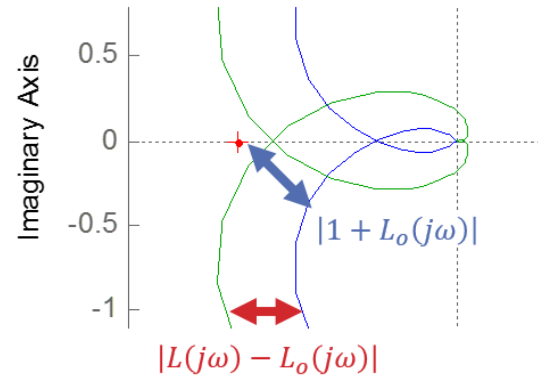
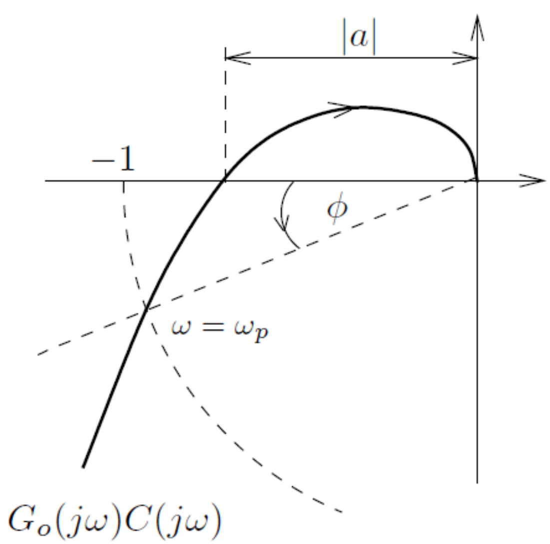
FROM LAST TIME

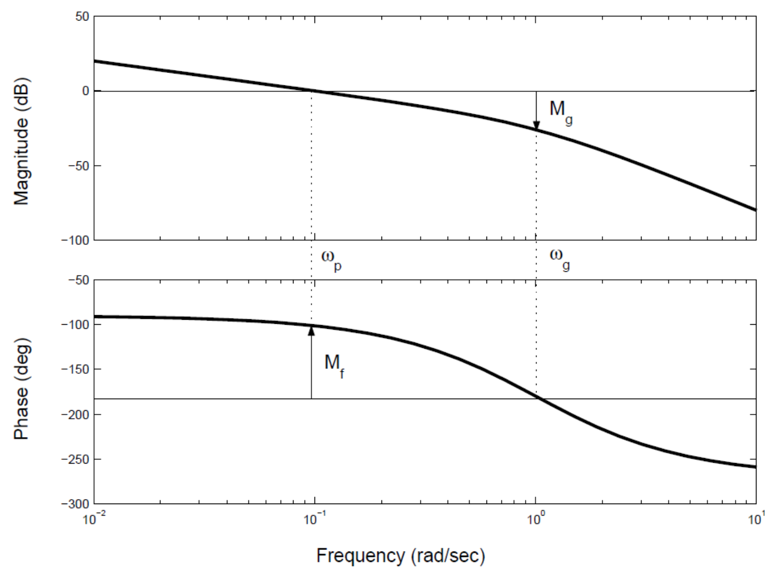
More Stability

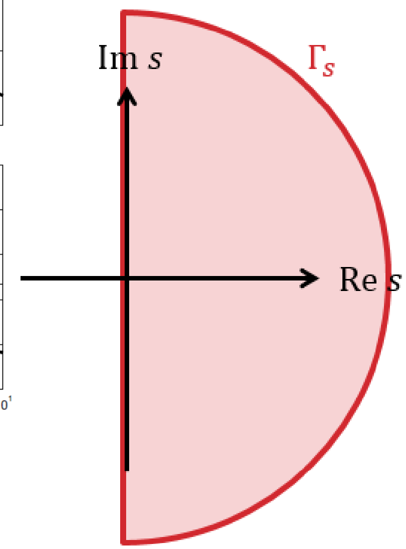
- Nyquist test for stability
- Relative stability
- Robust stability

$\rightarrow |T_o| \cdot |G_\Delta| < 1$









POLE PLACEMENT DESIGN

Topics

- Pole placement design
- Controller with integration
- PID via pole placement
- Smith predictor

At the end of this section, students should be able to:

- Design a controller using pole placement method.
- Describe effects of P, I, and D terms.
- Design PID controllers using pole placement.
- Describe the operation and benefits of a Smith predictor.

WHAT CONTROLLER DESIGN TECHNIQUES DO YOU KNOW FROM PREVIOUS COURSES?

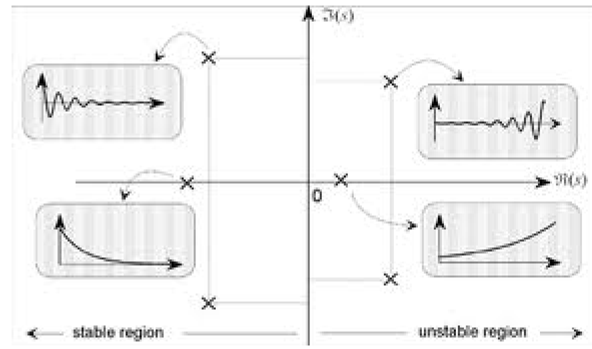
PID - K_i, K_p, K_d (SIMULINK & BY HAND)

LQR

LEAD/LAG IN A ROOT LOCUS

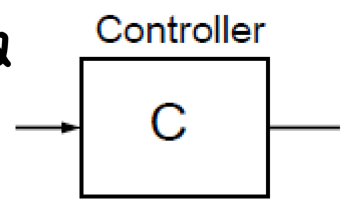
WE WANT A SYSTEMATIC PROCEDURE TO SYNTHESIZE A CONTROLLER FOR SISO LTI SYSTEMS

SINGLE INPUT SINGLE OUTPUT, LINEAR TIME INVARIANT

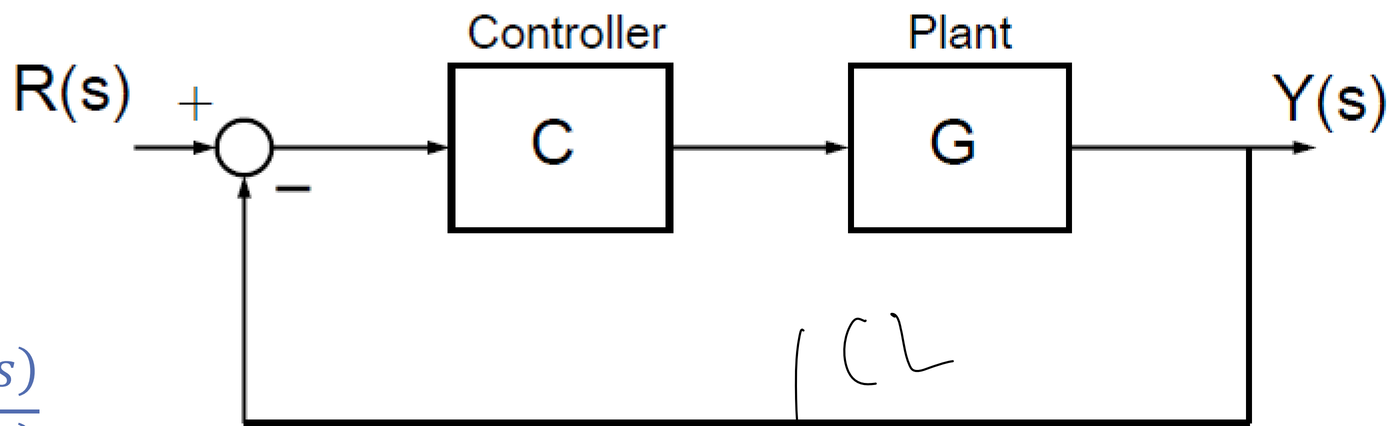


SPECIFICATIONS

DONE BY PLACING THE POLES



RECALL THE CLOSED-LOOP CHARACTERISTIC EQUATION



CL

Let

$$C(s) = \frac{N_C(s)}{D_C(s)}$$

$$G_o(s) = \frac{N_{G_o}(s)}{D_{G_o}(s)}$$

nominal

$$T_o(s) = \frac{G_o(s)C(s)}{1 + G_o(s)C(s)}$$

complementary sensitivity func.

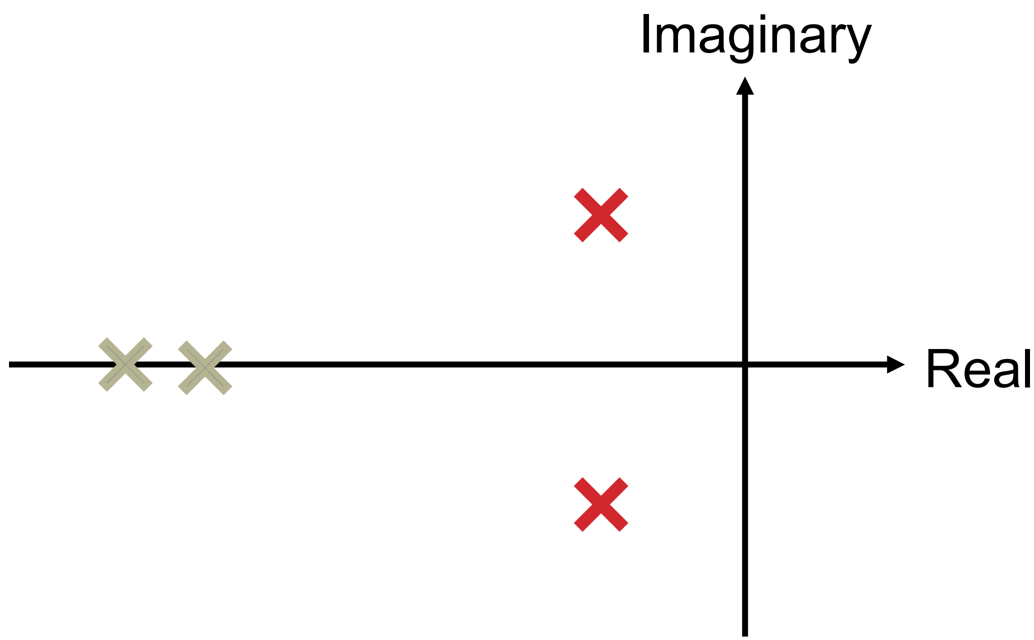
CL characteristic eq:

$$1 + G_o(s)C(s) = 0$$

$$D_{G_o}(s)D_c(s) + N_{G_o}(s)N_c(s) = 0$$

$$\frac{G_o(s)C(s)}{1 + \frac{N_{G_o}(s)\,N_c(s)}{D_{G_o}(s)\,D_c(s)}}$$

BASED ON DESIRED PERFORMANCE, SELECT DESIRED CLOSED-LOOP POLES



- Choose dominant poles first
- Place remaining poles far to the left of dominant poles
- Combine the desired closed-loop poles into a desired closed-loop characteristic polynomial

$$D_{CL}(s) = a^c_{n_{CL}}(s - p_1)(s - p_2)\cdots(s - p_{n_{CL}})$$

DEN. CL $\rightarrow 1 + G_{no}(s)C(s)$

POLE PLACEMENT EQUATES THE DESIRED AND ACTUAL CHARACTERISTIC POLYNOMIALS

$$D_{CL}(s) = a_{n_{CL}}^c (s - p_1)(s - p_2) \cdots (s - p_{n_{CL}})$$
$$= a_{n_{CL}}^c s^{n_{CL}} + a_{n_{CL}-1}^c s^{n_{CL}-1} + \cdots + a_1^c s^1 + a_0^c$$

MAKING POLES

CHAR. EQ

$$D_{CL}(s) = D_{G_o}(s) D_C(s) + N_{G_o}(s) N_C(s)$$

- This gives unknown coefficients due to controller $N_C(s)$ and $D_C(s)$
- Match coefficients and solve

EXAMPLE OF POLE PLACEMENT

$$T_o(S) = \frac{G_o(S)C(S)}{1 + G_o(S)C(S)}$$

Let

$$G_o(s) = \frac{1}{s^2 + 3s + 1} \qquad C(s) = \frac{b_{C1}s + b_{C0}}{a_{C1}s + a_{C0}}$$

Characteristic Equation:

$$(s^2 + 3s + 1)(a_{C1}s + a_{C0}) + (1)(b_{C1}s + b_{C0}) = 0$$

$$a_{c1}S^3 + (3a_{c1} + a_{c0})S^2 + (a_{c1} + 3a_{c0} + b_{c1})S + (a_{c0} + b_{c0}) = 0$$

Choose poles such that the characteristic polynomial is:

$$(s + 10)(s^2 + 6s + 25)$$

CHOSE BASED ON SPECS

$$S^3 + 16S^2 + 85S + 250 = 0$$

$a_{c1} = 1$ $\quad 3a_{c1} + a_{c0} = 16$ $\quad b_{c1} = 45$

$a_{c0} = 13$ $\quad b_{c0} = 237$

EXAMPLE OF POLE PLACEMENT

Solve for controller coefficients:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{c1} \\ a_{c0} \\ b_{c1} \\ b_{c0} \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \\ 85 \\ 250 \end{bmatrix}$$

$$\begin{bmatrix} a_{c1} \\ a_{c0} \\ b_{c1} \\ b_{c0} \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 16 \\ 85 \\ 250 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \\ 45 \\ 237 \end{bmatrix} \rightarrow C(s) = \frac{45s + 237}{s + 13}$$

WE CAN WRITE THE GENERALIZED PROBLEM AS:

Given:

n $\quad N_c$ $\quad m$ $\quad m_c$ $\qquad m \le n$

$$D_{G_o}(s)D_C(s) + N_{G_o}(s)N_C(s) = 0$$

$m_c \le n_c$

where

$$N_{G_o}(s) = b_{Gm}s^m + b_{G(m-1)}s^{m-1} + \cdots + b_{G1}s + b_{G0}$$
$$D_{G_o}(s) = a_{Gn}s^n + a_{G(n-1)}s^{n-1} + \cdots + a_{G1}s + a_{G0}$$
$$N_C(s) = b_{Cm_C}s^{m_C} + b_{C(m_C-1)}s^{m_C-1} + \cdots + b_{C1}s + b_{C0}$$
$$D_C(s) = a_{Cn_C}s^{n_C} + a_{C(n_C-1)}s^{n_C-1} + \cdots + a_{C1}s + a_{C0}$$

and

$$D_{CL}(s) = a^c_{n_{CL}}s^{n_{CL}} + a^c_{n_{CL}-1}s^{n_{CL}-1} + \cdots + a^c_1 s + a^c_0$$
$$= a^c_{n_{CL}}(s - p_1)(s - p_2)\cdots(s - p_{n_{CL}})$$

$n_{CL} = n + n_c$

Find:

$N_C(s), \qquad D_C(s)$

DOES A SOLUTION EXIST?

Given $G_o(s)$ and any desired CL pole locations, i.e., known n_{CL} and $D_{CL}(s)$ but with coefficients being arbitrarily specified, does there exist a proper $C(s)$ that can achieve the desired poles?

Lemma:

Assume that $N_{Go}(s)$ and $D_{Go}(s)$ are coprime (no common factor). Then, as long as order of desired CL polynomial $D_{CL}(s)$ is no less than $2n-1$, there always exists a proper controller $C(s)$ that solves the pole placement problem:

$$D_{Go}(s)D_C(s) + N_{Go}(s)N_C(s) = D_{CL}(s)$$

In fact, when $n_{CL} = 2n-1$, the solution is unique with $C(s)$ of order $n_C = n-1$.

NUM CONT

NUM. CL

THE GENERAL SOLUTION FOR THE CONTROLLER COEFFICIENTS IS GIVEN BY:

CONTROLLER COEFFS.

DESIRED CLOSED LOOP CHAREC. EQ. COEFFS.

$$\begin{bmatrix} a_{C(n-1)} \\ a_{C(n-2)} \\ \vdots \\ a_{C0} \\ b_{C(n-1)} \\ \vdots \\ b_{C0} \end{bmatrix} = \mathcal{S}^{-1} \begin{bmatrix} a_{2n-1}^{c} \\ a_{2n-2}^{c} \\ \vdots \\ a_{n}^{c} \\ a_{n-1}^{c} \\ \vdots \\ a_{0}^{c} \end{bmatrix}$$

PLANT DENOM. PLANT NUM

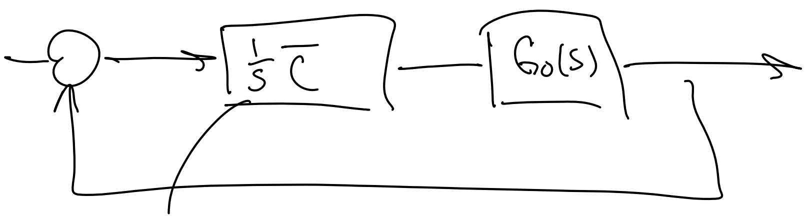
$$\mathcal{S} = \begin{bmatrix} a_{Gn} & & & b_{Gn} & & \\ a_{G(n-1)} & \ddots & & b_{G(n-1)} & \ddots & \\ \vdots & & \ddots & \vdots & & \ddots \\ a_{G0} & & a_{Gn} & b_{G0} & & b_{Gn} \\ & \ddots & a_{G(n-1)} & & \ddots & b_{G(n-1)} \\ & & \vdots & & & \vdots \\ & & a_{G0} & & & b_{G0} \end{bmatrix}$$

- $\mathcal{S}$ is called the eliminant or Sylvester matrix

DOWNLOAD DISK CONTENTS ON WEBSITE →
PAQ.m
P_elCERO.m

WHAT IF THE CONTROLLER MUST INCLUDE AN INTEGRATOR?

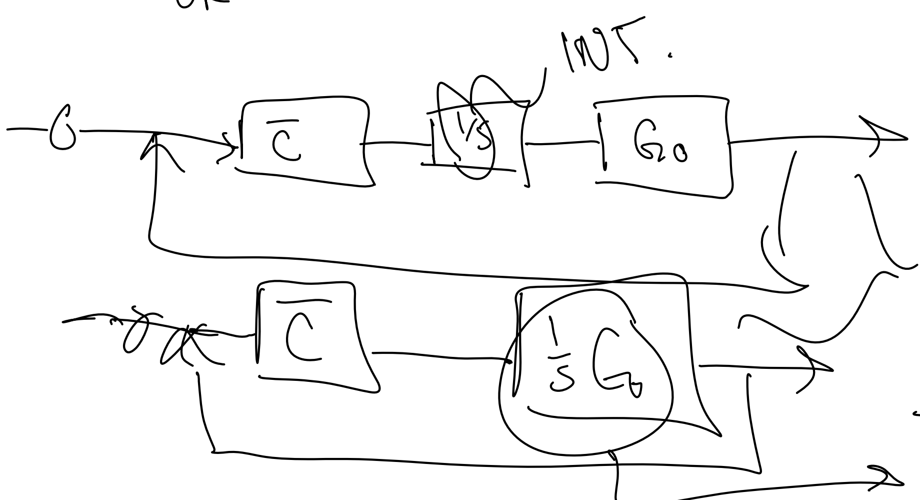
BIPROPER DOES NOT HAVE AN INTEGRATOR



MANIPULATE THE CONTROLLER

$\frac{1}{s}\bar{C} = C(s)$ — FOR NON-BIPROPER

OR



SAME CHAR. EQ.

$\bar{G}_o$

CONTROLLER WITH INTEGRATION

Want $D_C(s) = s\bar{D}_C(s)$

DENOMEN. OF CONTROLLER

- Pole placement problem

$$D_{Go}(s)\underbrace{s\bar{D}_C(s)}_{D_C(s)} + N_{Go}(s)N_C(s) = D_{CL}(s)$$

- Equivalent pole placement problem

$$\bar{D}_{Go}(s)\bar{D}_C(s) + N_{Go}(s)N_C(s) = D_{CL}(s)$$

WHERE DOES THE S Go

- Can be solved as before by assuming an equivalent fictitious plant of order $n + 1$ with a new denominator of

$$\bar{D}_{Go}(s) = sD_{Go}(s)$$

(L(Go $\frac{1}{s}$)

NUM. CLOSED LOOP

S

> Solution always exists if n_{CL} is no less than $2\tilde{n}$. When $n_{CL} = 2n$, the solution is unique with order of $\bar{D}_C(s)$ being $n - 1$ and order $N_C(s)$ of being n!

n IS THE HIGHEST POWER OF S

$$1 + \frac{N_C(s)}{D_C(s)} \frac{N_{Go}(s)}{D_{Go}(s)}$$

$s\bar{D}_C(s)$

WHAT IF WE WANT TO CANCEL SOME STABLE PLANT POLES OR ZEROS?

- Example

$$D_{Go}(s) = (s - p_C)\bar{D}_G(s)$$
$$N_{Go}(s) = (s - z_C)\bar{N}_G(s)$$

- Pole Placement Problem

$$(s - p_C)\bar{D}_G(s)(s - z_C)\bar{D}_C(s) + (s - z_C)\bar{N}_G(s)(s - p_C)\bar{N}_C(s) = D_{CL}(s)$$

- which has a solution only if $D_{CL}(s)$ contains the cancelled poles and zeros:

$$D_{CL}(s) = (s - p_C)(s - z_C)\bar{D}_{CL}(s)$$

- Equivalent Pole Placement Problem

$$\bar{D}_G(s)\bar{D}_C(s) + \bar{N}_G(s)\bar{N}_C(s) = \bar{D}_{CL}(s)$$

Cancelled poles/zeros remain as CL poles!

EXAMPLE

IMPROPER

We want to add an integrator

Let

$$\bar{G}_o(s) = \frac{1}{s(s^2 + 3s + 1)} \qquad \bar{C}(s) = \frac{b_{C2}s^2 + b_{C1}s + b_{C0}}{a_{C1}s + a_{C0}}$$

Characteristic Equation:

$$\underbrace{s(s^2 + 3s + 1)}_{\bar{D}_G}\overbrace{(a_{C1}s + a_{C0})}^{\bar{D}_C} + \overbrace{(1)}^{\bar{N}_G}\overbrace{(b_{C2}s^2 + b_{C1}s + b_{C0})}^{\bar{N}_C} = 0$$

$a_{C1}s^4 + (3a_{C1} + a_{C0})s^3 + (a_{C1} + 3a_{C0} + b_{C2})s^2 + (a_{C0} + b_{C1})s + b_0 = 0$

$s^4 \Rightarrow$ NEED 4 POLES

Let desired characteristic polynomial be:

$$(s + 10)^2(s^2 + 6s + 25)$$ ~ ADDED AN ADDIT. FAST POLE

$s^4 + 26s^3 + 245s^2 + 1100s + 2500 = 0$

EXAMPLE

Solve for controller coefficients:

$$
\begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} a_{c1} \\ a_{c0} \\ b_{c2} \\ b_{c1} \\ b_{c0} \end{bmatrix}
=
\begin{bmatrix} 1 \\ 26 \\ 245 \\ 1100 \\ 2500 \end{bmatrix}
$$

$$
\bar{C}(s) = \frac{175s^2 + 10775s + 2500}{s + 23}
$$

$$
C(s) = \frac{1}{s}\bar{C}(s) = 9 \times \frac{1}{s}
$$

PID CONTROL

PID CONTROLLER VIA POLE PLACEMENT

Proper PID Controller Structure

$$C(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau_D s + 1} \quad = \frac{(K_D + K_P\tau_D)s^2 + (K_P + K_I\tau_D)s + K_I}{\cancel{\tau_D} s^2 + \frac{s}{\tau_D}}$$

Equivalent Controller Form

$$C(s) = \frac{b_{C2}s^2 + b_{C1}s + b_{C0}}{s^2 + a_{C1}s}$$ – BIPROPER

where

NUM

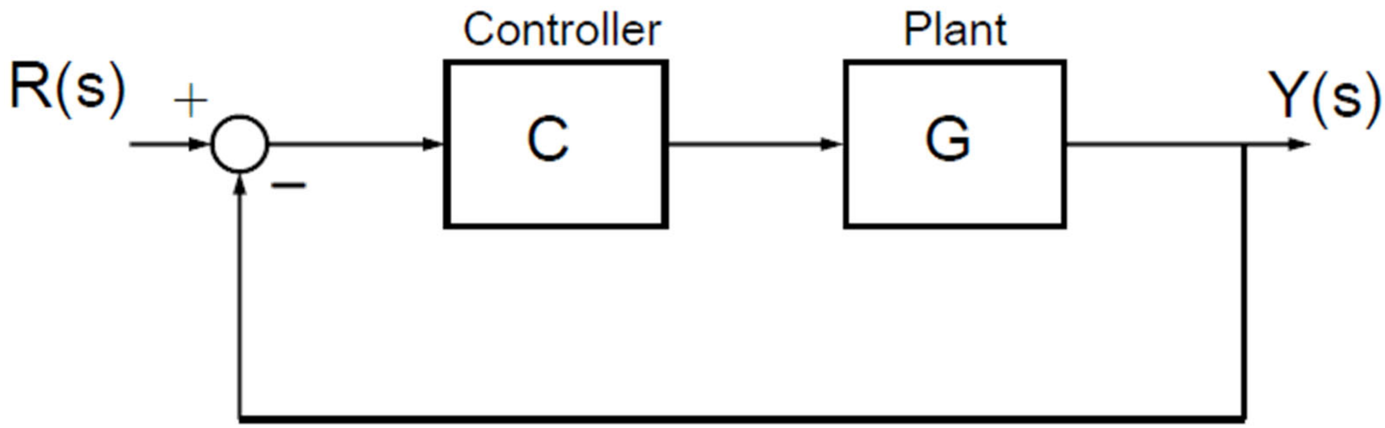
DENOM

$$b_{C2} = \frac{K_D + K_P\tau_D}{\tau_D}$$

$$b_{C1} = \frac{(K_P + K_I\tau_D)}{\tau_D} \qquad a_{C1} = \frac{1}{\tau_D}$$

$$b_{C0} = \frac{K_I}{\tau_D}$$

PID CONTROL



$$u(t) = K_P e(t) + K_I \int_0^t e(t)dt + K_D \dot{e}(t)$$

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s\right) E(s)$$

RECALL THE EFFECTS OF P, I, AND D

Proportional (P)

- Improves rise time
- Reduces steady-state error
- Reduces effect of modeling error
- May introduce oscillation

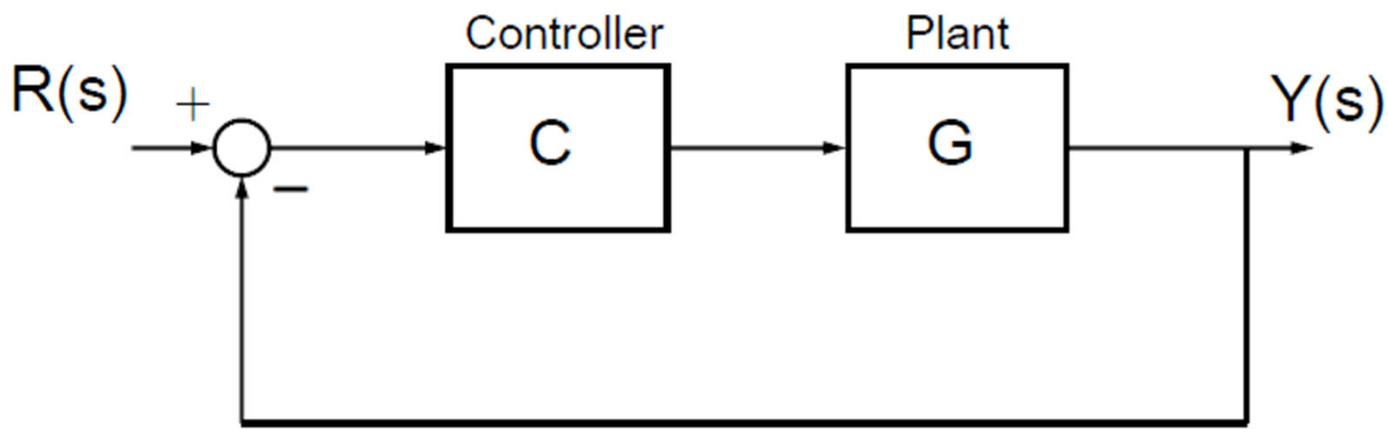
Integral (I)

- Eliminates steady-state error
- Increases system order
- May decrease stability margins

Derivative (D)

- Increases damping, may decrease settling time
- May increase overshoot

CONSIDER AN EXAMPLE



$$G_o(s) = \frac{4}{s(s+4)}$$

PID

$$C(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau_D s + 1} = \frac{b_{C2}s^2 + b_{C1}s + b_{C0}}{a_{C2}s^2 + a_{C1}s}$$

DESIGN A PID CONTROLLER FOR THE GIVEN POLE LOCATIONS

Case 1:

2nd one

$s_{1,2} = -6 \pm j6$ $D_{CL} = (s^2 + 12s + 72)(s+20)(s+70)$

$s_3 = -20$ ~ MAY BE TO SLOW

$s_4 = -70$ ~ MAY BE TO FAST

$s^4 + 102s^3 + 2552s^2 + 23280s + 100800$

$G_1(s) = \dfrac{540s^2 + 5820s + 25200}{s(s+98)}$

ZEROS = $s = -5.4 \pm j4.2$ ~ SLOW RELATIVE TO DOM. POLES

BUT LHP

WILL CAUSE OS

DESIGN A PID CONTROLLER FOR THE GIVEN POLE LOCATIONS

Case 2: $s_{1,2} = -6 \pm j6$

$s_3 = -0.1$ ~ DOMINATE 1st ORDER

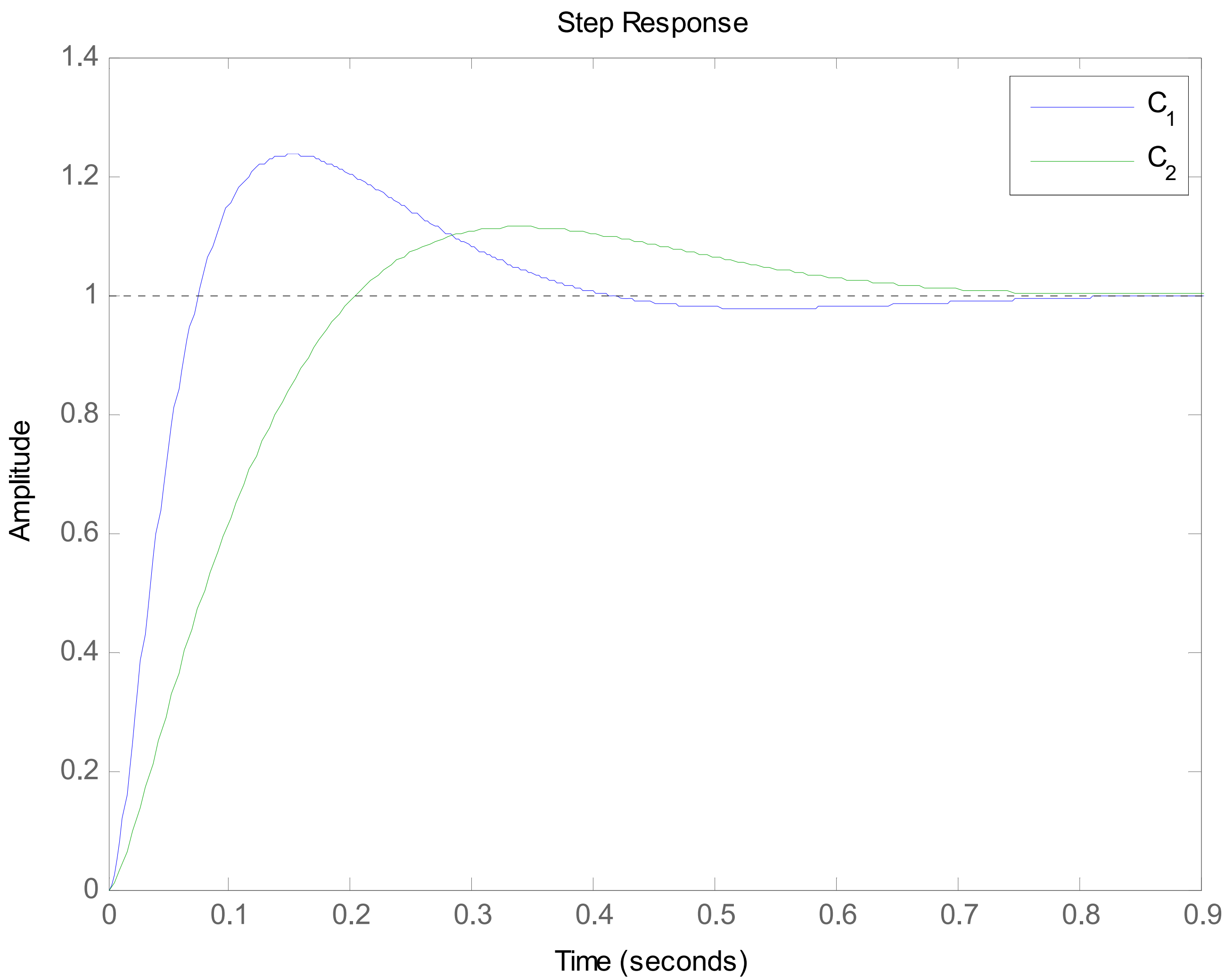
$s_4 = -90$

$$C_2(s) = \frac{192.4s^2 + 1649s + 162}{s(s+98.1)}$$

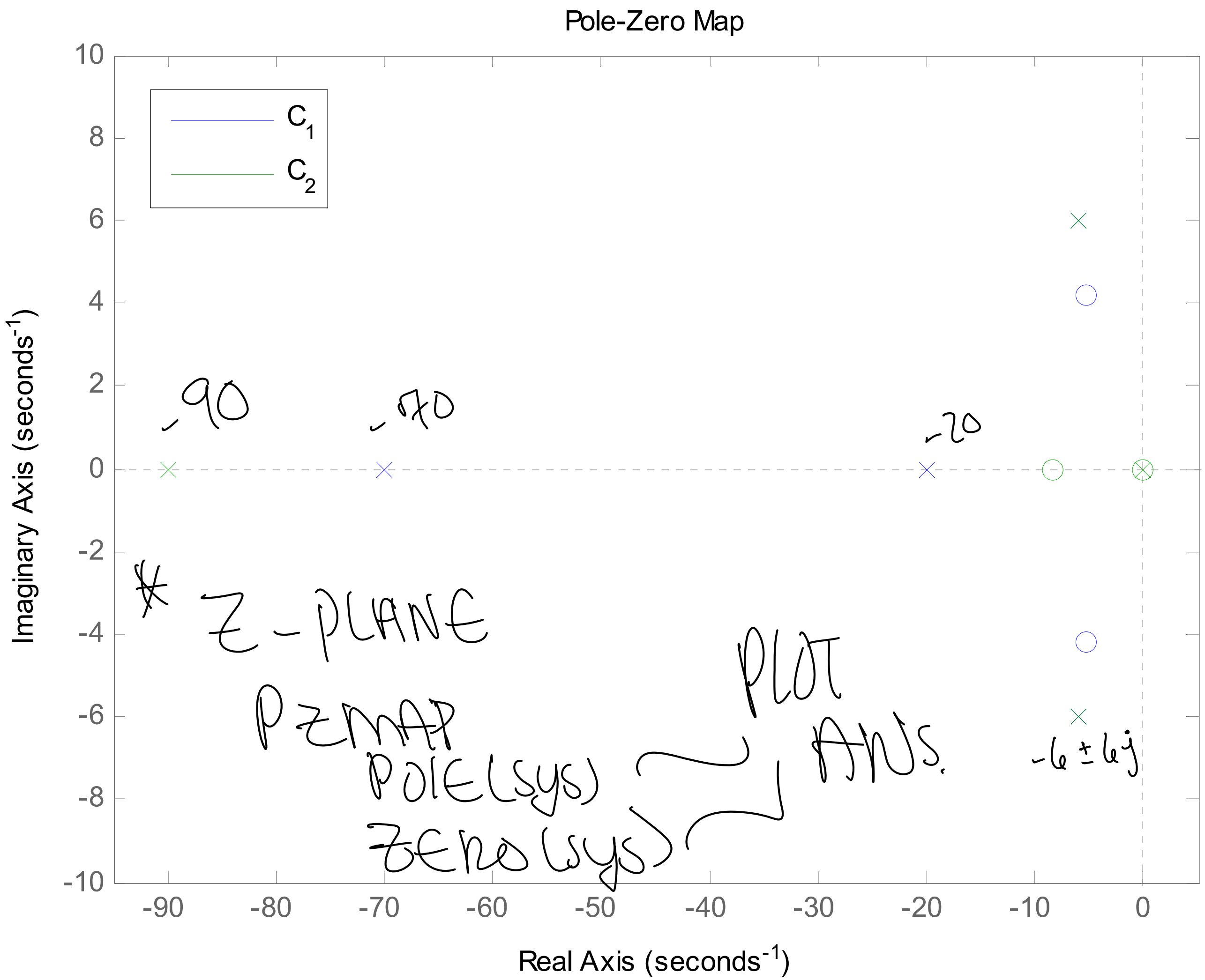
ZEROS $s = -8.5$

$-.099$

USE MATLAB TO PLOT THE CLOSED-LOOP STEP RESPONSES

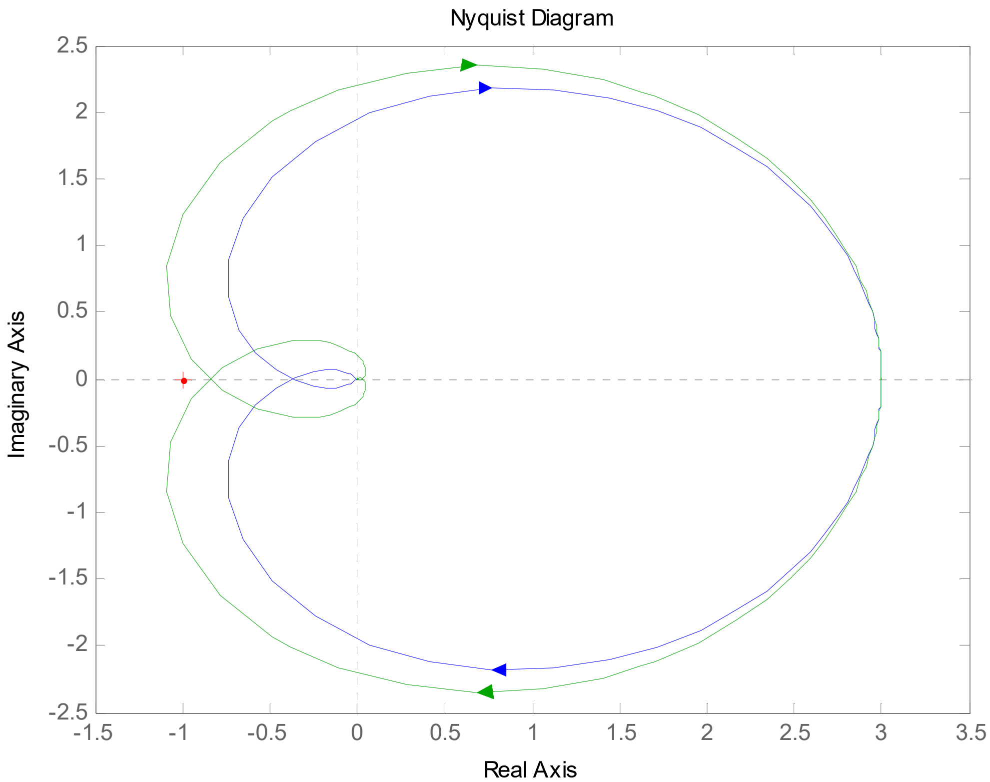


USE MATLAB TO PLOT THE CLOSED-LOOP POLES AND ZEROS



SMITH PREDICTOR

RECALL THE EFFECT OF A TIME DELAY ON THE NYQUIST STABILITY TEST

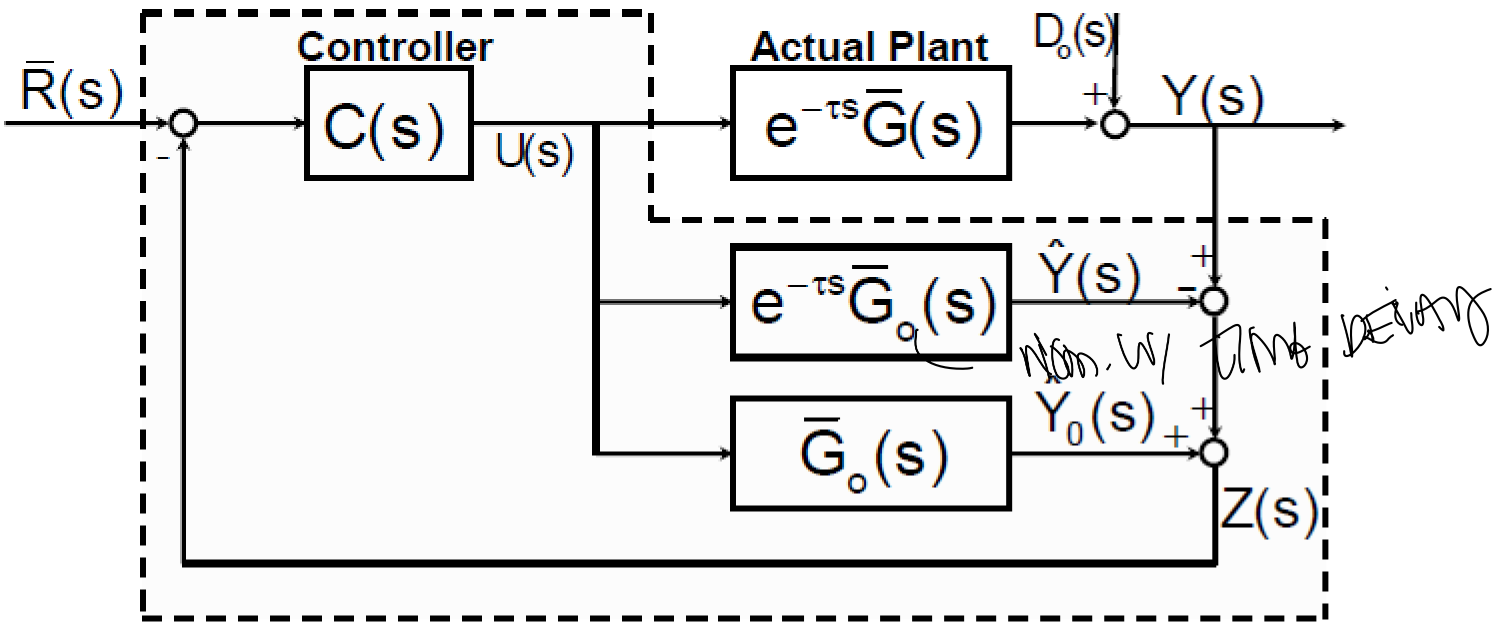


$$L_o(s) = \frac{3}{(s+1)^3}$$

$$L(s) = \frac{3}{(s+1)^3} e^{-0.5s}$$

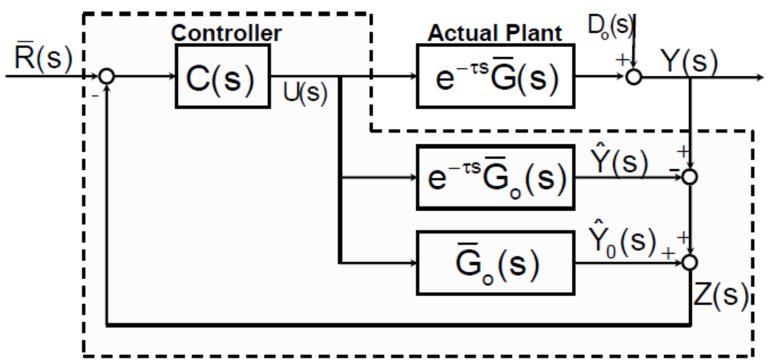
TIME DELAY

THE SMITH PREDICTOR USES A PARALLEL MODEL TO ~~CANCEL~~ THE DELAY EFFECT



- Requires stable open-loop system with known time-delay
- Controller can be designed for undelayed plant

TO VERIFY, DETERMINE THE TRANSFER FUNCTION FROM R TO Z



$$U(s) = C(s)[R(s) - Z(s)]$$

$$Y(s) = D_o(s) + e^{-\tau s}\bar{G}(s)U(s)$$

$$\bar{G}_o \approx \bar{G}$$

Nom. model Good model

$$Z(s) = Y(s) - \hat{Y}(s) + \hat{Y}_o(s)$$

$$= D_o(s) + \cancel{e^{-\tau s}\bar{G}(s)U(s) - e^{-\tau s}\bar{G}_o(s)U(s)} + \bar{G}_o(s)U(s)$$

$$\approx D_o(s) + \bar{G}_o(s)U(s)$$

$$T_{zr}(s) = \frac{\bar{G}_o(s)C(s)}{1 + \bar{G}_o(s)C(s)} \qquad T(s) = \frac{e^{-\tau s}\bar{G}_o(s)C(s)}{1 + \bar{G}_o(s)C(s)}$$

WHAT ARE THE LIMITATIONS OF THE SMITH PREDICTOR?

- Only works with stable plant!

- Significant robustness issues associated with the architecture

COMING UP…

SISO Design Limitations

- Free integrators
- Poles/Zeros

Frequency Domain Limitations

- Bode’s Integral Constraints on Sensitivity
- Integral Constraints on Complementary Sensitivity
- Poisson Integral Constraint on Sensitivity
- Poisson Integral Constraint on Complementary Sensitivity