FROM LAST TIME…

Solution of LTI State Equations

- State Transition Matrix
- Free Response
- Forced Response

$$x(t) = e^{At}x(0)$$

$$e^{At} = Te^{\lambda t}T^{-1} = T\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

$$x(t) = \overbrace{\Phi(t)x(0)}^{\text{Free}} + \overbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}^{\text{Force}}$$

CONTROLLABILITY

Topics

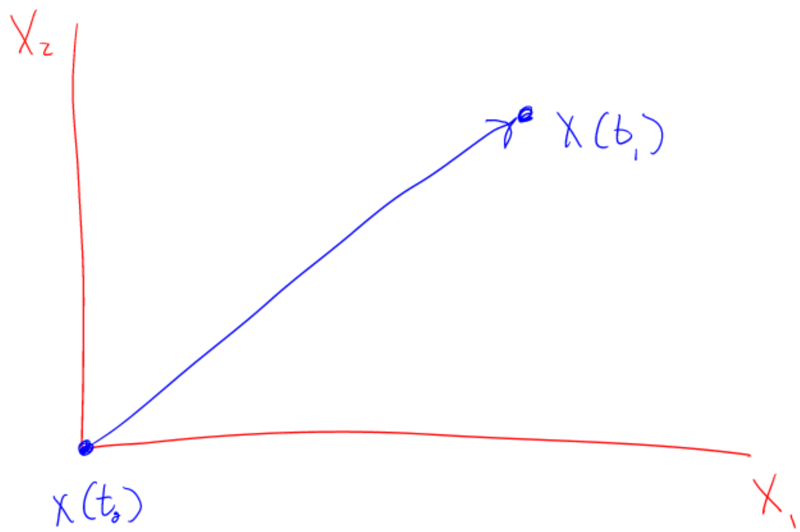
- Definition of Controllability
- Controllable Canonical Form
- Controllable Canonical Decomposition
- Stabilizability

At the end of this section, students should be able to:

- Determine controllability of a system.
- Transform a system into controllable canonical form.
- Decompose a system into controllable and uncontrollable subsystems.
- Determine if an uncontrollable system is stabilizable.

WHAT DOES IT MEAN FOR A SYSTEM TO BE CONTROLLABLE?

States → x_1, x_2

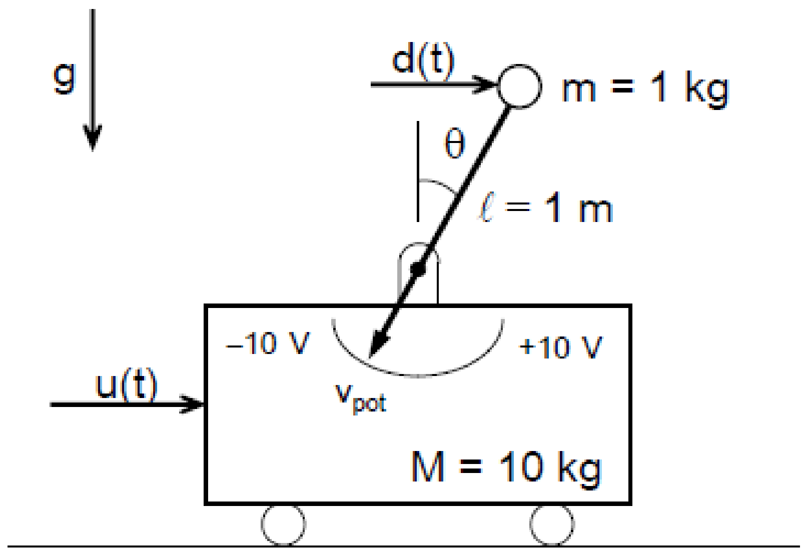


The system $\dot{x} = Ax + Bu$ is completely **state controllable** if a control $u(t)$ exists that will transfer the state of the system from any $x(t_o) = x_o$ to any $x(t_1) = x_1$ in a finite time $t_1 - t_o$.

RECALL THE SEGWAY CASE STUDY

- Rider leans forward to start driving forward
- Rider leans backward to start driving backward
- Wheels move to prevent the rider from falling over (maintain stability)

MODEL THE (UNREALISTIC) SEGWAY AND RIDER AS AN INVERTED PENDULUM ON A CART



Is it possible to achieve zero position of both the cart and the rod with only a single control input u? No

LET'S TRY A SIMPLE EXAMPLE USING NUMBERS

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Is it completely state controllable? Yes

Transform to diagonal form:

$\lambda = -1, -2$ $\quad T = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ $\quad T^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad T^{-1}ATz + T^{-1}Bu$

$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$

controller effects Both states

Controllability the combination of A (through T) and B.

DEFINITION OF CONTROLLABILITY

The system

$$\dot{x} = Ax + Bu$$

is completely state controllable iff the column vectors of the controllability matrix

$$W_C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

span the n-dimensional space (i.e., W_C has rank n).

$$W_c \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \checkmark$$

FOR A CONTROLLABLE SYSTEM, AN INPUT MUST EXIST TO MOVE FROM $\left(t_0, x(\cancel{t_0})\right)$ TO $\left(t_1, x(t_1)\right)$

0

Assume u is a scalar:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Let $x(0) = 0$, $x(t_1) = x_1$

$$x(t_1) = x_1 = \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

APPLYING THE SERIES REPRESENTATION OF THE STATE TRANSITION MATRIX…

$$\mathrm{e}^{\mathrm{A}(t_1-\tau)} = I + A(t_1-\tau) + \frac{1}{2!}A^2(t_1-\tau)^2 + \cdots$$

$$\mathrm{x}_1 = \mathrm{B}\int_0^{t_1} u(\tau)d\tau + \mathrm{AB}\int_0^{t_1}(t_1-\tau)u(\tau)d\tau$$
$$+\mathrm{A}^2\mathrm{B}\int_0^{t_1}\frac{1}{2}(t_1-\tau)^2u(\tau)d\tau + \cdots$$

- This is an infinite series, it would be better to have a finite series
- We need the Cayley-Hamilton Theorem

CAYLEY-HAMILTON THEOREM: MATRIX A SATISFIES ITS OWN CHARACTERISTIC EQUATION

Some matrix

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I = 0$$

- From this, we can show that any function of A can be written as a sum of $n-1$ powers of A
- In particular,

$$e^{A(t_1-\tau)} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$

APPLYING CAYLEY-HAMILTON THEOREM THIS PROBLEM...

$$e^{A(t_1-\tau)} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$

$$x_1 = B\int_0^{t_1} u(\tau)d\tau + AB\int_0^{t_1}(t_1-\tau)u(\tau)d\tau + A^2B\int_0^{t_1}\frac{1}{2}(t_1-\tau)^2 u(\tau)d\tau + \cdots$$

$$x_1 = \sum_{i=0}^{n-1} A^i B \int_0^{t_1} a_i(\tau)u(\tau)d\tau$$

$$= [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]\begin{bmatrix}\beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1}\end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \times$$

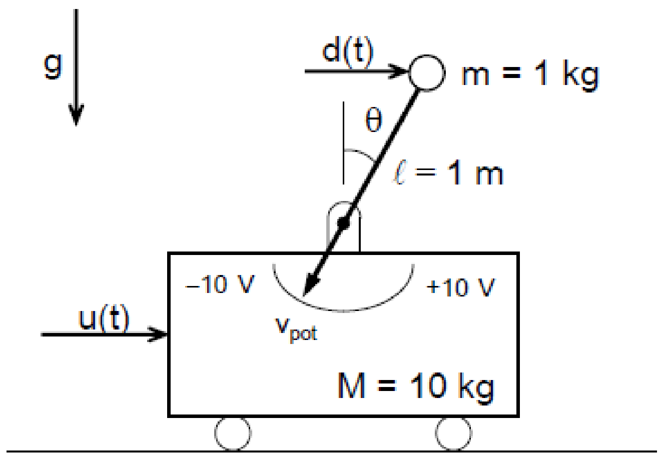
2×1^{st} row

$W_c =$

- For this to work, $[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$ must be full rank!

All columns are linear independent

RETURN TO THE INVERTED PENDULUM EXAMPLE



$$x = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10.79 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.98 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0.1 \end{bmatrix} u$$

CHECK CONTROLLABILITY FOR THE INVERTED PENDULUM

$$W_C = \begin{array}{cccc} [B & AB & A^2B & A^3B] \\ 1 & 2 & 3 & 4 \end{array}$$

4 rows
4 colums

$$W_C = \underbrace{\begin{bmatrix} 0 & -0.1 & 0 & -1.08 \\ -0.1 & 0 & -1.08 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.1 & 0 \end{bmatrix}}_{\text{Full rank}}$$

needs to be square and size of A

For some x_1 & x_2
u can be designed

Yes it is state controllable

CONSIDER ANOTHER NUMERICAL EXAMPLE

2×2 Jordan Block

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & a \end{bmatrix} x + \begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} u$$

1×1 J Block

This is in Jordan Canonical Form!

$$AB = \begin{bmatrix} b \\ -2b \\ a \end{bmatrix} \qquad A^2B = \begin{bmatrix} -4b \\ -4b \\ a^2 \end{bmatrix}$$

$$W_c = \begin{bmatrix} 0 & b & -4b \\ b & -2b & -4b \\ 1 & a & a^2 \end{bmatrix}$$

$b \neq 0$

$a \neq -2$

check $|W_c| \neq 0$

WE CAN ALSO CONSIDER OUTPUT CONTROLLABILITY

The system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is output controllable if a control u exists that will transfer the output of the system from any $y(t_o)$ to any $y(t_1)$ in a finite time $t_1 - t_o$.

This is true iff

$$\text{rank}[CB \quad CAB \quad CA^2B \quad \cdots \quad CA^{n-1}B \quad D] = m$$

If $D=0 \rightarrow \text{rank}\left[C[W_c]\right] = m$

CONSIDER A 3RD ORDER CONTROLLABLE CANONICAL FORM

could be $z \rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

B_c

$$A_c B_c = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix} \qquad A_c^2 B_c = \begin{bmatrix} 1 \\ -a_2 \\ -a_1 + a_2^2 \end{bmatrix}$$

$$W_c^c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$

rank 3 for any a_i

$\checkmark$ state controllable

A CONTROLLABLE SYSTEM CAN ALWAYS BE TRANSFORMED INTO CONTROLLABLE CANONICAL FORM

CCF

$$\dot{x} = Ax + Bu \quad \xRightarrow{x=Tz} \quad \dot{z} = A_C z + B_C u$$

$$A_C = T^{-1}AT$$
$$B_C = T^{-1}B$$

$$W_C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

$$W_C^C = [B_C \quad A_CB_C \quad A_C^2B_C \quad \cdots \quad A_C^{n-1}B_C]$$

$= [T^{-1}B \quad T^{-1}ATT^{-1}B \dots$

related by a T^{-1}

$= T^{-1}[W_c]$

$$T W_c^c = W_c$$

$$T = W_c [W_c^c]^{-1}$$

$$[W_c^c]^{-1} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Transforms original system to CCF

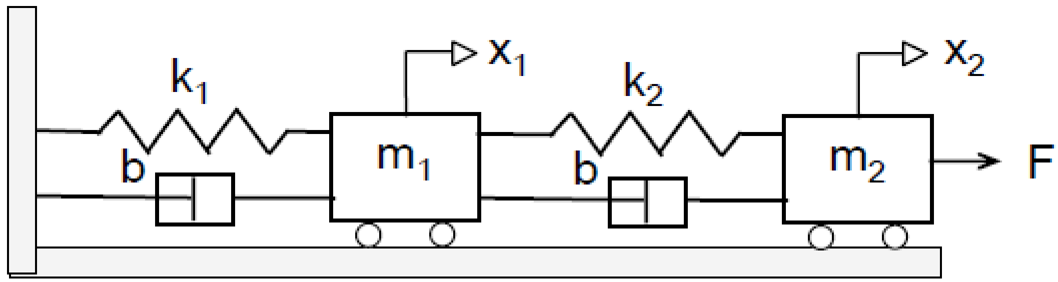
$$T = [B \quad AB \quad A^2B] \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

IN GENERAL, TO TRANSFORM TO CONTROLLABLE CANONICAL FORM:

$$T = W_C \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & a_4 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-1} & 1 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

WHAT DOES AN UNCONTROLLABLE SYSTEM LOOK LIKE?



$$\sum F_{m_1} = -k_1 x_1 - b\dot{x}_1 + k_2(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) = m_1 \ddot{x}_1$$

$$\sum F_{m_2} = -k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) + F = m_2 \ddot{x}_2$$

Assume that the block masses are negligible $\rightarrow M_1 = M_2 = 0$ bad assumption

$$k_2(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) = k_1 x_1 + b\dot{x}_1 = F$$

CHECK CONTROLLABILITY

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-k_1}{b} & 0\\ \frac{-k_1+k_2}{b} & \frac{-k_2}{b}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{b}\\ \frac{2}{b}\end{bmatrix}F$$

$$x_2 = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$W_c = \begin{bmatrix}\frac{1}{b} & -\frac{k_1}{b^2}\\ \frac{2}{b} & -\frac{k_1}{b^2}-\frac{k_2}{b^2}\end{bmatrix}$$

Controlable if $k_1 \neq k_2$

b can't be 0
but in this system
that can't really happen

CONVERT SYSTEM TO TRANSFER FUNCTION

$$X_2(s) = C(sI - A)^{-1}BU(s)$$

$$(sI - A) = \begin{bmatrix} s + \frac{k_1}{b} & 0 \\ -\frac{-k_1 + k_2}{b} & s + \frac{k_2}{b} \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s + \frac{k_2}{b} & 0 \\ \frac{-k_1 + k_2}{b} & s + \frac{k_1}{b} \end{bmatrix}}{\left(s + \frac{k_1}{b}\right)\left(s + \frac{k_2}{b}\right)}$$

CONVERT SYSTEM TO TRANSFER FUNCTION

$$X_2(s) = C(sI - A)^{-1}BU(s)$$

$$\frac{X_2(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s + \frac{k_2}{b} & 0 \\ \frac{-k_1 + k_2}{b} & s + \frac{k_1}{b} \end{bmatrix}}{\left(s + \frac{k_1}{b}\right)\left(s + \frac{k_2}{b}\right)} \begin{bmatrix} \frac{1}{b} \\ \frac{2}{b} \end{bmatrix}$$

$$= \frac{1}{b} \frac{2s + \frac{k_1 + k_2}{b}}{\left(s + \frac{k_1}{b}\right)\left(s + \frac{k_2}{b}\right)}$$

If $k_1 = k_2$

then $= \frac{2}{b} \frac{1}{\left(s + \frac{k}{b}\right)}$

a bunch of stuff cancels

MEANINGS OF UNCONTROLLABLE STATES

1. If state space system is realized from input-output data, then the realization is redundant. Some states have no relationship to either the input or the output.

2. If states are meaningful (physical) variables that need to be controlled, then the design of the actuators are deficient.

3. The effect of control is limited. There is also a possibility of instability

partitioned $\rightarrow \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]$

WHEN A SYSTEM IS *NOT CONTROLLABLE*, IT CAN BE PARTITIONED INTO CONTROLLABLE AND UNCONTROLLABLE PARTS.

Given

$$\dot{x} = Ax + Bu$$

$$W_C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

If $\operatorname{rank}[W_C] = \ell < n$, then W_C has only ℓ linearly independent vectors, and $n - \ell$ states are uncontrollable.

CONTROLLABLE DECOMPOSITION

$$\dot{x} = Ax + Bu \quad \xRightarrow{x=Tz} \quad \dot{z} = \hat{A}z + \hat{B}u$$

$$\hat{A} = T^{-1}AT$$
$$\hat{B} = T^{-1}B$$

$$T = [T_1 \quad T_2]$$

$$T_1 = [B \quad AB \quad A^2B \quad \cdots \quad A^{\ell-1}B]$$
$$T_2 = \text{any } n \times (n-\ell) \text{ matrix that makes T nonsingular}$$

controllable states
$\ell \times 1$

$$\begin{bmatrix} \dot{z}_C \\ \hline \dot{z}_{UC} \end{bmatrix} = \left[\begin{array}{c|c} \hat{A}_C & \hat{A}_{CU} \\ \hline 0 & \hat{A}_{UC} \end{array}\right] \begin{bmatrix} z_C \\ \hline z_{UC} \end{bmatrix} + \begin{bmatrix} \hat{B}_C \\ \hline 0 \end{bmatrix} u$$

uncontrollable states
$n-\ell \times 1$

CONSIDER ANOTHER NUMERICAL EXAMPLE

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

This is in Jordan Canonical Form!

check it later

STABILIZABILITY

The system

$$\dot{x} = Ax + Bu$$

is **stabilizable** if any uncontrollable states are open-loop stable, i.e., the eigenvalues of the uncontrollable subsystem $\hat{A}_{UC}$ have negative real part.

$$\dot{z}_{uc} = A_{uc} z_{uc} + 0u$$

If eigen values are $-\mathbb{R}$ then is stable

No effect

CONSIDER ANOTHER NUMERICAL EXAMPLE

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Z_c

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & -4 & 0 \\ 1 & -4 & 0 \\ \hline 0 & 0 & -2 \end{array}\right] \begin{bmatrix} z_1 \\ z_2 \\ \hline z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \hline 0 \end{bmatrix} u$$

A_{uc} Z_{uc}

$\dot{z}_3 = -2 z_3$

stable

COMING UP...

State Feedback Controller Design

- State Feedback Regulator
- Ackermann's Formula