}{3}\bar{y} + \frac{1}{12}\overline{y^2} \;\; \overset{?}{=}$ $\quad \bar{y} = 2(M+1)$

$= \begin{cases} T=\infty : ? \, \cancel{0.625} \\ T=0 : 1 \end{cases} \quad \gtrsim 0.75$ $\quad$ * $\overline{y}^2$ in $T=\infty$ limit $\quad = \begin{cases} M=0, T=\infty : 2 & \text{(half +ve)} \\ M=1, T=0 : 4 & \text{(all +ve)} \end{cases}$

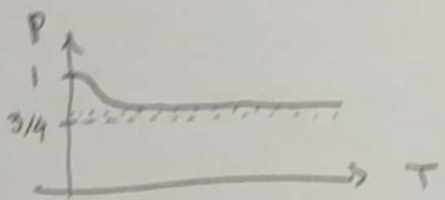
$\overline{y^2} = \frac{1}{N_v} \sum_i y_i^2$ $\quad$ We know $\langle \sum_i (-1)^{y_i} \rangle$, $\langle \sum_i y_i \rangle$

& $\quad z_i^+ + z_i^- = z_i$

$\quad y_i + w_i = z_i$

$$\sum_i (-1)^{y_i} = \sum_i \left(\delta_{y_i,0} - \delta_{y_i,1} + \delta_{y_i,2} - \delta_{y_i,3} + \delta_{y_i,4} \right)$$

Roughly, we have percolation $\gtrsim 0.75 \rightarrow 1$ as $T \to 0$



$\hookrightarrow$ Why not 0.5 @ $T = \infty$?

Total nº paths from a given point is

$N = 4 \cdot \#$ opposite of 1st $\sim \begin{cases} T=0 & 4 \cdot 0 \\ T=\infty & 4 \cdot 2 \end{cases}$ $\because$ can't double back

2 options on avg

$\Rightarrow$ ~~$\mathbb{P} \sim \frac{1 - 2/3}{1 - 0} \sim 1/3$~~ So it should be $\sim 1/3$

$$\mathbb{P} \sim 1 - \frac{4 \cdot 2}{4 \cdot 3} \sim 1 - \frac{2}{3} \sim \frac{1}{3}$$

If doubling back allowed, $\mathbb{P} \sim 1 - \frac{4 \cdot 2}{4 \cdot 4}$

$\sim 1 - \frac{1}{2} \sim \frac{1}{2}$

I ~~expect~~

$\langle z \rangle \sim$ ~~$(1-n(T))\cdot(1+M(T))$~~ $4(1-n(T))\frac{M(T)+1}{2}$

(*) $T = 0$ $\rightsquigarrow$ ONLY pair diffusion (noisy)

single diffusion percolation 100%

$\Rightarrow$ Expect RW. w/. $\alpha = 1$

& $\langle |\underline{x}(t) - \underline{x}(0)|^2 \rangle \sim 4Dt^{\alpha(=1)}$

Confirmed by $L \sim 1$ sim (need large t range to fit well...)

w/. $D \sim 1$ $\because$ just molecule pair hopping

For h large they're more likely $\Rightarrow$ pairs win?

BUT only for pairs $\to$ very rare & thermal diffusion suppressed?

(+) $T = \infty$ $\rightsquigarrow$ BOTH processes ~~diff~~ hop on + − + − paths

$\Rightarrow$ Two pictures: * 50% percolated.

Bad picture $\because$ prtcl mot$^{\bar{n}}$ modifies spins behind it $\to$ never get truly trapped.

* ~ RW on lattice w/.

$4D \sim \langle z \rangle \tilde{D}^{\sim 1}$

$\Rightarrow D \sim \frac{\langle z \rangle}{4} = (1-n(\lambda))\cdot\frac{1}{2}(1+M(H))$

$\sim 1/2 \cdot 1/2 = 1/4$.

& $\alpha = 1$

Better picture @ high T.

$\Rightarrow$ On the thermal diffusion side, K will have contributions from both

$\langle \dot{x}(t)\dot{x}(0) \rangle \sim A\delta_{t0} - B\delta_{t1}$

$\sum_{pq} \langle \dot{x}_p(t)\dot{x}_q(0) \rangle = \sum_{pq} (A\delta_{pq}^{p=q}\delta_{t0} - B\delta_{pq}^{p(\pm 1/2)}\delta_{t1}) \Rightarrow \sum (\text{''})(1 - \frac{\delta_{t0}}{2})$

$= \sum_p (A\delta_{t0} - B\delta_{t1})$

$= (\frac{1}{2}A - B)$.

Checks: ① Only pair / individual moves in thermal sim?

② Confirm these moves work as intended

↳ Why diffusion of pairs ~ const.?

$\cancel{\left(\frac{1}{n!}\sum_{\{x\}}(1)\sum_{pq}\right) <}$

single ~~pair~~ term $p=q$

pair term phos in q

$$\frac{1}{n!}\sum_{pq}\sum_{\{x\}} \dot{x}_p(t)\,\dot{x}_q(0) = n\cdot\binom{N}{n} A\,\delta_{t0} + n\cdot 4\binom{N}{n-1} B\,\delta_{t1}$$

$n \to \sum_p \delta_{pq}$

$4 \to$ choices for q

$$\frac{1}{n!}\left\{ {N \atop n} \right\} n$$

$$\binom{N}{n-1} = \frac{N!}{(N-n+1)!\,(n-1)!} = \frac{n}{n-1}\,\frac{(N-n)}{(N-n+1)}\binom{N}{n}$$

choose p · choose q

$n \cdot 4$

sum over $x_p, x_1, \dots x_{q-1}, x_{q+1}, \dots x_n$

↳ Fix x_p first w/. factor N, then permute the rest from $\{x\}\setminus x_q$

$$\Rightarrow N \cdot \left\{ {N-5 \atop n-2} \right\}$$

$$4nN \left\{ {N-5 \atop n-2} \right\} \frac{1}{n!}$$

$$= \frac{4nN}{n!}\,\frac{(N-5)!}{(N-3-n)!}$$

$$= \frac{4n}{n!}\,\frac{(N-n)!}{(N-3-n)!}\,\frac{(N-5)!}{N!}\left\{ {N \atop n} \right\}$$

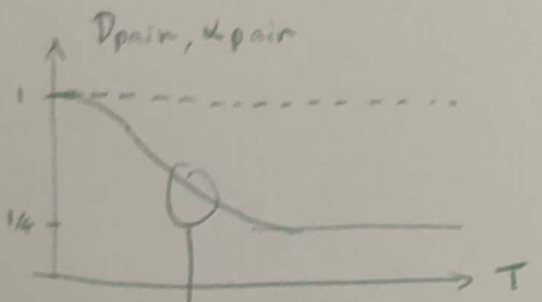
$$= 4n\,(\dots)(\dots)\left\{ {N \atop n} \right\}$$

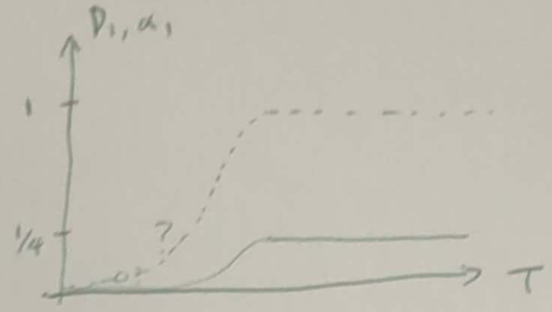
Relative factor

$$\eta = 4\,\frac{O(N^2)}{O(N^5)}$$

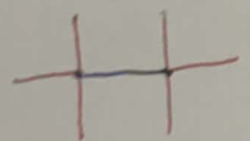
$\sim 1/N^2$ as we may expect!

$2\Lambda \longrightarrow 2\Lambda + \mathrm{HM(H)}$ $\stackrel{?}{=}$





This crossover confuses me—
Don't seem to see it?



No restriction on direction...

BUT I suppose:

* Factor ½ ∵ each prtcl moves 2x steps every <u>other</u> move

(* Factor (?) ∵ pairs rare?)

$$\frac{n(n-1)}{n!} N(N-5) \frac{(N-2)!}{(N-n)!} = \frac{n(n-1)}{n!} \frac{N!}{(N-n)!} \frac{N(N-1)}{N(N-5)}$$

$$\frac{n}{n!} N \cdot \frac{(N-1)!}{(N-n)!} = n \binom{N}{n}$$

$$\cancel{\langle n^2 \rangle = -\frac{\partial \ln Z}{\partial (2\Lambda)}} \qquad \langle n^2 \rangle_c = + \frac{\partial^2 \ln Z}{\partial (2\Lambda)^2} = \langle n^2 \rangle - \langle n \rangle^2$$

$$\langle n^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial (2\Lambda)^2} \qquad = -\frac{\partial \langle n \rangle}{\partial (2\Lambda)}$$

$$\langle n^2 \rangle = \langle n \rangle^2 - \frac{\partial \langle n \rangle}{\partial (2\Lambda)}$$

$$\cancel{Z} = -\frac{1}{Z} \frac{\partial Z}{\partial (2\Lambda)} = \frac{1}{2}(1 - \tanh(\Lambda))$$

$$= \frac{N}{1+e^{2\Lambda}}$$

$$\Rightarrow \langle n^2 \rangle = N \left(\frac{-\partial}{\partial (2\Lambda)} \left[\frac{(1+e^{-2\Lambda})^N}{1+e^{2\Lambda}} \right] \right) \frac{1}{Z}$$

$$\shortparallel$$

$$\neq \qquad e^{-2\Lambda} (1+e^{-2\Lambda})^{N-1}$$

$$\langle n^2 \rangle = N \left[(1+e^{-2\Lambda})^{N-1} + (N-1) e^{-4\Lambda} (1+e^{-2\Lambda}) \right.$$

$$\langle \rho^2 \rangle = \left(\frac{1}{N} \cancel{\frac{1}{2}} \frac{1}{1+e^{-2\Lambda}} + \frac{N-\cancel{1}}{\cancel{N}} \frac{e^{-4\Lambda}}{(1+e^{-2\Lambda})^2} \right)$$

Apparently $\underline{\langle n^2 \rangle = \langle n \rangle}$??

$$\sum_{t_0=0}^{t_{max}-1} (1 - \delta_{t_0}/2)\,(\,''\,) = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\lambda n} \left[\cancel{\binom{N}{n} \frac{A}{2} n} \right.$$

$$\cdot \frac{1}{n!} \left(\frac{A}{2} \left\{ {N \atop n} \right\} n + 4B \underbrace{\left\{ {N-5 \atop n-2} \right\}}_{\sim \frac{1}{N^2}\binom{N}{n}} N n \right)$$

$A \sim B \sim 2\gamma$

$$\sum_n ① + \sum_n ① \cdot \underbrace{8N \left\{ {N-5 \atop n-2} \right\} / \left\{ {N \atop n} \right\}}$$

$$= 8N \frac{(N-5)!}{(N-3-n)!} \frac{n!}{N!}$$

$$\frac{1}{n!} \underbrace{\sum_{pq}}_{\substack{n(n-1) \\ \because \text{ any too!}}} \underbrace{\sum_{\{x\} \setminus \{x_p, x_q\}}}_{\left\{ {N-2 \atop n-2} \right\}} \underbrace{\sum_{x_p \neq x_q}}_{N \cdot (N-5)} \; \& \; B \delta_{t1} = B\delta_{t1} \binom{N-2}{n-2} N \cdot (N-5)$$

$$= \binom{N-1}{n-1} \frac{n-1}{N-1}$$

$$= \binom{N}{n} \frac{n(n-1)}{N(N-1)} N(N-5)$$

If $p = q$

$n \cdot 1$ & $N(N-1)$

$A = B$

So

$$\Rightarrow \left[\frac{A}{2} n + B \frac{n(n-1)}{\cancel{n}} \cdot \frac{N-5}{N-1} \right] \binom{N}{n}$$

$$= A \binom{N}{n} n \left[\frac{A}{2} + A \left(\frac{N-5}{N-1} \right) (n-1) \right]$$

$$\sim A \binom{N}{n} n \left[\frac{\cancel{A}1}{2} + (n-1) \right]$$

$$\Rightarrow K \rightarrow \cancel{(2\lambda)^2}$$

Replace

$\langle n \rangle \rightarrow \langle n + 2n(n-1) \rangle$

$= 2\langle n^2 \rangle - \langle n \rangle$

$= \langle n \rangle \; \because \langle n^2 \rangle = \langle n \rangle$

$$K_1 \propto \langle n \rangle \cdot \frac{1-M}{2}$$

$$K_2 \propto \langle 2n(n-1) \rangle \cdot \frac{1+M}{2}$$

$$= \langle 0n \rangle \overset{!!\ \langle n^2 \rangle = \langle n \rangle}{\cdot} \frac{1+M}{2}$$

<u>How</u> is $\langle n^2 \rangle = \langle n \rangle$?

$$\langle n \rangle = \frac{-1}{N} \frac{\partial \ln Z}{\partial (2\Lambda)}$$

$$\langle n^2 \rangle_c = + \frac{1}{N^2} \frac{\partial^2 \ln Z}{\partial (2\Lambda)^2} = \langle n^2 \rangle - \langle n \rangle^2$$

$$\Rightarrow \langle n^2 \rangle = \langle n \rangle^2 - \frac{\partial \langle n \rangle}{\partial (2\Lambda)}$$

$$= \frac{1}{(1+e^{2\Lambda})^2} + \frac{e^{2\Lambda}}{(1+e^{2\Lambda})^2}$$

$$= \frac{1}{1+e^{2\Lambda}} = \langle n \rangle \quad \ldots \text{ Weird!}$$

Hypothesis: Pairs don't contribute to thermal conductivity ???

↳ They do, but barely!

Dominant contrib$^{\overline{n}}$ seems to be from individual spinons.

⇒ Would explain subdiffusivity @ low T.

$$\langle n^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial (2\Lambda)^2}$$

$$= \frac{Z''}{Z}$$

$$\langle n^2 \rangle_c = \frac{\partial^2 \ln Z}{\partial (2\Lambda)^2}$$

$$= \frac{\partial}{\partial (2\Lambda)} \left[Z'/Z \right] = Z''/Z - \left(Z'/Z \right)^2$$

$$= \langle n^2 \rangle - \langle n \rangle^2$$

$$Z = \sum_{\mu} e^{H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q}$$

$$= \sum_{\mu} e^{\sum_{pq} G_{pq} \mu_p \mu_q} \qquad G_{pq} = \begin{cases} H\eta_{pq} & \text{if } \langle pq \rangle \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \quad Z = \sum_{\mu} \int \mathcal{D}\varphi_p \; e^{\sum_{pq} \varphi_p G^{-1}_{pq} \varphi_q - \sum_p \varphi_p \mu_p} \cdot \underbrace{\left(\det(2\pi \underline{\underline{G}})\right)^{1/2}}_{A}$$

$$= A \int \mathcal{D}\varphi_p \; e^{\sum_{pq} \varphi_p G^{-1}_{pq} \varphi_q - \sum_p \ln(\cosh(\varphi_p))}$$

$$\approx A \int \mathcal{D}\varphi_p \; e^{\sum_{pq} \varphi_p G^{-1}_{pq} \varphi_q + \sum_p \frac{1}{2}\varphi_p^2 - \frac{1}{12}\varphi_p^4}$$

$$\phi_q = \frac{1}{\sqrt{N}} \sum_{v \in BZ} e^{iqv} \phi_v$$

$$\sum_{pq} \phi_p G_{pq} \phi_q = \sum_{uv} \frac{\phi_u \phi_v}{N} \underbrace{\sum_{pq} G_{pq} \, e^{iqv + ipu}}_{N G_{uv}}$$

$$\approx H \sum_{\langle pq \rangle} (\circ) -$$

$$G_{uv} = \overbrace{\sum_{\langle pq \rangle} H \eta_{pq}}\, e^{i(pu + qv)}$$

$$\approx \int d^2 p \int d^2 q \; H \eta(p, q) \, \delta(p - q) \, e^{i(pu + qv)}$$

$$= H \int d^2 p \; \eta(p, p) \, e^{ip(u+v)}$$

$$-\beta F = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha$$

$$\Rightarrow \quad A_i = \sum_{\alpha \in \partial^+ i} \sigma_\alpha$$

$$\approx \sum_{\alpha \in \partial^+ i} (\sigma + \delta\sigma_\alpha)$$

$$= \sigma^{z_i} + \sigma^{z_i - 1} \sum_\alpha \delta\sigma_\alpha \qquad \delta\sigma = \sigma_\alpha - \sigma$$

$$+ \sigma^{z_i - 2} \sum_{\alpha \neq \beta} \delta\sigma_\alpha \delta\sigma_\beta$$

$$= \text{const.} + (\sigma^{z-1} \overset{-}{\cancel{+}} 2\sigma^{z-2}) \sum_\alpha \sigma_\alpha$$

$$\sum_i \sum_{\substack{\alpha \in \partial^+ i \\ \beta = \partial^+ i \\ \beta \neq \alpha}} = \sum_{\langle \alpha\beta \rangle} \qquad + \sigma^{z-2} \sum_{\alpha \neq \beta} \sigma_\alpha \sigma_\beta .$$

$$\Rightarrow \quad -\beta F \approx \left(H + \tfrac{1}{2} A(\sigma^{z-1} - 2\sigma^{z-2})\right) \sum_\alpha \sigma_\alpha$$

$$+ \Lambda \sigma^{z-2} \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta$$

$$\overset{z=4}{\Rightarrow} \quad -\beta F \approx \left(H + \tfrac{1}{2}\Lambda(\sigma - 2)\sigma^2\right) \sum_\alpha \sigma_\alpha$$

$$+ \Lambda \sigma^2 \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta$$

$$= H_{eff}(\ldots) + J_{eff}(\ldots)$$

Doing H–S would give

$$Z \propto \int \mathcal{D}\psi \, e^{-\beta F}.$$

Assuming $G_{pq} \sim 2 J_{eff} \sum_{\dots} \cos(\dots)$

we'll get:

$$\beta F \approx \cancel{\tfrac{1}{a^2}}\, d \int \frac{d^2x}{a^2} \left(\frac{1}{2}\left(t|\psi|^2 + K|\nabla\psi|^2\right) + \lambda|\psi|^4 - h^+\psi \right)$$

where...

$$h = H_{eff}$$

$$\lambda = \frac{1}{12}$$

$$K = \frac{a^2}{8 J_{eff} d}$$

$$t = \frac{1}{2 J_{eff} d} - 1$$

$$= T/T_c - 1$$

$$J_{eff} = \frac{\lambda\sigma^2}{T}$$

$$\Rightarrow \quad T_c = 2\lambda \frac{\sigma^2}{\cancel{T}} d \quad ?$$

Mean-field: $\dfrac{\partial f}{\partial \psi} = 0 \quad \Rightarrow \quad \psi = 0$

OR $|\psi|^2 = -\dfrac{t}{2\lambda}$

$$\Rightarrow \quad f_0 = -\frac{t^2}{4\lambda} - \lambda\frac{t^2}{4\lambda} = -\frac{t^2}{8\lambda} \quad \text{if } h = 0.$$

Ising w/

$$-\beta F = J_{eff} \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta + H_{eff} \sum_{\alpha} \sigma_\lambda$$

Getting rid of B_- is <u>more</u> valid than throwing away V_ℓ, $|\ell|$ terms!

$$Z_\infty = \sum_\ell \Delta_\ell C_H^{N_E} t_H^{|\ell|} \left[\left(1+e^{-2\Lambda}\right)^{N-V-1} \left(1+e^{-2\Lambda(V+1)}\right) \right.$$

$$B_- \overset{N\to\infty}{\rightsquigarrow} 0$$

$$\therefore \; n \approx \cancel{} \, \frac{\cancel{1}}{\cancel{N_E}} \sum_\ell \Delta_\ell C_H^{N_E} t_H^{|\ell|} \left[\left(B_+ N_V / (1+e^{-J})\right) e^{-J} \right.$$

$$+ e^{-J} \left\{ (V+1) B_+ \cancel{\left(\frac{1}{1+e^{-J(V+1)}} - \cdots \right.} \right.$$

$$\sim 0$$

$$\approx \cancel{N_V} \frac{1}{\cancel{N_V}} \underbrace{\frac{1}{2}\left(1-\tanh(\Lambda)\right)}$$

@ the same level of approx^n,

$$C = \frac{1}{N_E}\left[(2\Lambda)^2 \frac{\partial^2 \ln Z}{\partial (2\Lambda)^2} + H^2 \frac{\partial^2 \ln Z}{\partial H^2} \right]$$

$$= \frac{1}{N_E}\Big[N_v (2\Lambda)^2 \underbrace{\tfrac{1}{2}\operatorname{sech}^2(\Lambda)\tfrac{1}{2}}_{\left(-\frac{\partial}{\partial(2\Lambda)}\frac{1}{2}(1-\tanh\Lambda)\right)} + N_E H^2 \tanh(H)\Big]$$

$$= \cancel{\tfrac{1}{2}\left(\Lambda^2 \operatorname{se}\right.}$$

$$\approx \frac{1}{2}\left(\Lambda^2 \operatorname{sech}^2(\Lambda) + 2H^2 \tan(H)\right)$$

& Likewise,

$K = ?$

$$\sum_{\mu} e^{H \sum_{\langle pq \rangle} \eta_{pq} \mu_p \mu_q} \approx Z_{Ising}\, e^{-2H\, W \odot L} \min\left\{ e^{-2H \sum_{pairs} l_{pair}} \right\}$$

min wrt. pairings

$$\sum_{W} \frac{1}{n!} \sum_{\{x\}} (\text{" } \ldots \text{ "}) \approx Z_{Ising}\left(1 + e^{-2HL_x} + e^{-2HL_y} + e^{-2HL_xL_y}\right) \cdot \binom{N}{n} \underbrace{\langle e^{-2H\, l_{pair}} \rangle^{n/2}}_{\sim \left(e^{-2H\langle l_{pair}\rangle}\right)^{n/2}}$$

$n/2 \to$ # pairs

$$N \sim n\, l_{pair}^2$$

$$\Rightarrow Z \approx Z_{Ising}(1 + \ldots) \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \binom{N}{n} e^{-H\langle l_{pair}\rangle n}$$

$$\rightarrow \quad 2\Lambda \rightarrow 2\Lambda + H\langle l_{pair} \rangle$$

$$= -(2H)^2 (2L) \frac{\partial}{\partial(2H)} \left(\frac{e^{-2HL} + e^{-4HL}}{1 + 2e^{-2HL} + e^{-4HL}} \right)$$

$$= +(2H)^2 (2L) \left(\frac{L(e^{-2HL} + 2e^{-4HL})(\ldots) - 2L(\ldots)^2}{(\ldots)^2} \right.$$

$$= \frac{(2H)^2 (2\cancel{L})}{\cancel{NE}} \left[\cancel{L} \frac{e^{-2HL} + 2e^{-4HL}}{1 + \ldots} - 2L \frac{(\ldots)^2}{(\ldots)^2} \right]$$

$$= \cap$$

Alright. Let's think ~~of~~ of this another way.

$$Z = \sum_{\{\sigma\}} e^{\Lambda \sum_i A_i + H \sum_i \sigma_i}$$

For H large, use mean-field

$$A_i = \prod_{\alpha \in \partial^+ i} \sigma_\alpha \approx \prod_{\alpha \in \partial^+ i} (\sigma + \delta\sigma_\alpha)$$

$$= \sigma^{z_i} + \sigma^{z_i - 1} \sum_{\alpha \in \partial^+ i} \delta\sigma_\alpha$$

$$+ \sigma^{z_i - 2} \sum_{\substack{\alpha \neq \beta \\ \alpha, \beta \in \partial^+ i}} \underbrace{\delta\sigma_\alpha \delta\sigma_\beta}_{(\sigma_\alpha - \sigma)(\sigma_\beta - \sigma)} \quad \text{to } O(\delta\sigma^2)$$

$$= \sigma^{z_i}(1 - z_i - z_i(z_i - 1))$$

$$+ \sigma^{z_i - 1} \sum_{\alpha \in \partial^+ i} \sigma_\alpha (1 - 2(z_i - 1))$$

$$+ \sigma^{z_i - 2} \sum_{\alpha \neq \beta} \sigma_\alpha \sigma_\beta .$$

$$Z = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \cancel{\beta} 2^{b_1} \underbrace{\sum_{\{z\}} C_H^{N_E} \sum_{FCG} (-1)^{\cdots} t_H^{|F|}}_{\sum_{FCG} \Delta_F C_H^{N_E} t_H^{|F|}}$$

$$\boxed{\begin{aligned} C &= T \frac{\partial S}{\partial T} \qquad S = -\frac{\partial f}{\partial T} \\ &= -T \frac{\partial^2 f}{\partial T^2} \qquad \frac{\partial}{\partial T} = -\beta^2 \frac{\partial}{\partial \beta} \\ &= \frac{1}{\beta} \frac{\partial}{\partial T}\left(\beta^2 \frac{\partial f}{\partial \beta}\right) \\ &= \beta \frac{\partial}{\partial \beta}\left(\beta^2 \frac{\partial f}{\partial \beta}\right) \end{aligned}}$$

$$C = -\beta^2 \frac{\partial^2 f}{\partial \beta^2}$$

@ $T \ll 1$, $Z \sim Z_{\text{prtcls}} \cdot \underline{\sum_{w} Z_{\text{Ising}}}$

$$\sim \underbrace{\left(1 + 2e^{-2HL} + e^{-4HL}\right)}_{Z_w} Z_{\text{Ising}}$$

$\cancel{C_w Z \frac{\partial}{\partial}}$

$\cancel{E\beta} =$

$$C_w = (2H)^2 \frac{\partial^2}{\partial (2H)^2} \ln(Z_w)$$

$$= (2H)^2 \frac{\partial}{\partial (2H)}\left(-\frac{2Le^{-2HL} + 2Le^{-4H\cdots}}{\cdots}\right.$$

$$\ln\lambda = \ln(2\cosh(2H)) + \frac{1}{\pi}\int_0^{\pi/2} d\omega \, \ln\left[\frac{1}{2}\left(1+\sqrt{1-K^2\sin^2\omega}\right)\right]$$

$$K = 2\,\mathrm{sech}(2H)\tanh(2H)$$

$$2H = J \qquad c^2 - s^2 = 1 \qquad 1 - t^2 = \frac{1}{c^2}$$

$$C = (2H)^2 \frac{\partial^2 \ln\lambda}{\partial(2H)^2}$$

$$= J^2 \frac{\partial^2 \ln\lambda}{\partial J^2}$$

$$\frac{\partial K}{\partial J} = 2\,\mathrm{sech}^3(J) - 2\,\mathrm{sech}(J)\tanh^2(J)$$

$$= 2\,\mathrm{sech}(J)$$

$$\frac{\partial \ln\lambda}{\partial J} = \tanh J + \frac{1}{\pi}\int_0^{\pi/2} d\omega \, \frac{+\cancel{\tfrac{1}{2}}}{\sqrt{1-K^2\sin^2\omega}} \cdot \frac{1}{1+\sqrt{1-K^2\sin^2\omega}}$$

$$\cdot - \cancel{2} K \sin^2\omega \; K'$$

$$\frac{\partial^2 \ln\lambda}{\partial J^2} = \mathrm{sech}^2 J + \ldots \; d\omega$$

$$K \sim 2\,\mathrm{sech}(2H) \quad (2H = J)$$

$$\Rightarrow \frac{\partial K}{\partial J} \sim -2\,\mathrm{sech}(J)$$

$$\sim -K$$

$$\frac{\partial \ln\lambda}{\partial J} \sim \cancel{\tanh J}^{\sim 1} + \frac{1}{\pi}\int_0^{\pi/2} d\omega \, \frac{K^2\sin^2\omega}{1-K^2\sin^2\omega + \sqrt{1-K^2\sin^2\omega}}$$

If $K(\cancel{J}) \sim 2\,\mathrm{sech}(\cancel{J})$ then

$$C \sim \mathrm{sech}^2(J) + \frac{1}{\pi}\int_0^{\pi/2} d\omega \, \frac{4 S_\omega^2 \, t_J / c_J^2}{1 - 4S_\omega^2/c_J^2 + \sqrt{1 - 4S_\omega^2/c_J^2}}$$

$$\Rightarrow \quad Z \approx \left(\sum_{n=0}^{N} \frac{1+(-1)^n}{2} \binom{N}{n} e^{-2\Lambda n} \right) \underbrace{\left(\sum_{FCG} e^{2H(N_E - |F|)} \right)}_{\equiv Z_H.}$$

$$C_H = \frac{H^2}{N_E} \frac{\partial}{\partial H} \left(\frac{1}{Z_H} \frac{\partial Z}{\partial H} \right)$$

$$= \frac{H^2}{N_E} \frac{\partial}{\partial H} \left(\frac{1}{Z_H} \sum_{FCG} 2(N_E - |F|) e^{2H(\dots)} \right)$$

$C_{Ising} \sim ?$ À la Onsager, cosh $\approx$ sinh for large arg.

$$-\beta f \approx \cancel{\ln 2 + \frac{1}{8\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln\left[\cosh(2\beta H)[1 + \cos\theta_1 + \cos]\right]}$$

$$\approx \ln 2 + \tfrac{1}{2} \ln \cosh(2H)$$

$$f \sim -T \ln(\cosh(2h/T))$$

$$\Rightarrow \quad S \sim \frac{\partial f}{\partial T} = -\ln(\cosh(2h/T)) + \cancel{T} \frac{2h}{T^{\cancel{2}}} \frac{1}{\sinh(2H)}$$

$$= \frac{2H}{\sinh(2H)} - \ln(\cosh(2H))$$

$$C \sim T \frac{\partial S}{\partial T} \sim \cancel{\frac{2h}{T} \frac{1}{\sinh(2H)}} - \cancel{\frac{2h}{T} \frac{1}{\sinh(2H)}}$$

$$+ \frac{2h}{T} \cdot \frac{2h}{T} \frac{1}{\sinh(2H) \tanh(2H)}$$

$$\sim \frac{(2H)^2}{S_{2H}\, t_{2H}}$$

$$Z = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\lambda n} \underbrace{\sum_{\underline{w}} \frac{1}{n!} \sum_{\{x\}} \cancel{\&} e^{\cancel{N_c H}} \sum_{FCG} e^{-2H|F\cup S|}}_{Y e^{-N_c H}}$$

If we ignore $2|F\cup S|$ wrt N_c,

$Z = Z_0 e^{N_c H}$

$C_H = \frac{H^2}{N_c}\frac{\partial}{\partial H}(N_c)$

$= 0 \ldots$

$\Rightarrow$ too crude!

$$Y = \underbrace{\sum_{\underline{w}} \frac{1}{n!} \sum_{\{x\}}}_{\text{determines } S.} \sum_{FCG} e^{-2H|F\cup S|}$$

$\rightarrow$ S can be any state with sector $\underline{w}$ & prtcls @ $\{x\}$.

$$= \sum_{FCG} \sum_{SCG}$$

How many such S?
If we knew, we could write

$\sum_{\underline{w}} \frac{1}{n!} \sum_{\{x\}} \cdot \Delta_{\text{?}} = \sum_{S}$

Suppose we can ignore the $\{x\}$ part of S in the $N \to \infty$ limit.

Then $Y \approx \binom{N}{n} \sum_{\underline{w}} \sum_{FCG} e^{-2H|F\cup S|}$ & $S \sim$ | or $\Rightarrow |S| \sim L$

For each F, $|F\cap S| \sim 0$ for most positions

$$Y \approx \binom{N}{n} \sum_{\underline{w}} \sum_{FCG} e^{+2H|F\cap S|} e^{-2H|F|} e^{-2H|S|}$$

$$\approx \binom{N}{n} \Big(\underbrace{\sum_{FCG} e^{-2H|F|}}_{\text{Ising.}} + e^{-2HL}(\ldots + \ldots) + e^{-4HL}(\ldots) \Big)$$

$$\tilde{Z} = 2e^{\cancel{H_E}(N_E - 2|S|)} \sum_{F \subset G} e^{-2H|F|} \, e^{+2H \underbrace{|F \cap S|}_{\text{shared edges.}}}$$

$$= 2e^{N_E H} \sum_{F \subset G} e^{+2H(|F \cap S| - |F| - |S|)}$$

$|F| + |S| = |F \cup S| + |F \cap S|$

$$= 2e^{N_E H} \sum_{F \subset G} e^{2H(\cancel{|F \cap S|} - \cancel{|F \cap S|} - |F \cup S|)}$$

$$= 2e^{N_E H} \sum_{F \subset G} e^{-2H|F \cup S|}$$

$\downarrow$

-ve contrib$^{\text{n}}$ from any

$$\tilde{Z} = \sum_{\underline{\mu}} e^{H\left(\sum_{\langle pq\rangle} \eta_{pq}\mu_p\mu_q\right)}$$

We want to expand about the all up state ~~but this is itself frustrated.~~

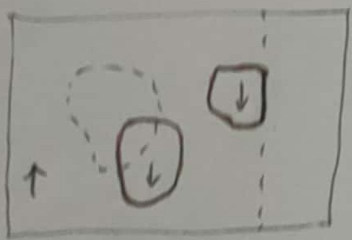
$S =$ set of $-$ve edges.

$$\sum_{\langle pq\rangle} \eta_{pq}\mu_p\mu_q \overset{\text{all up}}{=} N_E - 2|S| \quad \} \text{ all } \uparrow \text{ state}$$

$$\Rightarrow \tilde{Z} = 2e^{H(N_E - 2|S|)} \sum_{FCG} e^{-2H(\text{perimeter of } F \,-\, \#\text{ shared edges w/. } S)}$$

$\uparrow/\downarrow$ symm.

w/. F = closed loop state representing domain walls
$\Rightarrow$ lives on graph itself (p, q, etc. $\in V_{G^*}$)



$$\tilde{Z} = 2e^{H(N_E - 2|S|)} \sum_{FCG} e^{-2H(|F| - |F\cap S|)}$$

$$= 2e^{HN_E} \sum_{FCG} e^{-2H|F\cup S|}$$

@ ~~low H~~ large H we can instead expand around an ordered state $\mu = +1, -1$

$$Z = \sum_{\mu} e^{H(N_E - 2 \cdot \#\text{ unhappy edges})}$$

$$= \cancel{2}\, e^{H(N_E - 2|S|)} \sum_{FCG} e^{-2H|E_F|}\, e^{+2H|F \cap S|}$$

all ↑
or all ↓

@ H small,

$$Z = \cancel{2^{N_1}} \sum_{n=0}^{N} \frac{1+(-1)^n}{2}\, e^{-2\Lambda n}\, \cancel{2^{N_P}}\, C_H^{N_E} \sum_{\ell=0,4,6,\ldots}^{\sim L} t_H^{\ell} \sum_{L \,|\, |L| = \ell} \left(X_L \sum_{m=0}^{V_L} (-1)^m \binom{N - V_L}{n-m} \right)$$

cutoff

 $= 2N \sum_{m=0}^{3} (-1)^m \binom{N-3}{n-m}$

 $= \frac{N}{2} \sum_{m=0}^{4} (-1)^m \binom{N-4}{n-m}$

$$Y_\ell = 2^{b_1} X_\ell \sum_{m=0}^{|\ell|} (-1)^m \frac{(N-|\ell|)!}{(N-|\ell|-n+m)!} \frac{\cancel{n!}}{(n-m)!} \cancel{\frac{1}{n!}}$$

$$\boxed{= \cancel{2^{b_1}} X_\ell \sum_{m=0}^{|\ell|} (-1)^m \binom{N-|\ell|}{n-m}}$$

implicit

If non-cont, $(-1)^\Omega = +1$
(ill-defined ...)

X_ℓ = # pos^{ns}/ orientat^{ns}!

$\ell = \square$.

$$Y_\ell = N \sum_{m=0}^{1} (-1)^m \binom{N-1}{n-m}$$

$$= N\left(\binom{N-1}{n} - \binom{N-1}{n-1}\right)$$

$\ell = \square\ \square$ $\quad Y_\ell = N(N-5)\left(\ \uparrow\ \right)$

$\ell = \square\!\square$ $\quad Y_\ell = 2N\left(\binom{N-1}{n} - \binom{N-1}{n-1} + \binom{N-1}{n-2}\right)$

Works fine until $\underline{|E_\ell| \geq L_{min}}$

This gives a term in Z

$\sim C_H^{N_E} \cdot t_H^{L_{min}}$

$\sim H^{L_{min}}$

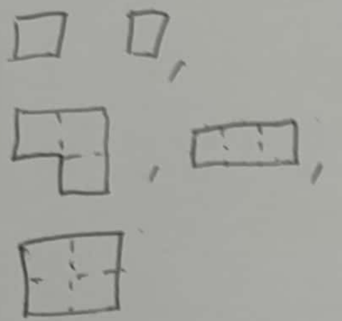
$\Rightarrow$ need $H \ll 1$
$\Leftrightarrow T \gg h$.

$l = 0$

$l = 4$

$l = 6$

$l = 8$



$l = 10$

$l = 12$

. . .

$$\binom{n}{k} = \oint_{\substack{|z|=a \\ a\in(0,1)}} đz \quad (1+z)^n \; z^{-(k+1)}$$

$$đz = \frac{dz}{2\pi i}$$

$$\sum_{m=0}^{M} (-1)^m \binom{N-V}{n-m} = \oint đz \; (1+z)^{N-V} \; z^{-(n+1)} \underbrace{\sum_{m=0}^{M} (-z)^m}_{\frac{1-(-z)^{M+1}}{1+z} \;=\; \frac{1+(-z)^M z}{1+z}}$$

$$= \oint đz \; (1+z)^{N-V-1} \left(z^{-(n+1)} + (-1)^M z^{-(n-M)} \right)$$

$$= \boxed{\binom{N-V-1}{n} + (-1)^M \binom{N-V-1}{n-M-1}}$$

$V=0$ $\quad \binom{N-1}{n} + \binom{N-1}{n-1} = \binom{N}{n} \; \dots$ ✓

$$C_\ell \sim \mathbb{P}(N) \cdot t_H^{|\ell|}$$

I <u>then</u> want

$$Z = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} \; e^{-2\Lambda n} \sum_\ell X_{\ell,n}$$

$$= \sum_\ell c_\ell \left(\sum_{n=0}^{N} \frac{1+(-1)^n}{2} \left(\binom{N-V-1}{n} e^{-2\Lambda n} \right) + (-1)^M \binom{N-V-1}{n-M-1} \right)$$

Z_2

Z_0

$$\cancel{Z = \sum_{\ell} \left(\sum_{n=0}^{v} \frac{1+(-1)^n}{2} e^{-2\lambda n} \right.}$$

$$Z_{①} = \sum_{\ell} \left(\frac{(1+e^{-2\lambda})^{N-v-1} + (1-\ldots)^{N-v-1}}{2} \right. \quad Z_{①A}$$

$$\left. + \sum_{n=N-v}^{N} \frac{1+(-1)^n}{2} e^{-2\lambda n} \binom{N-v-1}{n} \right) \quad Z_{①B}$$

$$\to 0$$

$$\equiv^{N}_{N-v}$$

$$\cancel{Z_{①B} = \sum_{m=0}^{v} \frac{1+(-1)^m(-1)^{N-v}}{2} e^{-2\lambda m} e^{-2\lambda(N-v)}} \quad \cancel{m = n + (N-v)} \quad m = n - (N-v)$$

$$\cancel{\binom{N-v-1}{m+(N-v)}}$$

$$Z_{①} = \frac{1}{2} \sum_{\ell} \left((1+e^{-2\lambda})^{N-v-1} + (1-\ldots)^{N-v-1} \right) \quad \cancel{\binom{n-1}{k+\ell} = ?}$$

$$Z_{②} = \frac{1}{2} \sum_{\ell} \sum_{n=0}^{N} (1+(-1)^n)(-1)^{\min(n,v)} \binom{N-v-1}{\min(n,v)-1} e^{-2\lambda n}$$

$$= \sum_{\ell} \left(\sum_{n=0}^{v} (1+(-1)^n)(-1)^n \binom{N-v-1}{-1} e^{-2\lambda n} \right.$$

$$\to 0$$

$$\left. + \sum_{n=v+1}^{N} (1+(-1)^n)(-1)^v \binom{N-v-1}{n-v-1} e^{-2\lambda n} \right)$$

$$m = n-v-1$$

$$= \sum_{\ell} \sum_{m=0}^{N-v-1} (1+(-1)^{m+v+1})(-1)^v \binom{N-v-1}{m} e^{-2\lambda m} e^{-2\lambda(v+1)}$$

$$= \sum_{\ell} (-1)^v e^{-2\lambda(v+1)} \sum_{m=0}^{N-v-1} (1-(-1)^{m+v}) \binom{N-v-1}{m} e^{-2\lambda m}$$

$$= \sum_{\ell} (-1)^v e^{-2\lambda(v+1)} \cdot \left((1+e^{-2\lambda})^{N-v-1} - (-1)^v (1-e^{-2\lambda})^{N-v-1} \right.$$

Alt deriv^{n},

$$\cancel{Z = \sum_{\lambda} c_\lambda \sum_{n=0}^{N} (1+(-1)^n)\, e^{-2\Lambda n}\, (-1)}$$

$$Z = \sum_{\lambda} c_\lambda \sum_{m=0}^{V} \sum_{n=0}^{N} (1+(-1)^n)(-1)^m \binom{N-V}{n-m} e^{-2\Lambda n}$$

$k = n-m$

$$= \sum_{\lambda} c_\lambda \sum_{m=0}^{V} \Big[\cancel{\sum_{n=0}^{m} (1+(-1)^n)(-1)^m \binom{N-}{}} \quad \cancel{\sum_{n=0}^{N-V} (1+(-1)^n)(-1)^m \binom{N-V}{n-m} e^{-2\Lambda n}}$$

$k = n-m$

$$= \sum_{\lambda} c_\lambda \sum_{m=0}^{V} \sum_{k=0}^{N-m} (1+(-1)^{k+m})(-1)^m e^{-2\Lambda k} e^{-2\Lambda m} \binom{N-V}{k}$$

$\rightarrow$ ($\because \binom{n}{k} = 0$ for $k > n$)

$$= \sum_{\lambda} c_\lambda \sum_{m=0}^{V} (-1)^m e^{-2\Lambda m} \sum_{k=0}^{N-V} (\text{''})$$

$$= \sum_{\lambda} c_\lambda \sum_{m=0}^{V} (-1)^m e^{-2\Lambda m} \left[(+)^{N-V} + (-1)^m (-)^{N-V} \right]$$

$$
\begin{aligned}
Z = \sum_{\ell} C_{\ell} \Big(& (+)^{N-v-1} + (-)^{N-v-1} \\
& - (-1)^{v} e^{-2\Lambda(v+1)} (-)^{N-v-1} \\
& + e^{-2\Lambda(v+1)} (+)^{N-v-1} \Big)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
= \sum_{\ell} C_{\ell} \Big(& [-] \left(1 - (-1)^{v} e^{-2\Lambda(v+1)} \right) \\
& + [+] \left(1 + e^{-2\Lambda(v+1)} \right) \Big)
\end{aligned}
}
$$

$C_\ell = \#$ loop pos$^{\text{ns}}$/orientat$^{\text{ns}}$ $\cdot\ C_H^{N_G}\, t_H^{|\ell|}$

For $\ell =$ no loop,

$$
Z(t_H = 0) = [+]\left(1 + e^{-2\Lambda}\right) + [-]\left(1 - e^{-2\Lambda}\right)
$$

$$
= Z_0 .
$$

In the large-N limit, $[+]$ "wins"

$$
\Rightarrow \quad Z \sim \sum_{\ell} C_{\ell}\, [+]^{N-v-1} \left(1 + e^{-2\Lambda(v+1)}\right)
$$

$$\sigma_{x_i}^2 = \frac{\tau}{\beta - 1}\,\overline{\eta^2}\,\left((t/\tau)^{\beta-1} - 1\right)$$

$\Rightarrow$ ~~$\langle x_i(t) x_i \dots$~~

~~$-2\langle x_i(t)\, x_j(0)\rangle = \langle (x_i(t) - x_i(0))^2 - \langle x_i^2\rangle - \langle x_j(0)^2\rangle$~~

Also verify β exponent?

Assuming prtcls ~ indep again,

$\gamma = \beta/2 - \beta$

$$K \sim \frac{\cancel{(2\Lambda)^2}}{2N}\cdot \sum_n \sum_{pq} \underbrace{\langle \dot{x}_p^i(t)\, \dot{x}_q^i(0)\rangle}_{\sim A(T)\, t^{\gamma(t)}\ \delta_{pq}}$$

$\sum_{pq}\delta_{pq} = n$

$$\Rightarrow Z\,K = \frac{(2\Lambda)^2}{N_E}\left(\sum_{t=0}^{t_{max}-1}\left(1 - \frac{\delta_{t0}}{2}\right) A(T)\, t^{\gamma(t)}\right)\underbrace{\left(\sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n}\, n \sum_l c_l \sum_m \dots\right.}_{Y}$$

$M = \min(V, n)$

$$Z = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_l c_l (l) \sum_m^M (-1)^m \binom{N-V}{n-m}$$

$$Y = \sum_{n=0}^{N} (\text{''}) \cdot n = \langle n \rangle_Z \cdot Z. \quad \sim \frac{\partial}{\partial\beta}(\dots)$$

$$\sim \frac{\partial}{\partial\lambda}(\dots)$$

$$Y = -\cancel{\frac{\partial}{\partial(2\Lambda)}}\, Z \left(\frac{1}{2\beta}\frac{\partial}{\partial\lambda} \ln Z\right.$$

$$\Rightarrow K = -\frac{1}{2\beta}\frac{(2\Lambda)^2}{N_E}\left(\ldots\right)\frac{\partial}{\partial \lambda}\ln Z.$$

$$= -\frac{(2\Lambda)^2}{N_E}\underbrace{\left(\ldots\right)}\frac{\partial}{\partial(2\Lambda)}\ln Z.$$

$$(\ldots) = \sum_{t_n}\left(1-\frac{\delta_{t_0}}{2}\right)A(\tau)\,t^{\gamma(\tau)}$$

$$\cancel{C = \frac{(2\Lambda)^2}{N_E}\,\partial}$$

$$C = \frac{\beta^2}{N_E}\frac{\partial^2}{\partial\beta^2}\ln Z.$$

$$Z = \sum_{l} C_l(H)\Big(\left(1+e^{-2\Lambda}\right)^{N-v-1}\left(1+e^{-2\Lambda(v+1)}\right)$$
$$+\left(1-e^{-2\Lambda}\right)^{N-v-1}\left(1-(-1)^v e^{-2\Lambda(v+1)}\right)\Big)$$

$$\overset{N\to\infty}{\approx} \sum_{l} C_l(H)\left(1+e^{-2\Lambda}\right)^{N-v-1}\left(1+e^{-2\Lambda(v+1)}\right)$$

$$-\frac{\partial Z}{\partial(2\Lambda)} \approx \sum_{l} C_l(H)\Big[(v+1)\left(1+e^{-2\Lambda}\right)^{N-v-1}$$
$$+(N-v-1)\left(1+e^{-2\Lambda}\right)^{N-v-2}\left(1+e^{-2\Lambda(v+1)}\right.$$

Summary

$V = Vol(\ell)$
$|\ell| = perim(\ell)$

$$Z \approx \sum_{\ell} C_\ell(H)\,(1+e^{-2\Lambda})^{N-V-1}\,(1+e^{-2\Lambda(V+1)})$$

$$C_\ell(H) = c_H^{N_E}\, t_H^{|\ell|}\cdot \Delta_\ell$$

$\Downarrow$ # of configs of ℓ.

$$C = \frac{\beta^2}{N_E}\frac{\partial^2}{\partial\beta^2}\ln Z$$

$$K = -\frac{(2\Lambda)^2}{N_E}\frac{\partial}{\partial(2\Lambda)}\ln Z \cdot \sum_{t=0}^{t_{max}-1}\left(1-\frac{\delta_{t0}}{2}\right)\underbrace{A(T)\,t^{\gamma(T)}}_{B(t;T)}$$

Which all reduces for $h\to 0$, although $B(t,T) \xrightarrow{h\to 0} \frac{\langle z\rangle}{2}\delta_{t0}$

$\therefore$ no correlat$^{\underline{n}}$s.

$$\frac{\partial}{\partial\beta} = 2\lambda\frac{\partial}{\partial(2\Lambda)} + h\frac{\partial}{\partial H}$$

I need $\frac{\partial}{\partial(2\Lambda)}\,Z)$ & $\frac{\partial}{\partial H}\,(Z)$

$$Z = \sum_\ell \Delta_\ell\, z_\ell$$

$$\frac{\partial z_\ell}{\partial H} = \frac{\partial}{\partial H}\left(\cosh^{N_E}(H)\tanh^{|\ell|}(H)\right)\ldots$$

$$= \left(\sinh(H)\cosh^{N_E-1}(H)\tanh^{|\ell|}(H)N_E + |\ell|\cosh^{N_E}(H)\,\mathrm{sech}^{2}(H)\tanh^{|\ell|-1}(H)\right)$$

$$\cancel{= \left(\cosh^{N_E}(H)\tanh^{|\ell|-1}(H) + \cosh^{N_E-2|\ell|}(H)\right)}\ldots$$

$$= \left(\cosh^{N_E}(H)\tanh^{|\ell|+1}(H) + \cosh^{N_E-2}(H)\tanh^{|\ell|-1}(H)\right)\ldots$$

$$= \underbrace{\cosh^{N_E}(H)\tanh^{|\ell|}(H)\ldots}_{z_\ell}\left(N_E\, t_H + |\ell|\, c_H^{-2}\, t_H^{-1}\right)$$

$$\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H} \approx \cancel{t_H + C_H^{-2}\, t_H^{-1}} \qquad sc^{-1} \qquad \frac{1}{c^2} \quad \frac{c}{s} \quad \frac{1}{cs}$$

$$= N_E\, t_H + |\ell|\, C_H^{-2}\, t_H^{-1}$$

$$= N_E\, S_H C_H^{-1} + |\ell|\, C_H^{-1} S_H^{-1}$$

$$= \left(N_E\, t_H + 2|\ell|\, S_{2H}^{-1}\right) / \text{other bits}$$

$$-\frac{\partial Z_\ell}{\partial(2\Lambda)} \approx e^{-2\Lambda}\left(N\left(\frac{1+e^{-2\Lambda(V+1)}}{1+e^{-2\Lambda}}\right) + (V+1) - (V+1)\left(\,''\,\right)\right) \Big/ (1+e^{-2\Lambda})$$

$$\Rightarrow -\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial(2\Lambda)} = \left(\frac{N-V-1}{1+e^{-2\Lambda}} + \frac{V+1}{1+e^{-2\Lambda(V+1)}}\right) e^{-2\Lambda} \Big/ \text{other bits.}$$

$$M = \frac{1}{N_E}\frac{\partial}{\partial H} \ln Z.$$

Is this subdiffusivity the same type as spin ice?

Percolation picture? → I need $D(M)$

We know ~~$M \approx \tanh(h)$~~ $M \approx \tanh(H)$

$M \sim$ percolation prob? not quite...

<u>All</u> hinges on: * What happens for $h > 1$?

* " " $T \gg h$?

NEED to get these nailed ASAP... { * Diffusion coeffs / exp-s

Side thing: Error analysis...

M vs Percolation

In the $M=0$ state, every bond has 50% chance to be $\pm$. In this case, the coordinatn @ each site is ~~50%~~ 2 as if $\mathbb{P} = 1/2$.

In the $|M| = 1$ state, all bonds +

$\Rightarrow$ no valid paths $\rightarrow$ $\mathbb{P} = 1$!

$\mathbb{P}$ = prob. of edge removal.

$\Rightarrow$ As a crude guess, $M \sim$ linear

$|M| = 2\mathbb{P} - 1$

$$Z = \sum_{\ell} \Delta_\ell \, Z_\ell^H \, Z_\ell^\Lambda$$

$$\frac{\partial Z}{\partial H} = \sum_\ell \Delta_\ell Z_\ell^\Lambda \frac{\partial Z_\ell^H}{\partial H}$$

$$= \sum_\ell \Delta_\ell Z_\ell^\Lambda \cdot Z_\ell^H \cdot \underbrace{\left(N_E + |\ell| \operatorname{cosech}^2(H)\right) \tanh(H)}_{\underset{N \gg |\ell|}{\sim} N_E \, t_H}$$

$$-\frac{\partial Z}{\partial (2\Lambda)} = \sum_\ell \Delta_\ell Z_\ell^\Lambda \cdot Z_\ell^H \cdot e^{-2\Lambda} \underbrace{\left(\frac{N-V-1}{1+e^{-2\Lambda}} + \frac{V+1}{1+e^{-2\Lambda(V+1)}} \right)}_{\underset{N \gg V}{\sim} \frac{N}{1+e^{-2\Lambda}}}$$

$$\frac{\partial Z}{\partial \beta} = \frac{\partial H}{\partial \beta} \frac{\partial Z}{\partial H} + \frac{\partial (2\Lambda)}{\partial \beta} \frac{\partial Z}{\partial (2\Lambda)}$$

$$= \sum_\ell \Delta_\ell Z_\ell^\Lambda Z_\ell^H \Big[h \left(N_E + |\ell|^2 s_H^{-2} \right) t_H$$

$$- 2\lambda \left(\frac{N-(V+1)}{1+e^{-2\Lambda}} + \frac{V+1}{1+e^{-2\Lambda(V+1)}} \right.$$

$$\underset{N \gg V, |\ell|}{\sim} \sum_\ell \Delta_\ell Z_\ell^\Lambda Z_\ell^H \left[2h N_V t_H - 2\lambda \frac{N_V}{1+e^{-2\Lambda}} \right]$$

$$\sim N_E \underbrace{\left(\sum_\ell \Delta_\ell Z_\ell^\Lambda Z_\ell^H \right)}_{Z} \left(h t_H - \frac{\lambda}{1+e^{-2\Lambda}} \right)$$

$$\Rightarrow \quad \frac{\partial \ln Z}{\partial \beta} \sim N_E \left(h t_H - \frac{\lambda}{1+e^{-2\Lambda}} \right)$$

$$\Rightarrow \quad C \approx \frac{1}{\cancel{N_E} T^2} \cancel{N_E} \frac{\partial}{\partial \beta} \left(h t_H - \frac{\lambda}{1+e^{-2\lambda}} \right)$$

$$\sim \frac{1}{T^2} \left[h^2 c_H^{-2} + \underbrace{\frac{2\lambda^2}{(1+e^{-2\lambda})^2} e^{-2\lambda}}_{\frac{1}{2}\operatorname{sech}^2(\Lambda)} \right]$$

$$M = \frac{1}{N_E} \frac{1}{Z} \frac{\partial Z}{\partial H} \sim \tanh(H)$$

$$K \sim -\frac{(2\Lambda)^2}{N_E} (\ldots) \underbrace{\frac{\partial}{\partial(2\Lambda)} \ln Z}_{= -\frac{N_E}{2} \frac{e^{-2\Lambda}}{1+e^{-2\Lambda}}}$$

$$\Rightarrow \quad K \sim \frac{1}{2} (2\Lambda)^2 \underbrace{\frac{e^{-2\Lambda}}{1+e^{-2\Lambda}}}_{\frac{1}{2}(1-\tanh\Lambda)} (\ldots)$$

$\Rightarrow$ K <u>only</u> changed ~~&~~ by a factor $(\ldots)$

$a \in (0,1)$

$$\binom{n}{k} = \oint_{|z|=a} \frac{dz}{2\pi i} (1+z)^n z^{-(k+1)}$$

$$\sum_{m=0}^{\min(V,n)} (-1)^m \binom{N-V}{n-m}$$ ↗ # ways to pick $n-m$ prtcl posns from $N-V$

~~Note assumes~~

$M_{max} = \min(V,n) = M$

$$= \sum_{m=0}^{\min(V,n) = m_{max}} (-1)^m \oint_{|z|=a} \frac{dz}{2\pi i} (1+z)^{N-V} z^{m-n-1}$$

$$= \oint \frac{dz}{2\pi i} (1+z)^{N-V} z^{-(n+1)} \underbrace{\sum_{m=0}^{M} \left(-\frac{1}{z}\right)^m}$$

$$= \frac{1+(-1)^{M+1} z^{M}}{1+z}$$

$$\frac{1 \mp (-z)^{M+1}}{z+1}$$

$$\frac{z^M + (-z)^M}{1+z^{-1}}$$

$$= \oint \frac{dz}{2\pi i} (1+z)^{N-V-1} \left(z^{-(n+1)} + z^{-(M+n)} (-1)^{M+1} \right)$$

$$= \binom{N-V-1}{n} + (-1)^{M+1} \binom{N-V-1}{n+M-1}$$

$\Rightarrow$ contribut^n from a loop of volume V for n prtcls is:

$$\sum_{m=0}^{M} (-1)^m \binom{N-V}{n-m} = \binom{N-V-1}{n} + (-1)^M \binom{N-V-1}{n+M-1}$$

w/. $M = \min(n,V)$ $\Rightarrow$ if $n=0$, just get $\binom{N-V}{0} = 1 = \binom{N-V-1}{0}$

$$Z = \cosh^{N_E}\left(H + \tfrac{1}{2}\Lambda M^3\right)$$

$$M = -\frac{1}{N_E}\frac{\partial \ln Z}{\partial H} = \tanh\left(H + \tfrac{1}{2}\Lambda M^3\right)$$

$$\sum_i n = -\frac{1}{\underset{\overset{\|}{N_E}}{N_V}}\frac{\partial \ln Z}{\partial(2\Lambda)} = -\frac{\cancel{N_E}}{\cancel{N_E}}\frac{M^3}{2}\ \ldots$$

$$n = \frac{M^3}{2}\underbrace{\tanh\left(H + \tfrac{1}{2}\Lambda M^3\right)}_{M}$$

$$= \frac{1}{2}M^4$$

$$\sum_i A_i \approx -\frac{\partial \ln Z}{\partial \Lambda} = \underline{N_V M^4}$$

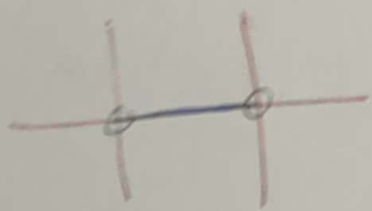
$$A_i = \begin{cases} -1 & \text{if prtcl} \\ 1 & \text{otherwise} \end{cases}$$

$$= (1 - 2\delta_i)$$

$$\Rightarrow N_V - 2\underbrace{\sum_i \delta_i}_{N_V n} = N_V M^4$$

$$\Rightarrow 1 - 2n = M^4$$

$$n = \frac{1}{2}(1 - M^4)$$



$\Delta \epsilon = 4\lambda + 2h$

$\Rightarrow$ <u># pairs $\propto e^{-4\lambda - 2h}$</u>

& isolated ptcls suppressed by $e^{-2h\ell}$ w/. ℓ their separat$\bar{n}$.

@ high T, we have an estimate for $n(T)$.

What is the percolation for each species as a function of magnetisation?

$$M(H) = \left\langle \frac{1}{N_E} \sum_\alpha \sigma_\alpha \right\rangle = \langle \langle \sigma_\alpha \rangle_{\text{lattice}} \rangle$$

So should be a good measure.

$$\mathbb{P}(\sigma_\alpha = \pm 1) = M$$

$$\cancel{M = \sum_\alpha P_\alpha \sigma_\alpha = P \sum_\alpha \sigma_\alpha}$$

We have $M = \tanh(H)$

Whereas mean-field gives $M = \tanh(H + \frac{1}{2} \Lambda M^{\overset{3}{\overset{\shortparallel}{z}}-1})$

Take this

$\Rightarrow \mathbb{P} \sim \tanh(H)$? For spins, yes. For $+-+-\ldots$ paths, no!

We should have that $\mathbb{P}_{①} + \mathbb{P}_{②} = 1$ $\because$ they're opposite conditions in some sense.

$\mathbb{P} \,\&\, M$ linearly $\because$ above.

$M(\mathbb{P}_{①} = 1) = M(\mathbb{P}_{②} = 0) = 0$

$M(\mathbb{P}_{①} = \mathbb{P}_{②} = 1/2) = 1$

$\Rightarrow$ propose $\underset{= 1 - \tanh(H)}{1 - M} = 2\mathbb{P}_{②} = 2(1 - \mathbb{P}_{①})$

$\rightarrow$

Overall, $\sum_i m_i$ conserved.

$= 2(N_E - (\sum_\alpha \sigma_\alpha))$

~~$\Rightarrow$ let's look @ the~~ values it

$\Rightarrow \langle m_i(t) \rangle = \langle m_i(0) \rangle = m$.

However, it can fluctuate when prtcls move!

Picture is:

Two transport processes: ① individual spinons
② spinon pairs

Low $T \ll h$: ① 100% percolated - can't move

② 0% percolated - free diffusion

High $T \gg h$: ① ② Both move ~indep. (besides collisions) on 50% percolated lattice $\because$ +-+- paths

collisions make it more like 1/4 factor in D

$\because$ $Z(T)$ term.

$$\Rightarrow Z_n = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} \binom{N}{n} e^{-2\lambda n}\, e^{-H\sqrt{Nn}}$$

$$\cancel{\frac{\partial Z_n}{\partial \lambda}} \qquad \langle n \rangle = \frac{1}{Z_n} \sum \text{''}\; n$$

$$= -\frac{\partial \ln Z_n}{\partial (2\lambda)}$$

Not good enough?

$$Z_n = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\lambda n}\, \underbrace{\frac{1}{n!} \sum_{\{\underline{x}\}} e^{-2H \sum_{\text{pairs}} l_{\text{pair}}}}_{\sim \binom{N}{n} \cdot \left(\langle e^{-2H l_{\text{pair}}} \rangle\right)^n}$$

l_{pair} → minimal dist. between them.

$$\sim \int d^2\underline{x}\, d^2\underline{y}\; e^{-2H|\underline{x}-\underline{y}|_1}$$

$$= \left(\int_0^L dx\, dy\, e^{-2H|x-y|}\right)^2 \rightarrow \because \text{square lattice}$$

$$= \left(\int_0^L dx \left(\int_0^x dy\, e^{-2H(x-y)} + \int_x^L dy\, e^{2H(x-y)}\right)\right)$$

$$= \frac{1}{2H}\int_0^L dx \left(e^{-2Hx}\left(e^{2Hx}-1\right) - e^{2Hx}\left(e^{-2HL} - e^{-2Hx}\right)\right.$$

$$= \frac{1}{2H}\int_0^L dx \left(2 - e^{-2Hx} - e^{-2HL} e^{2Hx}\right)$$

$$= \frac{1}{2H}\left(2L + \frac{1}{2H}\left(e^{-2HL}-1\right) - \frac{1}{2H} e^{-2HL}\left(e^{2HL} - \right.\right.$$

$$= \frac{1}{2H}\left(2L + \frac{1}{2H} e^{-2HL} - \frac{1}{2H} - \frac{1}{2H} + \frac{1}{2H} e^{-2\ldots}\right.$$

$$=$$

$$\Rightarrow Z = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\underline{w}} \frac{1}{n!} \sum_{\{\xi\}} \sum_{\underline{\mu}} e^{+H\sum_{\langle pq\rangle} \eta_{pq}\mu_p\mu_q}$$

@ small T, ~~most~~ n small $\Rightarrow$ most η's are +ve (excl. the odd topo sector) $\Rightarrow$

$$Z \approx \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \binom{N}{n} \sum_{\underline{w}} \underbrace{\sum_{\underline{\mu}} e^{+H\sum_{\langle pq\rangle} \mu_p\mu_q \eta_{pq}}}$$

n° frustrated bonds $\sim \underline{w} \odot \underline{L}$ + excitat$\bar{\text{n}}$s.

$$\Rightarrow Z \approx \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \binom{N}{n} \underbrace{\sum_{\underline{w}} \left(e^{-2H(\underline{w}\odot\underline{L})}\right)}_{Z_W} Z_{Ising}.$$

Which once again gives $n \sim \frac{1}{2}(1-\tanh\Lambda)$ & modifies C a little.

To include a cost for the prtcls, we bound by ~~cost $\propto$ edge~~ $2H\frac{n}{2}\ell$ — ℓ: typical bond length; $\frac{n}{2}$: num bonds

$\rightarrow \underline{N \sim n\ell^2} \Rightarrow 2H\sqrt{N}\, n^{\frac{1}{2}}$

$$\Rightarrow Z \approx Z_{Ising} Z_W \sum_{n=0}^{N} \frac{1+(-1)^n}{2} \binom{N}{n} e^{-2\Lambda n} e^{-H\sqrt{Nn}}$$

$$\frac{1}{(2H)^2}\left[\left((2H)L + e^{-2HL} - 1\right) + \left((2H)L + e^{-2HL} - 1\right)\right]$$

$$\Rightarrow (1) = \left(\frac{2}{(2H)^2}\left[e^{-2HL} - 1 + 2HL\right]\right)^2$$

$$= \frac{4}{(2H)^4}\left[e^{-2HL} - (1 - 2HL)\right]^2$$

$$= \langle e^{-2H l_{pair}} \rangle = \alpha_H$$

$$Z_n = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\lambda n} \binom{N}{n} \alpha_H^{\,n}$$

$$= \left(1 + \alpha_H e^{-2\lambda}\right)^N + \left(\ldots\right)^N$$

$$\approx \left(1 + \alpha_H e^{-2\lambda}\right)^N$$

$$\langle n \rangle = -\frac{\partial \ln Z}{\partial (2\lambda)} = \alpha_H e^{-2\lambda} \frac{\left(1 + \alpha_H e^{-2\lambda}\right)^{N-1}}{\left(\ldots\right)^N}$$

$$= \cancel{\alpha_H \cdot \frac{1}{2}\left(1 - \tanh\right.}$$

$$= \frac{\alpha_H e^{-2\lambda}}{1 + \alpha_H e^{-2\lambda}}$$

$$\int d^2x \int d^2y \; e^{-2H|x-y|_2}$$

$$= \underset{N}{\cancel{4}}\, 2\pi \int_0^L \cancel{\phi}\, dr\, r\, e^{-2Hr}$$

$$\sim 2\pi N \frac{1}{(2H)^2}\left(1 - e^{-2HL}(1 + 2HL)\right)$$

$$F = -\lambda \sum_i A_i \overset{-}{\cancel{+}} h \sum_\alpha \sigma_\alpha$$

$$= \sum_i \epsilon_i$$

$$\sum_\alpha \sigma_\alpha = \frac{1}{2} \sum_i \sum_{\alpha \in \partial^+ i} \sigma_\alpha$$

$$\epsilon_i = -\lambda A_i \overset{-}{\cancel{+}} \frac{h}{2} \sum_{\alpha \in \partial^+ i} \sigma_\alpha .$$

Now flip a spin β.

If $\beta \in \partial^+ i$, $\quad A_i \to -A_i$

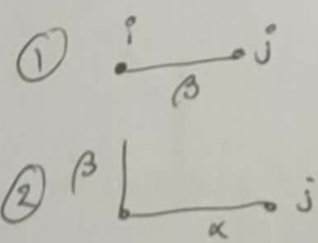
$$\sum_{\alpha \in \partial^+ i} \sigma_\alpha \to \sum_{\alpha \in \partial^+ i} \sigma_\alpha - 2\sigma_\beta .$$

$$\Rightarrow \cancel{\epsilon_i \to \epsilon_i' = +\lambda A_i}$$

$$\Delta \epsilon_i^{(\beta)} = 2\lambda A_i \, \delta_{\beta \in \partial^+ i} + h \sigma_\beta \, \delta_{\beta \in \partial^+ i}$$

$$= (2\lambda A_i + h\sigma_\beta)\, \delta_{\beta \in \partial^+ i}$$

$$\Delta \epsilon_i^{(\beta)} = -\sum_{j \in nn(i)} j_{ij}^{(\beta)}$$ cont.-eqn.



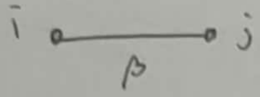
$$\Rightarrow \cancel{j_{ij}^{(\beta)} = (h\sigma_\beta}$$

$$\cancel{(1)\; j_{ij}^{(\beta)} =}$$

$$\Delta \epsilon_j^{(\beta)} = (2\lambda A_j + h\sigma_\beta)\, \delta_{\beta \in \partial^+ j}$$

$$= -\sum_{i \in nn(j)} j_{ji}^{(\beta)} = \sum_{i \in nn(j)} j_{ij}^{(\beta)}$$

$\cancel{j_{ij}^{(\beta)} = 2\lambda(A_i - A_j)\delta_{\beta\,\delta_j}}$



$\cancel{\dot{j}_{ij}^{(\beta)} = 2\lambda(}$

$$\Delta\epsilon_i^{(\beta)} = 2\lambda A_i + h\sigma_\beta$$
$$= - \cancel{\quad} \cancel{\dot{j}_{ij}^{(\beta)}}$$

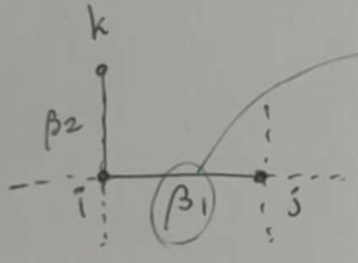
$$\cancel{\dot{j}_{ij}^{(\beta)}} = -h\sigma_\beta - 2\lambda A_i \ ?$$

<u>CAN'T</u> be right...

Reason: move doesn't conserve energy!

$\Rightarrow$ Cont. eqn. <u>fails</u> for 1 timestep.

Need to account for <u>both</u> timesteps.



Did I exclude β_1 from list of possible β_2?

$$\Delta\epsilon_i^{(\beta_n)} = (2\lambda A_i + h\sigma_\beta)\,\delta_{\beta_n \in \partial^+ i}$$
$$n \in \{1, 2\}$$

We demand that $\Delta E^{(\beta_1)} + \Delta E^{(\beta_2)} = 0$

$$\left(\Delta E^{(\beta)} = 2\lambda(A_i + A_j) + 2h\sigma_\beta\right) \quad i \overset{\beta}{-} j$$

$\Rightarrow$ ~~can~~ demand: $\Delta\epsilon_i^{(\beta_1)} + \Delta\epsilon_i^{(\beta_2)} = -\sum_{ij} j_{ij}^{(\beta_1+\beta_2)}$

$=$

$\cancel{\Rightarrow 2\lambda}$

We get $\Delta\epsilon_i^{(\beta_1,\beta_2)} = \cancel{2}\, h(\sigma_{\beta_1} + \sigma_{\beta_2})$ $\to 0$ if magnetisat^n cons.

$$\Delta\epsilon_j^{(\beta_1,\beta_2)} = 2\lambda A_j + h\sigma_{\beta_1}$$

$$\Delta\epsilon_k^{(\beta_1,\beta_2)} = 2\lambda A_k + h\sigma_{\beta_2}$$

& overall, $\Delta E^{(\beta_1,\beta_2)} = 2\lambda(A_j + A_k) + \overset{2}{\cancel{\ }}\, h(\sigma_{\beta_1} + \sigma_{\beta_2})$

$= 0$ $\quad (+A_i - A_i)$ ∵ flipped on β_1 move. $\quad \cancel{2}\sigma_{\beta_n} - \sigma_{\beta_n}$ from the shared vertex

$$\Delta\epsilon_j^{\beta} = -\sum_{\ell \in nnn(j)} j_{j\ell}$$

$$\dot{j}_{jk}^{\beta} = (h\sigma_{\beta_1} + 2\lambda A_j)\,\delta_j\delta_k \qquad \beta_1 \in j,\ \beta_2 \in k,\ \beta_1,\beta_2 \in i$$

& $\dot{j}_{kj}^{\beta} = (h\sigma_{\beta_2} + 2\lambda A_k)\,\delta_j\delta_k$

$j_{jk} = -j_{kj}$ by $\Delta E = 0$

Trick! Since h, λ are non-$\mathbb{Q}$ we can conserve these separately

$\Rightarrow$ $\underline{A_j = -A_k}$, $\underline{\sigma_{\beta_1} = -\sigma_{\beta_2}}$.

$j \overset{\beta_1}{\text{—}} i \overset{\beta_2}{\text{—}} k$

Manifestly asymm form: $\quad j_{jk}^{(\beta_1,\beta_2)} = \frac{h}{2}(\sigma_{\beta_2} - \sigma_{\beta_1}) + \lambda(A_k - A_j)$

no term 0 ∴ can't do 0-current moves w/ $\Delta E = 0$ $\quad = \underbrace{\lambda(A_k - A_j)}_{\in \pm 2\lambda} + \underbrace{\frac{h}{2}(\sigma_{\beta_2} - \sigma_{\beta_1})}_{\pm h}$

This resolves the issues of:

* Path-dependent currents $(\beta_1\beta_2 \neq \beta_2\beta_1)$
* The missing factor of 2? → Maybe? Maybe not actually an issue...?

Moves of Interest

① For any teleport,

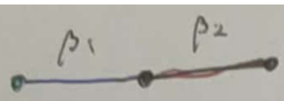
but be concrete:

$\simeq$ β_2 β_1

unlike my naive attempt...

$$\dot{j}^{(\beta_1,\beta_2)}_{jk} = -h\,\sigma^{-1}_{\beta_1} - 2\lambda A_j^{\lambda^{-1}} \in 2\lambda + h \neq 0$$

$$\dot{j}^{(\beta_1,\beta_2)}_{kj} = -h + 2\lambda$$

(2) 

$$j_{jk}^{\beta} = 2\lambda + h$$

(3) 

$\because \sigma_{\beta_1} \neq -\sigma_{\beta_2}.$

UNLESS $h = 0$, in which case

$$j_{jk}^{\beta} = -2\lambda A_j = 2\lambda$$

explains factor of two!
Why we're getting wrong results in current sims!

Question: Still have $\underline{J}^{(\beta)} = \frac{1}{2}\sum_{ij} (\underline{r}_i - \underline{r}_j)\, j_{ij}^{(\beta)}$
BUT do all the assumptions of the Kubo formula hold?

To summarise: $\quad i_1 \xrightarrow{\beta_1} i \xrightarrow{\beta_2} i_2$

$$\Delta E^{\beta_1\beta_2} = 2\lambda\,(A_{i_1} + A_{i_2}) + 2h\,(\sigma_{\beta_1} + \sigma_{\beta_2})$$

$$\Delta \epsilon_{i_n}^{\beta_1\beta_2} = 2\lambda A_{i_n} + h\,\sigma_{\beta_n}$$

$$\Delta \epsilon_{i}^{\beta_1\beta_2} = h(\sigma_{\beta_1} + \sigma_{\beta_2})$$

$h \neq 0$

$A_{i_1} = -A_{i_2}$ & $\sigma_{\beta_1} = -\sigma_{\beta_2}$ for $\Delta E = 0$

$\Rightarrow \Delta \epsilon_i = 0$

$\Delta \epsilon_{i_1} = -\Delta \epsilon_{i_2}$

& $\dot{j}_{i_1 i_2}^{\beta_1\beta_2} = \lambda(A_{i_2} - A_{i_1}) + h/2\,(\sigma_{\beta_2} - \sigma_{\beta_1})$

$h = 0$

$\Delta E = 0$ ~~still gives~~ $\cancel{\Delta\epsilon_{i_1} = \Delta\epsilon_{i_2}}$ only gives $A_{i_1} = -A_{i_2}$.

BUT we still recover $\Delta \epsilon_i = 0$

$\Delta \epsilon_{i_n} = 2\lambda A_{i_n}$.

$h/\lambda \in \mathbb{Q}$

Bad things! Stay away!

Approximate: $B(\Lambda, H) = \sum_{t_{\Omega}=0}^{t_{max}} \left(1 - \frac{\delta_{t_0}}{2}\right) A(H,\Lambda)\, t^{\ldots}$

$$A(H,\Lambda) = A(T,h) = A(M(T,h))$$

$M \sim M(H) = \tanh(H)$ (for

$Z \cancel{\approx (1+e)^2}$

$$Z = \sum_{\ell} \Delta_\ell\, C_H^{N_E}\, t_H^{|\ell|} \left[(1+e^{-2\Lambda})^{N-V-1} (1+e^{-2\Lambda(V+1)}) + (1-e^{-2\Lambda})^{N-V-1} (1-(-1)^{V} e^{-2\Lambda(V+1)}) \right]$$

$$M = \frac{1}{2N} \frac{\partial \ln Z}{\partial H}$$

$$\frac{\partial Z}{\partial H} = \sum_{\ell} \Delta_\ell [\ldots]_\ell \underbrace{\frac{\partial}{\partial H}\left(C_H^{2N} t_H^{|\ell|}\right)}$$

$$= 2N S_H C_H^{2N-1} t_H^{|\ell|} + |\ell|\, C_H^{2(N-1)} t_H^{|\ell|-1}$$

$$= C_H^{2N} t_H^{|\ell|} \left(2N t_H + |\ell| C_H^{-2} t_H^{-1}\right)$$

$$= C_H^{N_E} t_H^{|\ell|} \left(N_E t_H + |\ell| S_H^{-1} C_H^{-1}\right)$$

$$\Rightarrow \frac{\partial \ln Z}{\partial H} = N_E \tanh(H) + \frac{\sum_\ell \Delta_\ell [\ldots]_\ell\, C_H^{N_E} t_H^{|\ell|} \cdot |\ell| S_H^{-1} C_H^{-1}}{\sum_\ell \Delta_\ell [\ldots]_\ell\, C_H^{N_E} t_H^{|\ell|}}$$

$$= \underset{\sim H}{N_E \tanh(H)} + \underset{\sim \frac{1}{2H} O(1)}{2\,\mathrm{cosech}(2H)} \underbrace{\frac{\sum_\ell |\ell| \Delta_\ell C_H^{N_E} t_H^{|\ell|}}{\sum_\ell \Delta_\ell C_H^{N_E} t_H^{|\ell|}}}_{\sim \langle |\ell| \rangle_{\text{graphs } \ell}}$$

$M \xrightarrow{N \to \infty} \tanh(H)$ if we have a UV cutoff for $|\ell| \sim \sqrt{N_V}$

$\square$ $\quad \Delta_\ell = N_v \quad |\ell| = 4$

$\square\!\square$ $\quad \Delta_\ell = 2N_v \quad |\ell| = 6$

$\square$ $\square$ $\quad \Delta_\ell = N_v(N_v - 5) \quad |\ell| = 8$

$$n = -\frac{1}{N_v}\frac{\partial \ln Z}{\partial (2\lambda)}$$

$$J = 2\lambda$$

$$-\frac{\partial Z}{\partial J} = -\sum_\ell \underbrace{\Delta_\ell C_H^{N_E} t_H^{|\ell|}}_{\{\ldots\}} \frac{\partial}{\partial J}\underbrace{\Big[(1+e^{-J})^{N-V-1}(1+e^{-J(V+1)}) + (1-e^{-J})^{N-V-1}(1-(-1)^V e^{-J(V+1)}) \Big]}_{B}$$

$$\frac{\partial B}{\partial J} = (N-V-1)\Big[(1+e^{-J})^{N-V-2}(\ldots) - (1-e^{-J})^{N-V-2}(\ldots) \Big]$$

$$+ (V+1)\Big[(+)^{N-V-1} - (-1)^V (-)^{N-V-1} \Big] e^{-J}$$

$$= N\left[\frac{B_+}{1+e^{-J}} - \frac{B_-}{1-e^{-J}} \right] e^{-J}$$

$$+ (V+1)\left[\frac{B_+}{1+e^{-J(V+1)}} - (-1)^V \frac{B_-}{1-e^{-J(V+1)}} - \frac{B_+}{1+e^{-J}} - \frac{B_-}{1-e^{-J}} \right] e^{-J}$$

If $|\ell| \ll \sqrt{N_E}$, $V \sim |\ell|^2 \ll N_E \sim N_v$

$$\Rightarrow -\frac{\partial B}{\partial J} \approx \frac{(B_+ - B_-) - (B_+ + B_-)e^{-J}}{1 - e^{-2J}} e^{-J}$$

Recall that in the double-flip dynamics
(choose to also impose spin swap $\because$ easier!)

$$\epsilon_i = -\lambda A_i - \frac{h}{2} \sum_{\alpha \in \partial^+ i} \sigma_\alpha$$

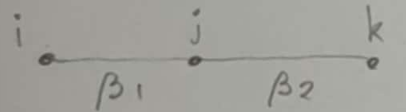
$$\Rightarrow \text{Offset}: \quad \epsilon_i = \lambda(1 - A_i) + \frac{h}{2}\Big(\overset{4 \text{ if square!}}{z_i} - \sum_{\alpha \in \partial^+ i} \sigma_\alpha\Big)$$

So that $\epsilon_i = 0 \;\forall i$ in the absolute GS.
(all spins $\uparrow$).

The cont. eqn. still applies, so long as one understands each timestep δt to be a spin swap.

$$\delta t \; \underline{J}(t) = \sum_i \underline{x}_i \Big[\underbrace{\epsilon_i[\sigma(t+1)] - \epsilon_i[\sigma(t)]}_{\Delta \epsilon_i[\sigma(t)]} \Big]$$

Recall that $\Delta \epsilon_i \in \{ \underset{\text{site hit twice}}{0}, \; \pm_1 2\lambda \pm_2 h \}$



$$\Delta \epsilon_i = 2\lambda A_i + h\sigma_{\beta_1}$$
$$\Delta \epsilon_j = h(\sigma_{\beta_1} + \sigma_{\beta_2}) = 0$$
$$\Delta \epsilon_k = 2\lambda A_k + h\sigma_{\beta_2}$$

$$\Delta \epsilon_i = - \sum_{\ell \in nn(i)} j_{i\ell}$$

$\Rightarrow$ since $\sigma_{\beta_1} = -\sigma_{\beta_2}$ enforced, we have $A_i = -A_k$ for $\Delta E = 0$

$$\Rightarrow \; j_{ik} = \lambda(A_i - A_k) + \frac{h}{2}(\sigma_{\beta_2} - \sigma_{\beta_1})$$

Current only well-defined between points joined by both spins. Fine $\because \sum_\ell j_{i\ell} = 0$ anyway.

Now, $K^{\mu\nu} = \frac{\beta^2}{N_E} \sum_{t=0}^{t_{max}-1} \langle J^\mu(t) J^\nu(0) \rangle_{P_{eq}} \left(1 - \delta_{t0}/2\right)$

$$\cancel{\delta t}^2 \langle J^\mu(t) J^\nu(0) \rangle = \sum_{ij} x_i^\mu x_j^\nu \left[A_{ij}(t+1,1) - A_{ij}(t+1,0) - A_{ij}(t,1) + A_{ij}(t,0) \right]$$

$$\epsilon_i = 2\lambda \cdot \underbrace{\delta_{\text{pred}@i}}_{\Theta_i} + h \cdot \underbrace{\#\text{ angry bonds} \in \partial^+ i}_{m_i}$$

$$A_{ij}(t,0) = \langle \epsilon_i[\Omega^t(\sigma)] \epsilon_j[\sigma] \rangle$$

$$= \frac{1}{Z} \sum_{\sigma} e^{-\beta F[\sigma]} \Big[\overset{①}{(2\lambda)^2 \Theta_i(t) \Theta_j(0)} + \overset{③}{h^2 m_i(t) m_j(0)} + 2\lambda h \big(\underset{ⓐ}{\Theta_i(t) m_j(0)} \overset{②}{+} \underset{ⓑ}{\Theta_j(0) m_i(t)} \big) \Big]$$

$$① = \frac{1}{Z} \sum_{n=0}^{N_v} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\{\Xi\}} \delta_{i \in \omega^t(\Xi)} \delta_{j \in \Xi} \underbrace{\sum_{\underline{\omega}} \sum_{\underline{\mu}} e^{H \sum_{\mu\nu} \eta_{\mu\nu} \mu_\mu \mu_\nu}}_{Z}$$

~~Let's try writing~~

$$\cancel{-\beta F = \Lambda \sum_i A_i + H \sum_\alpha \sigma_\alpha}$$

$$\cancel{= \Lambda \sum_i A_i + \frac{H}{2} \sum_i \sum_{\alpha \in \partial^+ i} \sigma_\alpha}$$

$$\cancel{= -\beta \sum_i \epsilon_i}$$

① is trivial (ish) & just as before, but with a (...) factor for H-dependence of $\langle \dot{x}(t) \dot{x}(0) \rangle$.

Let's look at ②:

$$②ⓐ = \frac{2\lambda h}{Z} \sum_{n=0}^{N_v} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\{\Xi\}} \delta_{i \in \omega^t(\Xi)} \sum_{\underline{\omega}} \sum_{\underline{\mu}} m_j(0)\, e^{H \sum_{\mu\nu} \eta_{\mu\nu} \mu_\mu \mu_\nu}$$

Try mean-field.

$m_i \longrightarrow \langle m_i \rangle$

$$\sum_i m_i = \sum_i \overset{\sum_{\alpha \in \partial^+ i}(1)}{z_i} - 2 \sum_\alpha \sigma_\alpha$$

where $\sum_i z_i = 2\sum_\alpha (1)$

$$m = 2\left(N_E/N_v - N_E/N_v\, M \right)$$

$$= \underline{4\left(1 - \tanh(H)\right)}$$

$M := \langle \sum_\alpha \sigma_\alpha \rangle / N_E$

$m := \sum m_i / N_v$

Out of interest, $\Theta = n$ on avg. but we're not doing mean-field in Θ

$$\Rightarrow A_{ij}(t,0) = (2\lambda)^2 \langle \Theta_i(t)\Theta_j(0)\rangle + h^2 m^2 + 2\lambda h m \underbrace{\langle \Theta_i(t) + \Theta_j(0)\rangle}$$

$$= \langle \Theta_i(0) + \Theta_j(0)\rangle = 2\langle \Theta_i(0)\rangle = 2\Theta$$

$$= \underbrace{(2\lambda)^2 \langle \Theta_i(t)\Theta_j(0)\rangle}_{\text{fluctuat}^{\text{ns}}} + \underbrace{h^2 m^2 + 4\lambda h m \Theta}_{\text{time-indep.}}$$

$\Rightarrow$ drop out of K.

So in the mean-field, just taking ① is justified.

$\Rightarrow$ Proceed as before...

$$K = \sum_{t=0}^{t_{max}-1} \left(1 - \frac{\delta_{t0}}{2}\right) \frac{(2\Lambda)^2}{N_E} \frac{1}{Z} \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\underline{w}} \sum_{\{x\}} \sum_{pq} \langle \dot{x}_p(t)\dot{x}_q(0)\rangle Z_{\underline{w}\underline{x}}$$

$$= \frac{(2\Lambda)^2}{N_E} \frac{1}{Z} \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\underline{w}} \sum_{\{x\}} Z_{\underline{w}\underline{x}} \sum_{t=0}^{t_{max}-1} \left(\sum_{pq} \langle \dot{x}_p(t)\dot{x}_q(0)\rangle\right) \cdot (1 - \delta_{t0}/2)$$

<u>If</u> $\langle \dot{x}_p(t)\dot{x}_q(0)\rangle \propto \delta_{pq}$ & indep. of p, we get

$$K = \frac{(2\Lambda)^2}{N_E} \langle n \rangle \sum_{t=0}^{t_{max}-1} (1 - \delta_{t0}/2) \langle \dot{x}(t)\dot{x}(0)\rangle$$

<u>BUT</u> this isn't true!

For free prtcls, have to move on $+-+-+-$ paths

$\Rightarrow$ Can get everywhere @ large T, just slowly.

@ low T, $\exists$ v. few such paths $\Rightarrow$ stuck

UNLESS in a pair, in which case you can move diffusively (same in either limit really, just a factor of 2 different $\because$ restricted to "opposite spin" paths).

totally free for bound pairs

$\Rightarrow$ Percolated for isolated prtcls @ low T

$\frac{1}{2}$ coordination @ high T (also <u>nn effects</u>)
(for all moves)

$\Rightarrow$ would expect <u>diff</u> results if initialised in a random state then thermalised $\because$ prtcls more likely to be sparse (@ short runtimes).

Q1 * Only allowing pair moves.

* Allowing both.

high T

We should see $\langle |\underline{x}(t) - \underline{x}(0)|^2 \rangle \sim 2\gamma t = 4Dt$ if ~RW.

$\sim \langle z \rangle \tilde{D} t$ $\quad \langle z \rangle \sim \frac{M(T)+1}{2}$

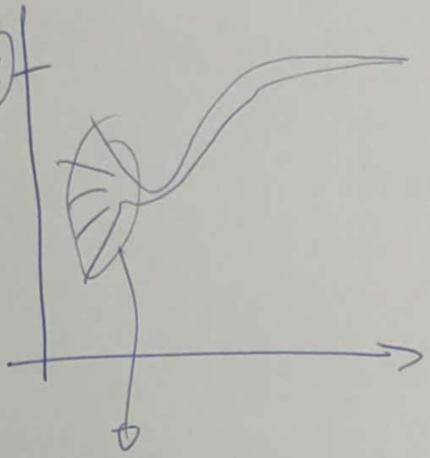
we get $\underline{D \sim 1/4}$ $\approx \frac{\langle z \rangle}{4} = \frac{1}{2}(1 - M(T)) \sim \frac{1}{4}$?

In fact, for $0\% < p < 50\%$ we get enhanced diffusion & $p = 1/2 = p_c$ where $\nexists$ no cts paths any more $\Rightarrow$ starts to be subdiffusive.

Not seen in my square lattice tests or the system ...

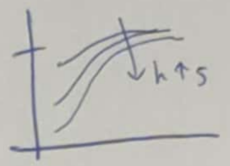
Rough results:

Overall, $\alpha \sim \lesssim 1$

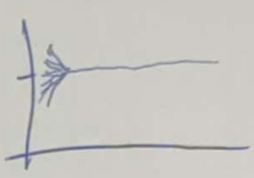


noise :: pairs

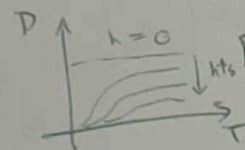
no pairs



only pairs



Weirdly, $T \to \infty$ limit <u>not</u> 1, hovers around 0.95 ish: Either : * Bad fit – not enough histories (x30 v small)



But we get $D > 0$ here? So should <u>be 1!</u>

* 50% <u>percolation</u>

↳ Also that would give $\alpha \sim 0.25$!

For pair $T \to 0$ limit : check if just runtimes by doing a sim w/. ℓ instead of T.

Q1

Next thing: * Are my <u>diffusive</u> high-T predictions valid? i.e.

Q2

↳ $\langle \dot{x}(t)\, \dot{x}(0) \rangle \sim \gamma\, \delta_{t0}$ along each dimens$\bar{a}$.

$$\gamma = \frac{\langle z \rangle}{2} \tilde{D}$$

1 for RW

$\sim \frac{1}{2} \cdot (1 - \langle n \rangle)$

:: +-+- paths

blocking.

Should be fine in both lim.s

:: $M = 0, 1.$ $\sim$ const.?

↳ Also discarded $M_i = \sum_{\alpha \in \partial T_i} \sigma_\alpha$ fluctuat^{ns}

$$\frac{(2H)^2}{N_E} \frac{\partial^2 Z_w}{\partial (2H)^2} \ln Z_w = \frac{(2H)^2}{N_E} \frac{\partial}{\partial (2H)} \left[\frac{-2\sqrt{N}\, e^{-2H\sqrt{N}} - N e^{-2HN}}{1 + 2e^{-2H\sqrt{N}} + e^{-2HN}} \right]$$

$$\prod_{d} \left(1 + e^{-2HLd}\right)$$

$$\downarrow$$

$$\varepsilon_d = \frac{(2H)^2}{N_E} \frac{\partial}{\partial (2H)} \left(-Ld \underbrace{\frac{e^{-2HLd}}{1 + e^{-2HLd}}}_{\frac{1}{1+e^{2HLd}}} \right)$$

$$= -Ld \cdot -Ld\, e^{+2HLd} \frac{(2H)^2}{N_E} \frac{1}{(\dots)^2}$$

$$= \frac{Ld^2}{N_E} (2H)^2 \frac{e^{+2HLd}}{(1+e^{+2HLd})^2}$$

$$\Rightarrow \quad Z \approx Z_{Ising} \underbrace{\left(1 + 2e^{-2HL} + e^{-2HL^2}\right)}_{Z_w} \underbrace{\frac{1}{2}\left(1 + e^{-2\Lambda} e^{-H\langle l_{pair}\rangle}\right)}_{Z_n}$$

$$\Rightarrow \quad \Lambda \rightarrow \Lambda + \frac{H}{2}\langle l_{pair}\rangle \approx \Lambda + \frac{H}{2}\sqrt{\frac{N}{n}}$$

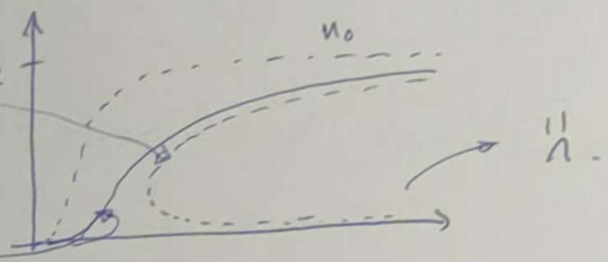
$$\approx \Lambda + \frac{H}{2}\frac{1}{\sqrt{\frac{1}{2}(1-\tanh\Lambda')}} \; 3 \; \sqrt{\rho}$$

$$\Rightarrow \quad \rho = \frac{1}{2}\left(1 - \tanh(\Lambda')\right)$$

$$= \frac{1}{2}\left(1 - \tanh\left(\Lambda + \frac{H}{2\sqrt{\rho}}\right)\right)$$

$$\rho = \frac{1}{1 + e^{2\Lambda + H/\sqrt{\rho}}} \quad \sim$$

$$\rho = \frac{1}{1 + e^{2\Lambda + H/\sqrt{\rho_0}}} \quad \sim$$



$$Z_n = \sum_{n=0}^{N} \frac{1+(-1)^n}{2} \binom{N}{n} \underbrace{e^{-2\Lambda n} \, e^{-H\sqrt{N}\sqrt{n}}}_{e^{-(An + B\sqrt{n})}}$$

$$\mathcal{X} = \sum_{n=0}^{N} \binom{N}{n} x^n y^{\sqrt{Nn}}$$

$$y = x^{h/2\lambda}$$

$$x = e^{-2\lambda}$$

I want $h \in [0, 2)$

$$n + \frac{h\sqrt{N}}{2\lambda}\sqrt{n} \overset{h \text{ small} < 2\lambda/\sqrt{N}}{\approx} \left(\sqrt{n} + \frac{h}{4\lambda}\sqrt{N}\right)$$

Large for $h \sim \lambda$

$$n + \frac{h\sqrt{N}}{2\lambda}\sqrt{n} = \underbrace{\left(\sqrt{n} + \frac{h\sqrt{N}}{2\lambda}\right)^2}_{?} - \left(\frac{h}{2\lambda}\right)^2 N$$

$$\mathcal{X}_2 = \sum_{n=0}^{N} \binom{N}{n} x^{\left(\sqrt{n} + \frac{h}{4\lambda}\sqrt{N}\right)^2}$$

$\Rightarrow$ @ low T, let's consider the H^2 term first.

$$Z K^{\mu\nu} = \frac{1}{N_E} \Big(\text{''} \qquad \text{''} \Big) \frac{1}{n!} \sum_{\{x\}} \sum_{t=0}^{t_{max}-1} \left(1 - \frac{\delta_{t0}}{2}\right)$$

$$\sum_{\{x\}} (\ldots) \sum_{pq} \Big[(2\Lambda)^2 \dot{x}^\mu_p(t) \dot{x}^\nu_q(0)$$

$$+ (2H)^2 \sigma_{\alpha_p}(t) \sigma_{\alpha_q}(0) \left(x^\mu_p(t) + x^\mu_p(t+1)\right)$$

$$\left(x^\nu_q(0) + x^\nu_q(1)\right)$$

$$+ (2\Lambda)(2H) \Big(\dot{x}^\mu_p(t) \sigma_{\alpha_q}(0) \left(x^\nu_q(0) + x^\nu_q(1)\right)$$

$$+ \dot{x}^\nu_q(0) \sigma_{\alpha_p}(t) \left(x^\mu_p(t) + x^\mu_p(t+1)\right)\Big)$$

= usual term

$$\sum_{t_0=0}^{t_{max}-1}$$

$$\sum_{\tau=0}^{\tau_{cutoff}} \frac{1}{t_{max}-\tau} \sum_{t_0=0}^{t_{max}-\tau-1} G_{t_0,\tau}$$

$$t_{max} \Big\langle \sum_\tau \quad \Big\rangle_{t_0}$$

$$ZK^{\mu\nu} = \frac{1}{N_E} \sum_{n=0}^{N} \frac{1+(-1)^n}{2} e^{-2\Lambda n} \sum_{\underline{w}} \frac{1}{n!} \sum_{\{x\}} \sum_{\{\mu\}} e^{H\sum_{\langle pq\rangle} \eta_{pq}\mu_p\mu_q}$$

$$\cdot \sum_{t=0}^{t_{max}-1} \left(1 - \frac{\delta_{t0}}{2}\right) \sum_{ij} x_i^{\mu} x_j^{\nu}$$

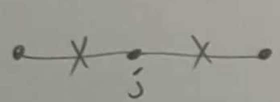
$$\cdot \Big((2\Lambda)^2 \dot{\Theta}_i(t)\dot{\Theta}_j(0) + (H)^2 \dot{m}_i(t)\dot{m}_j(0) + (2\Lambda)(H)\big(\dot{\Theta}_i(t)\dot{m}_j(0) + \dot{\Theta}_j(0)\dot{m}_i(t)\big) \Big)$$

Okay ... $m_j(t) := Z_j - \sum_{\alpha \in \partial^+ j} \sigma_\alpha(t)$

$\Rightarrow$ finite diff : $\dot{m}_j(t) \overset{\delta t=1}{:=} \sigma_{\alpha_j}\big(2\delta_{j\in w^t(x)} + 2\delta_{j\in w^{t+1}(x)}$

chosen spin to go in/out along...

$\delta_{j\in w^t(x)} = \Theta_j(t)$, $\quad \delta_{j\in w^{t+1}(x)} = \Theta_j(t+1)$

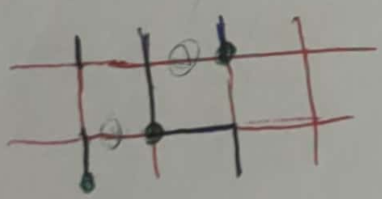


$\dot{m}_j = 0$

$\dot{m}_j \neq 0$
$\dot{m}_k \neq 0$

$\dot{m}_j = -\dot{m}_k \quad \because$ opposite spins.

$(\sum_j \dot{m}_j \overset{\delta t=1}{=} \Delta M = 0)$



@ Low T, $\sigma_{\alpha_j} = +1$ almost always

Let's not fix that yet though.

$$\Theta_j = \sum_p \delta_{j = w^t(x_p)}$$

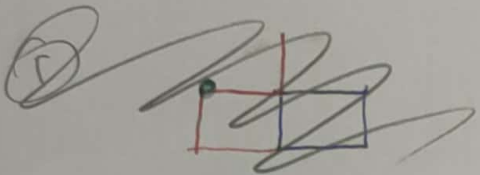
Let's look @ that again . . .

$$\epsilon_i(t) = (2\lambda)\,\theta_i + (2h)\,m_i$$

$$\Rightarrow \dot{\epsilon}_i(t) = (2\lambda)\,\dot{\theta}_i(t) + (2h)\,\dot{m}_i(t)$$

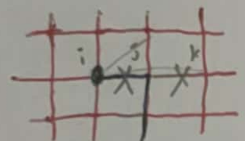
$$= (2\lambda)\,\underset{\substack{\| \\ (\theta_i(t+1)-\theta_i(t))}}{\dot{\theta}_i(t)} + (2h)\,\sigma_{(\text{flip}\in\partial^+ i)}(t)\,\big(\theta_i(t) + \theta_i(t+1)\big)$$

& individually consider diff. moves . . .

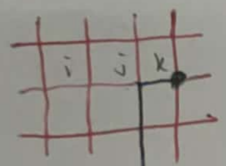


$$\frac{1}{2}\dot{\epsilon}_i = \theta_i(t+1)\big(h\,\sigma_{\text{in/out}} + \lambda\big) + \theta_i(t)\big(h\,\sigma_{\text{in/out}} - \lambda\big)$$

(1)



time t



$t+1$

$$|x_i(t) - x_i(0)|^2 \in \{0, \sqrt{2}, 2\}$$

$\Rightarrow$ on avg, $|\cdot\cdot| \sim 1 + 1/\sqrt{2}$ ~ 1.707

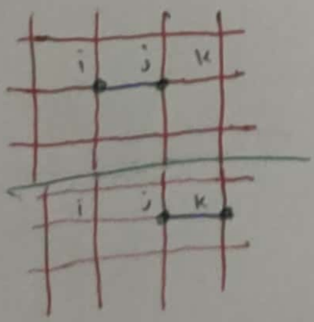
$$\dot{\epsilon}_i = 2\lambda(0-1) + 2h(-1)(1+0)$$
$$= -2\lambda - 2h$$

$$\dot{\epsilon}_j = 2\lambda(0-0) + 2h(\ldots)(0+0)$$
$$= 0$$

$$\dot{\epsilon}_k = 2\lambda(1-0) + 2h(+1)(0+1)$$
$$= 2\lambda + 2h$$

$\Rightarrow \exists$ current . . .

(2)



$$\dot{\epsilon}_i = 0 + 2(-h-\lambda) = -2h - 2\lambda$$
$$\dot{\epsilon}_j = 2(h+\lambda) + 2(-h-\lambda) = 0$$
$$\dot{\epsilon}_k = 2(h+\lambda) - 0 = 2h + 2\lambda$$

$\Rightarrow$ SAME current contrib^{n}!

For case 1 in this limit, we'll always have that the move must be undone on the next step, so even though the below still holds, the current fluctuat$^{\text{n}}$s will be small?

For 2, after n steps we'll have $\sum_{m=1}^{n} \dot{\epsilon}_i(t_m) = 0 \quad \forall i$ except the endpoints.

$$\rightsquigarrow \quad \dot{\epsilon}_{start}(t_1) = -2(h+\lambda)$$
$$\dot{\epsilon}_{end}(t_n) = +2(h+\lambda)$$

If this isn't thermal transport, what is?

$$\underline{J}(t) = \overbrace{\sum_i \underline{x}_i \dot{\epsilon}_i(t)}^{\dot{p}_i(t)}$$

↳ Current of mot$^{\text{n}}$ @ time t.

$$\langle 2\underline{J}(t)\cdot\underline{J}(0)\rangle = \langle|\underline{J}(t)-\underline{J}(0)|^2\rangle - 2\langle|\underline{J}(t)|^2\rangle^{\nearrow 0}$$

$$\Rightarrow \langle\underline{J}(t)\cdot\underline{J}(0)\rangle = +\frac{1}{2}\langle|\underline{J}(t)-\underline{J}(0)|^2\rangle$$
$$= +\frac{1}{2}\left\langle\left|\sum_i \underline{x}_i(\dot{\epsilon}_i(t)-\dot{\epsilon}_i(0))\right|^2\right\rangle$$
$$= +\frac{1}{2}\langle|+2(h+\lambda)[\underline{x}(t)-\underline{x}(0)]|^2\rangle$$
$$= -2(h+\lambda)^2\underbrace{\langle|\underline{x}(t)-\underline{x}(0)|^2\rangle}_{\sim\begin{cases} 2\tilde{D}t & (2) \\ \underset{\sim O(1)t^0}{\overset{?}{\text{const.}}} & (1)\end{cases}}$$

$\langle |\underline{J}(t)|^2 \rangle = 0$

I'm not convinced by Claudio's argument...

Why should the pairs <u>not</u> contribute?

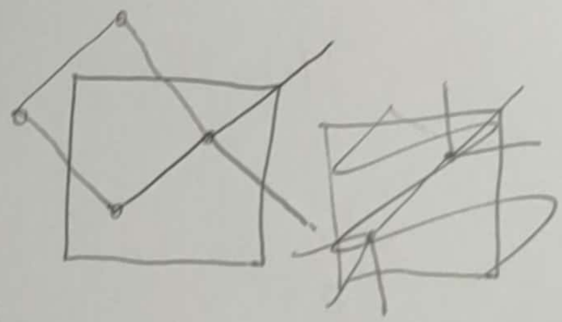
The $\dot{\epsilon}_i$'s generated are <u>exactly</u> the same!

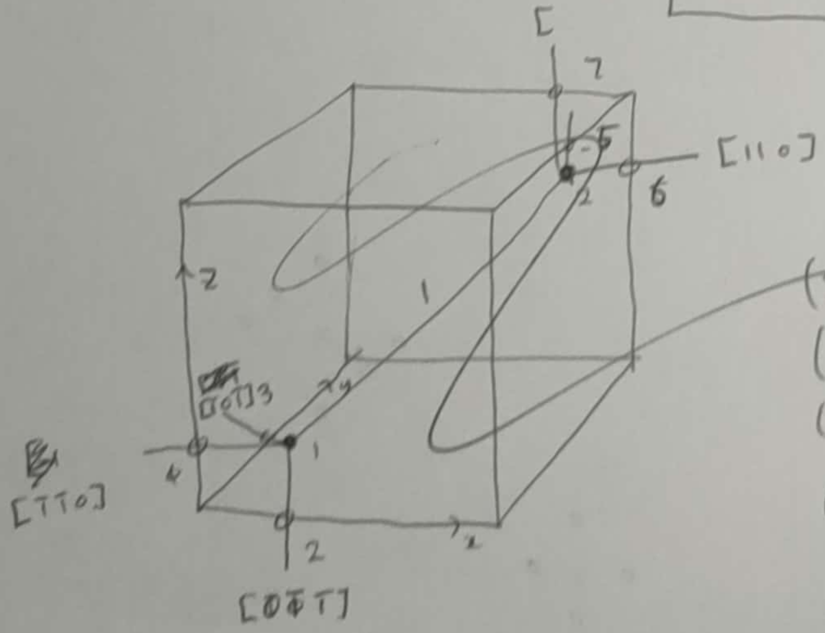
It has to be a density thing, but I'm not convinced.

And from my simulations allowing/disallowing pair moves, the <u>form</u> of $D(\tau)$ is seemingly retained.

Is the drop in $D(\tau)$ just $\because$ diff. peak locat^ns for C & K? If so - why?

Am I calculating sth wrong?





(2, 2)	$0\bar{1}\bar{1}$
(3, 2)	$\bar{1}0\bar{1}$
(4, 2)	$\bar{1}\bar{1}0$
(5, 1)	011
(6, 1)	110
(7, 1)	101

* Next Steps

↳ Different 6-vertex model lattice?

Can we do that?

$$\tilde{A}_i = \Big(\sum_{\alpha \in \partial^+ i} \sigma_\alpha\Big)^2 \longrightarrow \text{Enforces an equal in/out rule.}$$

On ~~triangular~~ any lattice,

$$\tilde{A}_i = \sum_{\alpha,\beta \in \partial^+ i} \sigma_\alpha \sigma_\beta$$

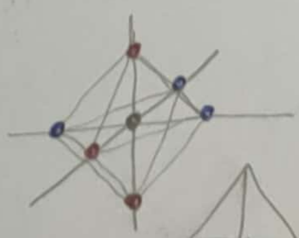
$$= \sum_{\alpha \in \partial^+ i} \cancelto{1}{\sigma_\alpha^2} + 2 \sum_{\substack{\alpha \in \partial^+ i \\ \beta \in \partial^+ i \\ \alpha \neq \beta}} \sigma_\alpha \sigma_\beta$$

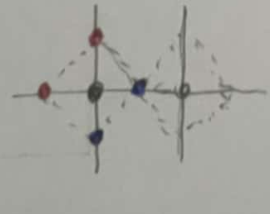
$$= z_i + 2 \sum_{\alpha \neq \beta} \sigma_\alpha \sigma_\beta$$

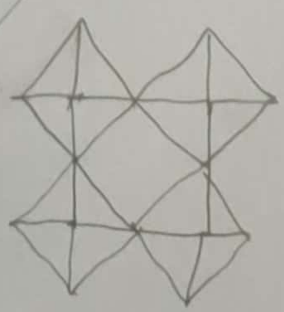
$$\Rightarrow \lambda \sum_i \tilde{A}_i = \lambda \underbrace{\sum_i z_i}_{2N_E} + 2\lambda \underbrace{\sum_i \sum_{\alpha \neq \beta} \sigma_\alpha \sigma_\beta}_{2\lambda \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta}$$

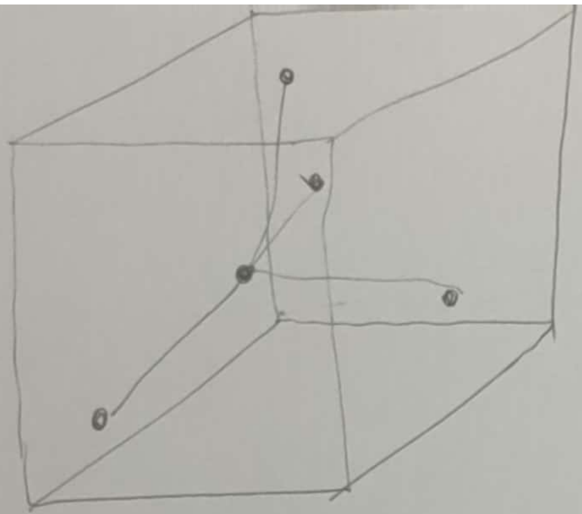
$$\Rightarrow -\beta F = \overbrace{2\Lambda}^{J.} \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta + H \sum_\alpha \sigma_\alpha .$$

etc.

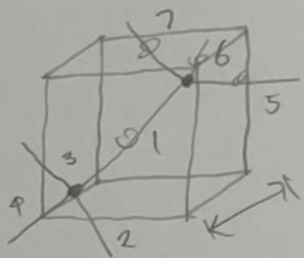








$(\frac{1}{4}, \frac{1}{4}; \frac{1}{4})$ & $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$



Vertices basis : 1 & 2.

$\underline{x} = \frac{1}{4}(\underline{1})$ $\quad \underline{x} = \frac{3}{4}(\underline{1})$

edges.: $(1 \to 1)$ $(1 \to 2)$

$(2 \to 2)$ $(3 \to 2)$ $(4 \to 2)$

$(5 \to 1)$ $(6 \to 1)$ $(7 \to 1)$

Will <u>need</u> to adjust diffusion scaling :

cube side length $c = 1$ $\Rightarrow$ $a = c \cdot \sqrt{3 \cdot ((\frac{3}{4})^2 - (\frac{1}{4})^2)}$

$$= c\sqrt{3\left(\frac{9-1}{16}\right)}$$

$$= c\sqrt{24/16}$$

$$= c\sqrt{3/2}$$

$$= c\,\frac{\sqrt{6}}{2}.$$

$\forall$ bonds!

12 8
6 4

To check: * <u>All</u> thermal tests for $h = 0$

* Diffusion <u>exponent</u>

$T = 0$:
* Spinons diffusive
* Visons confined to small/null subgraph if isolated

$T = \infty$:
* Spinons diffusive w/. collisns/blocking by other spinons/visons
* Visons move on $+-+-$ paths if isolated

Questions:
* Why don't we see subdiffusivity overall? ∵ always <u>sth</u> diffusive which dominates
* Key reason for $D \to 0$ is

Why field difficult.

Maybe say sth about what you can complete before it's done.

↳ Work in each magnetisation sector artificially.
  ↳ swap moves w/. locality
  ↳ Sectors @ least difficulty: $M = 0, 1$
  $\tau = \infty, 0$

① Count pairs & strings.

↳ Not immediately obvious <u>which</u> win ∵ entropy of string of length l → known!

vs. entropy of pairs w/. excluded volume

↳ What is the balance?

* Our locality enforcement is bad ∵ can kill a pair somewhere & create elsewhere

↳ So can $\circ\!\times\!\circ \longrightarrow \circ\!-\!\circ$

gives $D = \infty$!

↳ Pair annihilatⁿ / creatⁿ doesn't obey cont. eqn.

⇒ won't contribute ∵ carries entropy

↳ find argument for this (heat tspt in prtcl systems)

→ textbooks?

↳ facilitated process: pair collides w/. end of string ⤳ can move strings arb. far w/. these collisⁿs

⇒ ends of strings RW but w/. timescale ~ 1st passage time of pairs hitting it.

1st Passage Properties

Work a lot on these! { Sid Redner
Paul Kap... }

↳ Time taken for prtcl to return to origin if # pairs = 0.

↓

Scaling for time is known.

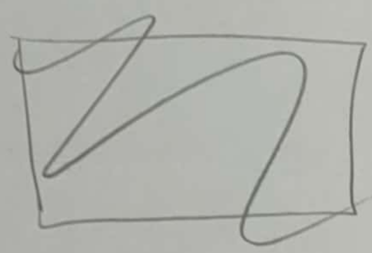
↳ May be $\sim L^2$

↳ Need to measure R^2 only for isolated ones when testing this facilitated process.

* Picking the prtd to track won't [illegible] exponent, but in that limit it's not obvious that the diffusion picture is well-defined
  ↳ Are they diffusive between collisions?
  ↳ Conductivity @ $H \neq 0$.

* $D(T)$ for a given magnetisatⁿ sector.

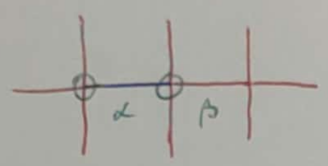
~~Suggestion: Set up a larger sim where we artificially set up:~~



Focus on this picture:
* High T approxⁿ
* Why pair creatⁿ / annihilatⁿ moves don't transport energy
  ↳
* Low T: How quickly does system settle down into motionless state?

Two Approaches

① Take $\Delta D_1 = -\Delta E^{(\beta_1)}$
$\Delta D_2 = -\Delta E^{(\beta_2)}$ excl. h-energy...



* α first: D_α provides energy for annihilatⁿ, D_β for creatⁿ

$$\Delta D_\alpha = +4\lambda \cancel{+2h}$$
$$\Delta D_\beta = -4\lambda \cancel{-2h}$$

* β first: $\Delta D_\beta = \cancel{-2h}$
$\Delta D_\alpha = \cancel{+2h}$

② $\Delta D_1 = \Delta D_2 = -\dfrac{\Delta E^{(\beta_1 \& \beta_2)}}{2}$

* α first: $\Delta D_1 = \Delta D_2 = 0$

* β " : " "

(1)

$$\epsilon_i = -\lambda A_i + 3B_i - \frac{h}{2}\sum_{\alpha\in\partial^+ i} \sigma_\alpha + \frac{1}{2}\sum_{\alpha\in\partial^+ i} D_\alpha .$$

Flip (1): σ_1

$A_0 \longrightarrow -A_0 = A_0'$

$A_1 \longrightarrow -A_1$

$Q_0 \longrightarrow Q_0 - 2\sigma_1 \Rightarrow B_0 \to B_0 - \underline{4Q_0\sigma_1 + 4}$

$Q_1 \longrightarrow Q_1 - 2\sigma_1 \Rightarrow B_1 \to B_1 - \underline{4Q_1\sigma_1 + 4}$

$\sigma_1 \longrightarrow -\sigma_1$

$$\Delta\epsilon_0^{(\sigma)} = +2\lambda A_0 + 43(1-\sigma_1 Q_0) + h\sigma_1$$

$$\Delta\epsilon_1^{(\sigma)} = +2\lambda A_1 + 43(1-\sigma_1 Q_1) + h\sigma_1$$

Flip (2): σ_2

$A_0' \longrightarrow -A_0'$

$A_2 \longrightarrow -A_2$

$Q_0' \longrightarrow Q_0' - 2\sigma_2 \Rightarrow B_0' \to B_0' + 4 - 4\sigma_2 Q_0'$

$Q_2 \longrightarrow Q_2 - 2\sigma_2 \Rightarrow B_2 \to B_2 + 4 - 4\sigma_2 Q_2$

$\sigma_2 \longrightarrow -\sigma_2$

$\Delta\epsilon_0^{(\sigma)} =$ " with A_0', Q_0'

$\Delta\epsilon_2^{(\sigma)} =$ " with A_2, Q_2.

Total energy change: (excl. demons).

$$\Delta E^{(\sigma)} = 2\lambda(A_1 + A_2) + 2h(\sigma_1+\sigma_2) + 43(4 - \sigma_1 Q_1 - \sigma_2 Q_2 - \sigma_1 Q_0 - \sigma_2 Q_0 + 2\sigma_1\sigma_2)$$

$43 \cdot 2(1-\sigma_1\sigma_2)$

$- 43(\sigma_1+\sigma_2) Q_0$

$$\overset{\sigma_1=-\sigma_2}{=} 2\lambda(A_1+A_2) + 43(2 - \sigma_1 Q_1 - \sigma_2 Q_2)$$

$$③\ +2\Delta\epsilon_0 = -2\lambda(A_2+A_1) + 4z\left(\sigma_2 Q_2 + \cancel{\sigma_2} Q_0 - 2 + \cancel{\sigma_1} Q_0 + \sigma_1 Q_1\right)$$

$$\Rightarrow \Delta\epsilon_0 = -\lambda(A_2+A_1) + 2z(\sigma_1 Q_1 + \sigma_2 Q_2 - 2)$$

$$\Delta\epsilon_1 = 2\lambda A_1 + 4z(\cancel{1} - \sigma_1 Q_1) + h\sigma_1$$
$$\cancel{+2z} - \lambda(A_0+A_1) - 2z(\cancel{1} - \sigma_1 Q_0 - \sigma_1 Q_1)$$

$$= \lambda(A_1 - A_0) - 2z(\sigma_1 Q_1 - \sigma_1 Q_0) + h\sigma_1$$
$$= \lambda(A_1 - A_0) - 2z\sigma_1(Q_1 - Q_0) + h\sigma_1$$

$$\Delta\epsilon_2 = 2\lambda A_2 + 4z(1 - \sigma_2 Q_2) + h\sigma_2$$
$$- \lambda(A_2 - A_0) + 2z(\sigma_2 Q_2 + \sigma_2 Q_0)$$
$$= \lambda(A_2 + A_0) - 2z\sigma_2(Q_2 - Q_0) + 4z + h\sigma_2$$

---

e.g. $z = 0$:

$$\Delta\epsilon_0 = -\lambda(A_1 + A_2)$$
$$\Delta\epsilon_1 = +\lambda(A_1 - A_0)$$
$$\Delta\epsilon_2 = +\lambda(A_2 + A_0)$$
$$\Rightarrow \sum \Delta\epsilon = 0 \quad \therefore \text{demons!}$$

④ Now for the currents!

~~If 3=a,~~ $\Delta E_1 = - \dot{j}_{1\to 0}$

$$\Delta E_2 = - \dot{j}_{2\to 0}$$

$$\Delta E_0 = - \dot{j}_{0\to 1} - \dot{j}_{0\to 2}$$

$$= \dot{j}_{1\to 0} + \dot{j}_{2\to 0}$$

$\Rightarrow$ $\dot{j}_{1\to 0} = \Delta E_0 + \Delta E_2$

$$\dot{j}_{0\to 2} = -\Delta E_0 - \Delta E_1$$

$$\Rightarrow \dot{j}_{1\to 0} = -\lambda(\cancel{A_2} + A_1) + 23(\sigma_1 Q_1 + \cancel{\sigma_2 Q_2} - \cancel{2})$$

$$+ \lambda(\cancel{A_2} + A_0) - 23(\cancel{\sigma_2 Q_2} - \sigma_2 Q_0) + \cancel{43} + h\sigma_2$$

$$= \lambda(A_0 - A_1) + 23(\sigma_1 Q_1 - \sigma_2 Q_0) + h\sigma_2$$

$$\dot{j}_{0\to 2} = \lambda(A_2 + \cancel{A_1}) - 23(\cancel{\sigma_1 Q_1} + \sigma_2 Q_2 - 2)$$

$$- \lambda(\cancel{A_1} - A_0) + 23(\cancel{\sigma_1 Q_1} - \sigma_1 Q_0) + h\sigma_1$$

$$= \lambda(A_2 + A_0) - 23(\sigma_2 Q_2 + \sigma_1 Q_0 - 2) + h\sigma_1$$

If I include h terms in the demons, I <u>gain</u> $\dot{j}$ indep. of h again but <u>lose</u> quantisation.

Overall,

$$\Delta \epsilon_1 = 2\lambda A_1 + 43(1-\sigma_1 Q_1) + h\sigma_1 + \tfrac{1}{2}\Delta D_1$$

$$\Delta \epsilon_2 = 2\lambda A_2 + 43(1-\sigma_2 Q_2) + h\sigma_2 + \tfrac{1}{2}\Delta D_2$$

$$\Delta \epsilon_0 = 2\lambda(A_0 + A_0') + 43(2 - \sigma_1 Q_0 - \sigma_2 Q_0') + h(\sigma_1+\sigma_2) + \tfrac{1}{2}(\Delta D_1 + \Delta D_2)$$

$$= 2\lambda(\cancel{A_0} - A_0) + 43(2 - \sigma_1 Q_0 - \sigma_2 Q_0 + 2\sigma_1\sigma_2) + h(\sigma_1+\sigma_2) + \tfrac{1}{2}(\Delta D_1 + \Delta D_2)$$

Impose $\sigma_1 = -\sigma_2$

$$\Rightarrow \Delta \epsilon_0 = -43(\sigma_1 \cancel{2} + \sigma_2)Q_0 + 0 + 0 - \frac{\Delta D_T}{2}$$

Take

$$\Delta D_1 = -2\lambda(A_0 + A_1) - 43(2 - \sigma_1(Q_0 + Q_1)) - 2h\sigma_1$$

$$\Delta D_2 = -2\lambda(A_0' + A_2) - 43(2 - \sigma_2(Q_0' + Q_2)) - 2h\sigma_2$$

$$= -2\lambda(A_2 - A_0) - 43(\cancel{2} - \sigma_2(Q_2 + Q_0) + 2\sigma_1\sigma$$

$$= -2\lambda(A_2 - A_0) + 43\,\sigma_2(Q_2 + Q_0)$$

Alt: $\Delta D_1 = \Delta D_2 = -\dfrac{\Delta E^{(\sigma)}}{2}$

$$= -\lambda(A_1 + A_2) - 23(2 - \sigma_1 Q_1 - \sigma_2 Q_2)$$

(3) Alt.

$$\Delta\epsilon_0 = -\cancel{4}\,2\lambda(A_1+A_2) -\cancel{4}\,4\mathfrak{z}(2-\sigma_1Q_1-\sigma_2Q_2)$$

$$\Delta\epsilon_2 = 2\lambda A_2 + 4\mathfrak{z}(1-\sigma_2Q_2) + h\sigma_2$$
$$\quad -\lambda(A_1+A_2) -\cancel{4}\,\cancel{2}\,\mathfrak{z}(2-\sigma_1Q_1-\sigma_2Q_2)$$
$$= \lambda(A_2-A_1) - 2\mathfrak{z}(\sigma_2Q_2-\sigma_1Q_1) + h\sigma_2$$

& $\Delta\epsilon_1 = \lambda(A_1-A_2) - 2\mathfrak{z}(\sigma_1Q_1-\sigma_2Q_2) + h\sigma_1$

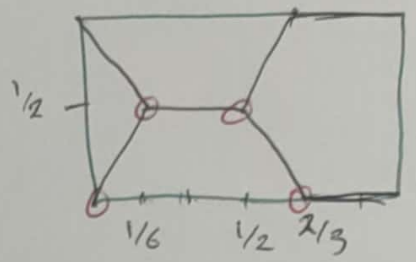
& $\sum\Delta\epsilon = 0$ ✓

This is <u>much</u> nicer!

(4) Alt. $\Delta\epsilon_1 = -\dot{j}_{1\to 0}$
$\Delta\epsilon_2 = -\dot{j}_{2\to 0} = \dot{j}_{0\to 2}$.
$\Delta\epsilon_0 = \cancel{-}\,\dot{j}_{1\to 0} - \dot{j}_{0\to 2}$

$$\Rightarrow \dot{j}_{1\to 0} = \lambda(A_2-A_1) -\cancel{+}\,2\mathfrak{z}(\sigma_2Q_2-\sigma_1Q_1) -\cancel{+}\,h\sigma_1$$
$$\dot{j}_{0\to 2} = \lambda(A_2-A_1) - 2\mathfrak{z}(\sigma_2Q_2-\sigma_1Q_1) + h\sigma_2$$

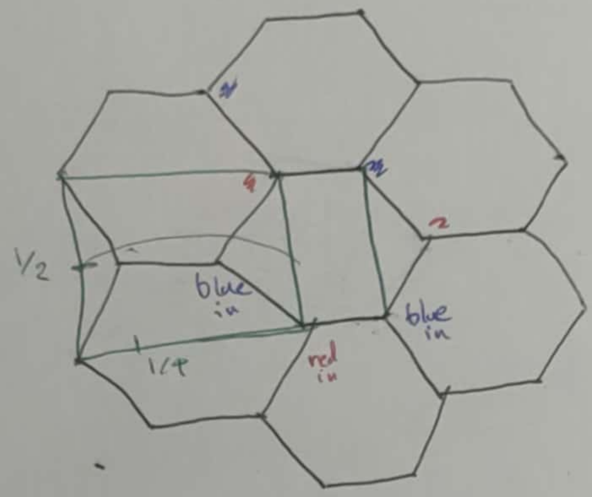
Excl. h terms, they're the same $\because\ \Delta^f Q_0 = \Delta^f A_0 = 0$ so prtcls <u>must</u> either skip or start & end @ central site.



$S_y = \sqrt{3}$

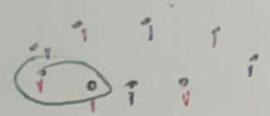
$S_x = 3$

$6 - 1/3$

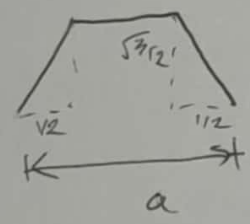


Consistent! ü.

→ All cont. loops must be of even length (bipartite).



↳ ALSO need all loops of even length to be truly consistent, inc. non-cont. ones



$S_x = 2$

$S_y = \sqrt{3}/2$

$S = 1/\sqrt{\frac{3}{16}} = \frac{4}{\sqrt{3}}$

Fine in thermodynamic limit?

Conductivity of holes <u>certainly</u> massively reduced.

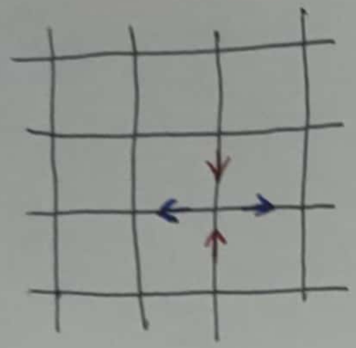
<u>BUT</u> <u>still</u> $D \to 0$ as $T \to 0$? :^

Maybe this is fine? Our approx^{n} matches it fine...

↳ In that case, we can <u>mention</u> percolation, but it doesn't affect things ~~&~~ $\therefore$ K defined in terms of VACF, <u>not</u> MSD

$\Rightarrow$ no critical percolat^{n} cluster stuff, <u>only</u> the independence of p.rtd direct^{ns} matters $\to$ isolated : move in  diamonds

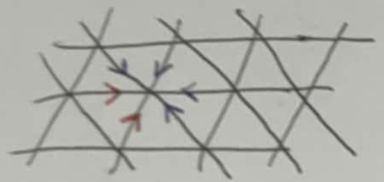
holes: diffusive $\to \delta_{\infty}$.



We can define red = in & blue = out & vice-versa on <u>each sublattice</u>

Does that still work for 3D spin ice? Sure! It's still a bipartite lattice...

<u>Wouldn't</u> work e.g. for:



Or would it? <u>No</u>

All I need is a <u>consistent</u> def$^{\underline{n}}$ for each edge $\because$ only borders 2 vertices

$\Rightarrow$ obvs doable. IF can be done consistently $\forall$ closed loops.

$\Rightarrow$ must be bipartite for mapping to work

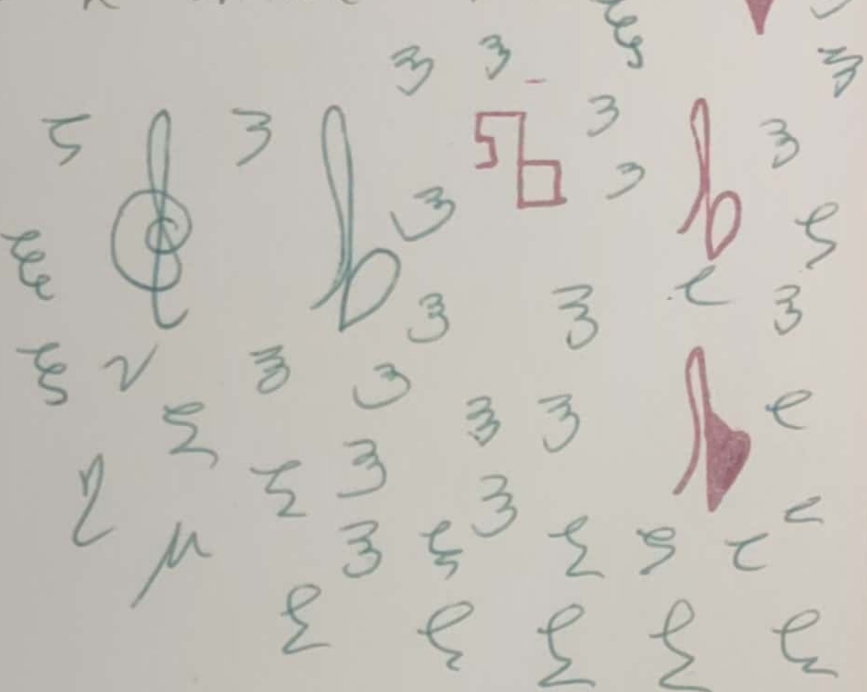
Single Pair

$E = \cancel{4\lambda} + h$

$\#\text{configs} = 2N_v \qquad S \sim \ln(\#\text{configs})$

$\Rightarrow F = E - TS = h - T\ln(2N_v)$

$\int d\mathfrak{z}\, f(\mathfrak{z})$



Chain

$E = \cancel{4\lambda} + \ell h$

$\#\text{configs} = ? = \#$ paths of length ℓ on the lattice.

I need to <u>disallow repeat</u> edges.

Let's relax that & <u>just</u> disallow <u>immediate</u> backtracking

$\rightarrow \#\text{configs} < 4 \cdot 3^{\ell-1} / \underset{\text{swap prods}}{2} \cdot \underset{\text{start point}}{N}$

$\rightarrow F = E - TS > \ell h - T\ln(2N_v \cdot 3^{\ell-1})$

$F_{pair} = h - T\ln(2N)$

$F_{string} = \ell(h-3T) - T\ln(4) + T\ln(3)$

The density goes like $e^{-F/T}$

$\Rightarrow \rho_{pair} \sim e^{\ln 2N} e^{-H} \sim (2N)e^{-H}$

$\rho_{string} \sim 3/4 \cdot e^{-(H-3)\ell}$

$\beta(F_{pair} - F_{string}) = H - \ln(2N) - \ell H + 3\ell + \ln(4) - \ln(3)$

$\beta F_{pair} = H - \ln(2N)$

$\beta F_{string} = \ell H - \ln(2N) + (1-\ell)\ln(3)$

$\Rightarrow \Delta\beta F = H(1-\ell) + (1-\ell)\ln(3)$

< 0 if $\ell > 1$!

$\ell > 1$ for strings!

So $\underline{F_{pair} < F_{string}}$ ~~always~~

$\varepsilon \mu \eta \rho \nu$

Pairs should <u>always</u> win.

$$\begin{cases} S_{string} = \ln(2N) + (\ell-1)\ln(3) \\ S_{pair} = \ln(2N) \end{cases}$$

@ low T,  __still__ should be diffusive

@ high T, convergence makes sense:

$n \sim 1/2$ $\Rightarrow$ both prtcls & holes move on + − + − paths w/. collisions

---

__Why__ would they be the same @ low T? Pair should be __diffusive!__ $\Rightarrow$ will have $D \not\to 0$ ...

UNLESS C diverges for some reason? Closed loops?

$D \underset{T \to 0}{\not\to} 0$ for pairs, surely...

Question: is the "subdiffusivity" just an artefact of small N?
Trying $L = 15 \to 50$ to check...

Hypothesis: It's because $\exists$ no pairs @ $T \to 0$ ... but that's fine for $h = 0$?

$$\beta F_{pair} \sim 4\Lambda + \ell H - \ln(2N_v) + (1-\ell)\ln(3)$$

$$\to \quad n_{pair} \sim e^{-\beta F_{pair}}$$

$$\cancel{\sim (2N_v) \cdot 3^{\ell-1} \cdot (N_v/2)} \quad \cancel{(N_v/2)}$$

$$\sim \cancel{(2N)} \cdot 3^{\ell-1} \cdot e^{-(4\Lambda + \ell H)}$$

$$n_{pair} \propto 3^{\ell-1} \cdot e^{-(4\Lambda + \ell H)}$$

$$n_{pair,\ell=1} \propto e^{-(4\Lambda + H)}$$

I'm a moron ...

Two moves:

α β γ δ

ν η μ

υ ζ

υ r

u h a

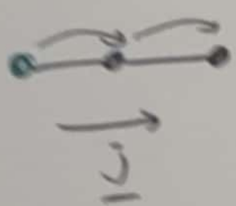
ϕ b u c

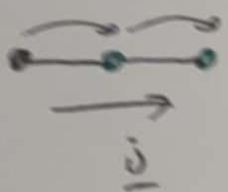
ϕ φ φ ρ q

(r, θ, ϕ)

(r, θ, ϕ)

(r, θ, φ)

①  $\sim +2\lambda$

②  $\sim \mp 2\lambda$

<u>Holes</u> move the same as prtcls?

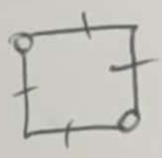
Not <u>quite</u> ∵ strings connect prtcls 

But, say, @ low T we can think of holes filling the lattice → can only move @ string ends <u>or</u> if there's a bubble of 2 prtcls.

⇒ If we were to <u>fix</u> the magnetisatⁿ or stop creatⁿ of closed loops then ...?

But no, spin swap doesn't allow motion of strings...

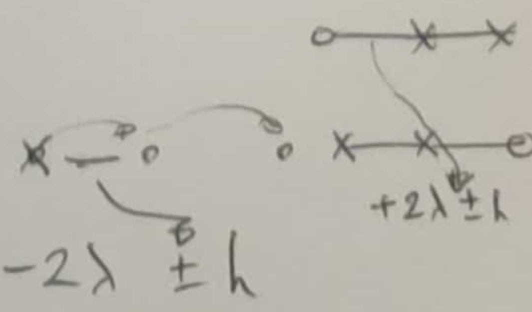
Pairs contribute <u>equally</u> to J as individual particles do! (except for $h=0$...)



$$\Delta J \in \mathbb{Z}(2\lambda \underset{\underset{2.5}{\parallel}}{\pm} h)$$

$$\Delta J = \lambda(A_j - A_i) + \frac{h}{2}(\sigma_{\beta j} - \sigma_{\beta i})$$

$$= \lambda A_j + h\sigma_{\beta j}$$



$h=0$ exceptⁿal ∴ GS will have prtd strings freely extended.

⇒ Random $A=1$ state

⇒ $M \sim 0$

⇒ Will have ~~spins~~ pairs stuck & ends diffusive along neighbouring closed ↓ loops etc.

@ $h \neq 0$ though, we don't see pairs being diffusive @ low T?

They both behave ~ the same...

Why pairs immobile @ low T?

$$\underbrace{D_{pairs} \gtrsim D_{ends}}_{\sim \frac{1}{2} D_{total}}$$

To me, this <u>doesn't</u> imply subdiffusivity. Instead it implies that the cause is just C peaking @ low T ∴ heat capacity stored in closed loops as well as stretched prtd strings.

Pair Diffusion vs. Isolated diffusion

* For pairs @ $T \approx 0$, merely need to pick one of the endpoints

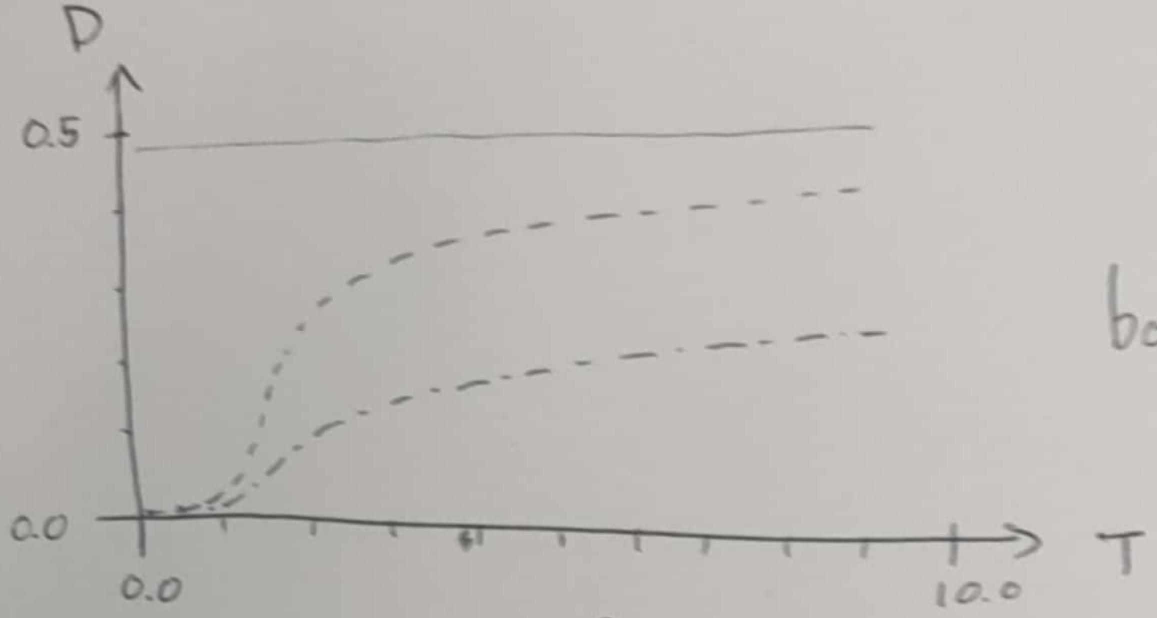
  ⇒ on avg. will happen twice to a given pair, once for each side?

  ⇒ On avg. goes → then ← ?

* For isolated ends of strings, we have almost 100% percolatn & ~~each site is hit~~ need to pick site next to endpoint to wiggle

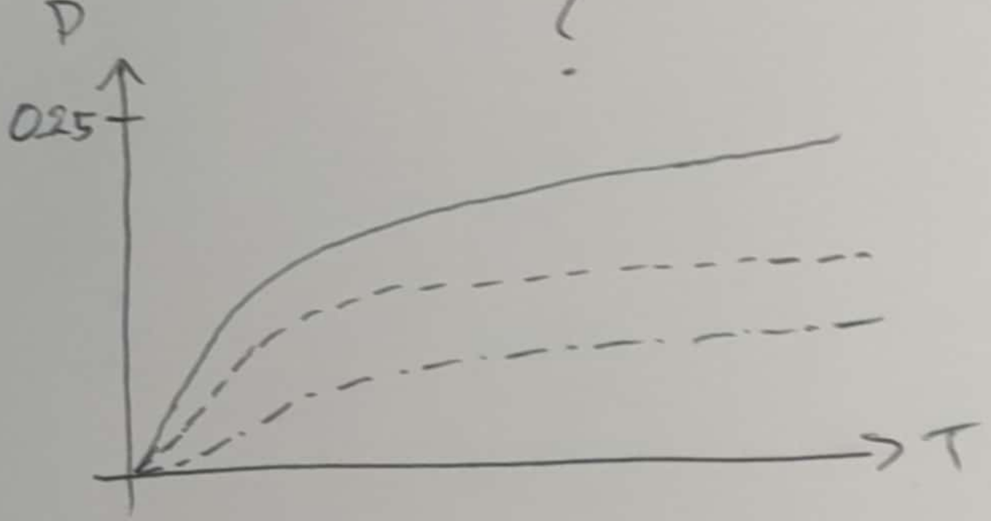
  ⇒ probability should be the same...

$h = 0$
$= 1/2$
$= 1$

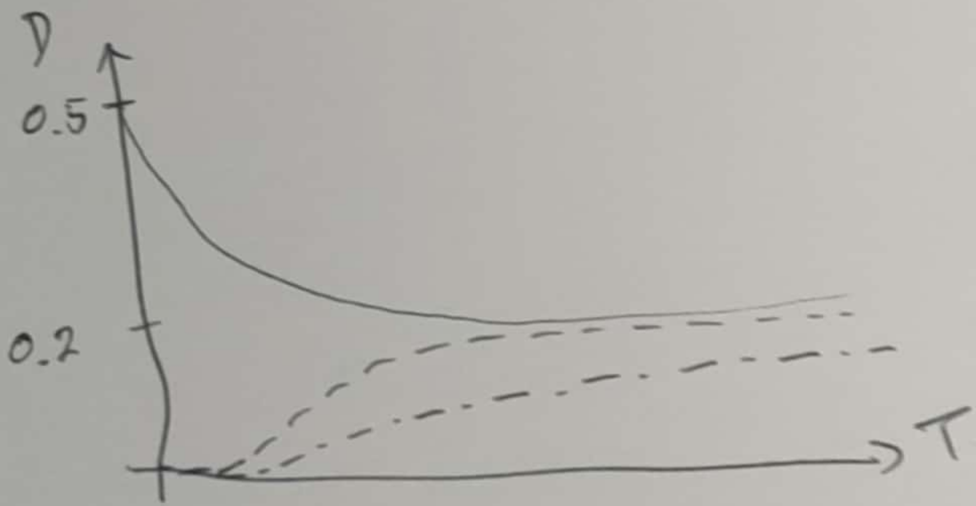


both

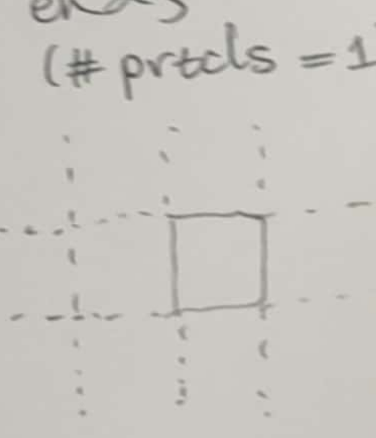
?



pairs
(# prtcls = 2)



ends
(# prtcls = 1)



3 no
2 pair
1 single
0 no.

In our shitty model,

$$n(\Lambda) = \frac{1}{2}(1-\tanh(\Lambda))$$

$$\cancel{C = \frac{1}{2T^2}\left(2h^2 \operatorname{sech}^2(H)\right)}$$

$$C = H^2 \operatorname{sech}^2(H) + \frac{1}{2}\Lambda^2 \operatorname{sech}^2(\Lambda)$$

$$K = (2\Lambda)^2 \cdot \underbrace{n(\Lambda) \cdot \frac{1}{2} \cdot (1-n(\Lambda))}_{= \frac{1}{4}(\operatorname{sech}^2(\Lambda))} \frac{1-M(H)}{2}$$

$$= \Lambda^2 \operatorname{sech}^2(\Lambda) \cdot \frac{1-M(H)}{4}$$

γ

$\alpha \; \beta \; \gamma \; \delta$

$$\underline{K \sim \left((2\Lambda)^2 n(\Lambda) + H^2 m(\Lambda)\right)}$$

① 3D Spin Ice Results

② Edge resampling?

③ Why pairs don't contribute to J?

Or seemingly don't...

Energy of a pair is $4\lambda + h$

$\Rightarrow$ # ~~quasi~~ pairs $\sim N e^{-(4\lambda+h)/T}$

For # pairs ~ 1, ~~k~~ $N/2 \sim e^{(4\lambda+h)/T}$

$$\Rightarrow T = \frac{4\lambda + h}{\ln(N/2)}$$

$n \sim \tanh(\Lambda + H/4)$

$\Rightarrow Nn < 1$?

* Whyyy pairs don't contribute (?) to thermal conductivity?

They should!

↳ Pairs win density battle, $\beta F_{pair} < \beta F_{string}$ $\forall H$.

* Edge resampling problem?

* Does the "fixed" error analysis definitely work @ low T?

Still need:

* 3D Spin Ice. Results.

↳ Is $\cancel{1/N_E}$ $1/|E|$ for K right?

No! Need to fix!

$$K \propto \frac{1}{V} \qquad C \propto \frac{1}{N_E}$$

$$V = \prod L_i = \frac{\text{length (vertices)}}{\text{size (basis)}}$$

$$= |V|/|B|$$

$$= |V|/8$$

3D spin ice.

N_E → Correct.

& also need to correct for bond lengths:

$$b = \sqrt{3 \cdot \left(\tfrac{1}{4}\right)^2}\, a$$

$$= \frac{\sqrt{3}}{4}\, a \qquad (a = 1)$$

At the very least I can confirm scalings of exponents & overall trends, though!