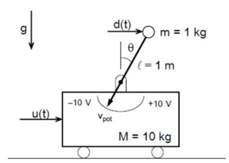
# FROM LAST TIME…

**Controllability**

- Definition of Controllability
- Controllable Canonical Form
- Controllable Decomposition
- Stabilizability

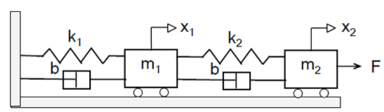
$$W_C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

BASED ON EIGEN VALUES




$$x = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10.79 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.98 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0.1 \end{bmatrix} u$$






# STATE FEEDBACK CONTROLLER

**Topics**

- State Feedback Regulator
- Ackermann’s Formula
- State Feedback for Uncontrollable System

**At the end of this section, students should be able to:**

- Design state feedback controllers using controllable canonical transformation.
- Design state feedback controllers using Ackermann’s formula.

# WHAT CONTROLLER DESIGN TECHNIQUES DO YOU KNOW?

**From previous courses**

- PID by trial and error
- PID by ZN tuning
- Lead/Lag
- Fuzzy control

**From this course**

- PID by pole placement



# STATE FEEDBACK IS A STATE-SPACE EQUIVALENT OF THE DIRECT POLE PLACEMENT DESIGN

Given $\dot{x} = Ax + Bu$

Eigenvalues are given by: $|sI - A| = 0$

LET $u = -Kx$

$\dot{x} = Ax + B(-Kx)$

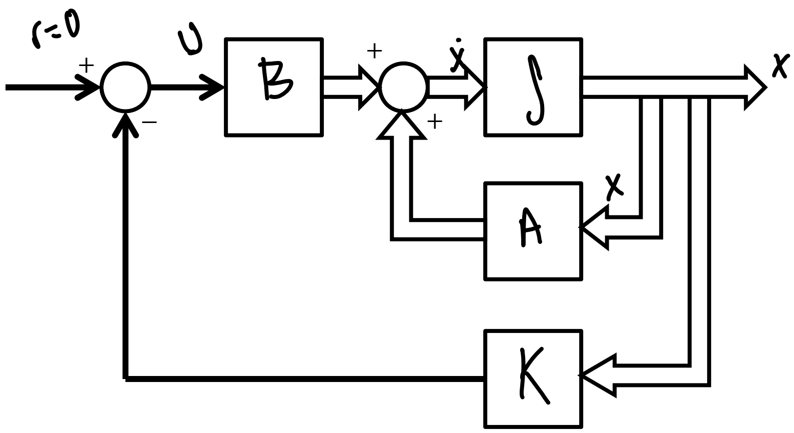
$\dot{x} = (A - BK)x$

$\dot{x} = A_{cL}x$

CONTROLLING ONLY INPUT STATES

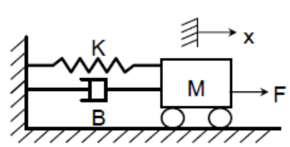
CL EIGENVALUES $|sI - A_{cL}| = 0$

REGULATION






## CONSIDER A MASS-SPRING-DAMPER




Choose states: $x_1 = x$
$x_2 = \dot{x}$

Choose parameters: $\underbrace{m = b = k = 1}_{\text{NON-REALISTIC}}$

CONTROLLABLE CANONICAL FORM

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$y = x = [1 \;\; 0]x + [0]u$

$$|sI - A| = s^2 + s + 1 \;\Rightarrow\; \zeta = 0.5 \quad \omega_n = 1 \text{ rad/sec}$$

We want to design a state feedback controller such that:

- $\zeta_{CL} = 0.7$ – INCREASE DAMPING
- $\omega_{n,CL} = 5$ rad/sec – SPEED UP

Let: $u = -Kx = -[k_1 \quad k_2]x$ $\leftarrow \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$A_{CL} = A - BK =$

$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1\ k_2]$

$|sI - A| = s^2 + (1+k_2)s + (1+k_1)$

DESIRED $= s^2 + 2(.7)5s + 5^2$

$K_1 = 24 \quad K_2 = 6$

$K = [24 \quad 6]$

$u = -24x - 6\dot{x}$



# IF A SYSTEM IS COMPLETELY CONTROLLABLE, A TRANSFORMATION EXISTS TO PUT IT IN CONTROLLABLE CANONICAL FORM:

$$\dot{x} = Ax + Bu \quad \overset{x=Tz}{\Longrightarrow} \quad \dot{z} = A_C z + B_C u$$

$\hat{K} = KT$

$$T = W_C \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & a_4 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-1} & 1 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

# APPLY THE TRANSFORMATION FOR CONTROLLABLE CANONICAL FORM TO THE STATE FEEDBACK DESIGN

Let: $u = -Kx = -\hat{K}z \quad = -[\hat{k}_1 \quad \hat{k}_2 \quad \cdots \quad \hat{k}_n]z = -KTx$

$$A_C - B_C\hat{K} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [\hat{k}_1 \quad \hat{k}_2 \quad \cdots \quad \hat{k}_n]$$

This problem should be easy to solve!



$$A_C - B_C\hat{K} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -(a_0 + \hat{k}_1) & -(a_1 + \hat{k}_2) & \cdots & -(a_{n-1} + \hat{k}_n) \end{bmatrix}$$

$\hat{A}_{CL}$ $\alpha_{n-1}$ → DESIRED TO BE CALCULATED

$$|sI - (A_C - B_C\,\hat{K})| = s^n + \overbrace{(a_{n-1} + \hat{k}_n)}^{\alpha_{n-1}} s^{n-1} + \cdots + (a_1 + \hat{k}_2)s + (a_0 + \hat{k}_1)$$

KNOWN

$$K = \hat{K}T^{-1} = [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})]T^{-1}$$

# CONSIDER A NUMERICAL EXAMPLE

Given

JORDAN BLOCK

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

↳ $A^2 = A \times A$

Find a controller to place all closed-loop poles at $-4$

$$W_c = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$ FULL RANK

$|sI-A| = (s+2)(s+2)(s+1) = s^3 + 5s^2 + 8s + 4$

$$|sI-A| = \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+2 & 0 \\ 0 & 0 & s+1 \end{bmatrix} \quad a_2 \;\; a_1 \;\; a_0$$

DET $\neq 0$



$$T = W_c \begin{bmatrix} 8 & 5 & 1 \\ 5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 4 & 4 & 1 \end{bmatrix}$$

↳ TRANSFORMATION MATRIX NEEDED FOR CONTROLLABLE CANONICAL FORM

MADE FROM COEFS OF CHAR. EQ. FEW SLIDES BACK

LET $U = -KX = [-K_1 \; -K_2 \; -K_3]X$ $\quad a_2 \;\; a_1 \;\; a_0$

$|sI-A| = (s+4)^3 = s^3 + 12s^2 + 48s + 64$

└ DESIRED, DOES NOT CHANGE THE ORDER OF THE SYSTEM

DESIRED – ACTUAL

$K = [64-4 \;\; 48-8 \;\; 12-5]T^{-1}$

$= [-8 \;\; -20 \;\; -27]$

$U = 8x_1 + 20x_2 - 27x_3$

L CONTROLLER DESIGN

# ANOTHER METHOD FOR COMPUTING GAINS IS GIVEN BY ACKERMANN'S FORMULA

$$\dot{x} = Ax + Bu \qquad \dot{x} = (A - BK)x$$
$$u = -Kx$$

We want: $|sI - A_{CL}| = (s - s_1)(s - s_2)\cdots(s - s_n)$

$$= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \ = \phi(s)$$

The gain matrix is given by

$$K = [0 \ \cdots \ 0 \ \ 1]\overbrace{[B \ \ AB \ \ \cdots \ \ A^{n-1}B]}^{W_c}{}^{-1}\phi(A)$$



# PROOF OF ACKERMANN'S FORMULA (CONSIDER ONLY $n = 3$ CASE)

**Cayley-Hamilton Theorem**

- A matrix satisfies its own characteristic equation

$$\phi(s) = 0 \quad \Rightarrow \quad \phi(A_{CL}) = 0$$

$$\phi(A_{CL}) = A_{CL}^3 + \alpha_2 A_{CL}^2 + \alpha_1 A_{CL} + \alpha_0 I = 0$$

$s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$$A_{CL} = A - BK$$

$$A_{CL}^2 = (A - BK)(A - BK)$$
$$= A^2 - ABK - BKA_{CL}$$

$$A_{CL}^3 = (A - BK)(A^2 - ABK - BKA_{CL})$$
$$= A^3 - A^2BK - ABKA_{CL} - BKA_{CL}^2$$

$$\phi(A_{CL}) = (A^3 - A^2BK - ABKA_{CL} - BKA_{CL}^2)$$
$$+\alpha_2(A^2 - ABK - BKA_{CL})$$
$$+\alpha_1(A - BK)$$
$$+\alpha_0 I = 0$$

POSITIVE

$$\underbrace{A^3 + \alpha_2 A^2 + \alpha_1 A + \alpha_0 I}_{\phi(A)} = B(KA_{CL}^2 + \alpha_2 KA_{CL} + \alpha_1 K)$$
$$+AB(KA_{CL} + \alpha_2 K)$$
$$+A^2B(K)$$

$W_c$

$$\phi(A) = [B \quad AB \quad A^2B]\begin{bmatrix} KA_{CL}^2 + \alpha_2 KA_{CL} + \alpha_1 K \\ KA_{CL} + \alpha_2 K \\ K \end{bmatrix}$$

↑ WANT

$K = [0\ 0\ 1]\ W_c^{-1}\phi(A)$

FOR $n=3$



# LET'S REVISIT THE PREVIOUS EXAMPLE: PLACE ALL POLES AT $-4$

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad W_C = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

OK

$A_{CL} = A - BK$

$(\lambda+2)(\lambda+2)(\lambda+1)$ $K = [-8\ \ -20\ \ -27]$

$\phi(A) = A^3 + 12A^2 + 48 + 64I$

$= \begin{bmatrix} 8 & 12 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

FROM ABOVE

$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}[-8\ \ -20\ \ -27]$

# HOW COULD WE SOLVE THIS USING MATLAB?

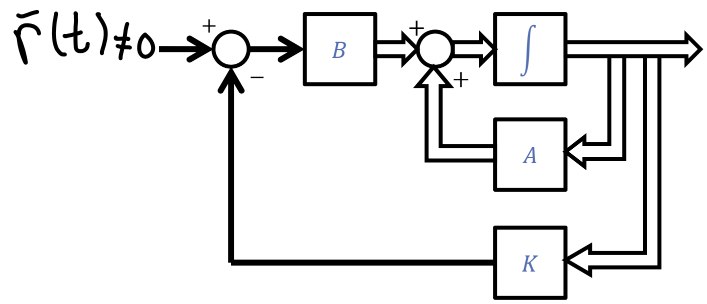
```
% Sample MATLAB script

» A = [-2 1 0; 0 -2 0; 0 0 -1];
» B = [0; 1; 1];
» Wc = [B A*B A^2*B]

Wc =
       0    1   -4
       1   -2    4
       1   -1    1
```



```
» phi=A^3+12*A^2+48*A+64
phi =
       8    76    64
       64    8    64
       64   64    27

» phi=A^3+12*A^2+48*A+64*eye(3)
phi =
       8    12     0
       0     8     0
       0     0    27

» K=[0 0 1]*inv(Wc)*phi
K =
       -8 -20 27

» eig(A-B*K)
ans =
       -3.9999
       -4.0000 + 0.0000i
       -4.0000 - 0.0000i
```

K= acker(A,B,[-4,-4,-4])
K= place= (A,B,....)

## WHAT IF WE WANT INPUT TRACKING, NOT REGULATION?




LECTURE 22



**Regulator Design:**

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

$\dot{x} = (A - BK)x$

**Tracking Design:**

$$\dot{x} = Ax + Bu + M\bar{r}$$

$$u = k_r\bar{r} - Kx$$

$\dot{x} = (A - BK)x$
$+ (M + Bk_r)\bar{r}$

SAME CL POLES
EIGENVALUES OF A-BK

# WHAT IF THE SYSTEM IS NOT COMPLETELY STATE CONTROLLABLE?

$$\underbrace{\mathrm{rank}(W_C) = \ell}_{\text{controllable states}} < n$$

n-ℓ → UNCONTROLLABLE STATES

Use the controllable decomposition transformation:

$$T = [\overset{\ell}{T_1} \quad \overset{n-\ell}{T_2}]$$

ℓxℓ   CONTROLLABLE

$$\hat{A} = T^{-1}AT = \begin{bmatrix} \hat{A}_C & \hat{A}_{CU} \\ 0 & \hat{A}_{UC} \end{bmatrix}, \qquad \hat{B} = T^{-1}B = \begin{bmatrix} \hat{B}_C \\ 0 \end{bmatrix}$$

UNCONTROLLABLE

Define $\hat{K} = KT = [\overset{\ell}{\hat{K}_1} \mid \hat{K}_2]$

$\hat{A}_C$   $\hat{A}_{UC}$



# FIND THE CHARACTERISTIC EQUATION FROM THE CLOSED-LOOP DETERMINANT

$$|sI - \overbrace{A + BK}^{A_{CL}}| = \left| sI - \begin{bmatrix} \hat{A}_C & \hat{A}_{CU} \\ 0 & \hat{A}_{UC} \end{bmatrix} + \begin{bmatrix} \hat{B}_C \\ 0 \end{bmatrix} [\hat{K}_1 \quad \hat{K}_2] \right|$$

$$= \begin{vmatrix} sI - \hat{A}_C + \hat{B}_C\hat{K}_1 & -\hat{A}_{CU} + \hat{B}_C\hat{K}_2 \\ 0 & sI - \hat{A}_{UC} \end{vmatrix}$$

$$= \underbrace{|sI - \overbrace{\hat{A}_c + \hat{B}\hat{K}_1}^{\hat{A}_{CL}}|}_{\text{poles from CL}} \cdot \underbrace{|sI - \hat{A}_{UC}|}_{\text{uncontrollable poles}}$$

Note: CL eigenvalues (poles) are independent of $\hat{K}_2$

# LET'S APPLY THIS TO AN UNCONTROLLABLE EXAMPLE

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \qquad \Rightarrow \quad W_C = \begin{bmatrix} 0 & -1 & -4 \\ 1 & -2 & 4 \\ 1 & -2 & 4 \end{bmatrix}$$

RANK($W_C$) = 2<3
ONE UNCONTROLLABLE MODE

- This system is not completely controllable (1 uncontrollable)
- Apply decomposition!



# APPLY THE STATE TRANSFORMATION FOR CONTROLLABLE DECOMPOSITION

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

TO MAKE INVERTABLE

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Let $u = -\underbrace{[\hat{k}_1 \quad \hat{k}_2 \quad 0]}_{K} z = -Kx = -KTz$

# FOCUS ON ONLY THE CONTROLLABLE SUBSYSTEM ($A_C$)

$$\dot{z}_C = \hat{A}_C z_C + \hat{B}_C u \qquad \hat{A}_C = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix}, \qquad \hat{B}_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

SET CL POLES TO -4

$$|sI - \hat{A}_c + \hat{B}_c \hat{K}_1| = (s+4)^2 = s^2 + 8s + 16 = \phi(s)$$

$$\hat{K}_1 = [0 \;\; 1][\hat{B}_c \;\; \hat{A}_c \hat{B}_c]^{-1} \ldots$$

$$\hat{A}_c^2 + 8\hat{A}_c + 16I$$

CHECK $|sI - A + BK|$
$= (s+4)^2(s+2)$

$$K_1 = [0 \;\; 1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 12 & -16 \\ 4 & -4 \end{bmatrix} = [4 \;\; -4]$$

$\hat{K} = [4 \;\; -4 \;\; 0]$ GAINS $\qquad K = \hat{K}T^{-1} = [4 \;\; 8 \;\; -4]$

# WHAT IF WE HAD USED A DIFFERENT SET OF EIGENVECTORS?

$$T_{alt} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \qquad T_{alt}^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$K_{alt} = [4 \quad -4 \quad 0] T_{alt}^{-1} = [4 \quad 4 \quad 0]$$

DIFFERENT GAIN VALUES!

By choosing a different third eigenvector, we have arrived at a different set of controller gains! The closed-loop eigenvalues are unchanged.

GET SAME POLES! ZEROS MAY DIFFER

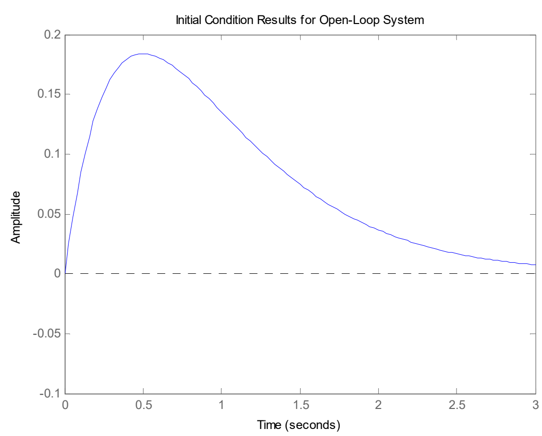


# SIMULATION RESULTS

**Choose:**

$$y = x_1$$

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

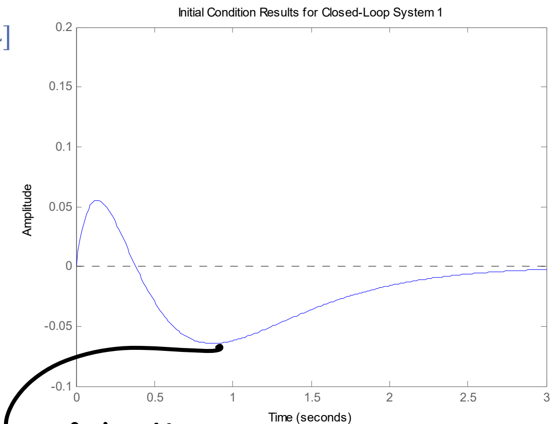


# SIMULATION RESULTS

**Choose:**

$$K = K_1 = [4 \quad 8 \quad -4]$$

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$




IF K VALUES ARE DIFF. → CHANGES ZEROS → CHANGES THE RESPONSE

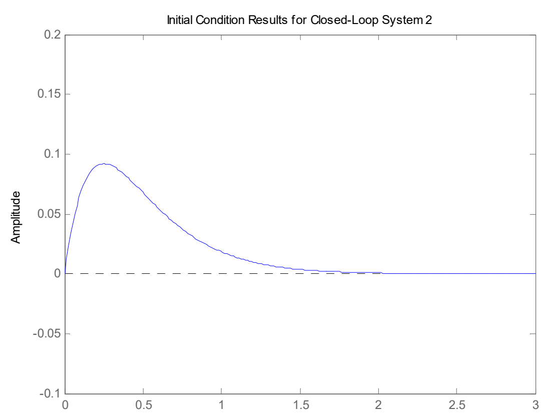


# SIMULATION RESULTS

**Choose:**

$$K = K_2 = [4 \quad 4 \quad 0]$$

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



# COMING UP…

**Observability**

- Observable Canonical Form
- Observable Canonical Decomposition
- General Decomposition

**State Observers**

- Rationale
- Observer Design