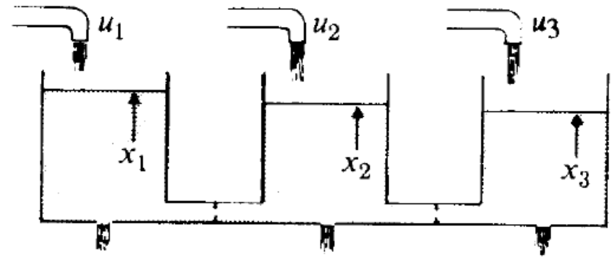
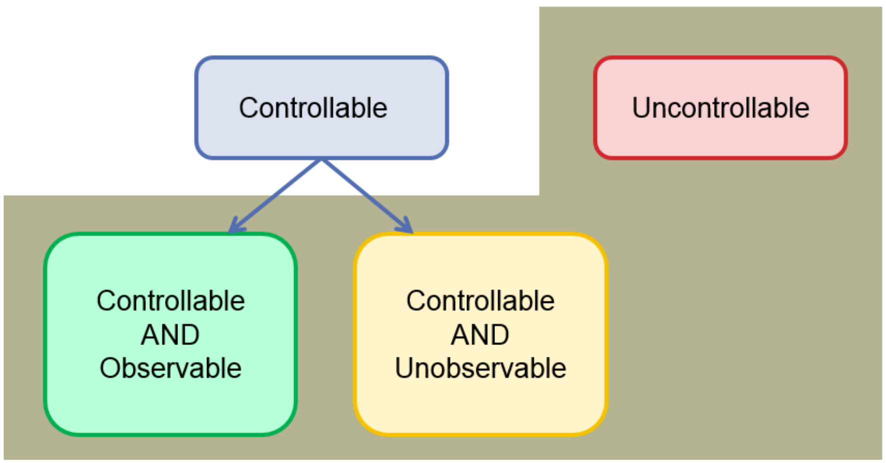
FROM LAST TIME...

Observability

- Observable Canonical Form
- Observable Canonical Decomposition
- General Decomposition



$$W_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$



STATE OBSERVER DESIGN

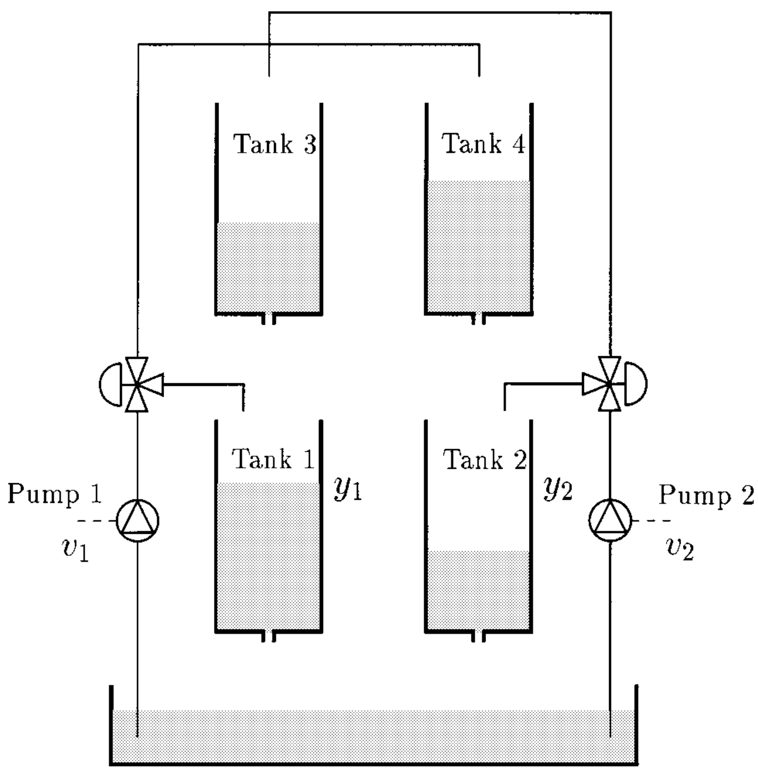
State Observers

- Rationale
- Observer Design
- Method 1
- Method 2 (Ackermann's Formula)

At the end of this section, students should be able to:

- Explain the benefit of a state observer.
- Design a full-order state observer.

WHEN NOT ALL THE STATES ARE MEASURABLE, WE WANT TO ESTIMATE THE STATES



We know:

$\dot{x} = Ax + Bu$

$y = Cx$

Measure:

u and y

INPUT OUTPUT

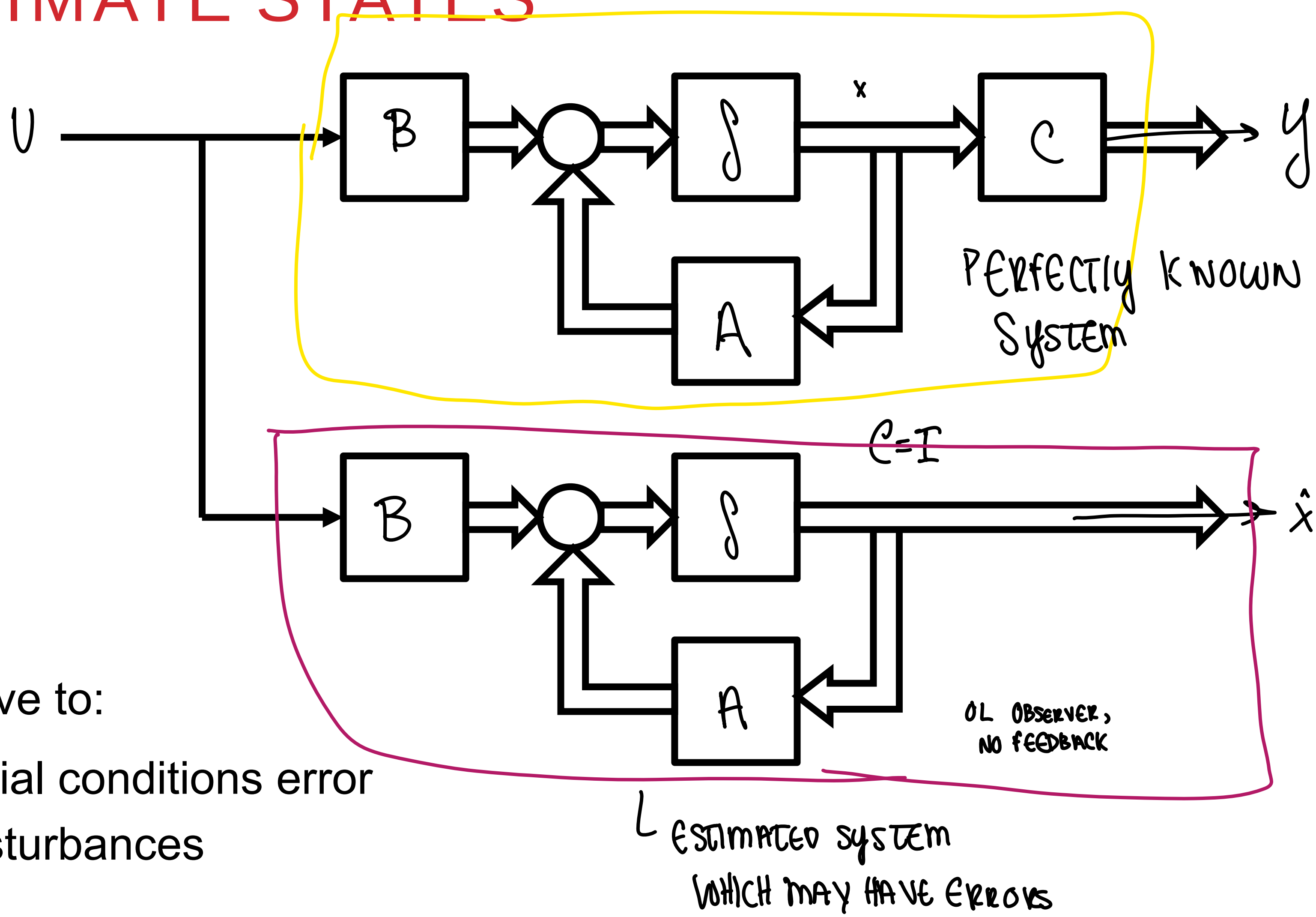
Calculate:

x

STATES

This is called a state observer (or state estimator).

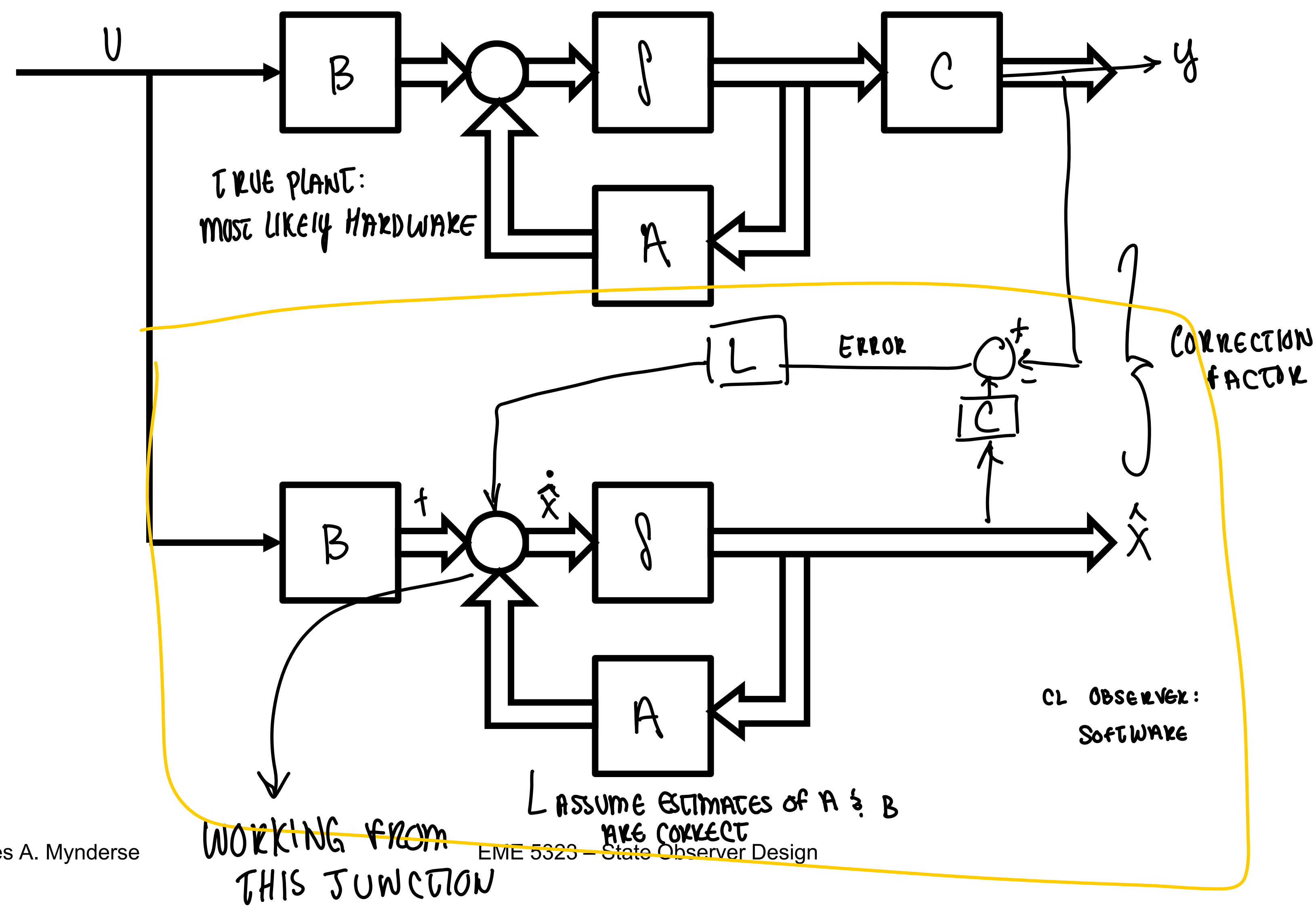
AN *OPEN-LOOP* STATE OBSERVER DESIGN USES SYSTEM DYNAMICS TO ESTIMATE STATES



Sensitive to:

- Initial conditions error
- Disturbances

A *CLOSED-LOOP* STATE OBSERVER DESIGN IMPROVES THE STATE ESTIMATION WITH FEEDBACK



CLOSED-LOOP STATE OBSERVER DESIGN:

$$\dot{\hat{x}} = A\hat{x} + Bu + \boxed{L(y - C\hat{x})}$$ ~ FEEDBACK TERM

$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$

DEFINE STATE ESTIMATION ERROR

ERROR $= e = x - \hat{x}$

$\dot{e} = \dot{x} - \dot{\hat{x}} = (Ax - Bu) + ((A - LC)\hat{x} + Bu + Ly)$

$= Ae - LCe = \underbrace{(A - LC)}_{\text{CONTROLS THE FREE RESPONSE}} e$

State observeration is a dual to state feedback!

CHOOSE HOW FAST THE ERROR SHOULD CONVERGE (ERROR SETTLING TIMES)

POLES (EIG) $\rightarrow L$

OBSERVER DESIGN

With L, the eigenvalues of $A - LC$ can be placed arbitrarily, therefore controlling the behavior of e. Thus, even if $e(0) \neq 0$, over time, the estimation error will decay to zero.

Rule of Thumb:

- Pick observer poles at least as fast as the desired closed-loop poles for the state feedback regulator.

  └ B/C WE WANT THE ERROR TO DECAY BEFORE THE RESPONSE

CONSIDER A SYSTEM IN OBSERVABLE CANONICAL FORM

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}}_{A_o} x + \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}}_{B_o} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C_o} x$$

STATE OBSERVER $\dot{\hat{x}} = A_o\hat{x} + B_o u + L(y - C_o\hat{x})$

$= (A_o - LC_o)\hat{x} + Bu + Ly$

WANT POLES $|sI - A_o + \underbrace{LC_o|}_{3\times3} = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$

$$A_O - LC_O = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -(a_0 + l_1) \\ 1 & 0 & -(a_1 + l_2) \\ 0 & 1 & -(a_2 + l_3) \end{bmatrix}$$

$$|sI - A_0 + LC_0| = s^3 + (a_2 + l_3)s^2 + (a_1 + l_2)s + (a_0 + l_1)$$

$$L = \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \\ \alpha_2 - a_2 \end{bmatrix}$$

STATE OBSERVER DESIGN: METHOD 1

1. **Convert system to observable canonical form**

$$\dot{x} = Ax + Bu \qquad \xRightarrow{x=Tz} \qquad \dot{z} = A_O z + B_O u$$
$$y = Cx \qquad\qquad\qquad\qquad y = C_O z$$

$$\boxed{\begin{aligned} A_O &= T^{-1}AT \\ B_O &= T^{-1}B \\ C_O &= CT \end{aligned}}$$

SOLVE FOR T
USING W₀...

2. **Choose poles for convergence of state estimation error**

$$|sI - A_{EST}| = \left|sI - A_O + \hat{L}C_O\right|$$

$$= (s - p_1)\cdots(s - p_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$$

3. Convert back to x

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$\hat{x} = T\hat{z} \qquad \dot{\hat{z}} = \underbrace{T^{-1}AT}_{A_o}\hat{z} + \underbrace{T^{-1}B}_{B_o}u + \underbrace{T^{-1}L}_{L_o}(y - \underbrace{CT}_{C_o}\hat{z})$$

$$L_o = T^{-1}L$$

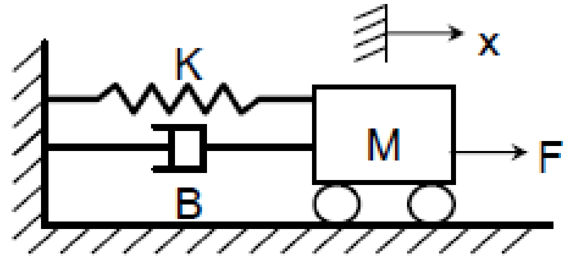
$$L = TL_o$$

$$L = T\begin{bmatrix} \alpha_0 - a_0 \\ \vdots \\ \alpha_{n-1} - a_{n-1} \end{bmatrix}$$

Where T transforms the system to observable canonical form

APPLY METHOD 1 TO AN EXAMPLE

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



$$|sI - A| = s^2 + \overset{a_1 = 1}{s} + \overset{a_0 = 1}{1}$$

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{RANK}(W_o) = 2, \quad a_1 = 1$$

$$T^{-1} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} W_o = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & -1 \\ -1 & a_1 \end{bmatrix} \frac{1}{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

PLACE CONTROLLER POLES @ $s = -3.5 \pm j3.57$

PLACE OBSERVER POLES @ $s = -18$) QUICKER, X5 IS USUALLY RECOMMENDED

$$|sI - A_o + LC_o| = (s+18)^2 = s^2 + \underset{\alpha_1}{36}s + \underset{\alpha_0}{324}$$

From $|sI - A|$

$$L = T \begin{bmatrix} 324 - 1 \\ 36 - 1 \end{bmatrix} = \begin{bmatrix} 35 \\ 288 \end{bmatrix}$$

λ

STATE OBSERVER DESIGN: METHOD 2 (ACKERMANN'S FORMULA)

To place eigenvalues of $(A - BK)$, let

$$K = [0 \quad 0 \quad \cdots \quad 1] W_C^{-1} \phi(A)$$ — For Controller

where $\phi(s)$ is the desired characteristic polynomial.

Since the eigenvalues of $(A - LC)^T$ are the same as those of $(A - LC)$, we have

$$(A - LC)^T = A^T - C^T L^T$$

which is analogous to the form for K above.

APPLYING ACKERMANN'S FORMULA...

$$\boxed{K = [0 \quad 0 \quad \cdots \quad 1] W_C^{-1} \phi(A)}$$

$$L^T = [0 \quad 0 \quad \cdots \quad 1][C^T \quad A^T C^T \quad \cdots \quad (A^T)^{n-1} C^T]^{-1} \phi(A^T)$$

$$L = [\phi(A^T)]^T \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{W_O}{}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad L = \phi(A) W_O^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

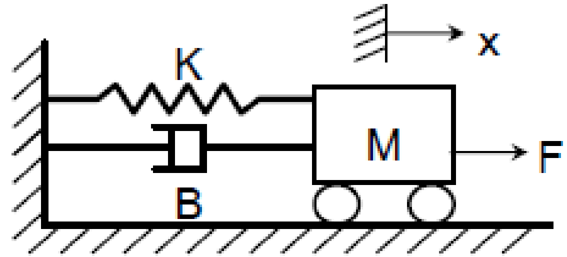
This is a dual of the state feedback solution!

Use MATLAB acker() command.

acker(A, B, poles) → controller
acker(A', C', poles)' → observer

APPLY METHOD 2 TO AN EXAMPLE

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



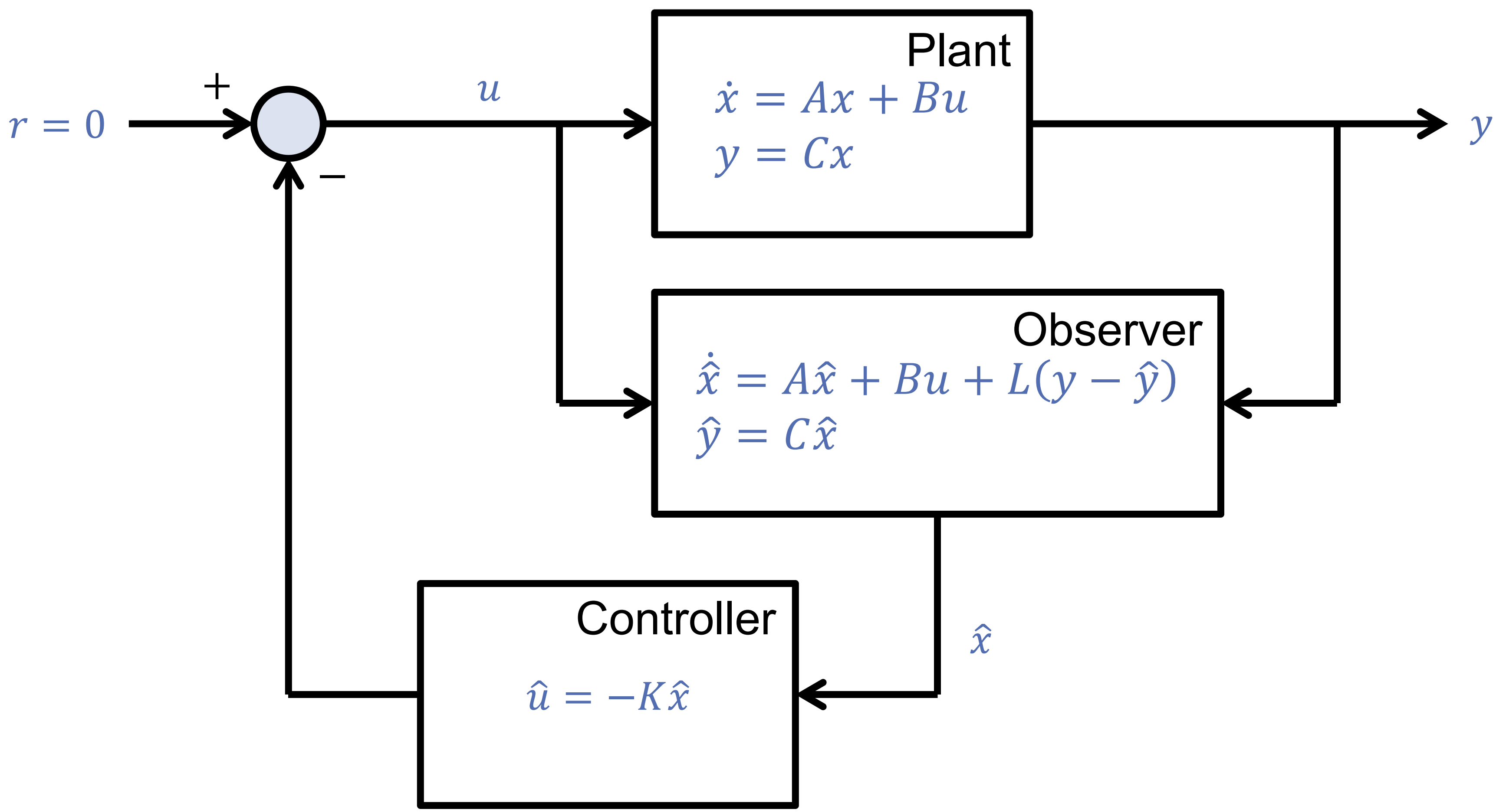
$$W_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(s+18)^2 \rightarrow \phi(s) = s^2 + \overset{\alpha_1}{36}s + \overset{\alpha_0}{324}$$

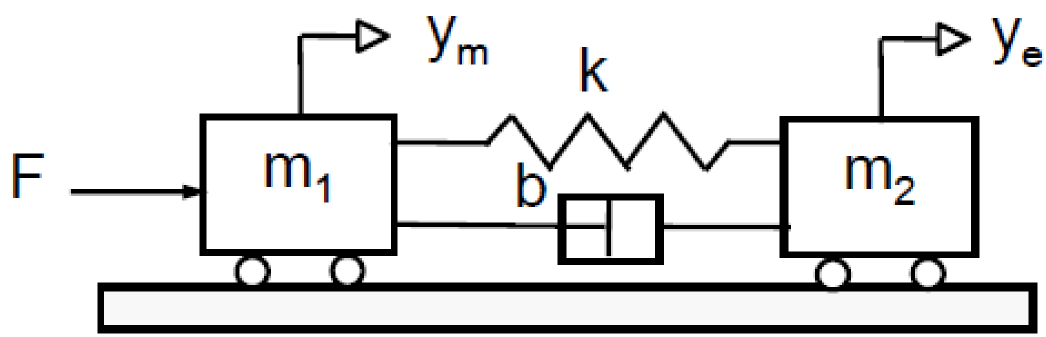
$$\phi(A) = A^2 + 36A + 324I = \begin{bmatrix} 323 & 35 \\ -35 & 288 \end{bmatrix}$$

$$L = \begin{bmatrix} 323 & 35 \\ -35 & 288 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 35 \\ 288 \end{bmatrix}$$

LET'S COMBINE THE STATE OBSERVER WITH STATE FEEDBACK



CONSIDER THE PREVIOUS ROBOTIC WELDING CASE STUDY



$$m_1 = 1\ \text{kg}$$
$$m_2 = 2\ \text{kg}$$
$$k = 36\ \text{N/m}$$
$$b = 0.6\ \text{Ns/m}$$

- Design a state feedback controller with poles at:

$$s = -2 \pm j2\sqrt{3}, -10, -10$$

- Design a full-order state observer with poles at:

$$s = -16$$

DOING THE MATH GIVES THE FOLLOWING RESULTS:

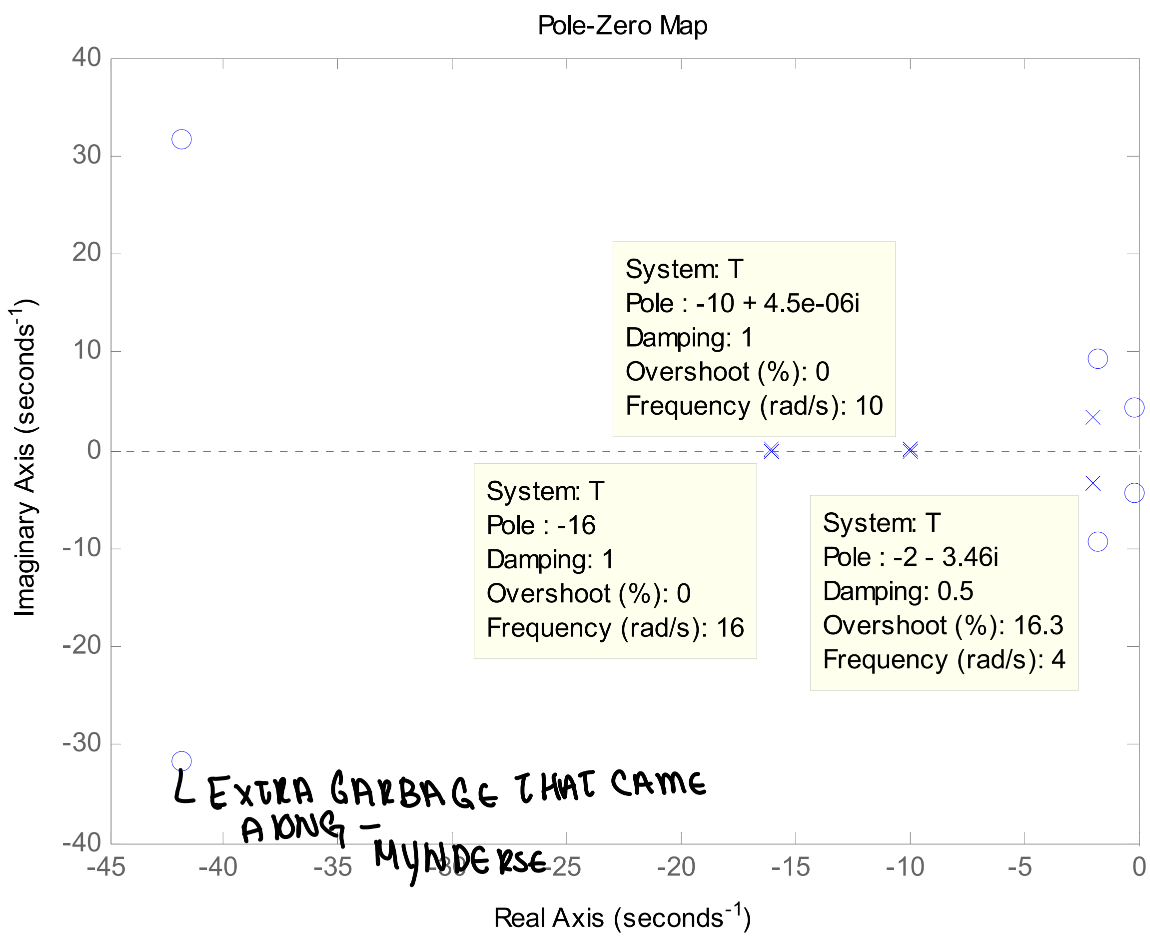
$$K = [130.44 \quad -41.56 \quad 23.10 \quad 15.42]$$

$$L = \begin{bmatrix} 63.1 \\ 393.2 \\ 1452 \\ 1108 \end{bmatrix}$$

$$\dot{x}_C = (A - BK - LC)x_C + Ly_m$$
$$u = Kx_C$$

- Validate in MATLAB (trick for this will be shown next time)

PLOTTING THE CLOSED-LOOP POLES AND ZEROS VALIDATES THAT THE DESIGN ACHIEVED SPECIFICATIONS



COMING UP…

Output Feedback

- Separation Principle
- Comparison of Output Feedback and State Feedback

Reduced-Order Observer