FROM LAST TIME...

State Observers

- Rationale
- Observer Design
- Method 1 – "POLE PLACEMENT"
- Method 2 (Ackermann's Formula)

We know:

$\dot{x} = Ax + Bu$

$y = Cx$

Measure:

u and y

Calculate:

x

A-LC

$$L = T\begin{bmatrix} \alpha_0 - a_0 \\ \vdots \\ \alpha_{n-1} - a_{n-1} \end{bmatrix}$$

FROM |SI-A| CHARACTERISTIC EQUATION

"POLE PLACEMENT"

$$L = \phi(A)W_O^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

ACKERMAN'S FORMULA

STATE OBSERVER + FEEDBACK

Topics

- Output feedback
- Reduced-order observer

At the end of this section, students should be able to:

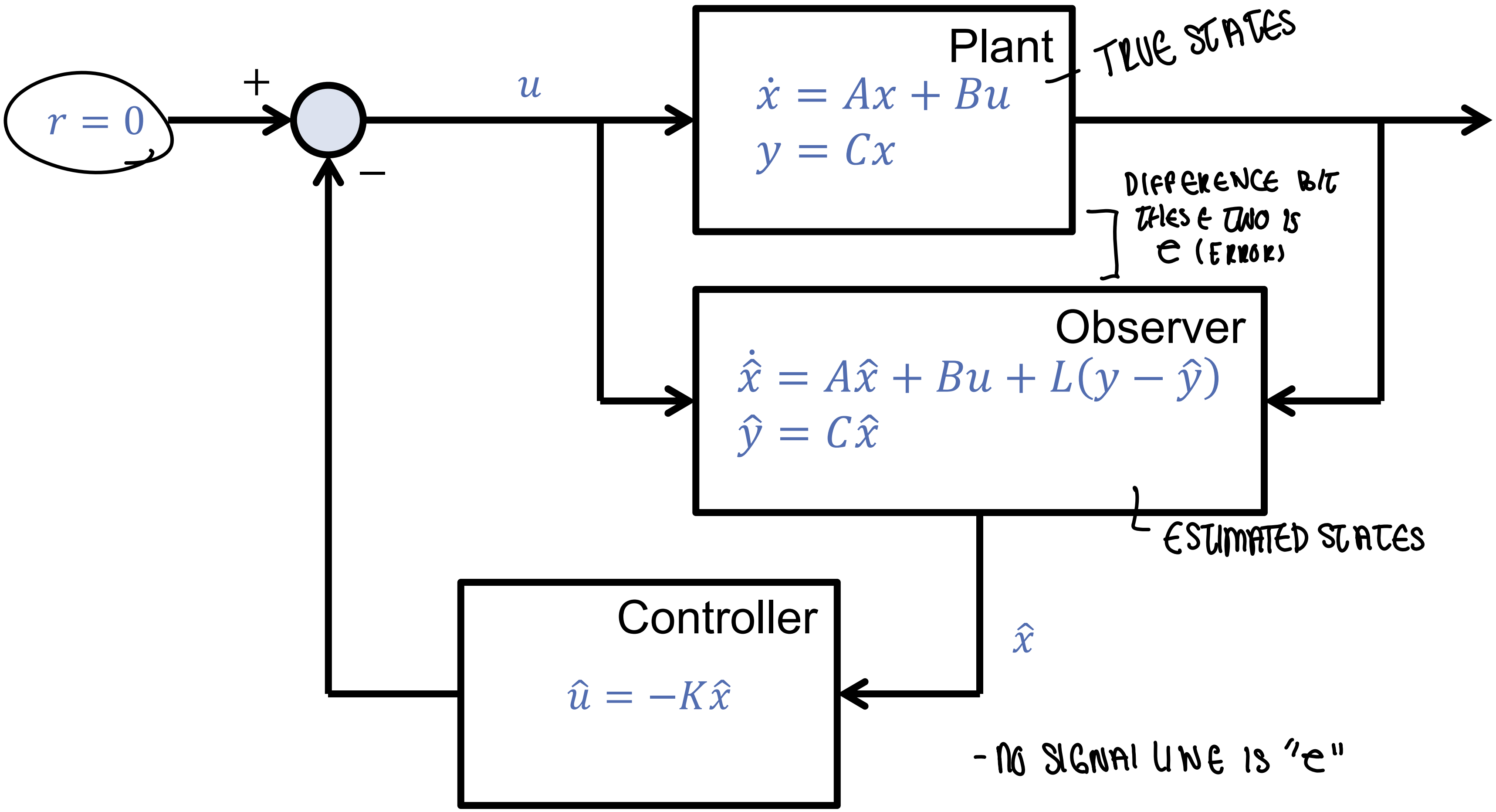
- Combine state observer and feedback into output feedback.
- Explain the benefit of a reduced-order state observer.
- Design a reduced-order state observer.

STATE OBSERVER + FEEDBACK

OUTPUT FEEDBACK

MEASURE INPUT
MEASURE OUTPUT
ESTIMATE STATES

LET'S COMBINE THE STATE OBSERVER WITH STATE FEEDBACK TO CREATE OUTPUT FEEDBACK



$\hat{}$ STATE

DEFINE ESTIMATION ERROR AS

$$e = x - \hat{x}$$

$U = -K\hat{X}$

└ STATE FEEDBACK BASED ON ESTIMATED STATES

"WE NEED TO INVENT MORE LETTERS" - MYNDERSE

$\dot{X} = AX + BU = AX - BK\hat{X}$

$+ BKX - BKX$

$\dot{X} = (A - BK)_X + BKe$

FROM OBSERVER DYNAMICS $\dot{e} = (A - LC)e$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

REGULATOR CLOSED LOOP DYNAMICS W/ OBSERVER

└ CLOSED-LOOP FREE RESPONSE

CLOSED LOOP EIGEN VALUES

$$\left| SI - \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \right| = |SI - A + BK| \cdot |SI - A + LC|$$

└ POLES FROM STATE FEEDBACK

└ POLES FROM STATE OBSERVER

SYSTEM WILL BE A COMBINATION OF POLES

$$\begin{matrix} SI - A + BK & -BK \\ 0 & SI - A + LC \end{matrix}$$

$\det(SI - A) =$

$(SI - A + BK)(SI - A + LC)$

FIND THE CLOSED-LOOP EIGENVALUES

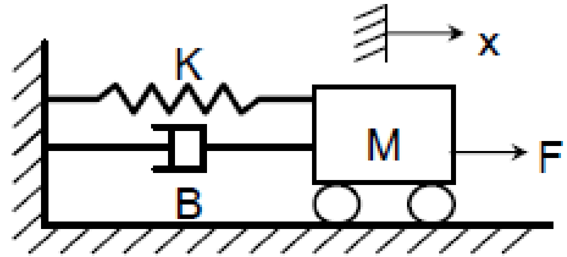
SEPARATION PROPERTY ALLOWS US TO SEPARATE THE STATE OBSERVER AND STATE FEEDBACK DESIGNS

In an output feedback controller, the state feedback controller and state observer can be independently designed provided that . . .

1. A, B, and C are known exactly
2. (A, B) controllable
3. (A, C) observable

APPLY SEPARATION PROPERTY TO AN EXAMPLE

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



State feedback gains to give $\zeta_{CL} = 0.7$ and $\omega_{nCL} = 5\ \mathrm{r}/s$:

$$K = \begin{bmatrix} 24 & 6 \end{bmatrix}$$

Observer poles at $s = -18$ requires observer gains:

$$L = \begin{bmatrix} 35 \\ 288 \end{bmatrix}$$

$$\dot{x} = (A-BK)x + BKe$$

$$= \left(\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}24 & 6\end{bmatrix}\right)x \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}24 & 6\end{bmatrix}e$$

DUE TO OBSERVER

$$\dot{e} = (A - LC)e$$

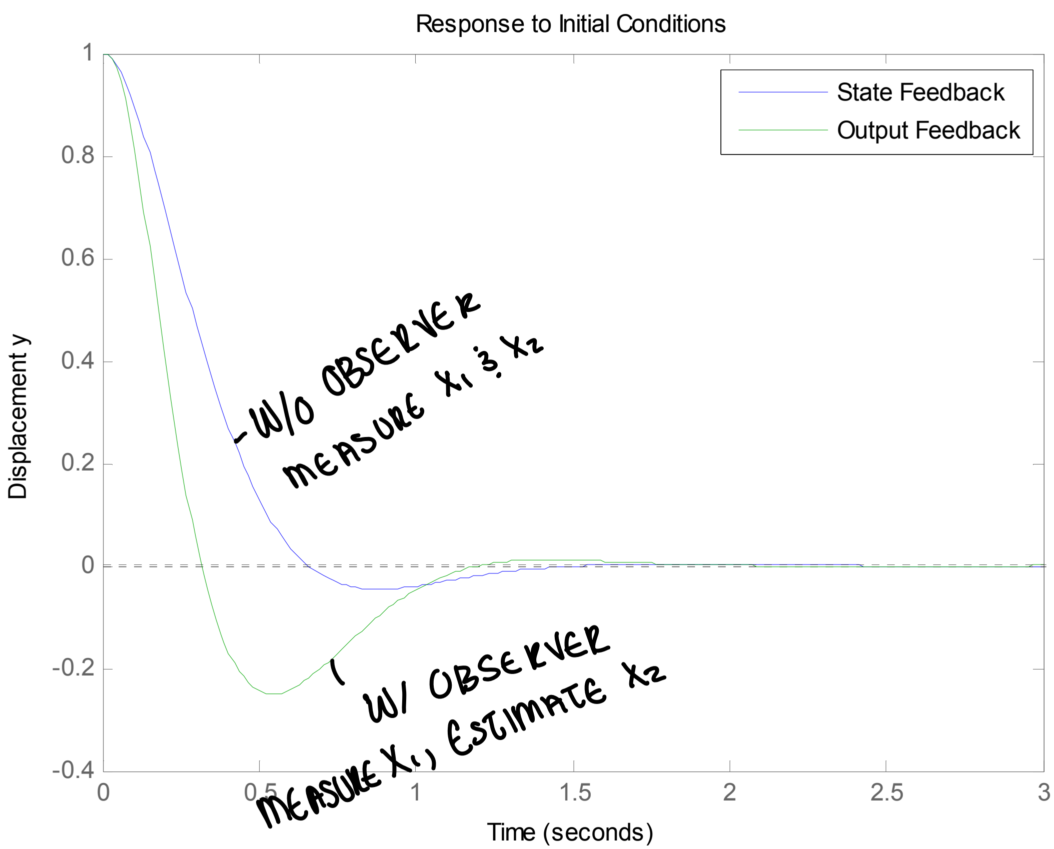
$$= \left(\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} - \begin{bmatrix}35\\ 288\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)e$$

$e = x - \hat{x}$

FREE RESPONSE OF THE CLOSED LOOP SYSTEM

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{e}_1\\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -25 & -7 & 24 & 6\\ 0 & 0 & -35 & 1\\ 0 & 0 & -289 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ e_1\\ e_2\end{bmatrix}$$

SIMULATE THE CLOSED-LOOP SYSTEM

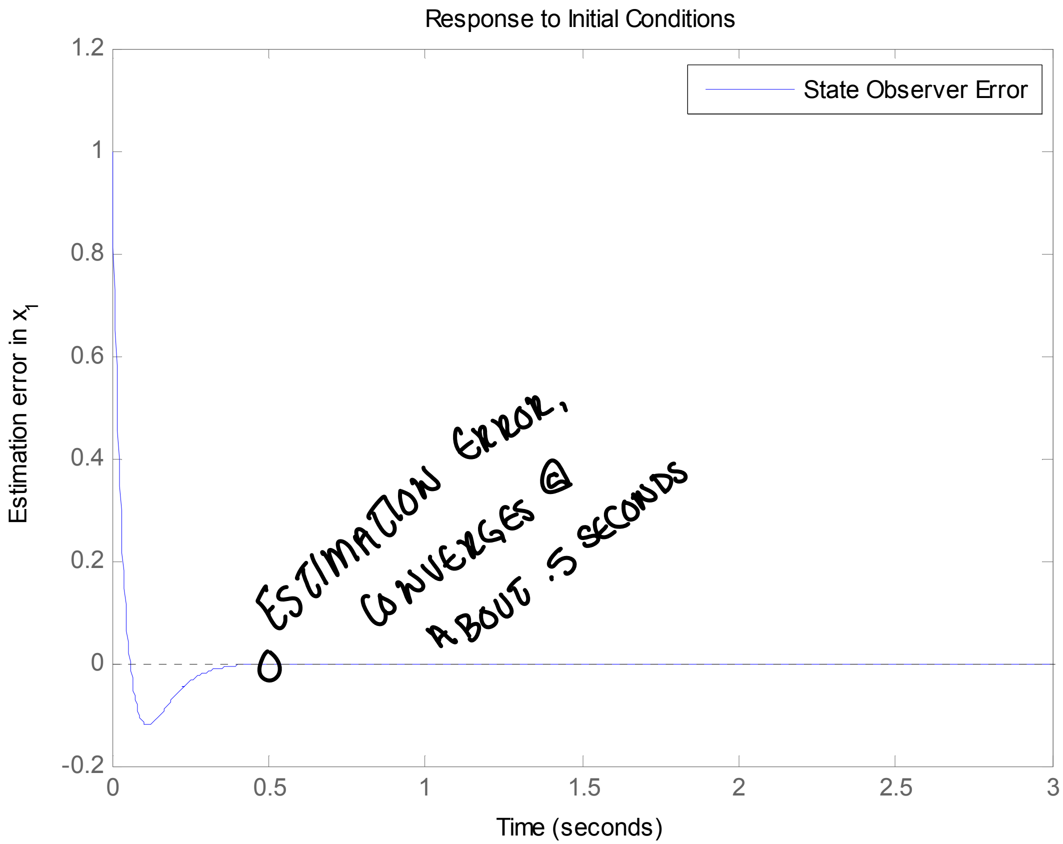


$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

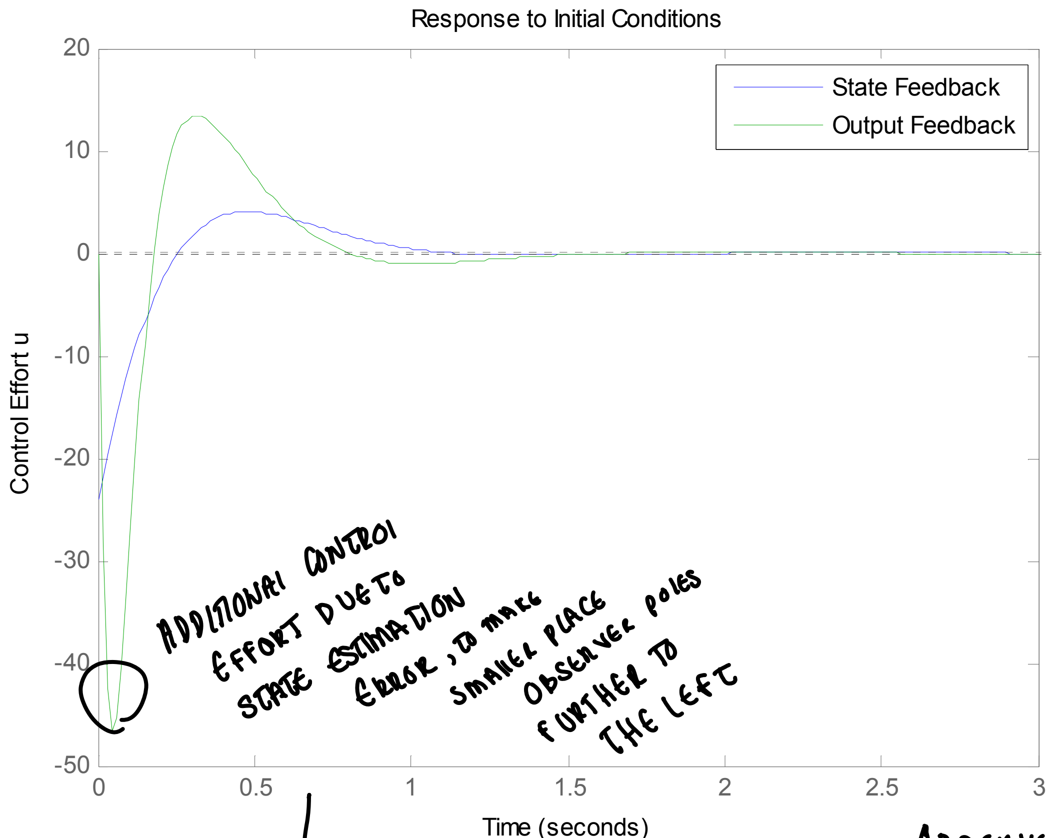
$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$e(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

HOW WELL DID THE STATE OBSERVER PERFORM?



WHAT CONTROL EFFORT WAS REQUIRED?



$$u = -K\hat{x}$$

$$= -K(x - e)$$

$$= \begin{bmatrix} -K & K \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

PLOTTED USING INITIAL(SYS, X_0) — SYS: STATE SPACE; X_0: INITIAL CONDITIONS

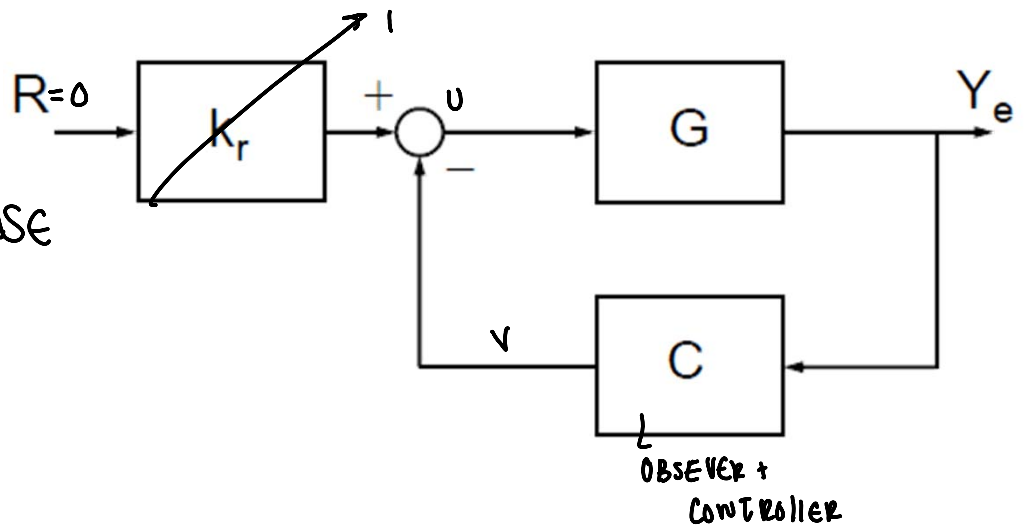
$A = [\quad]$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C = [-k_1 \; -k_2 \; k_1 \; k_2]\, x$

- OBSERVER POLES SHOULD BE 5X FASTER THAN THE CONTROLLER POLES

CAN WE CONVERT THE COMBINED OBSERVER + CONTROLLER TO TRANSFER FUNCTION?

Free Response



$U = -K\hat{x}$

$V = -U = K\hat{x}$

Observer Dynamics

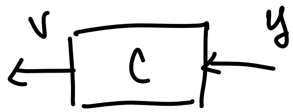
$$s\hat{x}(s) = (A - LC)\hat{x}(s) + BU(s) + Ly(s)$$

$$\hat{x}(s) = (sI - A + LC + BK)^{-1} L y(s)$$

where $s\hat{x} = s(I - \hat{x})$

eg $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

THE FEEDBACK COMPENSATOR IS GIVEN BY…



$$\dot{x}_C = (A - BK - LC)x_C + Ly$$
$$v = Kx_C$$

← OUTPUT / INPUT OF THE OUTPUT FEEBACK COMPENSATOR

$$C(s) = \frac{V(s)}{Y(s)} = K(sI - A + LC + BK)^{-1}L$$

$\dot{X} = AX + BU$
$Y = CX$
$\frac{Y}{U} = C(sI - A)^{-1}B$

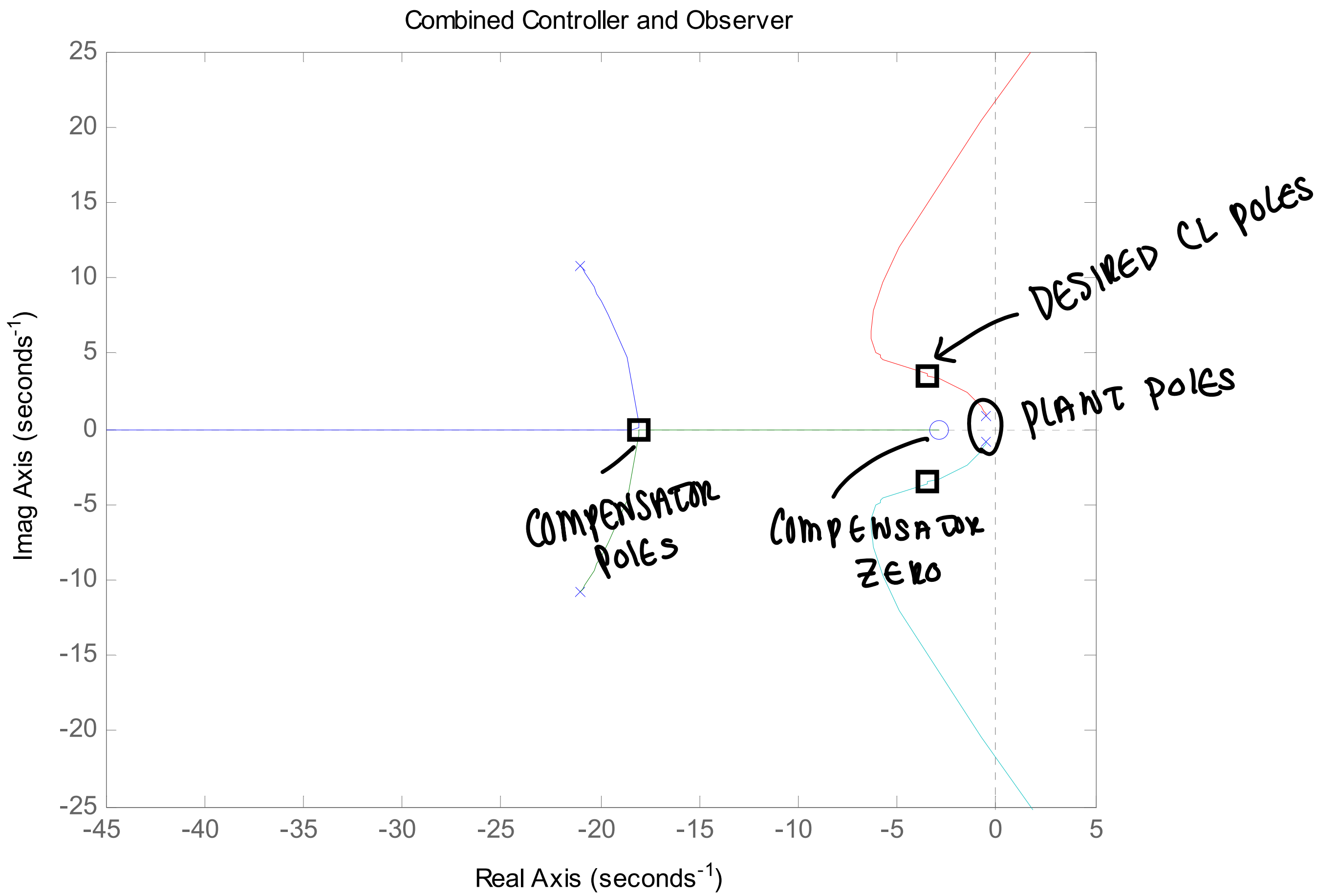
$$= \begin{bmatrix} 24 & 6 \end{bmatrix} \begin{bmatrix} s+35 & -1 \\ 313 & s+7 \end{bmatrix}^{-1} \begin{bmatrix} 35 \\ 288 \end{bmatrix}$$

$$= \frac{2568s + 7542}{s^2 + 42s + 588}$$

WHY DO WE CARE

- IT'S A GOOD WAY TO LOOK @ THINGS - MYNDERSE

EXAMINE THE ROOT LOCUS OF THE CLOSED-LOOP SYSTEM



LET'S COMPARE STATE FEEDBACK AND OUTPUT FEEDBACK COMPENSATORS FURTHER

$$C(s) = 6s + 24$$

MEASURE
x_1 & x_2

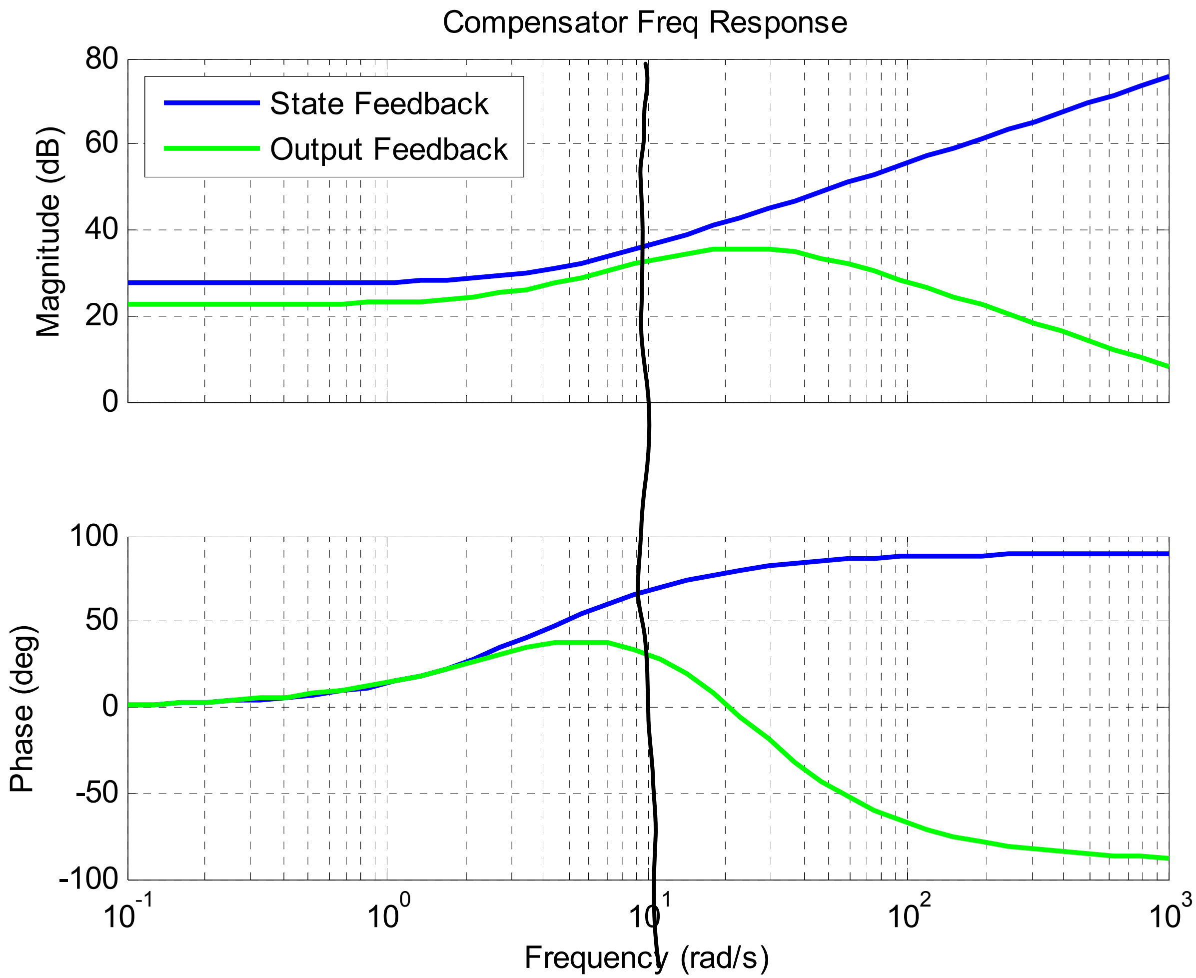
w/ OBSERVER

$$C(s) = \frac{2568s + 7542}{s^2 + 42s + 558}$$

MEASURE ONLY OUTPUT y

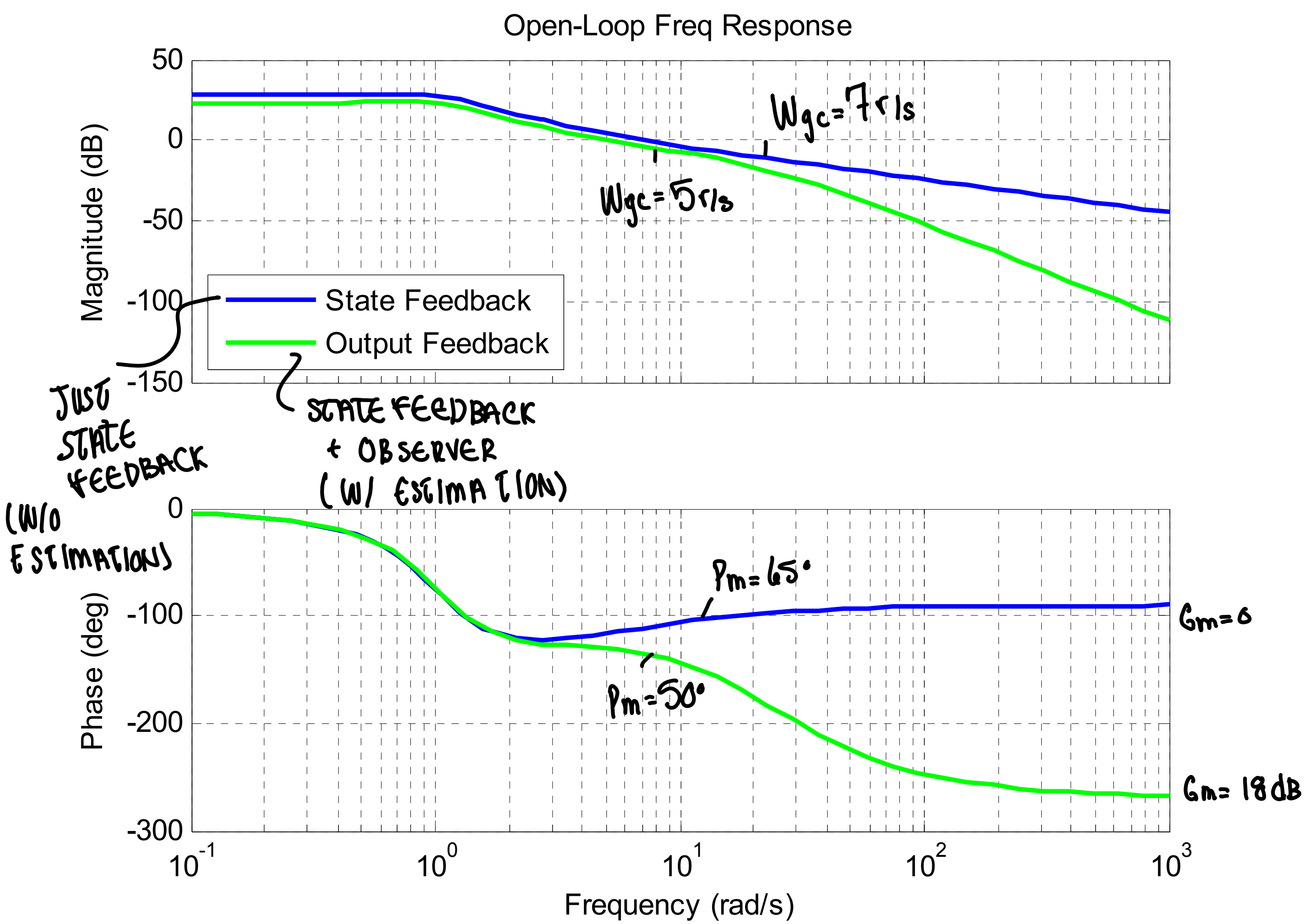
The output feedback compensator is strictly proper

COMPENSATOR FREQUENCY RESPONSE IS SIMILAR BELOW $10\ r/s$



SIMILARLY, THE OPEN LOOP ($L = CG$) FREQUENCY RESPONSES ARE COMPARABLE BELOW $10\ r/s$.

CONTROLLER & OBSERVER



WHAT IS THE EFFECT OF THE OBSERVER POLES?

$(s+18)^2$
$p=-18$
OR $(s-(-p))$

Let: $|sI - A + LC| = (s+p)^2 = s^2 + 2ps + p^2$

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = T\begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p^2 - 1 \\ 2p - 1 \end{bmatrix} = \begin{bmatrix} 2p - 1 \\ p^2 - 2p \end{bmatrix}$$

L

L IS DIRECTLY RELATED TO P

$$C(s) = \begin{bmatrix} 24 & 6 \end{bmatrix}\begin{bmatrix} s+\ell_1 & -1 \\ \ell_2 + 25 & s+7 \end{bmatrix}^{-1}\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$$

$$= \frac{(24\ell_1 + 6\ell_2)s + (18\ell_1 + 24\ell_2)}{s^2 + (\ell_1 + 7)s + (7\ell_1 + \ell_2 + 25)}$$

Next, substitute the known observer gains to observe pole effects!

$$C(s) = \frac{6(p^2+6p-4)s + (24p^2-12p-18)}{s^2+(2p+6)s+(p^2+12p+18)} \cdot \frac{\frac{1}{p^2}}{\frac{1}{p^2}}$$

- I CAN'T SPELL

NOT NESSASARY JUST MAKES THE MATH EASIER TO SEE

$$C(s) = \frac{6\left(1+\frac{6}{p}-\frac{4}{p^2}\right)s + \left(24-\frac{12}{p}-\frac{18}{p^2}\right)}{\frac{1}{p^2}s^2+\left(\frac{2}{p}+\frac{6}{p^2}\right)s+\left(1+\frac{12}{p}+\frac{18}{p^2}\right)}$$

$p \to \infty$

$$p \to \infty, \qquad C(s) \to 6s + 24$$

As the observer poles become infinitely fast, the observer error converges to zero in infinitesimal time.

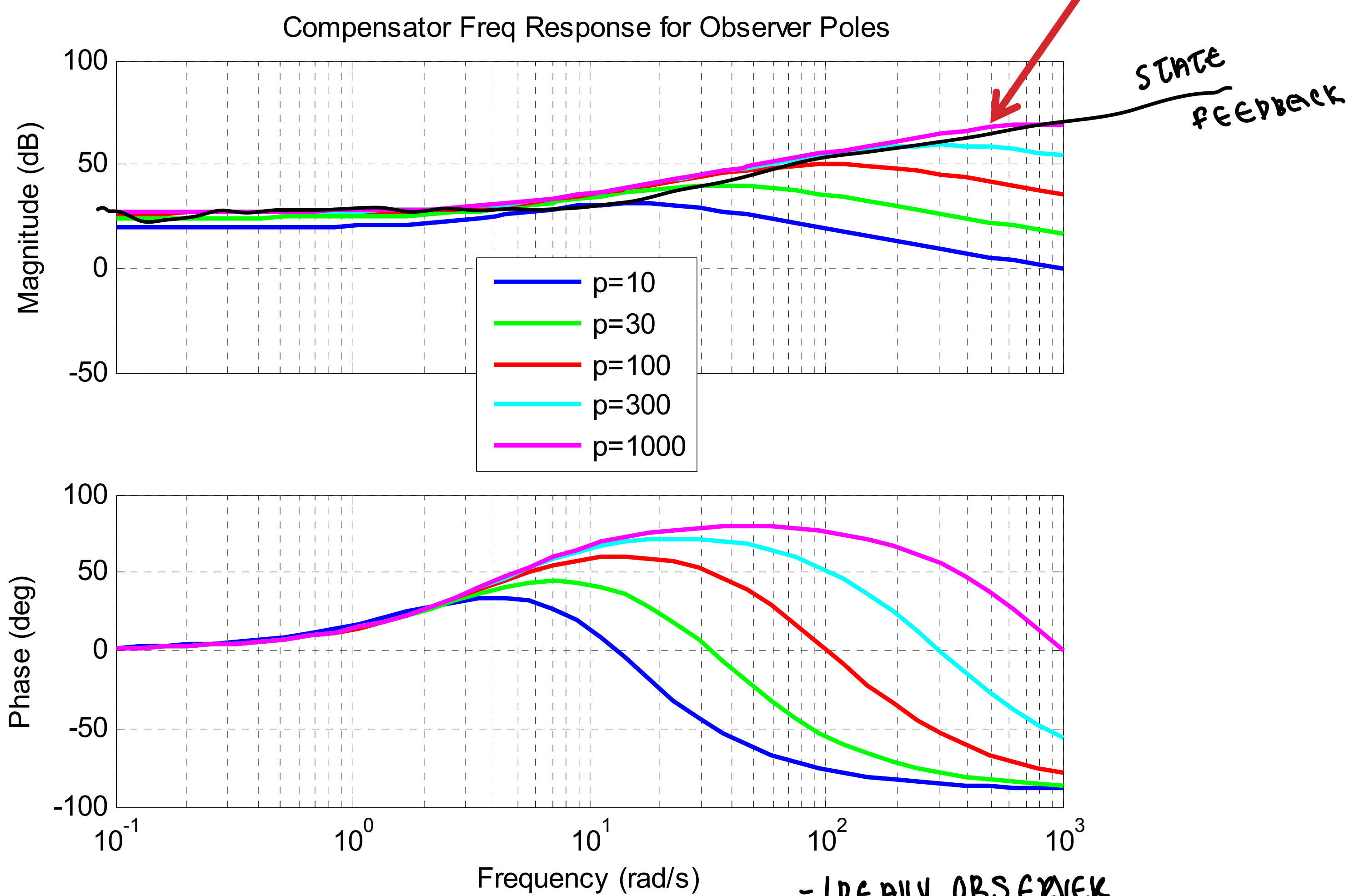
ERROR WILL BE SMALL BETWEEN TRUE & ESTIMATED MODEL

Then we have state feedback!

LIKE W/O AN OBSERVER

LET THE VALUE OF p VARY

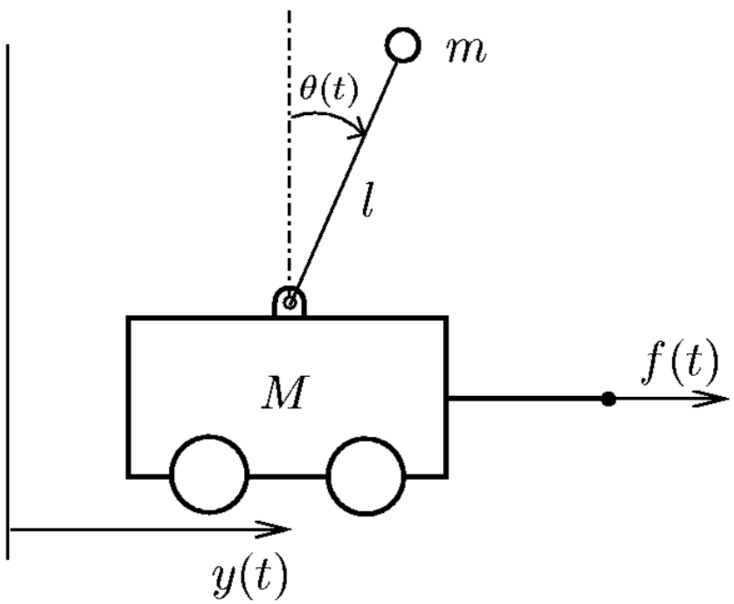
Getting closer to state feedback!



- IDEALLY OBSERVER SHOULD HAVE FAST POLES

REDUCED-ORDER OBSERVER

RECALL THE ROBOTIC WELDING PROBLEM



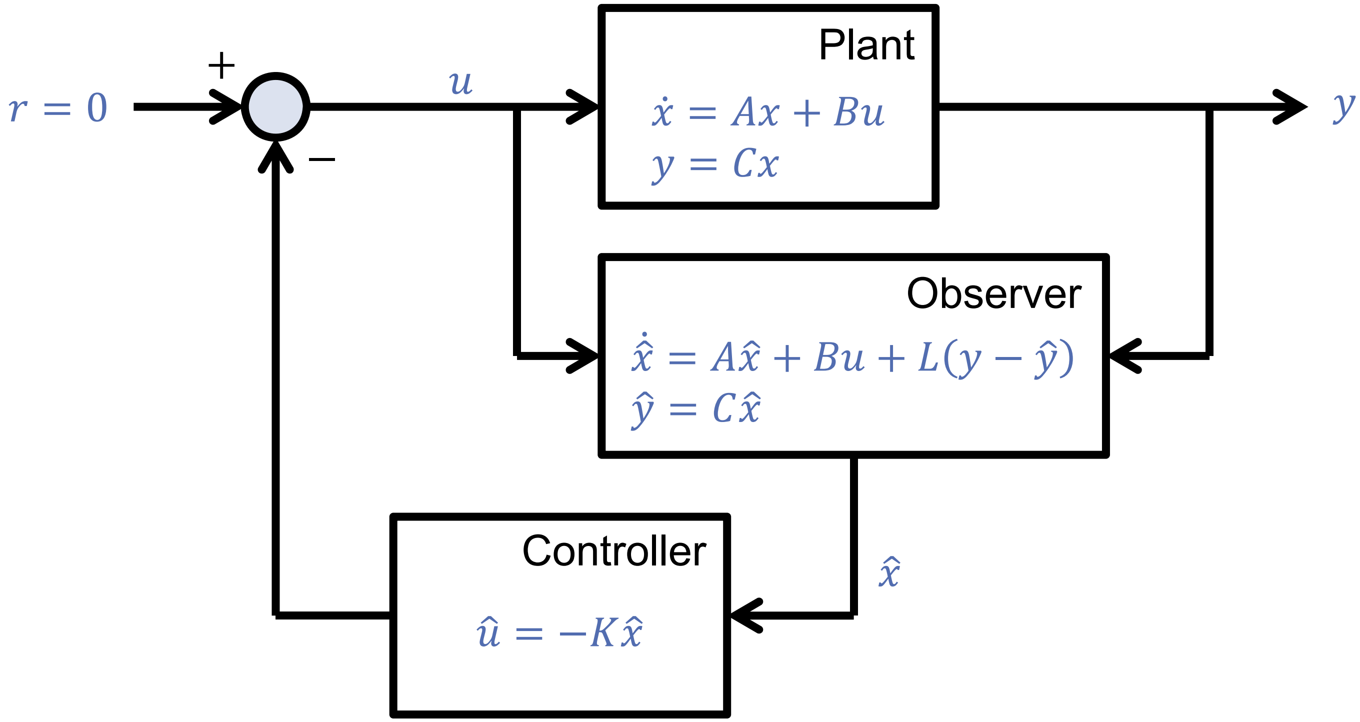
$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 20 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} f(t)$$

$$\begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

System is fully state observable with output y

- Choose y for output feedback design

DESIGN A STATE OBSERVER AND STATE FEEDBACK



Notice that we can measure output $x_1 = y$, but our state feedback instead uses $\hat{x}_1 = \hat{y}$.

Why not use the measurement instead of the estimate?

DEFINE A REDUCED ORDER OBSERVER USING m LINEARLY INDEPENDENT MEASURED OUTPUTS

n	states
m	measured outputs
$n-m$	estimated states

Given: $\dot{x} = Ax + Bu$

(assuming $m = 1$) $y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x$

1 STATE

Partition the state vector: $x = \left[\begin{array}{c} x_1 \\ \hline x_e \end{array}\right]$

n-1 STATES

Measured state

Estimated states

PARTITION THE STATE SPACE BASED ON STATE VECTOR PARTITION

1x1 1xn1

$$\begin{bmatrix} \dot{x}_1 \\ \hline \dot{x}_e \end{bmatrix} = \left[\begin{array}{c|c} a_{11} & A_{1e} \\ \hline A_{e1} & A_{ee} \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_e \end{bmatrix} + \begin{bmatrix} b_1 \\ \hline B_e \end{bmatrix} u$$

n-1x1 n-1xn1

MEASURED STATES

ESTIMATED STATES

Express as plant in terms of states x_e:

$$\dot{x}_e = A_{ee}x_e + (A_{e1}x_1 + B_e u)$$

ALMOST AN INPUT LIKE ≈ BU

OUTPUT ≈ y

$$(\dot{x}_1 - a_{11}x_1 - b_1 u) = A_{1e}x_e$$

$\dot{X} = AX + BU$

$y = CX$

Reduced-order observer equation:

$$\dot{\hat{x}}_e = \underbrace{A_{ee}\hat{x}_e}_{A\hat{X}} + \underbrace{(A_{e1}x_1 + B_e u)}_{BU} + L(\underbrace{\dot{x}_1 - a_{11}x_1 - b_1 u}_{y} - \underbrace{A_{1e}\hat{x}_e}_{C\hat{X}})$$

$$\dot{\hat{x}}_e = (A_{ee} - LA_{1e})\hat{x}_e + A_{e1}x_1 + B_e u + L(\dot{x}_1 - a_{11}x_1 - b_1 u)$$

HOW DO WE DESIGN THE REDUCED-ORDER OBSERVER GAINS?

Define $e = x_e - \hat{x}_e$

ESTIMATED STATES ONLY

$$\dot{e} = \dot{x}_e - \dot{\hat{x}}_e = (A_{ee} - LA_{1e})e$$

$$L = \phi(A_{ee}) \begin{bmatrix} A_{1e} \\ A_{1e}A_{ee} \\ \vdots \\ A_{1e}A_{ee}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

WILL BE OF SIZE n-1X1

ACKERMANS METHOD

This is a standard observer design method!

CONSIDER THE IMPLEMENTATION: THIS METHOD REQUIRES KNOWLEDGE OF THE OUTPUT DERIVATIVE

$$\dot{\hat{x}}_e = (A_{ee} - LA_{1e})\hat{x}_e + A_{e1}x_1 + B_e u + L(\boxed{\dot{x}_1} - a_{11}x_1 - b_1 u)$$

Differentiation amplifies measurement noise.

We want to eliminate differentiation!

DEFINE A CHANGE OF VARIABLES:

$$\hat{x}_p = \hat{x}_e - Ly \qquad \Rightarrow \qquad \hat{x}_e = \hat{x}_p + Ly$$

$$\dot{\hat{x}}_e = \dot{\hat{x}}_p + L\dot{y} \rightarrow L\dot{x}_1$$

L

p DOESN'T STAND FOR ANYTHING SIGNIFICANT

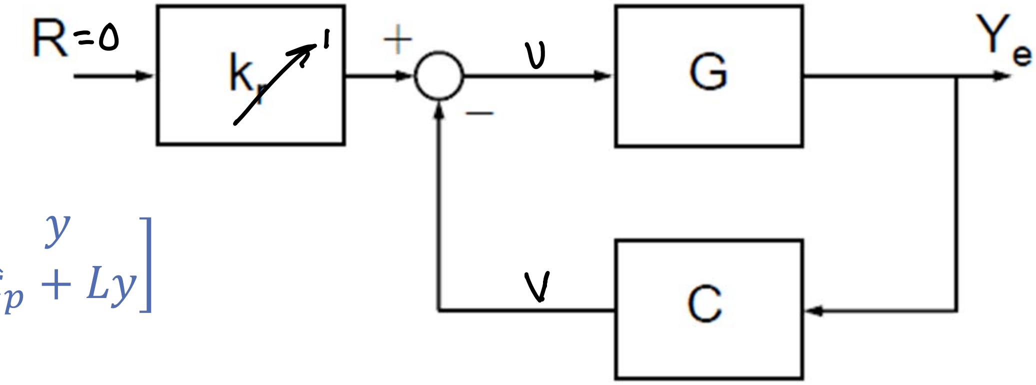
$$\dot{\hat{x}}_p + \cancel{L\dot{y}} = (A_{ee} - LA_{1e})(\hat{x}_p + Ly) + A_{e1}y + B_e u + L(\cancel{\dot{y}} - a_{11}y - b_1 u)$$

$$\dot{\hat{x}}_p = \underbrace{(A_{ee} - LA_{1e})}_{A-LC}\hat{x}_p + \underbrace{(A_{e1} - La_{11} + A_{ee}L - LA_{1e}L)}_{L}y + \underbrace{(B_e - Lb_1)}_{B}u$$

Recall that the full state vector is given by (A - LC) xp + LY + BU

$$\hat{x} = \begin{bmatrix} x_1 \\ \hat{x}_e \end{bmatrix} = \begin{bmatrix} y \\ \hat{x}_p + Ly \end{bmatrix} = \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix} \hat{x}_p + \begin{bmatrix} 1 \\ L \end{bmatrix} y$$

CAN WE MAKE A COMBINED REDUCED-ORDER OBSERVER + CONTROLLER SYSTEM?

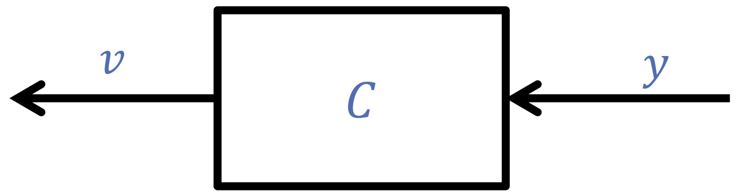


$$u = -K\hat{x} \quad = [-K_y \quad -K_p]\begin{bmatrix} y \\ \hat{x}_p + Ly \end{bmatrix}$$

$$\dot{\hat{x}}_p = (A_{ee} - LA_{1e})\hat{x}_p + (A_{e1} - La_{11} + A_{ee}L - LA_{1e}L)y + (B_e - Lb_1)u$$

$$\begin{aligned}\dot{\hat{x}}_p = (A_{ee} - LA_{1e})\hat{x}_p &+ (A_{e1} - La_{11} + A_{ee}L - LA_{1e}L)y \\ &+(B_e - Lb_1)\left[-\left(K_y + K_pL\right)y - K_p\hat{x}_p\right]\end{aligned}$$

FEEDBACK CONTROLLER C IS A STATE-SPACE SYSTEM!



$$\dot{\hat{x}}_p = \left(\overset{-1}{\cancel{A}_{ee}} - L\overset{1}{\cancel{A}_{1e}} - (\overset{1}{\cancel{B}_e} - L\overset{0}{\cancel{b}_1})K_p\right)\hat{x}_p$$
$$+\Big(A_{e1} - La_{11} + A_{ee}L - LA_{1e}L - (B_e - Lb_1)\big(K_y + K_pL\big)\Big)y$$
$$v = K_p\hat{x}_p + \big(K_y + K_pL\big)y$$

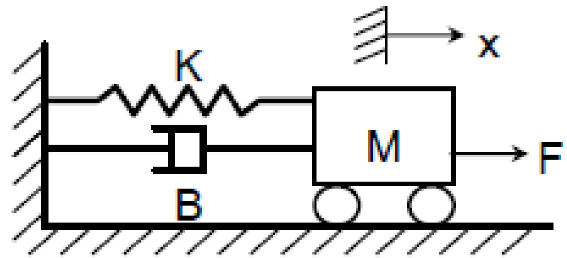
- CRIES

It may be more convenient to short-hand this controller as:

$$\dot{x}_C = A_C x_C + B_C y$$
$$v = C_C x_c + D_C y$$

DEVELOP A REDUCED-ORDER OBSERVER FOR THE PREVIOUS EXAMPLE:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



e DENOTES ESTIMATE

$A = \begin{bmatrix} a_{11} & A_{1e} \\ A_{e1} & A_{ee} \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -1 & -1 \end{array}\right]$ $B = \begin{bmatrix} b_1 \\ B_e \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix}$

$\hat{x}_z = \hat{x}_p + L y$

$\dot{\hat{x}}_p = (-1 - L)\hat{x}_p + (-1 - L \cdot 0 + (-1)\cdot(L - L^2(1)))y + (1 - L \cdot 0)u$

$\dot{e} = (-1 - L)e$

POLES @ -18

$|sI - A + LC| = (s+18)$

$|sI - (-1+L)| = s + 18$

$s + 1 + L = s + 18$

$L = 17$

SCALAR OBSER. $\dot{\hat{x}}_p = -18\hat{x}_p - 307y + u$

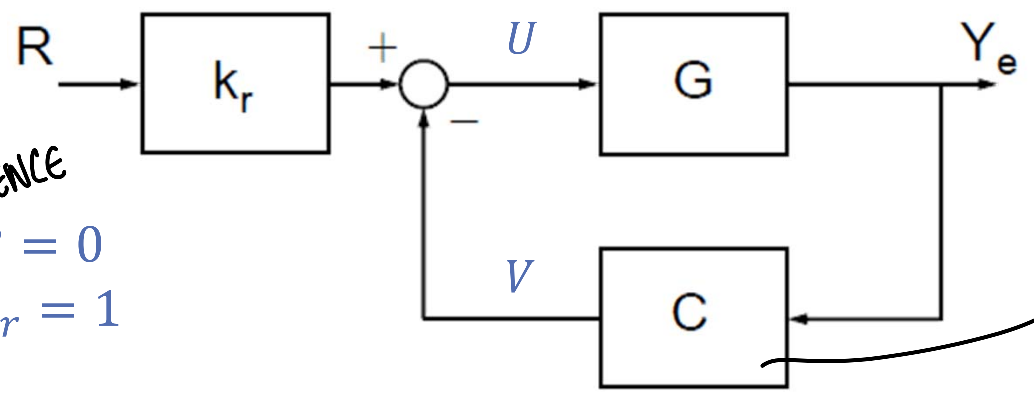
$\hat{x}_z = \hat{x}_p + 17y$

ESTIMATION OF THE 2nd STATE

CONTROLLER $u = -24y - 6\hat{x}_z$

CHOOSE THE REDUCED-ORDER OBSERVER POLE AT $s = -18$

CAN WE CONVERT THE REDUCED ORDER OBSERVER + CONTROLLER AS A COMPENSATOR?



REFERENCE

$R = 0$

$k_r = 1$

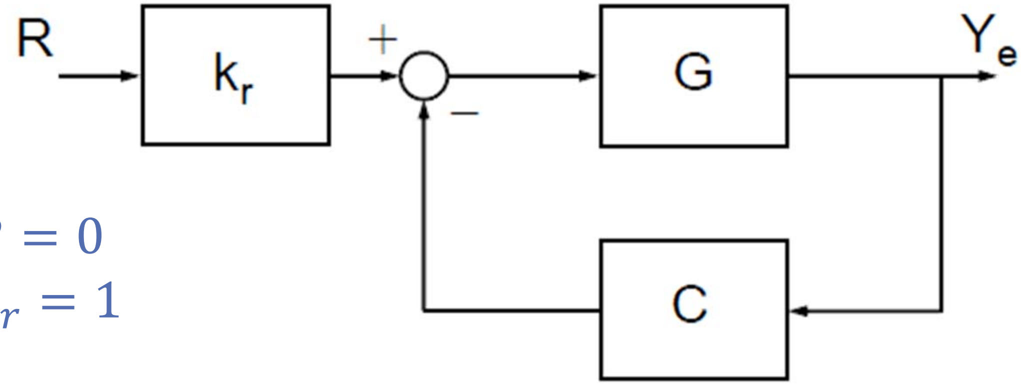
STATE ESTIMATE

$$U(s) = -24Y(s) - 6\hat{X}_2(s) = -V(s)$$

MEASUREMENT

$$\hat{X}_2 = \hat{X}_p(s) + 17Y(s)$$

$$\hat{X}_p(s) = \frac{-307Y(s) + U(s)}{s+18} \quad \Rightarrow \quad \hat{X}_2(s) = \frac{(17s-1)Y(s) + U(s)}{s+18}$$



$R = 0$

$k_r = 1$

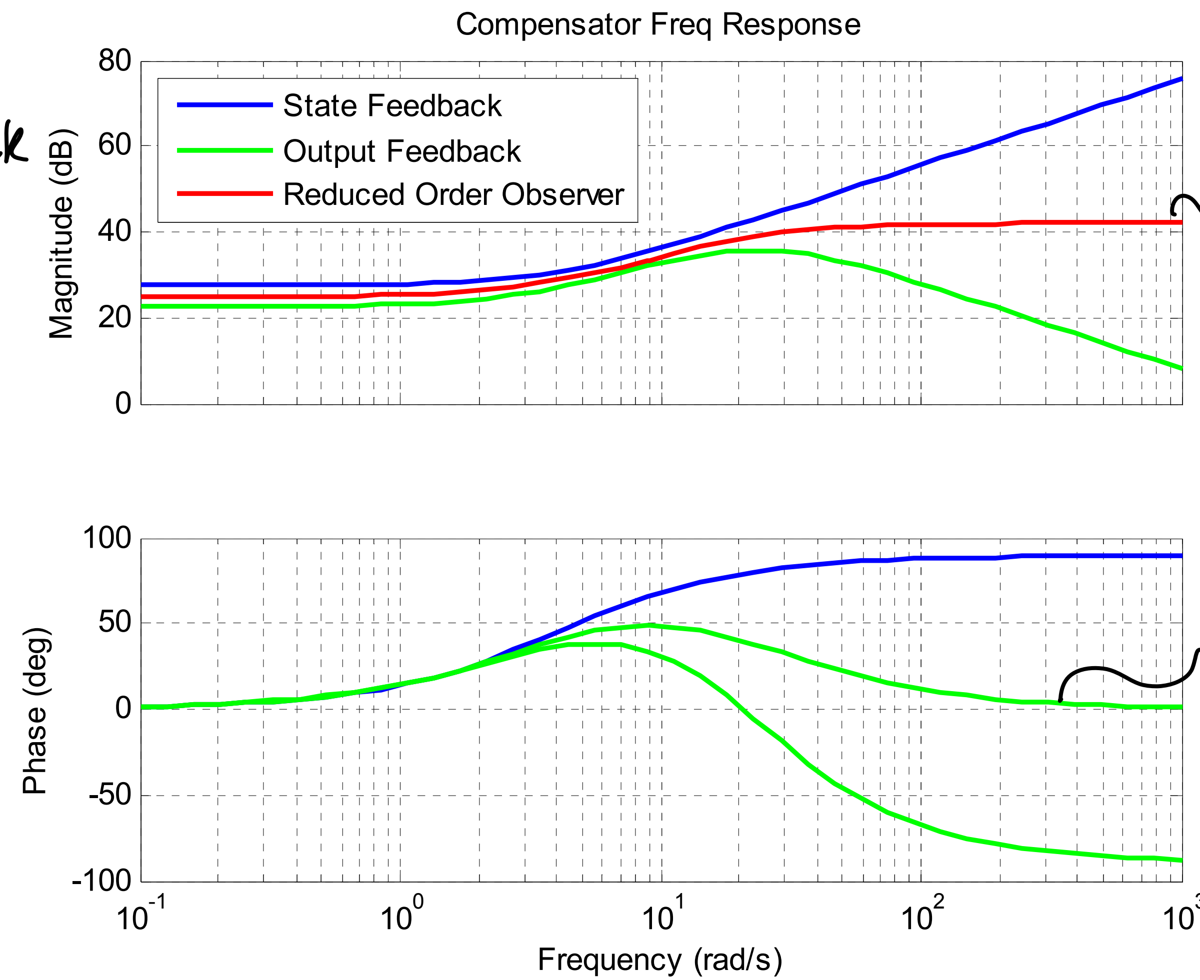
$$C(s) = \frac{V(s)}{Y(s)} = \frac{6(21s + 71)}{s + 24}$$

~ OUTPUT FEEDBACK COMPENSATOR IN TRANSFER FUNCTION FORM

COMPENSATOR FREQUENCY RESPONSE LIES BETWEEN STATE AND OUTPUT FEEDBACK

FULL ORDER
- STRICTLY PROPER

REDUCED ORDER
- BIPROPER



~ BIPROPER WILL FLATTEN

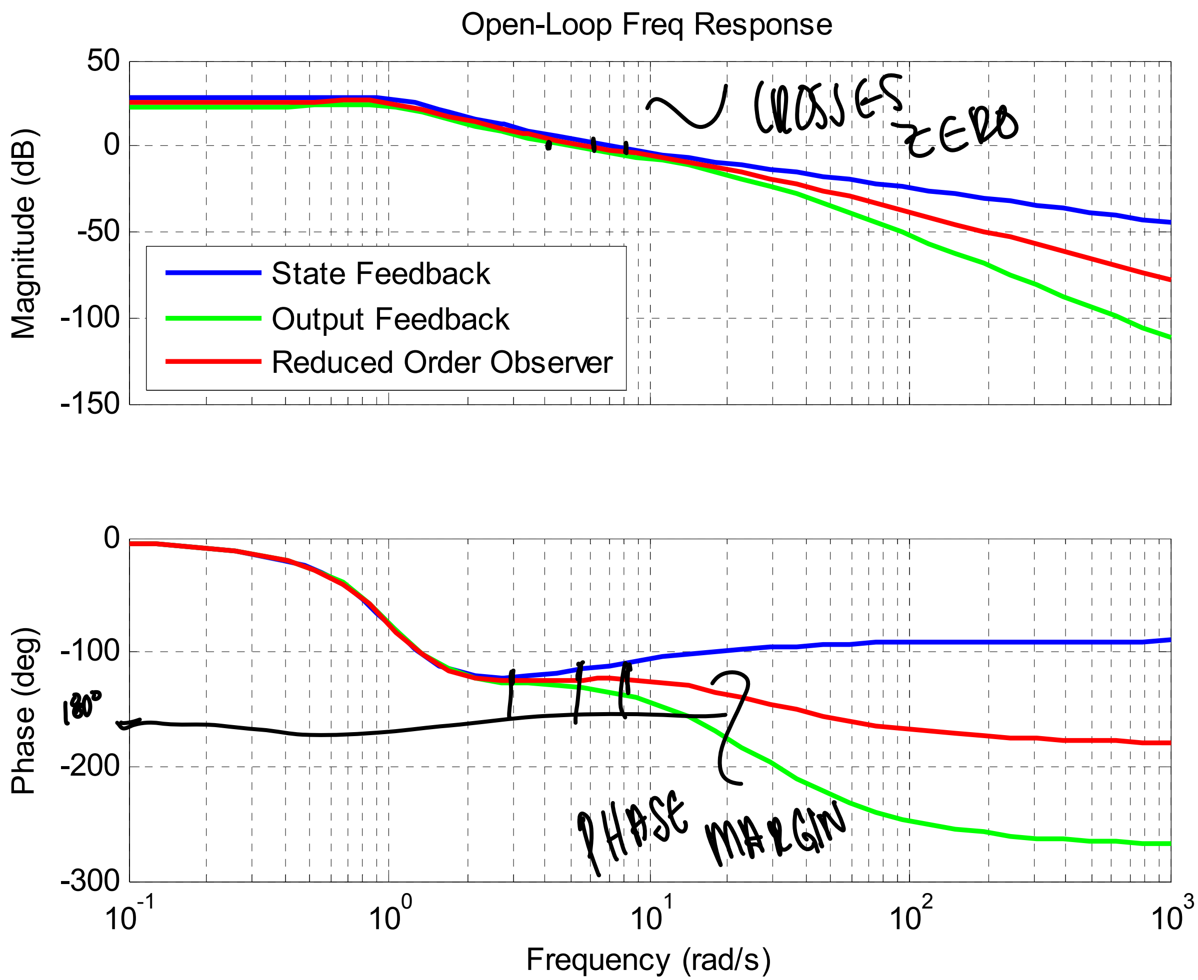
$\frac{\text{OUT}}{\text{IN}}$

RED

SIMILAR BEHAVIOR FOR THE OPEN LOOP ($L = CG$)

C PLANT

COMPENSATOR



COMING UP…

Case Study

Tracking and integral control