FROM LAST TIME…

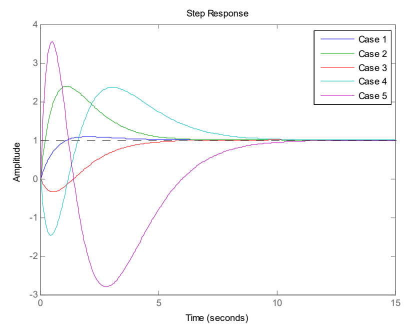
SISO Design Limitations

- System type
- Open-loop integrators
- Open-loop poles/zeros
- Imaginary poles/zeros

$$\int_0^{\infty} e(t)e^{-z_0 t}dt = \frac{1}{z_0}$$

$$\int_0^{\infty} e(t)e^{-p_0 t}dt = 0$$

	Steady-State System Error		
Type	**Step**	**Ramp**	**Parabolic**
0	finite	∞	∞
1	0	finite	∞
2	0	0	finite
3	0	0	0



FREQUENCY-DOMAIN DESIGN LIMITATIONS

Topics

- Bode’s Integral Constraints on Sensitivity
- Integral Constraints on Complementary Sensitivity
- Poisson Integral Constraint on Sensitivity
- Poisson Integral Constraint on Complementary Sensitivity

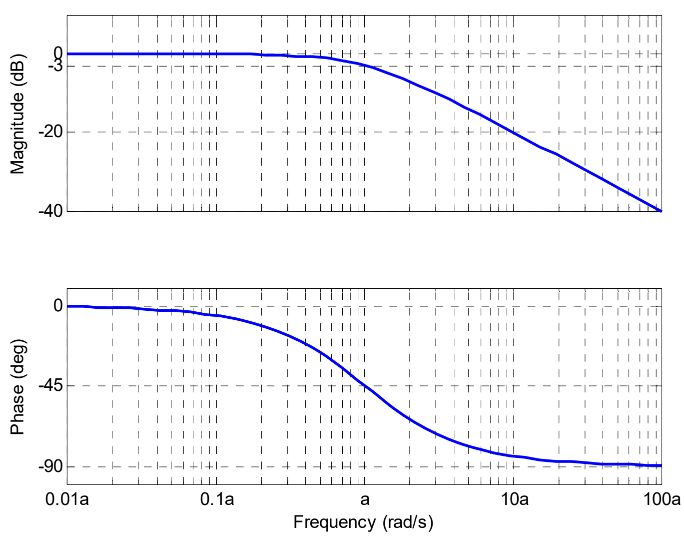
At the end of this section, students should be able to:

- Describe the waterbed effect with respect to sensitivity functions.
- Identify causes of large sensitivity peaks.
- Determine appropriate CL bandwidth values based on frequency-domain limitations.

BANDWIDTH IS THE FREQUENCY AT WHICH THE MAGNITUDE IS -3 dB BELOW DC GAIN

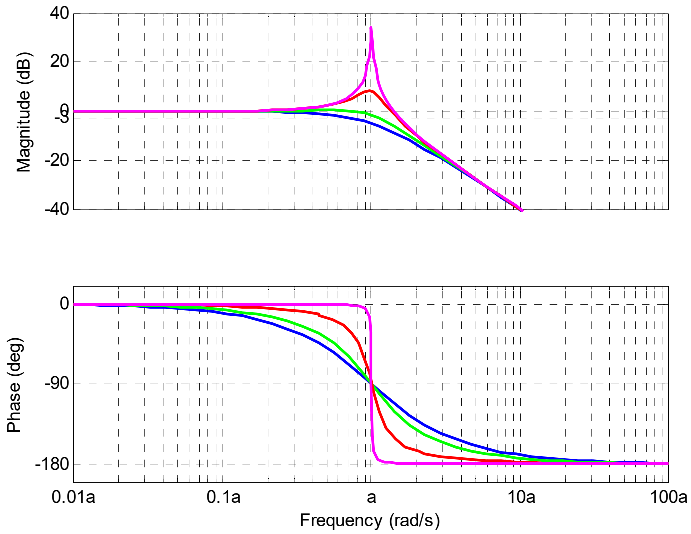
EXAMPLE: 1ST ORDER BANDWIDTH

$s = -a \qquad \omega_{BW} = a$



EXAMPLE: 2ND ORDER BANDWIDTH

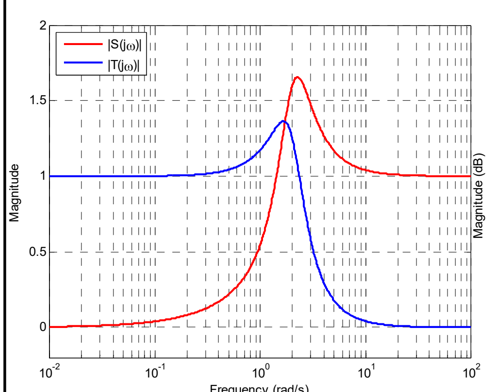
$$s = -\zeta\omega_n \pm j\omega_d \qquad \omega_{BW} = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

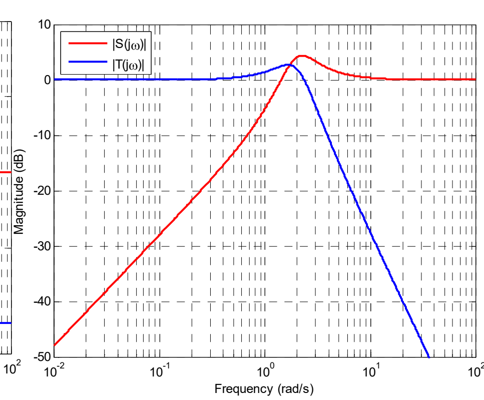


CONSIDER A TYPICAL PERFORMANCE SPECIFICATION

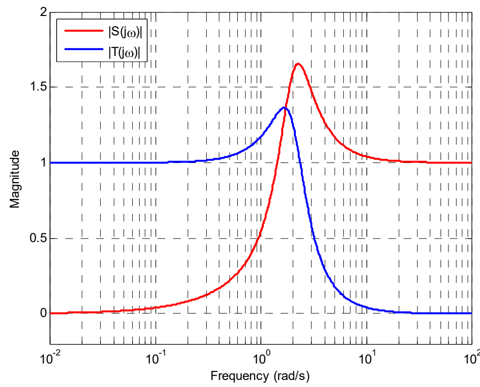
$$|S_o(j\omega)| < \varepsilon_S < 1, \qquad \text{for } \omega < \omega_l$$

$$|T_o(j\omega)| < \varepsilon_T < 1, \qquad \text{for } \omega > \omega_h$$





BASED ON PERFORMANCE SPECIFICATIONS, WHAT DO WE EXPECT TO SEE FROM $S(j\omega)$ AND $T(j\omega)$?



- How will these affect step response?

FREQUENCY DOMAIN LIMITATIONS

BODE'S INTEGRAL CONSTRAINTS

BODE'S INTEGRAL CONSTRAINTS ON SENSITIVITY

Lemma 9.1 (Water Bed Effect)

- Consider stable CL system with one-DOF controller configuration and open loop TF given by

$$L(s) = G(s)C(s) = e^{-s\tau}\bar{L}(s), \qquad \tau \geq 0$$

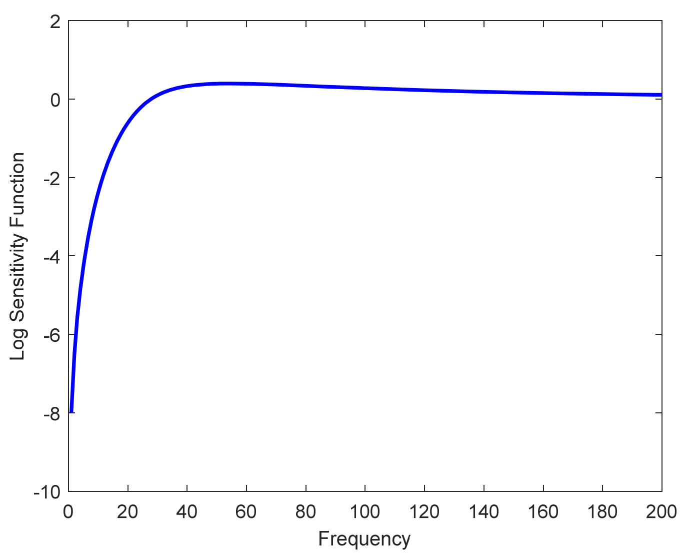
where $\bar{L}(s)$ is a rational TF of relative degree $r = n_{\bar{L}} - m_{\bar{L}} > 0$. Assume that $\bar{L}(s)$ has no open-loop poles in open RHP. Then, the sensitivity function satisfies

$$\int_0^\infty \ln|S(j\omega)|\, d\omega = \begin{cases} 0 & \text{for } r > 1 \\ -\kappa\dfrac{\pi}{2} & \text{for } r = 1 \end{cases}$$

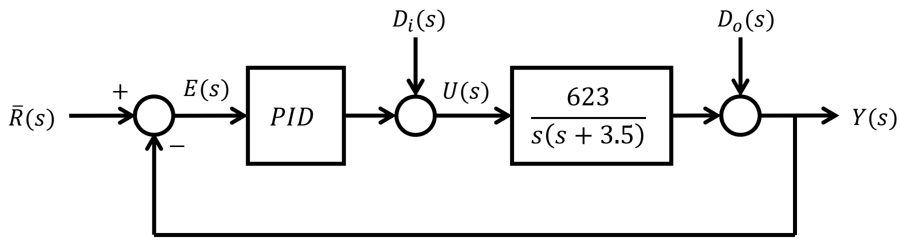
where

$$\kappa = \lim_{s\to\infty} s\bar{L}(s)$$

WHAT DOES IT MEAN TO PLOT THE LOG OF THE SENSITIVITY FUNCTION?



LET'S VERIFY WITH A SIMPLE EXAMPLE



PID 1 Controller

$$C_1(s) = \frac{8.3s^2 + 205s + 2054}{s(s+117)}$$

PID 3 Controller

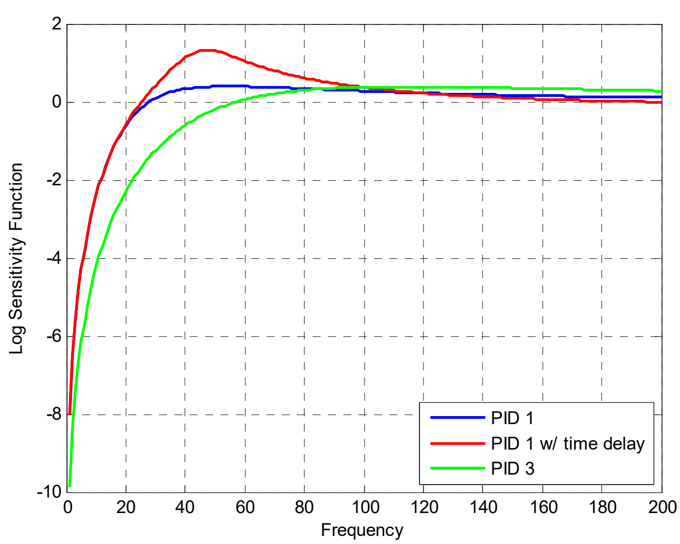
$$C_3(s) = \frac{34s^2 + 1550s + 26729}{s(s+237)}$$

Without Delay

$$L_1(s) = G_o(s)C_1(s)$$

With actuation delay

$$L_{1T}(s) = e^{-\tau s}G_o(s)C_1(s)$$
$$\tau = 0.01 \text{ sec}$$



All log sensitivity functions have equal negative and positive integral areas!

BODE'S INTEGRAL CONSTRAINTS ON SENSITIVITY

Lemma 9.2 (Unstable Open Loop Poles)

- Consider stable CL system with one-DOF controller configuration and open loop TF given by

$$L(s) = G(s)C(s) = e^{-s\tau}\bar{L}(s), \qquad \tau \geq 0$$

where $\bar{L}(s)$ is a rational TF of relative degree $r = n_{\bar{L}} - m_{\bar{L}} > 0$. Assume that $\bar{L}(s)$ has unstable open-loop poles at $p_1, \dots, p_n$. Then, the sensitivity function satisfies

$$\int_0^\infty \ln|S(j\omega)|\, d\omega = \begin{cases} \pi \sum_{i=1}^{N} Re\{p_i\} & \text{for } r > 1 \\ \pi \sum_{i=1}^{N} Re\{p_i\} - \kappa \frac{\pi}{2} & \text{for } r = 1 \end{cases}$$

where

$$\kappa = \lim_{s\to\infty} s\bar{L}(s)$$

WHAT CONCLUSIONS CAN WE DRAW FROM THESE CONSTRAINTS?

1. Independent of controller design, low sensitivity ($|S(j\omega)| \ll 1$) in certain prescribed frequency bands will result in a sensitivity larger than one ($|S(j\omega)| > 1$) in other frequency bands

2. With unstable open-loop poles, the integral of log sensitivity is required to be greater than zero, which makes sensitivity minimization more difficult

INTEGRAL CONSTRAINTS ON COMPLEMENTARY SENSITIVITY

Lemma 9.3 (Minimum Phase Systems or No RHP Zeros)

- Consider stable CL system with one-DOF controller configuration and open loop TF given by

$$L(s) = G(s)C(s) = e^{-s\tau}\bar{L}(s), \qquad \tau \geq 0$$

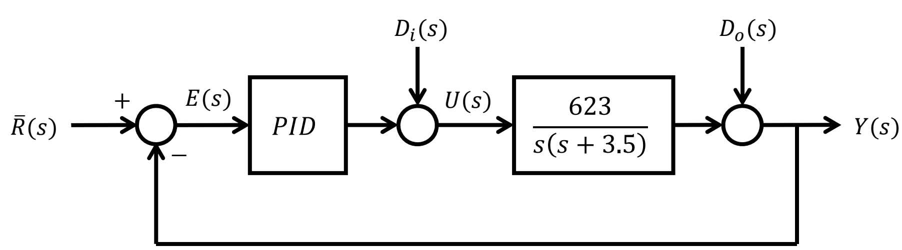
where $\bar{L}(s)$ is a rational TF of relative degree $r = n_{\bar{L}} - m_{\bar{L}} > 1$ and has at least one free integrator (i.e. $\bar{L}(0)^{-1} = 0$). Assume that $\bar{L}(s)$ has no open-loop zeros in the open RHP. Then, the complementary sensitivity function satisfies

$$\int_{0^-}^{\infty} \frac{1}{\omega^2} \ln|T(j\omega)|\, d\omega = \frac{\pi\tau}{2} - \frac{\pi}{2K_V}$$

where $\quad K_V \equiv \lim_{s\to 0} s\bar{L}(s)$

Note: $\int_{0^-}^{\infty} \frac{1}{\omega^2} \ln|T(j\omega)|\, d\omega = \int_{0}^{\infty} \ln\left|T\left(j\frac{1}{v}\right)\right| dv\,, \qquad v = \frac{1}{\omega}$

BACK TO THE DC MOTOR EXAMPLE



PID 1 Controller

$$C_1(s) = \frac{8.3s^2 + 205s + 2054}{s(s+117)}$$

PID 3 Controller

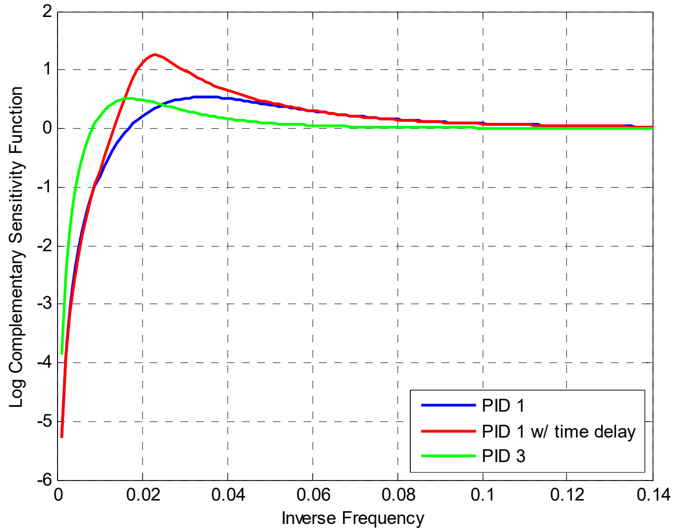
$$C_3(s) = \frac{34s^2 + 1550s + 26729}{s(s+237)}$$

Without Delay

$$L_1(s) = G_o(s)C_1(s)$$

With actuation delay

$$L_{1T}(s) = e^{-\tau s}G_o(s)C_1(s)$$
$$\tau = 0.01 \text{ sec}$$



All log complementary sensitivity functions have equal negative and positive integral areas w.r.t. $v = 1/\omega$ when $\tau = 0$, and larger positive integral area when $\tau > 0$!

INTEGRAL CONSTRAINTS ON COMPLEMENTARY SENSITIVITY

Lemma 9.4 (Nonminimum Phase Systems or RHP Zeros)

- Consider stable CL system with one-DOF controller configuration and open loop TF given by

$$L(s) = G(s)C(s) = e^{-s\tau}\bar{L}(s), \qquad \tau \geq 0$$

where $\bar{L}(s)$ is a rational TF of relative degree $r = n_{\bar{L}} - m_{\bar{L}} > 1$ and has at least one free integrator (i.e. $\bar{L}(0)^{-1} = 0$). Assume that $\bar{L}(s)$ has RHP open-loop zeros $z_1, \dots, z_m$. Then, the complementary sensitivity function satisfies

$$\int_{0^-}^{\infty} \frac{1}{\omega^2} \ln|T(j\omega)|\, d\omega = \pi \sum_{i=1}^{M} Re\left\{\frac{1}{z_i}\right\} + \frac{\pi\tau}{2} - \frac{\pi}{2K_V}$$

where $K_V \equiv \lim_{s\to 0} s\bar{L}(s)$

Note: $\int_{0^-}^{\infty} \frac{1}{\omega^2} \ln|T(j\omega)|\, d\omega = \int_{0}^{\infty} \ln\left|T\left(j\frac{1}{v}\right)\right| dv, \qquad v = \frac{1}{\omega}$

WHAT CONCLUSIONS CAN WE DRAW FROM THESE CONSTRAINTS?

1. Independent of controller design, low complementary sensitivity in certain prescribed frequency bands will result in a complementary sensitivity larger than one in other frequency bands

2. In general, negative integral is unavoidable at high frequency range for noise attenuation while zero value is desirable at low frequency range for command following and disturbance rejection

3. The appearance of RHP open-loop zeros adds more positive value to the integral of log complementary sensitivity, and thus makes the allocation of complementary sensitivity in the frequency domain more difficult.

FREQUENCY DOMAIN LIMITATIONS

POISSON'S INTEGRAL CONSTRAINTS

POISSON INTEGRAL CONSTRAINT ON SENSITIVITY

Lemma 9.4 (Nonminimum Phase Systems or RHP Zeros)

- Consider stable CL system with one-DOF controller configuration and open loop TF given by

$$L(s) = G(s)C(s) = e^{-s\tau}\bar{L}(s), \qquad \tau \geq 0$$

- where $\bar{L}(s)$ is a rational TF. Assume that $\bar{L}(s)$ has RHP open-loop zeros $z_1, \ldots, z_m$, where $z_k = \gamma_k + j\delta_k$. Then, when $\bar{L}(s)$ has no unstable open-loop poles, the sensitivity function satisfies

$$\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{\gamma_k}{\gamma_k^2 + (\omega - \delta_k)^2} d\omega = 0, \qquad \forall k = 1, \ldots, m$$

- when $\bar{L}(s)$ has unstable open-loop poles at $p_1, \ldots, p_n$

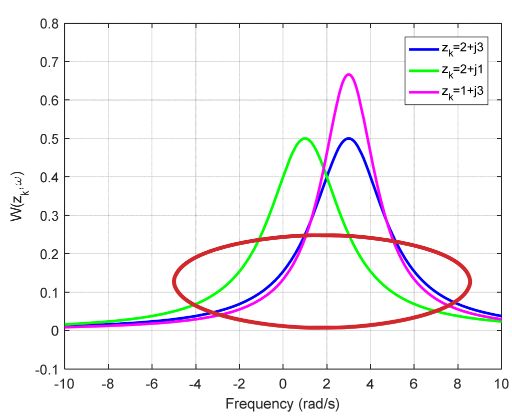
$$\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{\gamma_k}{\gamma_k^2 + (\omega - \delta_k)^2} d\omega = -\pi \ln|B_P(z_k)| \qquad B_P \triangleq \prod_{i=1}^{N} \frac{s - p_i}{s + p_i^*}$$

WHAT IS THAT COMPLICATED WEIGHTING TERM?

$$\int_{-\infty}^{\infty} \ln|S(j\omega)| \boxed{\frac{\gamma_k}{\gamma_k^2 + (\omega - \delta_k)^2}} d\omega = 0$$

$$z_k = \gamma_k + j\delta_k$$

$$W(z_k, \omega) \triangleq \frac{\gamma_k}{\gamma_k^2 + (\omega - \delta_k)^2}$$



Compensation has to be achieved over a finite frequency band

THE WEIGHTING TERM CAN BE USED TO DEFINE A WEIGHTED LENGTH OF FREQUENCY AXIS

$$\int_{-\infty}^{\infty} W(z_k, \omega) d\omega = \pi$$

$$\Omega(z_k, \omega_c) \triangleq 2\int_{-\omega_c}^{\omega_c} W(z_k, \omega) d\omega = 2\tan^{-1}\left(\frac{\omega_c - \delta_k}{\gamma_k}\right) + 2\tan^{-1}\left(\frac{\omega_c + \delta_k}{\gamma_k}\right)$$

$$\int_{\omega_1}^{\omega_2} W(z_k, \omega) d\omega =$$

WHAT IS SPECIAL ABOUT THAT BLASHKE PRODUCT?

$$B_P \triangleq \prod_{i=1}^{N} \frac{s - p_i}{s + p_i^*}$$

What if we choose a controller zero to nearly cancel one unstable plant pole?

$$\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{\gamma_k}{\gamma_k^2 + (\omega - \delta_k)^2} d\omega = -\pi \ln|B_P(z_k)|$$

WHAT CONCLUSIONS CAN WE DRAW FROM POISSON'S CONSTRAINT?

1. The appearance of unstable open-loop poles adds more positive value to the total integral, since

$$\ln|B_P(z_k)| < 0, \qquad \forall \text{ RHP } z_k$$

and thus makes the allocation of sensitivity in frequency domain more difficult

2. When one RHP zero approaches an unstable open-loop pole, $|\ln|B_p(z_k)||$ grows without bound, which would make the allocation almost impossible.

APPLY POISSON INTEGRAL CONSTRAINT FOR NMP ZEROS TO FIND THE SENSITIVITY PEAK

$$\int_{-\infty}^{\infty} \ln|S(j\omega)|\, W(z_k,\omega)d\omega = -\pi \ln|B_P(z_K)|\,, \qquad z_k = \gamma_k + j\delta_k$$

Lower bound for sensitivity peak $S_{max} \geq |S(j\omega)|$:

$$\ln S_{max} > \frac{1}{\Omega(z_k,\omega_h) - \Omega(z_k,\omega_l)}\left[\begin{array}{c} |\pi \ln|B_P(z_K)|| + \\ |(\ln \varepsilon_S)\Omega(z_k,\omega_l)| - \big(\pi - \Omega(z_k,\omega_h)\big)\ln(1+\varepsilon_T) \end{array}\right]$$

WHAT CAN WE OBSERVE FROM THIS?

1. When CL bandwidth is large when compared to the speed of NMP zero (e.g., $\omega_l = 2\gamma_k$), even without considering the effect of any possible open-loop unstable poles and the performance constraints on $T(s)$, it is easy to verify that there will be a huge sensitivity peak

Example: Assume $\omega_l = 2\gamma_k, \delta_k = 0, \varepsilon_S = 0.3$

$$\begin{aligned}\ln S_{max} &\geq \frac{1}{\Omega(z_k,\infty) - \Omega(z_k,\omega_l)}|(\ln \varepsilon_S)\Omega(z_k,\omega_l)| \\ &= \frac{1}{\pi - \Omega(z_k,2\gamma_k)}|(\ln 0.3)\Omega(z_k,2\gamma_k)|, \qquad \Omega(z_k,2\gamma_k) = 2.21 \\ &= 2.86\end{aligned}$$

2. Sharp transitions in the sensitivity frequency response, i.e., ω_l close to ω_h, will contribute to large sensitivity peaks.

POISSON INTEGRAL CONSTRAINT ON SENSITIVITY

Lemma 9.6 (RHP Open-Loop Poles and Zeros**)**

- Consider stable CL system with one-DOF controller configuration and open loop TF given by

$$L(s) = G(s)C(s) = e^{-s\tau}\bar{L}(s), \qquad \tau \geq 0$$

- where $\bar{L}(s)$ is a rational TF. Assume that $\bar{L}(s)$ has open-loop unstable poles at $p_1, \dots, p_n$, where $p_i = \alpha_i + j\beta_i$. Then, when $\bar{L}(s)$ has no RHP open-loop zeros,

$$\int_{-\infty}^{\infty} \ln|T(j\omega)| \frac{\alpha_i}{\alpha_i^2 + (\omega - \beta_i)^2} d\omega = \pi\tau\alpha_i, \qquad \forall i = 1, \dots, n$$

- when $\bar{L}(s)$ has RHP open-loop zeros at $z_1, \dots, z_m$

$$\int_{-\infty}^{\infty} \ln|T(j\omega)| \frac{\alpha_i}{\alpha_i^2 + (\omega - \beta_i)^2} d\omega = -\pi \ln|B_z(p_i)| + \pi\tau\alpha_i \qquad B_z \triangleq \prod_{k=1}^{m} \frac{s - z_k}{s + z_k^*}$$

APPLY CONSTRAINT TO FIND THE COMPLEMENTARY SENSITIVITY PEAK

$$\int_{-\infty}^{\infty} \ln|T(j\omega)| \frac{\alpha_i}{\alpha_i^2 + (\omega - \beta_i)^2} d\omega = -\pi \ln|B_z(p_i)| + \pi\tau\alpha_i, \qquad p_i = \alpha_i + j\beta_i$$

Lower bound for complementary sensitivity peak $T_{max} \geq |T(j\omega)|$:

$$\ln T_{max} > \frac{1}{\Omega(p_i, \omega_h)} \left[|\pi \ln|B_z(p_i)|| + \pi\tau\alpha_i + \big(\pi - \Omega(\alpha_i, \omega_h)\big)|\ln(\varepsilon_T)| \right]$$

WHAT CAN WE OBSERVE FROM THIS?

1. The lower bound on complementary sensitivity peak is larger for systems with pure delays and the influence of a delay increases for faster unstable poles (i.e., large α_i)
2. The lower bound grows unbounded when a NMP zero approaches an unstable pole, because then $|\ln|B_z(p_i)||$ grows unbounded
3. When CL bandwidth is much smaller compared to unstable poles (i.e., $\omega_h \ll \alpha_i$), $\Omega(p_i, \omega_h)$ will be very small, leading to very large complementary sensitivity peak. Therefore, to avoid large transient response, CL bandwidth should be chosen larger than unstable poles
4. Large sensitivity peaks leads to large deviations in transient response in time domain and small stability margins in frequency domain. Thus the design problem becomes more difficult when the system has *fast unstable open-loop poles* and *slow NMP zeros*. The notions of *fast* and *slow* are relative to CL bandwidth

CONSIDER TWO OPEN-LOOP TFS, WHICH DIFFER BY A LHP/RHP ZERO

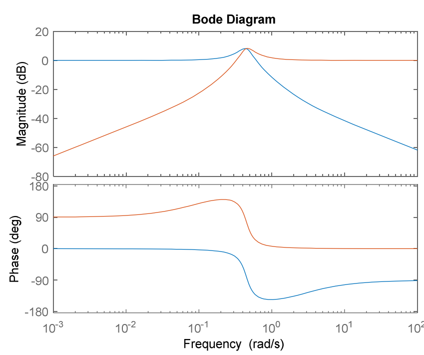
$$
\begin{aligned}
L_1(s) &= \frac{2(0.4s+1)}{s(10s+1)}, \\
L_2(s) &= \frac{2(-0.4s+1)}{s(10s+1)},
\end{aligned}
$$

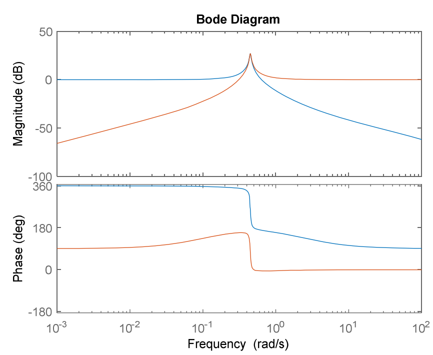
$$
S_1(s) = \frac{s(10s+1)}{(10s^2+1.8s+2)} = \frac{s(s+0.1)}{(s^2+0.18s+0.2)}
$$

$$
S_2(s) = \frac{s(10s+1)}{(10s^2+0.2s+2)} = \frac{s(s+0.1)}{(s^2+0.02s+0.2)}
$$

- The presence of the RHP zero (no canceling) makes things more difficult (larger positive integral of log of S)
- Expect a higher sensitivity peak

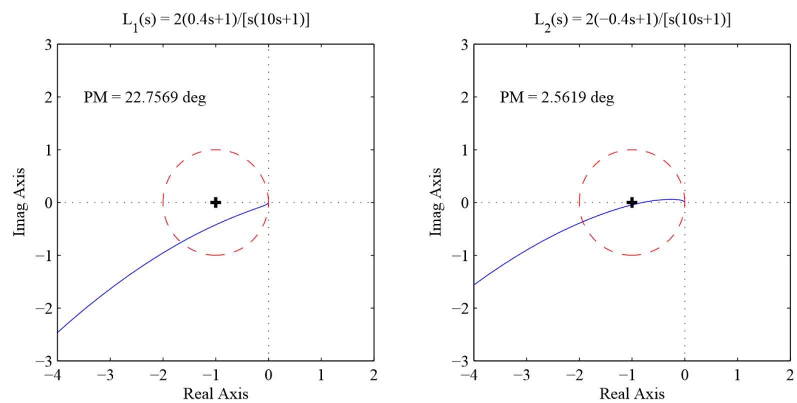
ACTUAL SENSITIVITY CURVES…



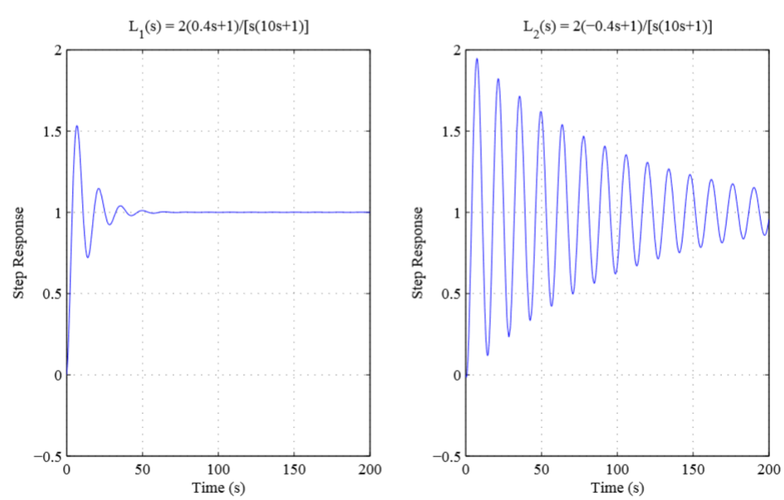


- As expected, sensitivity peak is higher for RHP zero
- This means that the Nyquist plot will approach -1 more closely (less gain and phase margin)

NYQUIST PLOT, AS EXPECTED…



WHAT ABOUT STEP RESPONSE?



COMING UP…

Summary of Design Limitations

Architectural Issues

- Internal Model Principle
- Feedforward
- Cascade Control

Intro to State-Space

- TF to SS
- Controllable Canonical Form
- Observable Canonical Form

Midterm Exam!