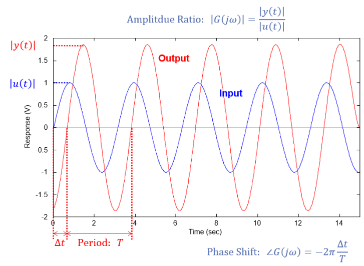
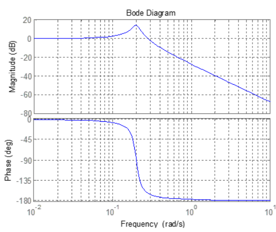
FROM LAST TIME…

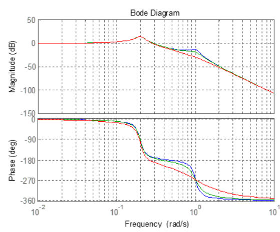
Frequency Response

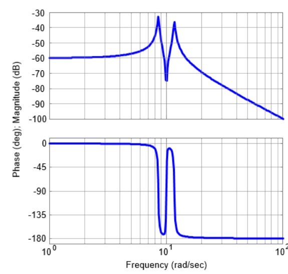
- Forced response to sinusoidal inputs
- Frequency response of LTI systems
- Bode plots
- Modeling errors in frequency domain



$$y_{ss}(t) = |G(j\omega)| \sin\big(\omega t + \angle G(j\omega)\big)$$







ANALYSIS OF FEEDBACK SYSTEMS

Topics

- Feedback controller structure
- Nominal sensitivity functions
- Stability of nominal feedback system
- Root locus

At the end of this section, students should be able to:

- Explain the meaning of sensitivity functions.
- Derive sensitivity functions for a given system.
- Determine system stability using Routh-Hurwitz.
- Sketch a root locus.

VIDEO

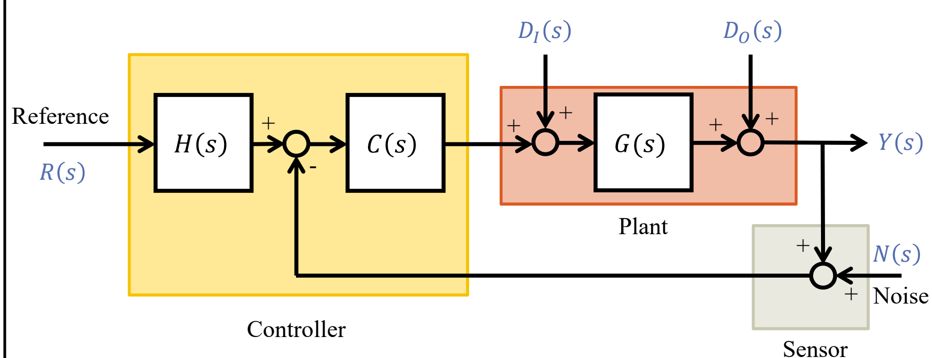
Building Model Test

http://youtu.be/0IpC2DE3nJE

Questions to consider

- What is happening in the video?
- What type of input is applied to the building model?
- Is stability of the structure important?

CONSIDER A TWO DEGREE-OF-FREEDOM CLOSED-LOOP SYSTEM



- $R(s)$ is reference input
- $D_I(s)$ and $D_O(s)$ are input and output disturbances
- $N(s)$ is noise input
- $Y(s)$ is output

DERIVE THE CLOSED-LOOP TRANSFER FUNCTIONS…

$U(s) =$

$Y(s) =$

DERIVE THE CLOSED-LOOP TRANSFER FUNCTIONS…

FROM THE CLTFS WE DEFINE THE SENSITIVITY FUNCTIONS

$$T(s) = \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{N_G(s)N_C(s)}{D_G(s)D_C(s)+N_G(s)N_C(s)}$$ Complementary sensitivity function

$$S(s) = \frac{1}{1+G(s)C(s)} = \frac{D_G(s)D_C(s)}{D_G(s)D_C(s)+N_G(s)N_C(s)}$$ Sensitivity function

$$S_i(s) = \frac{G(s)}{1+G(s)C(s)} = \frac{N_G(s)D_C(s)}{D_G(s)D_C(s)+N_G(s)N_C(s)}$$ Input disturbance sensitivity function

$$S_u(s) = \frac{C(s)}{1+G(s)C(s)} = \frac{D_G(s)N_C(s)}{D_G(s)D_C(s)+N_G(s)N_C(s)}$$ Control sensitivity function

ANOTHER VIEW OF SENSITIVITY FUNCTIONS

$$T(s) = \frac{Y(s)}{R(s)} = \frac{-Y(s)}{N(s)}$$

$$S(s) = \frac{E(s)}{R(s)} = \frac{Y(s)}{D_O(s)}$$

$$S_i(s) = \frac{Y(s)}{D_I(s)} = \frac{Y(s)}{U(s)}$$

$$S_u(s) = \frac{U(s)}{R(s)} = \frac{-U(s)}{N(s)}$$

WHAT IS A SENSITIVITY FUNCTION?

1. The sensitivity function S describes how error signal E is related to the reference input R.

2. The sensitivity function S also describes how much a change in the plant G (perhaps due to parameter errors) affects the closed-loop transfer function T (a.k.a. complementary sensitivity).

3. A controller C with high gain will make CG large, therefore making S small, exactly as desired.

WE CAN EASILY RELATE THE SENSITIVITY FUNCTIONS TO EACH OTHER

$$S(s) + T(s) = 1$$
$$S_i(s) = S(s)G(s)$$
$$S_u(s) = S(s)C(s)$$

- To minimize effects of noise, keep $T(s)$ small
- To minimize effects of disturbance, keep $S(s)$ small

ANALYSIS OF FEEDBACK SYSTEMS

TESTING FOR STABILITY

USING SENSITIVITY FUNCTIONS, RELATE SYSTEM INPUTS TO OUTPUTS

$$\begin{bmatrix} E(s) \\ U_D(s) \\ Y_m(s) \end{bmatrix} = \frac{1}{1+CG}\left[\qquad\qquad\qquad \right]\begin{bmatrix} \bar{R}(s) \\ D_i(s) \\ D_o(s) \\ N(s) \end{bmatrix}$$

- This system is internally stable if all of the internal signals decay to zero when any of the inputs are impulses

INTERNAL STABILITY

Theorem:

- The feedback system is internally stable if and only if the sensitivity function S has no poles in the RHP ($s \geq 0$) and there is no cancellation of unstable poles between the controller C and the plant G.

(equivalent to Definition 5.1 in text)

EXAMPLE: ARE THESE SYSTEMS INTERNALLY STABLE?

$$G_1(s) = \frac{1}{s(0.1s+1)}$$

$$C_1(s) = \frac{440(s+14.73)}{s^2+62s+1072}$$

$$T_2(s) = \frac{s-1}{(s-1)(s+1)}$$

A QUICK WAY TO CHECK STABILITY WITHOUT SOLVING FOR POLES…

QUICK TEST: Hurwitz Necessary Condition

- Given the characteristic equation

$$p(s) = \sum_{k=0}^{n} a_k s^k = 0$$

a stable system must have:

1.

2.

IS HURWITZ NECESSARY CONDITION GOOD ENOUGH?

$$p(s) = s^2 - 3s + 2$$

$$p(s) = s^3 + s^2 + 2s + 8$$

We need a sufficient condition!

A ROUTH ARRAY IS USED TO GENERATE A SUFFICIENT CONDITION

Given the characteristic equation

$$p(s) = \sum_{k=0}^{n} a_k s^k = 0$$

Create the Routh Array

$$\begin{array}{c|ccccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ \hline s^{n-2} & b_1 & b_2 & b_3 & & \\ s^{n-3} & c_1 & c_2 & c_3 & & \\ \vdots & & & & & \\ s^0 & & & & & \end{array}$$

$$b_1 = \frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{-a_{n-1}}$$

$$b_2 = \frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{-a_{n-1}}$$

$$c_1 = \frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{-b_1}$$

ROUTH SUFFICIENT CONDITION

The number of roots of $p(s)$ having positive real parts (in the RHP) equals the number of sign changes in the 1st column of the Routh array.

Example: $p(s) = s^3 + s^2 + 2s + 8$

ANALYSIS OF FEEDBACK SYSTEMS

ROOT LOCUS

THE ROOT LOCUS PROVIDES A VISUAL FOR THE EFFECT OF ONE PARAMETER VARYING

The Root Locus is the set of all points in the complex s plane that satisfy

$$1 + KC_a(s)G(s) = 0$$

when the parameter K **varies from** $0 \rightarrow \infty$

WE CAN REWRITE THE PRODUCT OF CONTROLLER AND PLANT AS FACTORIZED POLYNOMIALS

$$C_a(s)G(s) = \frac{N(s)}{D(s)}$$

$$N(s) = (s - z_1)(s - z_2)\cdots\left(s - z_{N_z}\right)$$
$$D(s) = (s - p_1)(s - p_2)\cdots\left(s - p_{N_p}\right)$$

$$D(s) + KN(s) = 0$$

- $z_1,\, z_2,\, \ldots,\, z_{N_z}$, are called the finite open-loop zeros.
- $p_1,\, p_2,\, \ldots,\, p_{N_p}$, are called the finite open-loop poles.

DR. MYNDERSE, HOW DO I PLOT A ROOT LOCUS?

1. **Given a value of K, numerically solve the characteristic equation for a set of roots. Repeat this for a set of K values and plot the corresponding roots on the complex plane.**
2. **Use MATLAB.**
3. **Apply the sketching rules to obtain an approximate root locus plot.**

USING MATLAB FOR ROOT LOCUS

Example

- Open-loop system given by $L(s) = \dfrac{800}{s(s+5)}$
- Characteristic equation given by

$$1 + K \cdot \frac{800}{s(s+5)} = 0$$

- MATLAB code:

```
>   L = tf(800,[1 5 0]);
>   rlocus(L);
>   [K, poles]=rlocfind(L);
```


SKETCHING ROOT LOCUS

Recall the characteristic equation

$$1 + K \cdot \frac{N(s)}{D(s)} = 0$$

$$K \cdot \frac{N(s)}{D(s)} = -1$$

This gives two necessary and sufficient conditions for a point in the s-plane to be on the root locus:

- Magnitude Condition $\left| K \cdot \dfrac{N(s)}{D(s)} \right| = 1$
- Phase Condition $\angle \left[K \cdot \dfrac{N(s)}{D(s)} \right] = 180°$

Magnitude Condition

$$\left|K \cdot \frac{N(s)}{D(s)}\right| = 1 \Rightarrow K = \frac{|D(s)|}{|N(s)|} \Rightarrow K = \frac{|s-p_1||s-p_2|\cdots\left|s-p_{N_p}\right|}{|s-z_1||s-z_2|\cdots\left|s-z_{N_z}\right|}$$

Phase Condition

$$\angle\left[K \cdot \frac{N(s)}{D(s)}\right] = 180° \Rightarrow \angle K + \angle N(s) - \angle D(s) = 180°$$

$$\angle(s-z_1) + \cdots + \angle\left(s-z_{N_z}\right) - \angle(s-p_1) - \cdots - \angle\left(s-p_{N_p}\right) = 180°$$

ROOT LOCUS SKETCHING RULES

Rule 1: There is a branch of the root locus for each root of the characteristic equation. The number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater.

Rule 2: Root locus starts at open-loop poles (when $K = 0$) and ends at open-loop zeros (when $K \to \infty$). If the number of poles is greater than the number of zeros, roots start at the excess poles and terminate at zeros at infinity. If the reverse is true, branches will start at poles at infinity and terminate at the excess zeros.

Rule 3: Root locus is symmetric about the real axis, i.e., closed-loop poles appear in complex conjugate pairs.

Rule 4: Along the real axis, the root locus includes all points to the left of an odd number of real poles and zeros.

ROOT LOCUS SKETCHING RULES

Rule 5: If number of poles N_P exceeds the number of zeros N_Z, then as $K \to \infty$, $(N_P - N_Z)$ branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles θ_k at σ_0.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_z} = \frac{\text{sum of open-loop poles} - \text{sum of open-loop zeros}}{\text{\# of open-loop poles} - \text{\# of open-loop zeros}}$$

$$\theta_k = \frac{(2k+1)\pi}{N_P - N_Z} \qquad k = 0, 1, 2, \ldots$$

If $N_Z > N_P$, then as $K \to 0$, $(N_Z - N_P)$ branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points can be obtained by solving s in the following equations:

$$\frac{d}{ds}\big(K(s)\big) = 0$$

ROOT LOCUS SKETCHING RULES

Rule 7: The departure (arrival) angle for a pole p_i (zero z_i) can be calculated by slightly modifying the following equation:

$$\angle(s - z_1) + \cdots + \angle\left(s - z_{N_z}\right) - \angle(s - p_1) - \cdots - \angle\left(s - p_{N_p}\right) = 180°$$

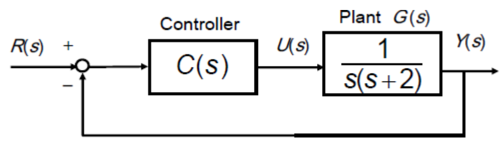
The departure angle θ_n from the pole p_n can be calculated by replacing the term $\angle(s - p_n)$ with θ_n and replacing all the s's with p_n in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found as follows:

- Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.
- Set the real and imaginary parts to zero, and solve for ω and K. This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

ROOT LOCUS EXAMPLE

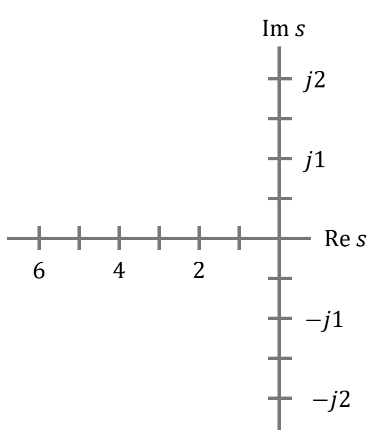
A feedback control system is proposed. The corresponding block diagram is:



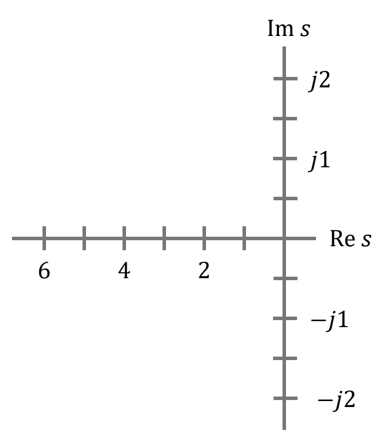
Let $C(s) = K$.

- Sketch the root locus of the closed-loop poles as the controller gain K varies from $0 \to \infty$.
- Find closed-loop characteristic equation:

$C(s) = K$



$C(s) = K(s+3)$



COMING UP…

More Stability

- Nyquist test for stability
- Relative stability
- Robust stability

Pole Placement Controller Design

- Pole placement design
- Controller with integration
- PID via pole placement