FROM LAST TIME…

Analysis of Feedback Systems

- Feedback controller structure
- Nominal sensitivity functions
- Stability of nominal feedback system
- Root locus

$$\begin{array}{c|ccccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ \hline s^{n-2} & b_1 & b_2 & b_3 & & \\ s^{n-3} & c_1 & c_2 & c_3 & & \\ \vdots & & & & & \\ s^0 & & & & & \end{array}$$

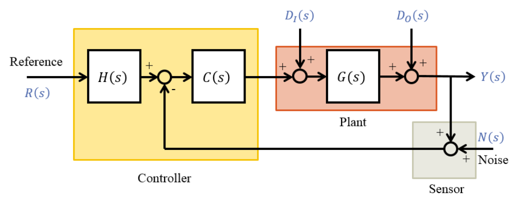
$$1 + KC_a(s)G(s) = 0$$

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

$$S(s) = \frac{1}{1 + G(s)C(s)}$$

$$S_i(s) = \frac{G(s)}{1 + G(s)C(s)}$$

$$S_u(s) = \frac{C(s)}{1 + G(s)C(s)}$$



MORE STABILITY

Topics

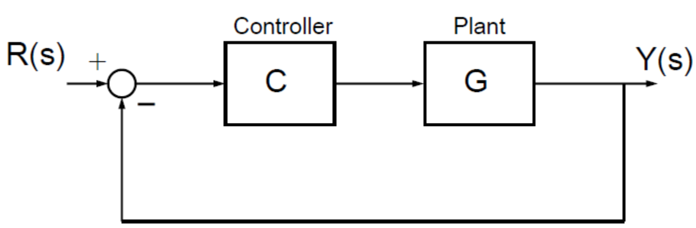
- Nyquist test for stability
- Relative stability
- Robust stability

At the end of this section, students should be able to:

- Apply the Nyquist stability theorem.
- Quantify relative stability using gain and phase margins.
- Apply the robust stability theorem.

MORE STABILITY

NYQUIST STABILITY



Recall the open-loop transfer function

$$L(s) = C(s)G(s) = \frac{N_L(s)}{D_L(s)}$$

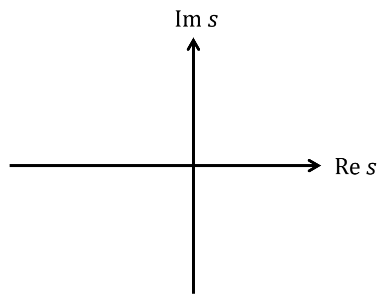
The closed-loop characteristic equation is given by

$$F(s) = 1 + L(s) = 0$$

STABILITY IN FREQUENCY DOMAIN

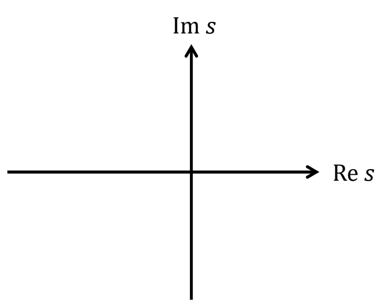
We want to know where the zeros of $F(s)$ are.

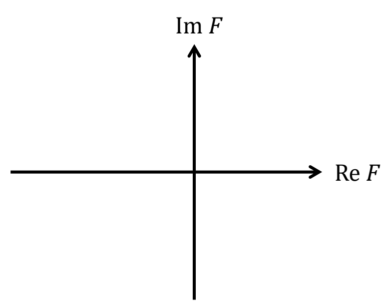
- Use a MAPPING between the s-plane (where the roots are) and the $F(s)$-plane.
- Since we want stability, isolate the RHP with a geometrically simple directed contour Γ_s.



EFFECT OF ZEROS OF $F(s)$

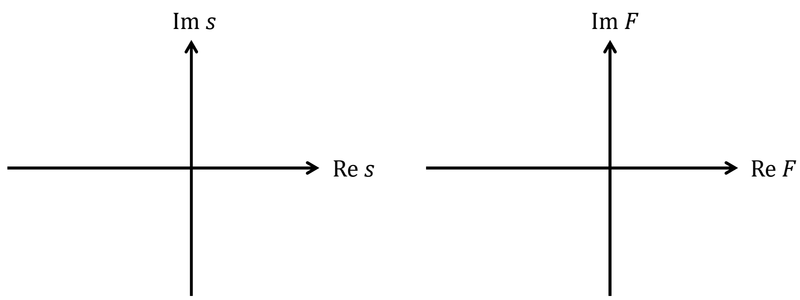
1. Zero outside contour $F(s) = s + a, \quad a > 0$





EFFECT OF ZEROS OF $F(s)$

1. Zero inside contour $F(s) = s - a, \quad a > 0$



PRINCIPLE OF THE ARGUMENT (CAUCHY)

Let $F(s)$ be a single-valued function that has a finite number of poles in the s-plane. Choose a closed path Γ_s in the s-plane such that it avoids any poles or zeros of $F(s)$. Then the corresponding contour Γ_F mapped in the $F(s)$-plane will encircle the origin N_{CW} times in a clockwise direction.

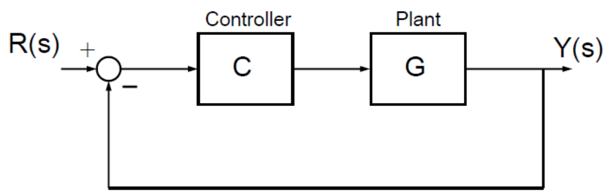
$$N_{CW} = N_Z - N_P$$

$N_Z = \#$ of zeros of $F(s)$ encircled by Γ_s
$N_P = \#$ of poles of $F(s)$ encircled by Γ_s

APPLICATION TO STABILITY

Recall:

$$L(s) = C(s)G(s) = \frac{N_L(s)}{D_L(s)}$$



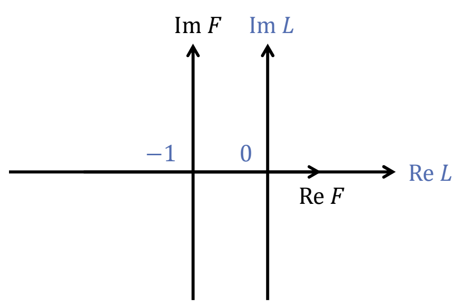
$$F(s) = 1 + L(s) = 0 \quad = 1 + \frac{N_L(s)}{D_L(s)} = \frac{D_L(s) + N_L(s)}{D_L(s)}$$

zeros of $F(s)$**:** roots of characteristic equation (closed-loop poles)

poles of $F(s)$**:** open-loop poles

WE STILL NEED TO RELATE $F(s)$ TO $L(s)$

$F(s) = 1 + L(s)$ ⇨ $L(s) = F(s) - 1$



NYQUIST STABILITY CRITERION

$$N_{CW} = N_Z - N_P$$

$$\begin{aligned} N_{CW} &= \text{\# of CW encirclements of } -1 \text{ by } \Gamma_L \\ N_Z &= \text{\# of closed-loop poles encircled by } \Gamma_s \\ N_P &= \text{\# of open-loop poles encircled by } \Gamma_s \end{aligned}$$

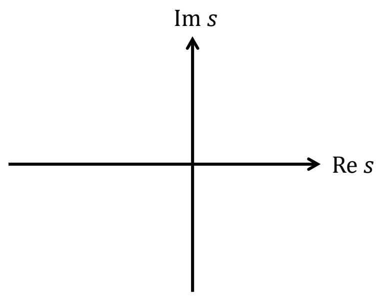
A feedback system having N_P open-loop poles in the RHP is stable if and only if the Nyquist plot of $L(s)$ encircles -1 N_P times in a counterclockwise direction.

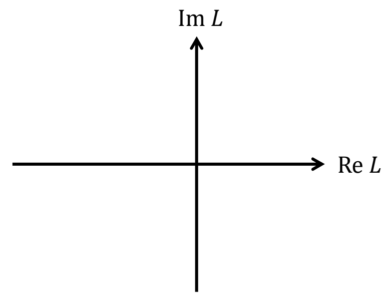
STEPS IN SKETCHING A NYQUIST PLOT

1. Plot poles of $L(s)$ in the s-plane.
2. Draw the Nyquist contour Γ_s, indenting to the right of any poles of $L(s)$ on the imaginary axis.
3. Map contour Γ_s to $L(s)$-plane.
4. Apply encirclement condition.

NYQUIST EXAMPLE

$$L(s) = \frac{2K}{(2s+1)(s+1)\left(\frac{s}{2}+1\right)}$$





NYQUIST PLOT

Curve 1: $s = j\omega$

$$L(j\omega) = \frac{2}{(1+j2\omega)(1+j\omega)\left(1+j\frac{\omega}{2}\right)}$$

$$|L(j\omega)| = \frac{2}{\sqrt{1+(2\omega)^2}\sqrt{1+\omega^2}\sqrt{1+\left(\frac{\omega}{2}\right)^2}}$$

$$\angle L(j\omega) = -\tan^{-1}(2\omega) - \tan^{-1}\omega - \tan^{-1}\left(\frac{\omega}{2}\right)$$

ω	$\boldsymbol{|L(j\omega)|}$	$\boldsymbol{\angle L(j\omega)}$
0		
1		
2		
$\omega \to \infty$		

Curve 3: $s = -j\omega$**: Complex conjugate of Curve 1**

NYQUIST PLOT

Curve 2: $s = Re^{j\phi}$

$$L(Re^{j\phi}) = \frac{2}{(1+2Re^{j\phi})(1+Re^{j\phi})\left(1+\frac{1}{2}Re^{j\phi}\right)}$$

Let $R \to \infty$

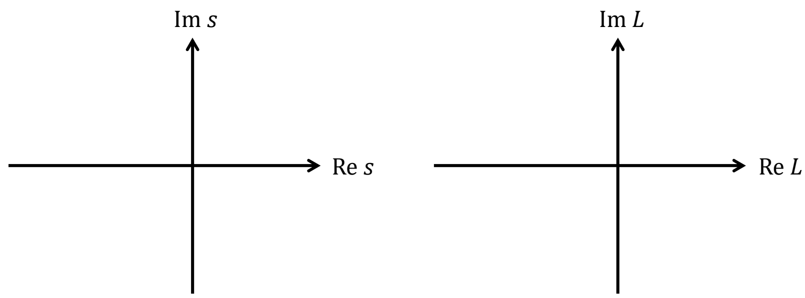
$$L(Re^{j\phi}) \approx \frac{2}{2Re^{j\phi}Re^{j\phi}\frac{1}{2}Re^{j\phi}}$$

$$L(Re^{j\phi}) \approx \frac{2}{R^2}e^{-3j\phi}$$

$$\lim_{R\to\infty}|L(s)| \to 0$$

NYQUIST STABILITY CRITERION

NYQUIST PLOT FOR ARBITRARY K



MORE STABILITY

RELATIVE STABILITY

SO FAR WE'VE ONLY DISCUSSED STABILITY AS A BINARY CONDITION

Poses in LHP

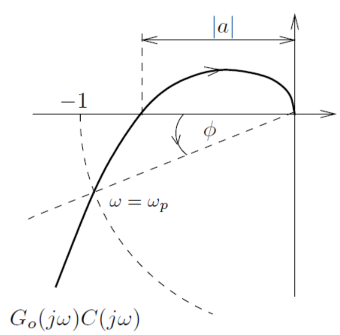
Poles in LHP
Passed Routh-Hurwitz test
Passed Nyquist test

Poles in RHP
Failed Routh-Hurwitz test
Failed Nyquist test

- For a given system, how far apart are these regions?
- Could we give it a little push from stability to instability?

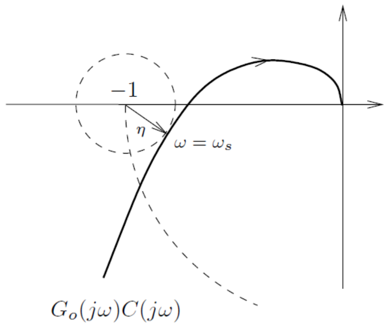
- Relative stability measures how far from instability a system is currently
- Relative stability is measured in magnitude and phase

PROXIMITY TO ENCIRCLEMENT OF −1 IS A RELATIVE STABILITY.



- Gain Margin – the factor by which the open-loop gain can be increased at a phase of −180° before the system goes unstable.
- Phase Margin – the amount by which open-loop phase can be decreased at unity magnitude before system goes unstable

THE CLOSEST APPROACH OF THE NYQUIST PATH TO -1 GIVES US THE SENSITIVITY PEAK

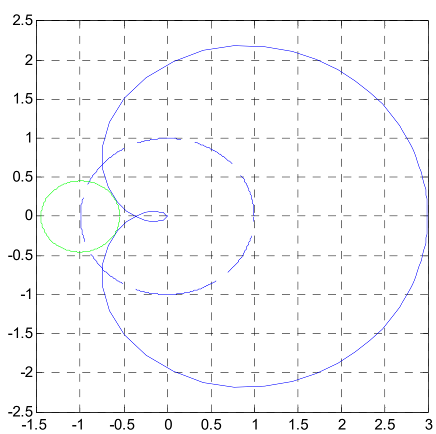


$$M_s = \frac{1}{\eta} = \max|S_o(j\omega)|$$

$$= \max\left|\frac{1}{1 + G_o(j\omega)C(j\omega)}\right|$$

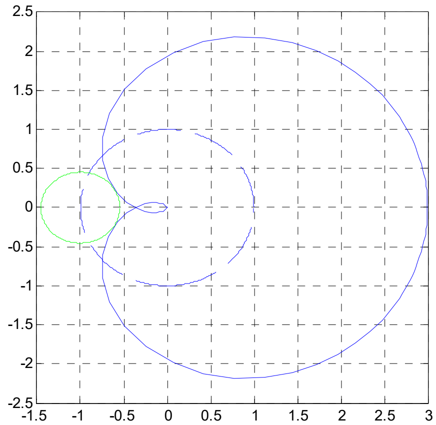
- M_s is the nominal sensitivity peak
- Larger M_s means closer to instability

EXAMPLE:



$$L_o(s) = \frac{3}{(s+1)^3}$$

EXAMPLE:

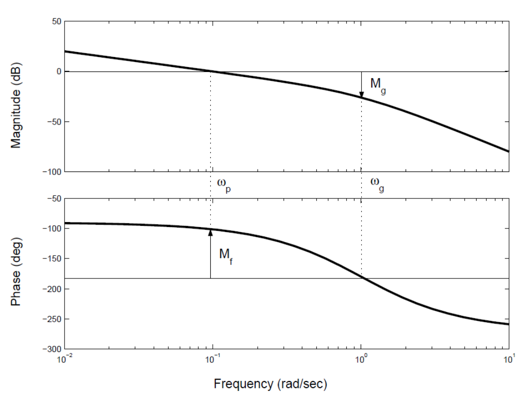


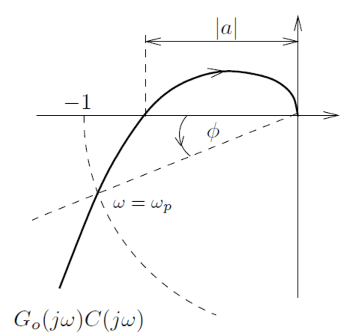
$$L_o(s) = \frac{3}{(s+1)^3}$$

$$|1 + L_o(j\omega)| > \eta$$

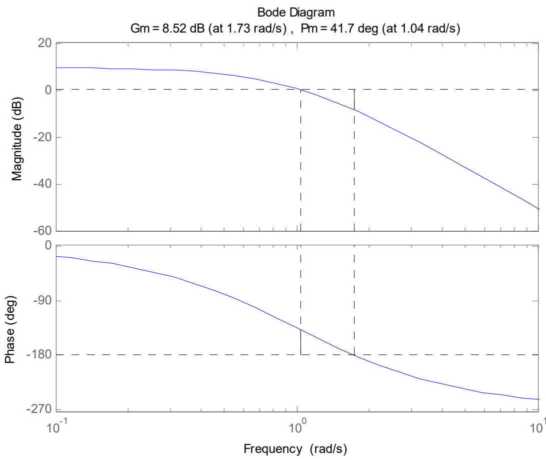
$$|S_o(j\omega)| = \left|\frac{1}{1 + L_o(j\omega)}\right| < \frac{1}{\eta}$$

WE CAN READ THE SAME INFORMATION DIRECTLY FROM THE OPEN-LOOP BODE PLOT





EXAMPLE:



$$L_o(s) = \frac{3}{(s+1)^3}$$

$$\left|L_o(j\omega_{gc})\right| = 1$$

$$\angle L_o(j\omega_{pc}) = -180°$$

SUMMARY OF MARGINS

For stability, we want no encirclement of −1 (for minimum-phase systems):

- $GM > 1$ or $GM_{dB} > 0$
- $PM > 0°$

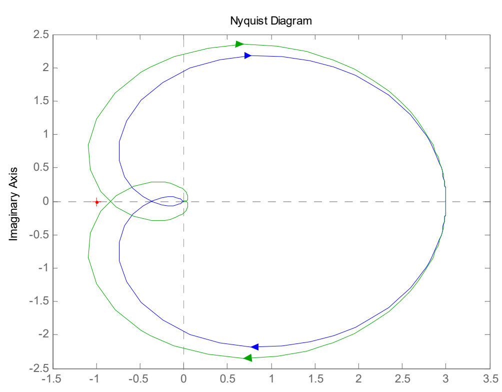
As measures of relative stability, more positive GM & PM imply farther away from instability:

- GM indicates allowable extra gain
- PM indicates allowable extra phase lag (time delay)

MORE STABILITY

ROBUST STABILITY

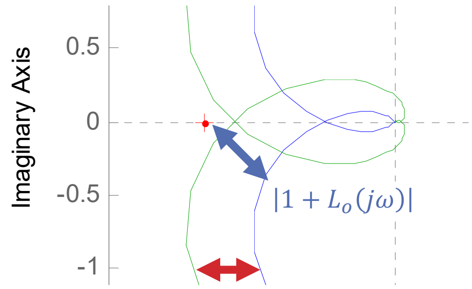
THAT'S ALL WELL AND GOOD, BUT WHAT HAPPENS WHEN ACTUAL SYSTEM ISN'T NOMINAL?



$$L_o(s) = \frac{3}{(s+1)^3}$$

$$L(s) = \frac{3}{(s+1)^3} e^{-0.5s}$$

COMPARE THE MODEL ERROR TO THE SENSITIVITY PEAK



If the difference between nominal and actual is less than the inverse of nominal sensitivity peak, the system is still stable!

$$|L(j\omega) - L_o(j\omega)| < |1 + L_o(j\omega)|$$

WE CAN DO SOME MANIPULATION TO MAKE THIS MORE USEFUL

$$|L(j\omega) - L_o(j\omega)| < |1 + L_o(j\omega)|$$

$$|C(j\omega)G(j\omega) - C(j\omega)G_o(j\omega)| < |1 + L_o(j\omega)|$$

$$|C(j\omega)G_o(j\omega)| \cdot \left|\frac{G(j\omega) - G_o(j\omega)}{G_o(j\omega)}\right| < |1 + L_o(j\omega)|$$

$$\frac{|L_o(j\omega)|}{|1 + L_o(j\omega)|} \cdot \left|\frac{G(j\omega) - G_o(j\omega)}{G_o(j\omega)}\right| < 1$$

$$|T_o(j\omega)||G_\Delta(j\omega)| < 1$$

ROBUST STABILITY THEOREM

Consider a plant with a nominal TF of $G_o(s)$ and a true TF of $G(s)$.

- Assume that a controller $C(s)$ has been designed to achieve nominal internal stability (i.e., no unstable pole/zero cancellation and $L_o(s) = G_o(s)C(s)$ is stable).
- Also assume that $L_o(s) = G_o(s)C(s)$ and $L(s) = G(s)C(s)$ have the same number of unstable poles.

Then, a sufficient condition for stability of the actual feedback loop obtained by applying the controller to the true plant is that

$$|T_o(j\omega)||G_\Delta(j\omega)| = \left|\frac{L_o(j\omega)}{1 + L_o(j\omega)}\right| |G_\Delta(j\omega)| < 1$$

where $G_\Delta(j\omega)$ is the frequency response of the multiplicative modeling error.

ROBUST STABILITY EXAMPLE

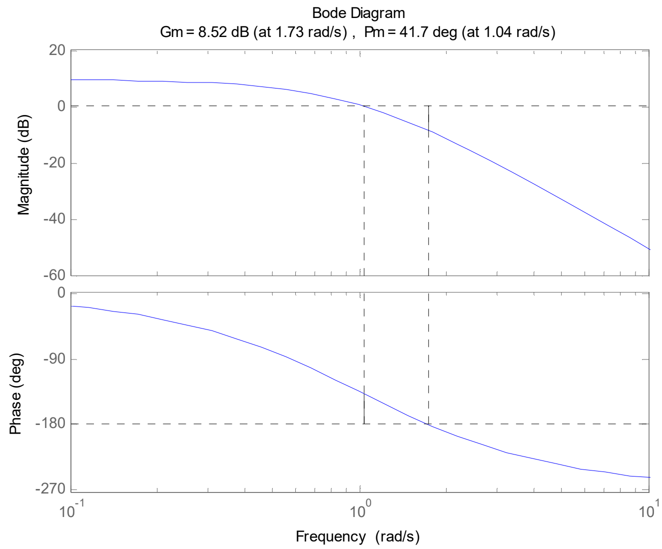
Problem Formulation

Nominal System	Actual System	MME
$L_o(s) = \dfrac{3}{(s+1)^3}$	$L(s) = \dfrac{3}{(s+1)^3} e^{-T_d s}$	$G_\Delta(s) = e^{-T_d s} - 1$

Find exact value of time delay that leads to instability.

$$\left|e^{-T_d j\omega}\right| = 1 \qquad \angle e^{-T_d j\omega} = -T_d \omega \ \ [\text{rad}]$$

BODE PLOTS OF NOMINAL SYSTEM



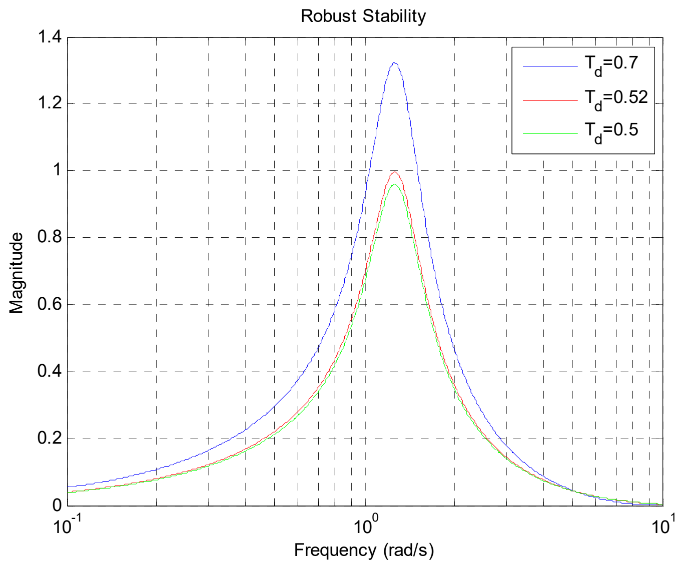
ROBUST STABILITY EXAMPLE (CONT.)

Estimate the critical time delay using Robust Stability

$$T_o(s) = \frac{L_o(s)}{1 + L_o(s)} = \frac{\frac{3}{(s+1)^3}}{1 + \frac{3}{(s+1)^3}} = \frac{3}{(s+1)^3 + 3}$$

$$G_\Delta(s) = e^{-T_d s} - 1$$

PLOTS OF $|T_o(j\omega)||G_\Delta(j\omega)|$



ROBUST STABILITY EXAMPLE

Comment on any differences in the two values for time delay

ACTUAL AND NOMINAL SENSITIVITY

Actual Achieved Sensitivity Functions

$$S(s) = S_o(s)S_\Delta(s)$$
$$T(s) = T_o(s)\big(1 + G_\Delta(s)\big)S_\Delta(s)$$
$$S_i(s) = S_{io}(s)\big(1 + G_\Delta(s)\big)S_\Delta(s)$$
$$S_u(s) = S_{uo}(s)S_\Delta(s)$$

where

$$S_\Delta(s) = \frac{1}{1 + T_o(s)G_\Delta(s)}, \qquad \text{Error Sensitivity}$$
$$G_\Delta(s) = \frac{G(s) - G_o(s)}{G_o(s)}, \qquad \text{MME}$$

HOW DID YOU DO THAT?

Sensitivity functions all have the same term in the denominator

$$\frac{1}{1 + GC}$$

We can rewrite this term to include a comparison of the nominal plant model and true plant

But it would be more useful if we could connect this to the MME

$$G_\Delta(s) = \frac{G(s) - G_o(s)}{G_o(s)}$$

HOW DID YOU DO THAT?

$$\frac{1+G_oC}{1+GC} = \frac{1}{\left(\frac{1+GC}{1+G_oC}\right)}$$

$$= \frac{1}{1+T_oG_\Delta}$$

PERFORMANCE ROBUSTNESS

To ensure that achieved performance is close to nominal performance, we need

$$S_\Delta(j\omega) \approx 1$$

$$\boxed{S_\Delta(s) = \frac{1}{1+T_o(s)G_\Delta(s)}}$$

COMING UP…

Pole Placement Controller Design

- Pole placement design
- Controller with integration
- PID via pole placement

PID Control via Pole Placement

- P, PD, PI, PID controllers
- Smith predictor