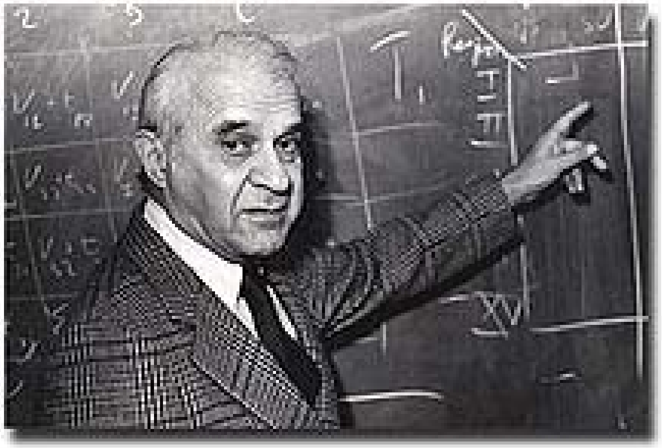
FROM LAST TIME...

Linear Algebra Primer

- Matrix Inverses
- Eigenvalues and Eigenvectors
- Jordan Canonical Form



$$A^{-1} = \frac{\mathrm{adj}(A)}{|A|} \quad = \frac{[c_{ij}]^T}{|A|}$$

$$\frac{Y(s)}{U(s)} = \frac{C\,\mathrm{adj}(sI - A)\,B}{|sI - A|} + D$$

EIGENVALUES AND EIGENVECTORS

SOLUTION OF LTI STATE EQUATIONS

Topics

- State Transition Matrix
- Free Response
- Forced Response

At the end of this section, students should be able to:

- Calculate the state transition matrix for a system.
- Calculate the free response of a system.
- Calculate the forced response of a system.

PREVIOUSLY, WE USED INVERSE LAPLACE TO SOLVE FOR SYSTEM FREE RESPONSE

function of IC

$$Y(s) = \cancel{\frac{N_G(s)}{D_G(s)}U(s)} + \frac{F(s)}{D_G(s)}$$

$$\Big\Downarrow \; \mathcal{L}^{-1}$$

$$y(t) = \text{Free Response}$$

We need an different process for state-space systems

- State ~~Transformation~~ Transition Matrix

START WITH THE FREE RESPONSE ($u = 0$) OF A STATE-SPACE SYSTEM

$$\dot{x} = Ax + \cancel{Bu}$$

- Assume $x(t)$ is a polynomial function of t: MaClausin Series (Taylor series about 0)

$$x(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_k t^k + \cdots$$

- Then it follows that $\dot{x}(t)$ is also a polynomial function of t:

$$\dot{x}(t) = \gamma_1 + 2\gamma_2 t + 3\gamma_3 t^2 + \cdots + k\gamma_k t^{k-1} + \cdots$$

- Substitute $x(t)$ and $\dot{x}(t)$ into our free response:

$$\gamma_1 + 2\gamma_2 t + 3\gamma_3 t^2 + \cdots + k\gamma_k t^{k-1} + \cdots = A(\gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_k t^k + \cdots)$$

$\dot{x} = Ax$

- Equate like powers of t and solve for coefficients.

$$x(0) = \gamma_0$$

$$\gamma_1 = A\gamma_0$$

$$\gamma_2 = \frac{1}{2}A\gamma_1 \quad = \frac{1}{2}A(A\gamma_0) = \frac{1}{2}A^2\gamma_0$$

$$\gamma_3 = \frac{1}{3}A\gamma_2 \quad = \frac{1}{3}A\left(\frac{1}{2}A^2\gamma_0\right) = \frac{1}{3\cdot 2}A^3\gamma_0$$

$$\vdots$$

$$\gamma_k = \frac{1}{k!}A^k\gamma_0$$

$$x(t) = e^{At}x(0)$$

$$x(t) = \boxed{\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{k!}A^kt^k + \cdots\right)}x(0)$$

Identity matrix

Matrix Exponential e^{At}

STATE TRANSITION MATRIX

For

$$\dot{x} = Ax$$

$\rightarrow x(t)$

The solution can be written in terms of the $n \times n$ state transition matrix $\Phi(t)$:

$$x(t) = \Phi(t)x(0)$$

$$\boxed{\Phi(t) = e^{At}}$$

$\dot{x} = \dot{\Phi}(t)x(0) = Ae^{At}x(0) = A\Phi(t)x(0) = Ax(t)$

which has the following properties:

$$\dot{\Phi}(t) = A(t)\Phi(t)$$

$$\Phi(0) = I$$

$$\Phi^{-1}(t) = \Phi(-t)$$

DR. MYNDERSE, HOW DO I COMPUTE THE STATE TRANSITION MATRIX?

1. Series approximation

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$
$$= \sum_{i=0}^{\infty} \frac{1}{i!}A^it^i$$

Tedious

DR. MYNDERSE, HOW DO I COMPUTE THE STATE TRANSITION MATRIX?

2. If A is diagonalizable

$$e^{At} = Te^{\lambda t}T^{-1} = T\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

$A = T\lambda T^{-1} \longrightarrow e^{At} = Te^{\lambda t}T^{-1}$

$\longrightarrow A^2 = T\lambda T^{-1} T\lambda T^{-1}$

$= T\lambda^2 T \ldots$

DR. MYNDERSE, HOW DO I COMPUTE THE STATE TRANSITION MATRIX?

3. If A can be put in Jordan canonical form:

$$e^{At} = Te^{Jt}T^{-1} = T\begin{bmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_n t} \end{bmatrix}T^{-1}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{1}{2}t^2 e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & 0 & e^{\lambda_i t} \end{bmatrix}$$

DR. MYNDERSE, HOW DO I COMPUTE THE STATE TRANSITION MATRIX?

4. Laplace transform approach:

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

$A^n = 0$

$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n$

$A = X + N$

$e^{At} = e^{Xt}e^{Nt}$

Seperat from this slide

$AN = NA$

$\dot{x} = Ax \rightarrow \mathcal{L} \rightarrow sX(s) - x(0) = AX(s)$

$(sI - A)X(s) = X(0)$

$\mathcal{L}^{-1} \quad X(s) = (sI - A)^{-1}X(0)$

LET'S REVISIT AN EXAMPLE FROM LAST TIME

$$\dot{x} = Ax$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\lambda_1 = -1, \qquad x^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = -2, \qquad x^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- Solve for e^{At} using diagonalization
- Solve for e^{At} using Laplace

FIRST, DIAGONALIZE TO GET e^{At}

$$T = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$e^{At} = Te^{\lambda t}T^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

NEXT, USE LAPLACE TO GET e^{At}

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

$$\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

$$\frac{s+3}{(s+2)(s+1)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1} = 2e^{-t} - e^{-2t}$$

FINALLY, USE THE STATE TRANSITION MATRIX TO DETERMINE THE FREE RESPONSE

$$x(t) = e^{At}x(0)$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2x_{10} + x_{20})e^{-t} - (x_{10} + x_{20})e^{-2t} \\ -(2x_{10} + x_{20})e^{-t} + 2(x_{10} + x_{20})e^{-2t} \end{bmatrix}$$

Notice that $x_1(t)$ and $x_2(t)$ are linear combinations of the responses associated with the eigenvalues.

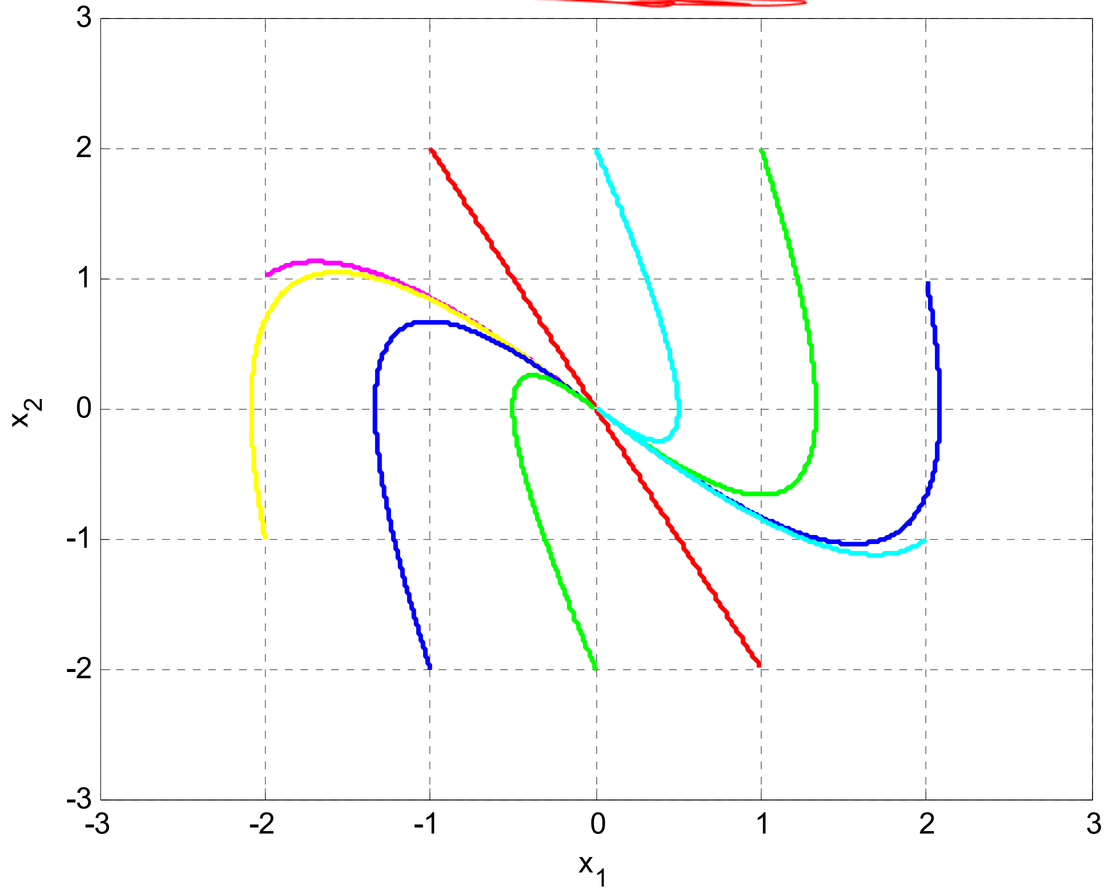
STABILITY CONDITION: FREE RESPONSE CONVERGES TO ZERO

$x(t) = e^{At} x(0) \to 0$

$e^{At} \to 0$ as $t \to \infty$

$\forall \ \mathrm{Re}[\lambda_i] < 0$

Stable Phase Portrait



HOW DO WE SOLVE FOR THE FORCED RESPONSE ($u \neq 0$)?

$$\dot{x} = Ax + Bu$$

$$\dot{x} - Ax = Bu$$

$$e^{-At}\left[\dot{x} - Ax\right] = e^{-At}Bu$$

$$\underbrace{e^{-At}\dot{x} + \underbrace{e^{-At}(-A}_{\frac{d}{dx}\left[e^{-At}\right]}x)}_{\frac{d}{dx}\left[e^{-At}x\right]} = e^{-At}Bu$$

$$e^{-At}x(t) - e^{At}x(0) = \int_0^t e^{-At}Bu\,dt$$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$ (Forced responce)

WHAT IF THE INITIAL TIME IS NOT ZERO ($t_0 \neq 0$)?

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathrm{A}(t-\tau)} Bu(\tau)d\tau$$

The proof is omitted for brevity. You would need to use the following fact:

$$\frac{d}{dt}\int_{k(t)}^{g(t)} H(t,\tau)d\tau = H(t,\tau)\Big|_{\tau=g(t)} \dot{g}(t) - H(t,\tau)\Big|_{\tau=k(t)} \dot{k}(t) + \int_{k(t)}^{g(t)} \frac{\partial}{\partial t} H(t,\tau)d\tau$$

LET'S REVISIT THE SAME EXAMPLE: FIND $x(t)$ FOR A UNIT STEP INPUT AND $x(0) = 0$:

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u \qquad u = 1$$

$$x(t) = \cancel{\Phi(t)x(0)} + \int_0^t \overbrace{\Phi(t-\tau)}^{e^{At}} Bu(\tau)d\tau$$

$$\int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right]_0^t \\ \left[-e^{-(t-\tau)} + e^{-2(t-\tau)} \right]_0^t \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ -1 + 1 + e^{-t} - e^{-2t} \end{bmatrix}$$

if $y = x_1$

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

COMING UP…

Controllability

- Definition of Controllability
- Controllable Canonical Form
- Controllable Decomposition
- Stabilizability

State Feedback Controller Design

- State Feedback Regulator
- Ackermann’s Formula