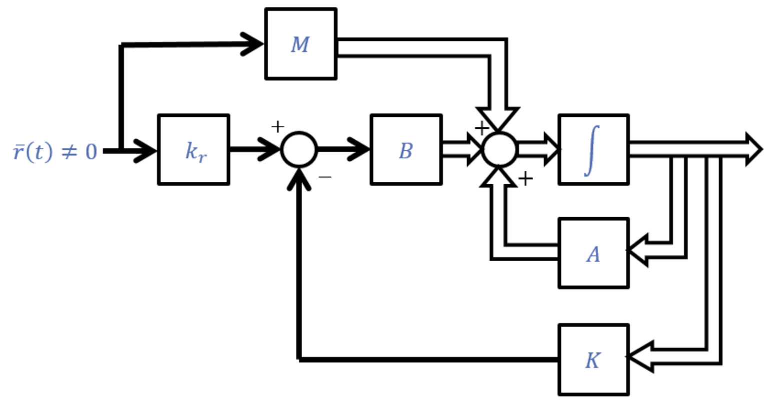
FROM LAST TIME…

Tracking and integral control

- Tracking Systems
- State Feedback with Integration



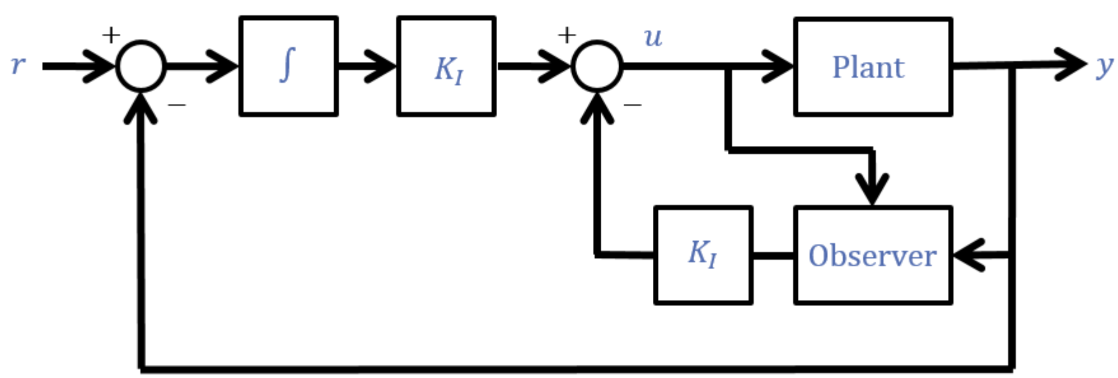
$\dot{x} = (A - BK)x$

$\dot{x} = (A - BK)x + (M + Bk_r)\bar{r}$

Same CL poles
(eigenvalues of $A - BK$)

$u = -Kx = K_I \int (r - y)dt$

$-\dot{x} = r - y$



OPTIMAL CONTROL

Topics

- Calculus of Variations
- Optimal Control
- Linear Quadratic Regulator

At the end of this section, students should be able to:

- Design state feedback controllers using the LQR method.

K = Lqr(sys, ...)

ASIDE: OPTIMAL CONTROL ORIGINATED WITH THE BRACHISTOCHRONE PROBLEM

BRACHISTOCHRONE PROBLEM

HOW DO WE DESIGN THE "BEST" CONTROLLER FOR A GIVEN APPLICATION?

1. Specify a performance index (objective function)
2. Optimize controller using performance index

This is called optimal control

OPTIMAL w.r.t THE "COST FUNCTION"

CAPABILITIES / PERFORMANCE

PERFORMANCE INDICES TYPICALLY INVOLVE AN INTEGRAL EXPRESSION

$$J = \int_{t_i}^{t_f} (y_d - y)^2 dt$$

MAKES POSITIVE

Integral of square error

$$J = \int_{t_i}^{t_f} (x_d - x)^T Q (x_d - x) dt$$

WEIGHTING FUNCTION

State error penalty

$$J = \int_{t_i}^{t_f} u^T R u \, dt$$

Control effort penalty

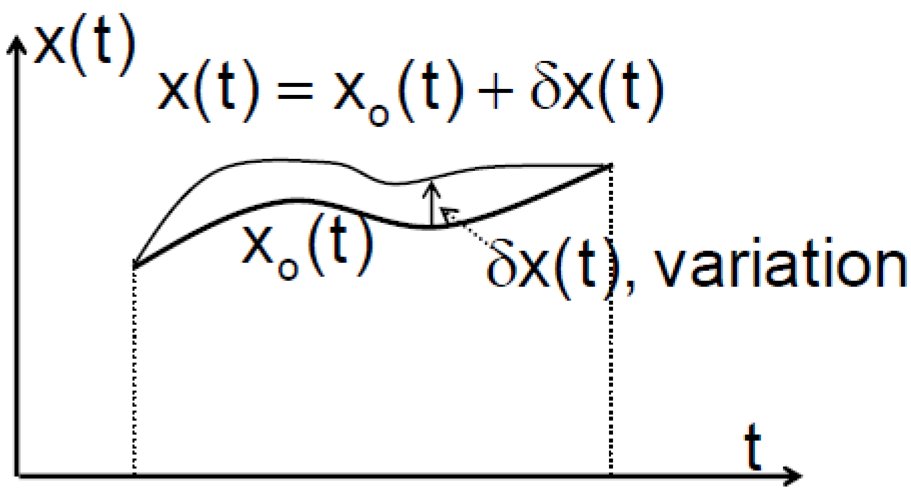
IN GENERAL, WE WANT TO FIND A SUFFICIENTLY SMOOTH FUNCTION $x_o(t)$ THAT MINIMIZES A PERFORMANCE INDEX

"O" DENOTES OPTIMAL STATE TRAJECTORY

$$J(x(t)) = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt$$

This leads to calculus of variation.

CONSIDER SCALAR CASE WITH FIXED END-POINTS



t_i t_f

$x(t_i) = 0$
$x(t_f) = 0$

VECTOR → SCALAR

The variation of the functional, J, is given by

$$\Delta J = J(x) - J(x_o) = J(x_o + \delta x) - J(x_o)$$

NON-OPTIMAL OPTIMAL

$$\Delta J = J(x) - J(x_o) = J(x_o + \delta x) - J(x_o)$$

$$= \cancel{J(x_o)} + \frac{\partial J}{\partial x}\Big|_{x=x_o} \Delta x + \frac{\partial^2 J}{\partial x^2}\Big|_{x=x_o} (\Delta x)^2 + \cdots$$

$$- \cancel{J(x_o)}$$

$$\Delta J = \frac{\partial J}{\partial x}\Big|_{x=x_o} \Delta x^2 + \frac{\partial^2 J}{\partial x^2}\Big|_{x=x_o} (\Delta x)^2 + \cdots$$

$$\approx \frac{\partial J}{\partial x}\Big|_{x=x_o} \Delta x$$

For J to have a minimum at x_o, it is necessary for the first variation to be zero:

$$\delta J = \frac{\partial J(x)}{\partial x}\Big|_{x=x_o} \delta x = 0$$

CONSIDER THE FIRST VARIATION OF THE PERFORMANCE INDEX

$$\Delta J = J\big(x_o(t) + \delta x(t)\big) - J\big(x_o(t)\big)$$

$$= \int_{t_i}^{t_f} [L(x_o(t) + \delta x(t), \dot{x}_o(t) + \delta \dot{x}(t), t) - L(x_o(t), \dot{x}_o(t), t)]dt$$

First variation:

$$\delta J = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right] dt$$

$$\int \underbrace{\frac{\partial L}{\partial \dot{x}}}_{u} \Delta \underbrace{\dot{x}\,dt}_{dv}$$

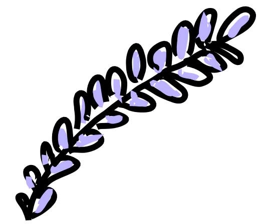
Integrate by parts:

$$= \overset{0}{\cancel{\frac{\partial L}{\partial \dot{x}} \Delta x \Big|_{t_i}^{t_f}}} - \int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \Delta x\, dt$$

$$\delta J = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x\; dt$$

ASIDE: THE FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS APPLIES HERE

FLOCOV



If M is continuous and

$$\int_a^b M(x)h(x)dx = 0$$

for all infinitely differentiable $h(x)$ then

$$M(x) = 0$$

on the open interval (a, b)

APPLYING THE FLOCOV TO OUR CASE:

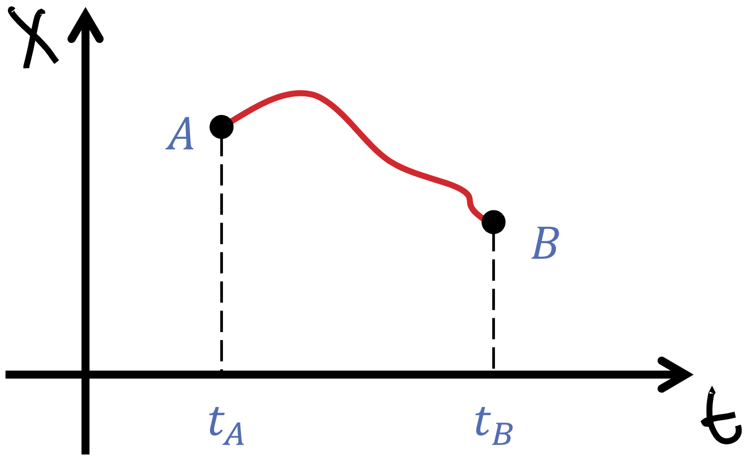
$$\int_{t_i}^{t_f} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x \, dt = 0 \quad \Longrightarrow \quad \frac{\partial L}{\partial x} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = 0$$

For the vector case:

$$x^T = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

$$\Rightarrow \quad \underbrace{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_i}\right] = 0,}_{\text{EULER-LAGRANGE EQ}} \qquad i = 1,2,\dots n$$

AS AN EXAMPLE, FIND THE SHORTEST PATH BETWEEN TWO POINTS IN A PLANE



$$ds = \sqrt{(dt)^2 + (dx)^2}$$

$$= \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \sqrt{1 + \dot{x}^2}\, dt$$

$$L = \sqrt{1 + \dot{x}^2}$$

$$J = \int_{t_A}^{t_B} ds = \int_{t_A}^{t_B} \sqrt{1 + \dot{x}^2}\, dt$$

(WHERE t IS time REP. IN TERMS OF SPACE

APPLY THE EULER-LAGRANGE EQUATION:

$L = \sqrt{1 + \dot{x}^2}$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\cancel{2}\,\dot{x}}{\cancel{2}\sqrt{1+\dot{x}^2}}$$

$$0 - \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0$$

MUST BE A CONSTANT

$$\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = C_0 \longrightarrow$$

$$\dot{x} = \frac{C_0}{\sqrt{1-C_0^2}} = C_1$$

$$\dot{x} = C_1$$

SLOPE IS CONSTANT

→ LINE

$x(t) = C_1(t) + C_2$ WHERE C_2 IS CONSTANT

OPTIMAL PATH IS A STRAIGHT LINE

CONSIDER THE PROBLEM OF OPTIMAL CONTROL

Given a plant is described by:

$$\underbrace{\dot{x} = f(x, u, t)}_{\text{NON-LTI}}$$

$$\dot{X} = \underbrace{AX + BU}_{\text{LTI}}$$

and a performance Index:

$$J = \int_{t_i}^{t_f} L(x, u, t)dt$$

We seek the optimal control $u(t)$ or control law $u(x)$ that will take x from $x(t_i)$ to $x(t_f)$ and minimize J.

OPTIMAL CONTROL

We could solve the plant equation for u in terms of x and $\dot{x}$, then substitute into J and solve, but this is cumbersome.

Instead, use Lagrange multipliers (λ_i) to remove dynamic constraints:

ADDITION OF CONSTRAINTS TO BE OPTIMIZED AS WELL

$$g_i(x, \dot{x}, u, t) = f_i(x, u, t) - \dot{x}_i = 0$$

$$L^*(x, \dot{x}, u, \lambda, t) = L(x, u, t) + \lambda^T(f(x, u, t) - \dot{x}) = L + \sum_{j=1}^{n} \lambda_j(f_j - \dot{x}_j)$$

x that minimizes L^* also minimizes L

NECESARRY CONDITION

THE EULER-LAGRANGE EQUATIONS STILL HOLD, BUT NOW WE MUST ALSO CONSIDER CONTROL EFFORT u

$$\frac{\partial L^*}{\partial x_i} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial \dot{x}_i}\right] = 0$$

$$\frac{\partial L^*}{\partial u_i} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial \dot{u}_i}\right] = 0$$

$$\frac{\partial L^*}{\partial \lambda_i} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial \dot{\lambda}_i}\right] = 0$$

APPLY THE EULER-LAGRANGE EQUATION TO THE STATE:

$$\frac{\partial}{\partial x_i}\left[L(x,u,t) + \sum_{j=1}^{n} \lambda_j \left[f_j(x,u,t) - \cancel{\dot{x}_j}\right]\right]$$
$$- \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left[\cancel{L(x,u,t)} + \sum_{j=1}^{n} \lambda_j \left[f_j(x,u,t) - \dot{x}_j\right]\right] = 0$$

$$\frac{\partial}{\partial x_i}\underbrace{\left[L(x,u,t) + \sum_{j=1}^{n} \lambda_j f_j(x,u,t)\right]}_{H} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left(-\lambda^T \dot{x}\right) = 0$$

APPLY THE EULER-LAGRANGE EQUATION TO THE INPUT:

$$\frac{\partial}{\partial u_i}\left[L(x,u,t) + \sum_{j=1}^{n} \lambda_j \left[f_j(x,u,t) - \dot{x}_j\right]\right]$$
$$- \frac{d}{dt}\frac{\partial}{\partial \dot{u}_i}\left[L(x,u,t) + \sum_{j=1}^{n} \lambda_j \left[f_j(x,u,t) - \dot{x}_j\right]\right] = 0$$

$$\frac{\partial}{\partial u_i}\underbrace{\boxed{\left[L(x,u,t) + \sum_{j=1}^{n} \lambda_j f_j(x,u,t)\right]}}_{H} - 0 = 0$$

CONSIDER THE DEFINITION OF H RELATIVE TO THE MULTIPLIER

$$H = L(x, u, t) + \sum_{j=1}^{n} \lambda_j f_j(x, u, t) = L + \lambda^T f$$

$$\frac{\partial H}{\partial \lambda} = 0 + f^T = \dot{x}^T$$

USING OPTIMAL METHODS WE DEFINE THE LINEAR QUADRATIC REGULATOR (LQR) DESIGN METHOD

Given:

$$\dot{x} = Ax + Bu$$
$$x(0) \text{ known}$$

Find:

u_o to minimize

SAME SIZE AS A

BASED ON INPUT

$$J = \frac{1}{2}\int_0^\infty \underbrace{[x^T Q x + u^T R u]}_{L} dt$$

WE CHOOSE THESE

All EIG ≥ 0

Q is symmetric, positive semi-definite

R is symmetric, positive definite

All EIG > 0

CONSTRAINTS

LQR SOLUTION

Step 1:

$$H = L + \lambda^T f \quad = \underbrace{\frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru}_{L} + \lambda^T \underbrace{[Ax + Bu]}_{f = \dot{x}}$$

Step 2:

$$\frac{\partial H}{\partial u_i} = 0 \quad \Rightarrow \quad Ru + B^T\lambda = 0$$

SINGLE INPUT ASSUMED SO $\neq U_i$

$$u_o = -R^{-1}B^T\lambda$$

OPTIMAL (NEED λ)

Step 3:

$$H_o = \frac{1}{2}x^T Qx + \frac{1}{2}\lambda^T BR^{-1}B^T\lambda + \lambda^T[Ax - BR^{-1}B^T\lambda]$$

Step 4:

$$\dot{x} = \frac{\partial H_o}{\partial \lambda} \qquad = Ax - BR^{-1}B^T\lambda$$

$$\dot{\lambda} = -\left(\frac{\partial H_o}{\partial x}\right)^T \qquad = -Qx - A^T\lambda$$

Let $\lambda = Px(t)$

UNKNOWN MATRIX

Step 5:

$$u_o = -R^{-1}B^TPx = -Kx$$

$$K = R^{-1}B^TP$$

THIS RESULTS IN THE ALGEBRAIC RICCATI EQUATION (ARE)

$$\lambda = Px \quad \Rightarrow \quad \dot{\lambda} = P\dot{x}$$

$$\dot{\lambda} = -Qx - \underbrace{\lambda^T A}_{\lambda^T P^T} = P(Ax - BR^{-1}B^T \underbrace{\lambda}_{Px})$$

$$PAx + A^T Px - PBR^{-1}B^T P + Qx = 0$$

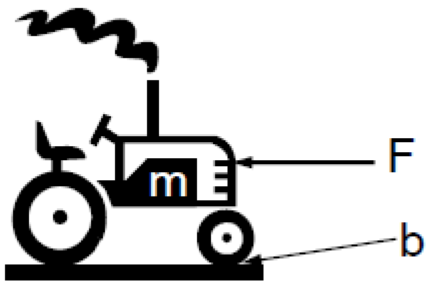
$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

A.R.E

P is symmetric and positive definite

$P > 0$ EIG > 0

CONSIDER THE ENERGY OPTIMAL DECELERATION OF A TRACTOR



$$m\dot{v} + bv = -F$$

$m = b = 1$

$x = v$ (SPEED)

$u = F$

$\dot{x} = -1x - 1u$

$A = [-1]$ $B = [-1]$

STATE

CONTROL

$$J = \frac{1}{2}\int_0^{\infty} [x^T Q x + u^T R u]dt = \frac{1}{2}\int_0^{\infty} u^2 dt$$

MIN ENERGY

$Q = 0$ $R = 1$

WEIGHT ON INPUT, HIGH PENALTY FOR LARGE INPUT

2

CONTROL EFFORT

The optimal control law is given by

$$u_o = -Kx$$

Where

$$K = R^{-1}B^T P$$

and P is the solution of

A.R.E

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$-P - P - P^2 + 0 = 0$$

$$2P + P^2 = 0$$

$A = 1$

$B = -1$

$Q = 0$

$R = 1$

$$P(P+2)=0$$

$$P=\begin{cases}0\\-2\end{cases}$$

$P=-2$ $\qquad U_0=-(1^{-1}\cdot -1\cdot -2)X=-2X$

$\dot{X}=1X$

$A=1$

$EIG=1$

$$\dot{X}=-X-(-2X)=X$$

UNSTABLE

$\dot{X}-X=0$

$X(s-1)$

$P=0$ $\qquad U_0=-(1^{-1}\cdot -1\cdot 0)X=0$

In the absence of a penalty on x, the minimum energy solution is to do nothing!

We should add a nonzero Q

ADD A NONZERO Q WEIGHTING

$$J = \frac{1}{2}\int_0^\infty [Qx^2 + u^2]dt$$

The new ARE becomes:

$$-2P - P^2 + Q = 0$$

CONSIDER THE EFFECT OF A MORE GENERAL FORMULATION

$$J = \frac{1}{2}\int_0^\infty [Qx^2 + Ru^2]dt$$

$A = -1$

$B = -1$

$U_0 = -R^{-1} B^T P X$

$B = -1$

$= R^{-1} P X$

$$u = \frac{P}{R}x$$

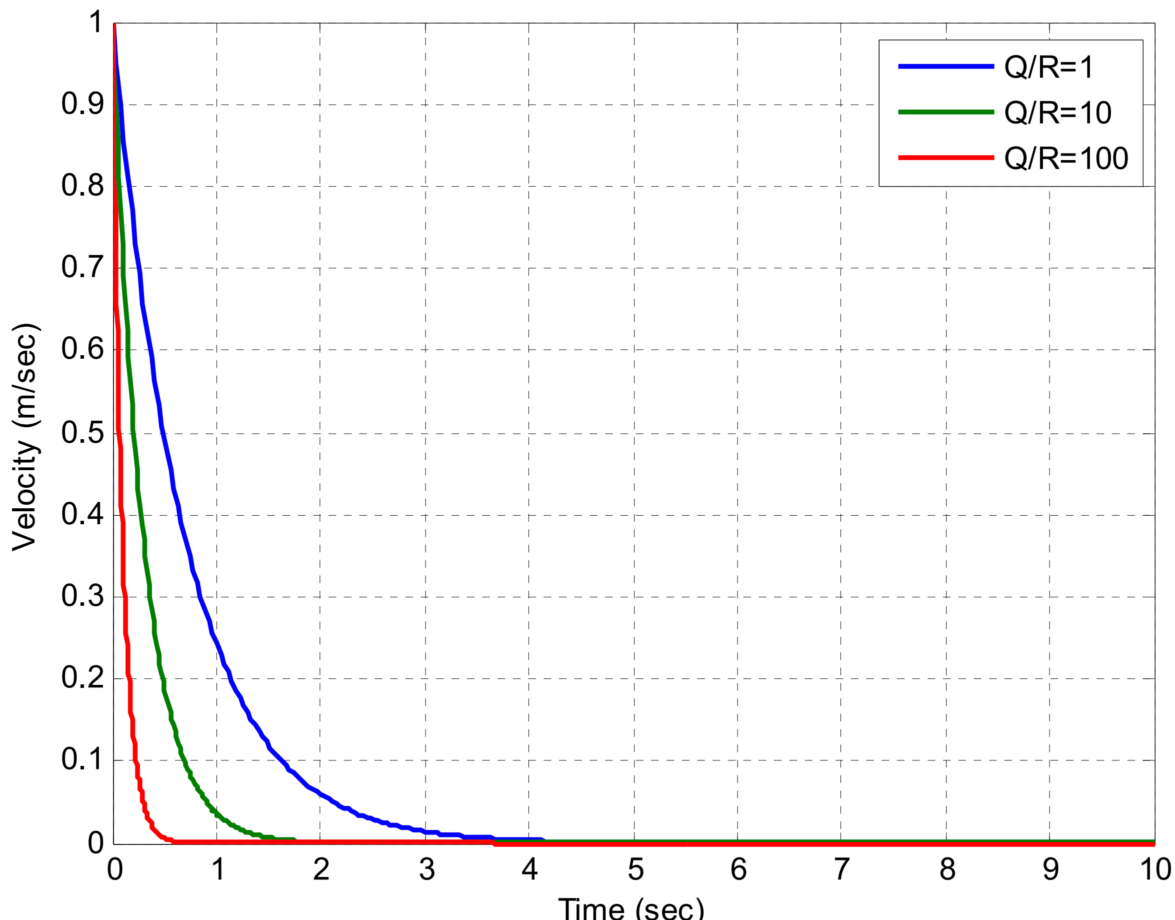
$$-2P - \frac{P^2}{R} + Q = 0$$

$$\left(\frac{P}{R}\right)^2 + 2\left(\frac{P}{R}\right) - \frac{Q}{R} = 0$$

$$\frac{P}{R} = -1 + \sqrt{1 + \frac{Q}{R}}$$

$$K = 1 - \sqrt{1 + \frac{Q}{R}}$$

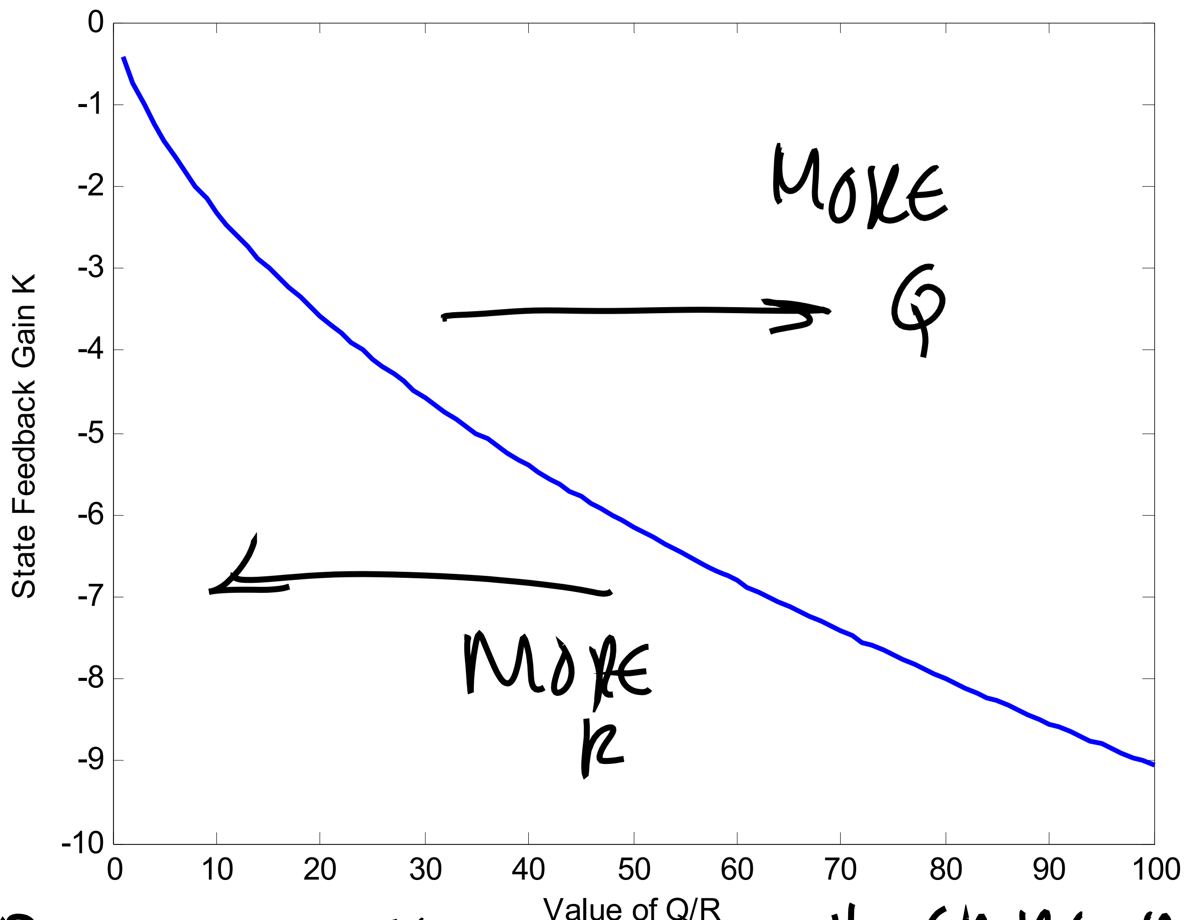
HOW DOES THE TRACTOR DECELERATION CHANGE WITH Q/R?



decelerate

SLOWER FOR LESS EFFORT

HOW DOES THE STATE FEEDBACK GAIN CHANGE WITH Q/R?



RATIO OF HOW MUCH YOU CARE ABOUT THE STATES VS. CONTROL EFFORT

COMING UP...

Case Study

More LQR

Linear Matrix Inequalities (LMIs)

Review

Final Exam