FROM LAST TIME...

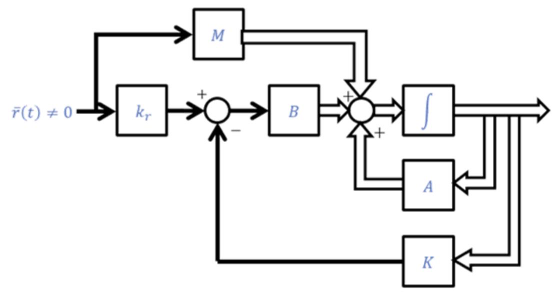
State Feedback Design

$$\dot{x} = (A - BK)x$$

- State Feedback Regulator
- Ackermann's Formula
- State Feedback for Uncontrollable System

$$K = \hat{K}T^{-1} = [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})]T^{-1}$$

$$K = [0 \quad \cdots \quad 0 \quad 1][B \quad AB \quad \cdots \quad A^{n-1}B]^{-1}\phi(A)$$



OBSERVABILITY

Topics

- Definition of Observability
- Observable Canonical Form
- Observable Canonical Decomposition
- General Decomposition

At the end of this section, students should be able to:

- Determine observability of a system.
- Transform a system into observable canonical form.
- Decompose a system into observable and unobservable subsystems.

IN OUR DESIGN EXAMPLE, WE HAD TO HAVE ACCESS TO THE OUTPUT AND THREE DERIVATIVES:

NEW CONTROLLABLE CANONICAL FORM STATES

y_e	end effector position
$\dot{y}_e$	end effector velocity
$\ddot{y}_e$	end effector acceleration
$\dddot{y}_e$	end effector jerk

y_m, $\dot{y}_m$, y_e, $\dot{y}_e$ U STATES

What if the measurements were not available?

- Online derivative estimation
- Simple finite difference approximations to the derivative are sensitive to noise

- 4 STATES IDEALLY USES 4 SENSORS

There must be a better way!

- An observer reconstruct all the states from available measurements
- This will requires observability (dual to controllability)

SENSOR REDUCTION, TO 'GUESS' STATES

- IF CONTROLLABLE → OBSERVABLE

WHAT DOES IT MEAN FOR A SYSTEM TO BE OBSERVABLE?

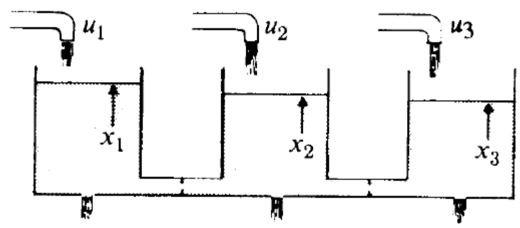
MIN. AMOUNT OF VARIBLES TO DESCRIBE A SYSTEM & ITS EQNS

$y = Cx + Du$

The system $\dot{x} = Ax + Bu$ is completely **state observable** if every state $x(t_0)$ can be determined from the observations of $y(t)$ over a finite time interval, $t_0 \leq t \leq t_1$.

CONSIDER A SYSTEM WITH THREE WATER TANKS FILLED BY THREE INLETS

- NO SPECIFIC SENSOR



We want to observe all three tank depths at any time.

MODEL THE TANK DYNAMICS, CONSIDERING THE COUPLING BETWEEN TANKS AND THE OUTFLOW

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

INLET FLOWS

STATES/ DEPTHS OF THE THREE TANKS

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + 0 [u]$$

MIMO, S.S ONLY NO TF

This is a multi-input multi-output system

We measure all three tank depths (output)

AS WITH CONTROLLABILITY, DIAGONALIZE THE A MATRIX

* CANONICAL FORMS
* WHICH IS THE STATE MATRIX

← EIGEN VECTORS $\lambda = -1, -3, -5$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

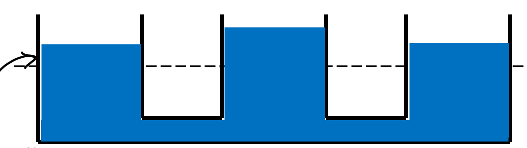
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \overset{x=Tz}{\Longrightarrow} \quad \begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}}_{\hat{C}} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}}_{z} \qquad \begin{aligned} z_1 + z_2 + z_3 &= 0 \\ z_1 - z_3 &= 0 \\ z_1 - z_2 + z_3 &= 0 \end{aligned}$$

Measurement y_2 can not detect state z_2 (2nd mode)!

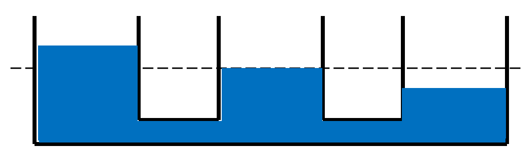
PICTURE THE MODE SHAPES

$\lambda_1 = -1, \quad v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

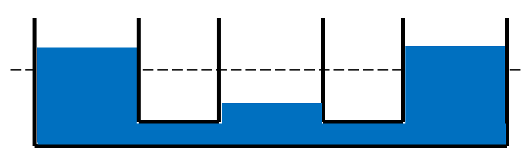


ZERO POINT (REF)

$\lambda_2 = -3, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$



$\lambda_3 = -5, \quad v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$



OUTPUT = SOME SUM OF THE WEIGHTED MODE SHAPES

LET'S TAKE A CLOSER LOOK:

PLOTTING OUTPUT NOT MODE

DEPTH OF TANK

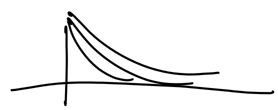
FROM z

$$x_1 = y_1(t) = z_1(t) + z_2(t) + z_3(t)$$

CAN BE RELATED BACK TO X

$$= e^{-t} z_1(0) + e^{-3t} z_2(0) + e^{-5t} z_3(0)$$

$\lambda = -1$ $\lambda = -3$ $\lambda = -5$



All three modes can be detected because of their different time responses (decay rates)

ALTERNATIVELY, PAST HISTORY CAN BE CAPTURED BY LOOKING AT $n-1$ DERIVATIVES OF A GIVEN OUTPUT:

$$\begin{aligned} y &= Cx \\ \dot{y} &= C\dot{x} = CAx \\ \ddot{y} &= C\ddot{x} = CA^2x \\ &\vdots \\ y^{(n-1)} &= CA^{n-1}x \end{aligned}$$

$$\Rightarrow \quad x(0) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix}$$

Must be invertible,

NO ZERO DET.,

FULL RANK

TEST FOR OBSERVABILITY

The system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is completely state observable iff the row vectors of the **observability matrix**

$$W_c = [B \ \ AB \ \ A^2B \cdots]$$

$$W_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

span the n-dimensional space (i.e., W_O has rank n).

OBSERVABILITY OF DIAGONAL SYSTEM

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\lambda_1 \neq \lambda_2$$
$$\lambda_1 \neq \lambda_3$$
$$\lambda_2 \neq \lambda_3$$

$$\hat{C} = [c_1 \ \ c_2 \ \ c_3]$$

$$\hat{W}_o = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \hat{C}\hat{A}^2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1\lambda_1 & c_2\lambda_2 & c_3\lambda_3 \\ c_1\lambda_1^2 & c_2\lambda_2^2 & c_3\lambda_3^2 \end{bmatrix}$$

- ONLY ONE OUTPUT NEEDS TO BE OBSERVABLE

b/c THE DET. WILL BE ZERO

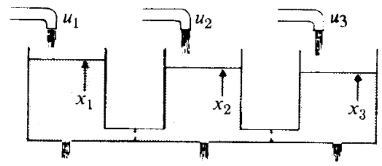
Observable as long as $c_1 \neq 0$, $c_2 \neq 0$, $c_3 \neq 0$

RECALL THE THREE-TANK EXAMPLE:

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

SHOULD ALL BE NEG.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Focus on y_2:

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -3 & 2 \\ -12 & 13 & -12 \end{bmatrix} \sim \mathrm{DET}[W_o] = 0$$

NOT INVERTIBLE
NOT FULLY OBSERVABLE

$\mathrm{RANK}[W_o] = 2$

1 UNOBSERVABLE STATE

OBSERVABILITY OF JORDAN FORM

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$C = [c \quad 0 \quad 1]$$

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} c & 0 & 1 \\ -2c & c & a \\ 4c & -4c & a^2 \end{bmatrix}$$

FOR FULL RANK $c \neq 0$
$a \neq -2$

DET →

$c((c)(a^2) - (a)(-4c)) + 1((-2c)(-4c) - (4c)(c))$

$c^2a^2 + 4ac^2 + 8c^2 - 4c^2$

$= c^2a^2 + 4ac^2 + 4c^2 = 0$

@ $c = 0$

@ $a = -2$

OBSERVABLE CANONICAL FORM

$$A_O = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} = A_C^T$$

$$C_O = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = B_C^T$$

TRANSPOSE OF CONTROLLABLE CANONICAL FORM B MATRIX

$$W_O^O = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1+a_2^2 \end{bmatrix} = W_C^C$$

DET = 1

RANK 3 FOR any a_i

A CONTROLLABLE SYSTEM CAN ALWAYS BE TRANSFORMED INTO OBSERVABLE CANONICAL FORM

(1 INPUT TO 1 OUTPUT CASE)

$$\begin{matrix} \dot{x} = Ax \\ y = Cx \end{matrix} \quad \xRightarrow{x=Tz} \quad \begin{matrix} \dot{z} = A_O z \\ y = C_O z \end{matrix}$$

$$A_O = T^{-1}AT$$
$$C_O = CT$$

$$W_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad W_O^O = \begin{bmatrix} C_O \\ C_O A_O \\ C_O A_O^2 \\ \vdots \\ C_O A_O^{n-1} \end{bmatrix} = \begin{bmatrix} CT \\ CTT^{-1}AT \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$W_o^o = W_o T$$

$$T = W_o^{-1} W_o^o$$

$$T^{-1} = (W_o^o)^{-1} W_o$$

$$T^{-1} = \underbrace{\begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\text{DET OF TRANSFER FUNCTION}} W_o$$

IN GENERAL, TO TRANSFORM TO OBSERVABLE CANONICAL FORM:

$$T^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & a_4 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-1} & 1 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} W_O$$

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

WHEN A SYSTEM IS NOT OBSERVABLE, IT CAN BE PARTITIONED INTO OBSERVABLE AND UNOBSERVABLE PARTS.

Given $\dot{x} = Ax + Bu$
$y = Cx$

$$W_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If $\text{rank}[W_O] = \ell < n$, then W_O has only ℓ linearly independent vectors, and $n - \ell$ states are un~~controllable~~. OBSERVABLE ℓ RANK IS NOT FULL RANK

OBSERVABLE DECOMPOSITION

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xRightarrow{x=Tz} \quad \begin{aligned} \dot{z} &= \hat{A}z + \hat{B}u \\ y &= \hat{C}z \end{aligned}$$

$$\begin{aligned} \hat{A} &= T^{-1}AT \\ \hat{B} &= T^{-1}B \\ \hat{C} &= CT \end{aligned}$$

set

$$T^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \Rightarrow T = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{-1} = [T_1 \mid T_2]$$

where

$$U_1 = \begin{bmatrix} C \\ \vdots \\ CA^{\ell-1} \end{bmatrix}$$

U_2 = any $n \times (n - \ell)$ matrix that makes T^−1 nonsingular

OBSERVABLE

$$\begin{bmatrix} \dot{z}_O \\ \dot{z}_{UO} \end{bmatrix} = \begin{bmatrix} \hat{A}_O & 0 \\ \hat{A}_{OU} & \hat{A}_{UO} \end{bmatrix} \begin{bmatrix} z_O \\ z_{UO} \end{bmatrix} + \begin{bmatrix} \hat{B}_O \\ \hat{B}_{UO} \end{bmatrix} u$$

$$y = [\hat{C}_O \mid 0] \begin{bmatrix} z_O \\ z_{UO} \end{bmatrix}$$

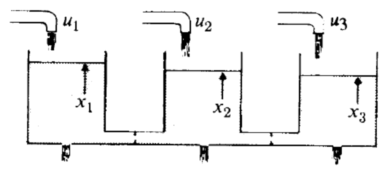
NOT OBSERVABLE

RECALL THE THREE-TANK EXAMPLE:

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$



Decompose into observable and unobservable subsystems:

$$W_o = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -3 & 2 \\ -12 & 13 & -12 \end{bmatrix}$$

RANK 2

$\downarrow$

1 UNOBSERVABLE STATE

$$T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -3 & 2 \\ \hline 0 & 0 & 1 \end{bmatrix} \Longrightarrow T = \begin{bmatrix} 1.5 & .5 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

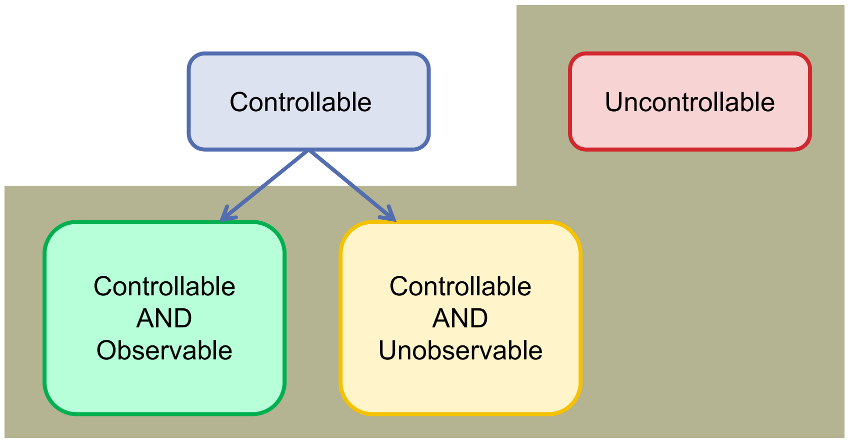
ANYTHING TO MAKE IT INVERTIBLE

$$\dot{z} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -5 & -6 & 0 \\ \hline 1 & 0 & -3 \end{array}\right] z + \begin{bmatrix} 0 & 1 & 0 \\ 2 & -3 & 2 \\ \hline 0 & 0 & 1 \end{bmatrix} u$$

$$y = \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right] z$$

z_3 IS UNOBSERVABLE

CAN WE COMBINE THE CONCEPTS OF CONTROLLABILITY AND OBSERVABILITY INTO A SINGLE DECOMPOSITION?



TO PUT THIS INTO STATE-SPACE:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned} \quad \overset{x=Tz}{\Longrightarrow} \quad \begin{aligned}\dot{z} &= \hat{A}z + \hat{B}u \\ y &= \hat{C}z\end{aligned}$$

$$\begin{bmatrix} \dot{z}_{CUO} \\ \dot{z}_{CO} \\ \dot{z}_{UC} \end{bmatrix} = \begin{bmatrix} \hat{A}_{CUO} & \hat{A}_{12} & \hat{A}_{13} \\ 0 & \hat{A}_{CO} & \hat{A}_{23} \\ 0 & 0 & \hat{A}_{UC} \end{bmatrix} \begin{bmatrix} z_{CUO} \\ z_{CO} \\ z_{UC} \end{bmatrix} + \begin{bmatrix} \hat{B}_{CUO} \\ \hat{B}_{CO} \\ 0 \end{bmatrix} u$$

UNCONTROLLABLE NOT EFFECTED BY INPUT

$$y = \begin{bmatrix} 0 & \hat{C}_{CO} & \hat{C}_{UC} \end{bmatrix} \begin{bmatrix} z_{CUO} \\ z_{CO} \\ z_{UC} \end{bmatrix} + Du$$

UNOBSERVABLE CAN NOT BE SEEN IN THE OUTPUT

Why not decompose z_{UC}?

COMING UP…

State Observer Design

- Method 1
- Method 2 (Ackermann's Formula)

Output Feedback

- Separation Principle
- Reduced-Order Observer