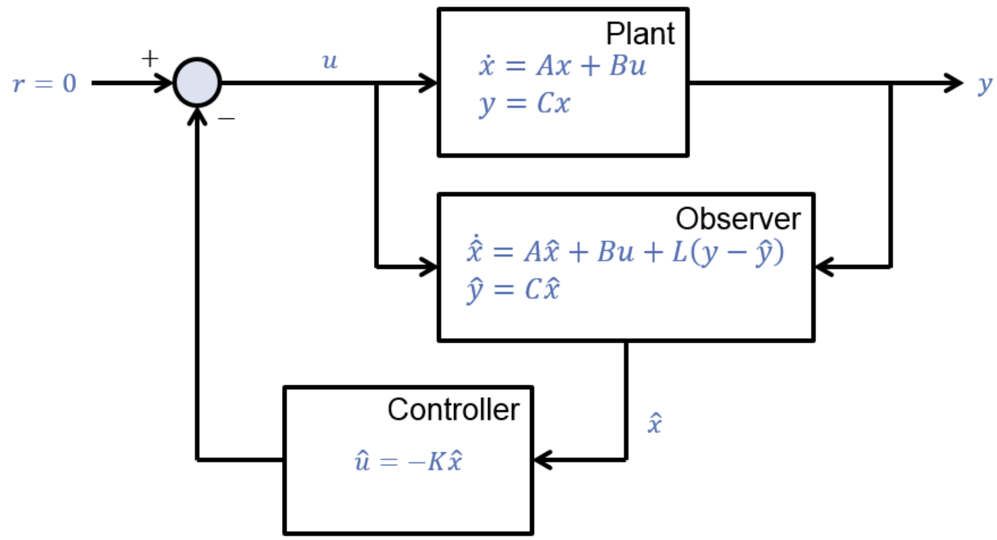
FROM LAST TIME…

State observer + Feedback

- Output feedback
- Reduced-order observer



$$|sI - A + BK + LC| = |sI - A + BK||sI - A + LC|$$

Measured state

Partition the state vector: $x = \begin{bmatrix} x_1 \\ x_e \end{bmatrix}$

Estimated states

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} a_{11} & A_{1e} \\ A_{e1} & A_{ee} \end{bmatrix} \begin{bmatrix} x_1 \\ x_e \end{bmatrix} + \begin{bmatrix} b_1 \\ B_e \end{bmatrix} u$$

TRACKING AND INTEGRAL CONTROL

Topics

- Tracking Systems
- State Feedback with Integration

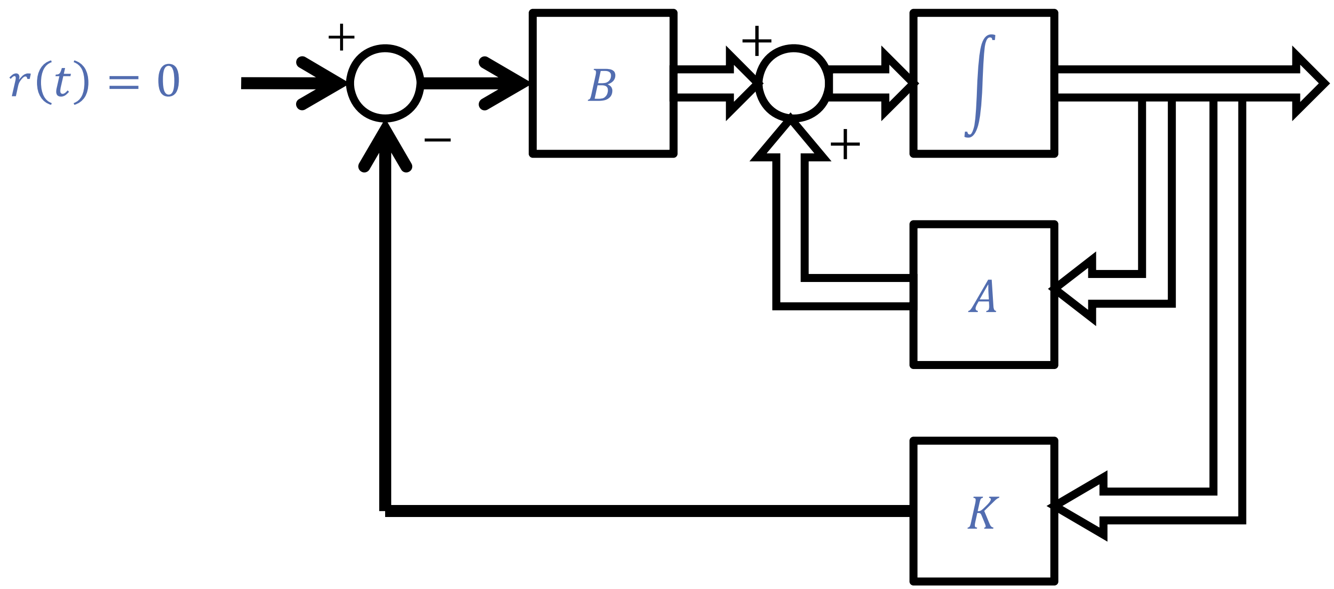
At the end of this section, students should be able to:

- Apply state observer and state feedback designs to tracking control problems.
- Develop state feedback gains for systems with integration.

Tracking the reference

TRACKING CONTROL

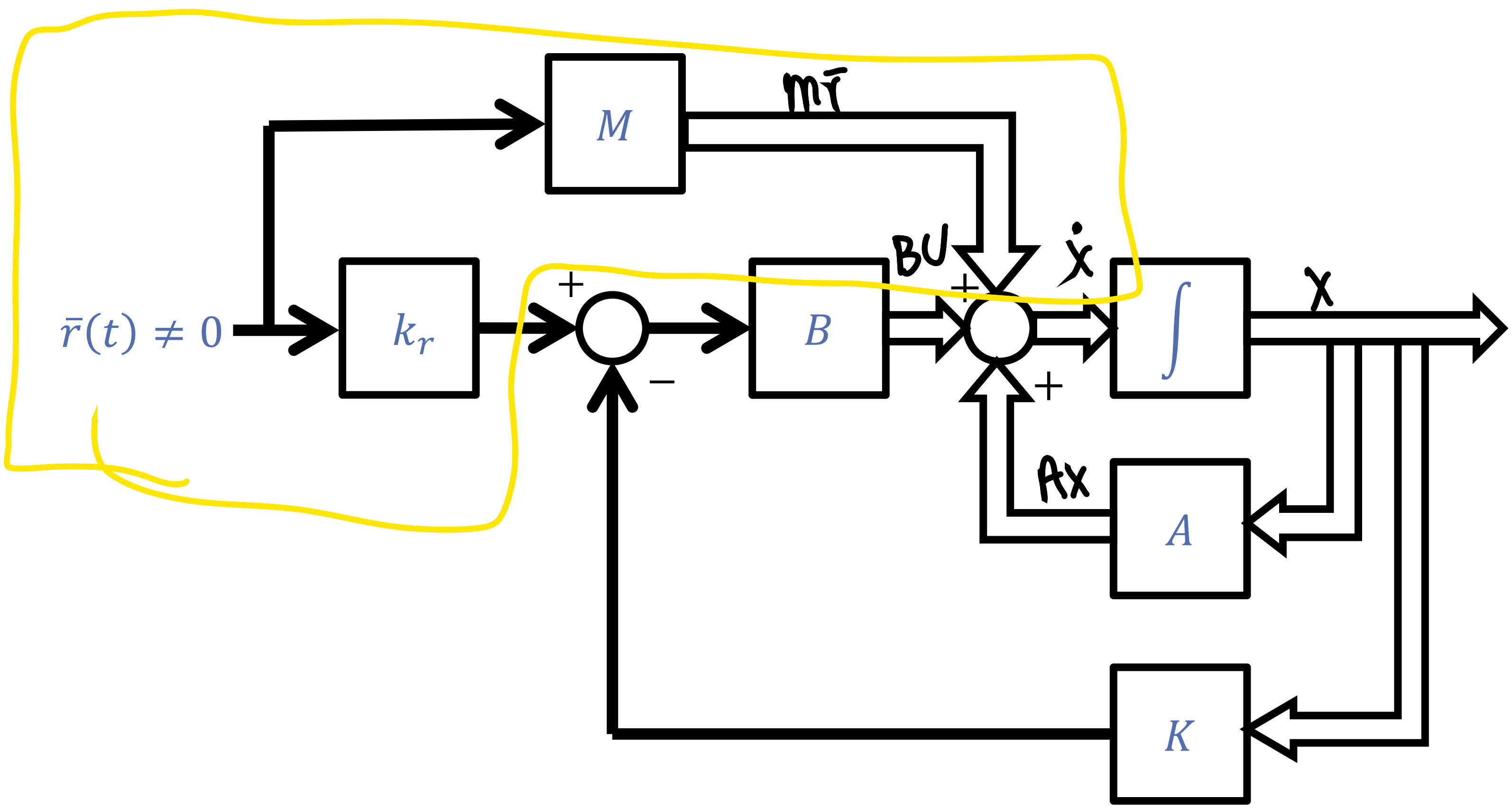
RECALL THAT MOST OF OUR PREVIOUS RESULTS WERE DERIVED FOR REGULATORS



← REFERENCE = 0

- Regulators can be useful, for instance the inverted pendulum
- We usually want the system to track a reference

FOR INPUT TRACKING, ALLOW THE REFERENCE TO BE NON-ZERO



- Previously we design only n gains (K)
- Now we must design $n + 1$ additional gains (M, k_r)

FOR STATE FEEDBACK, REGULATOR AND TRACKING DESIGNS ARE SIMILAR

Regulator Design:

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

$$\dot{x} = (A - BK)x$$

Tracking Design:

$$\dot{x} = Ax + Bu + \underline{M\bar{r}}$$

$$u = \underbrace{k_r\bar{r}}_{\text{REFERENCE}} - Kx$$

$$\dot{x} = (A - BK)x + \underbrace{(M + Bk_r)}_{\text{NEW}}\bar{r}$$

SAME CL POLES $(A - BK)$

TWO SPECIAL CASES EXIST FOR THE OBSERVER DESIGN

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly - M\bar{r}$$
$$u = k_r\bar{r} - K\hat{x}$$

1. Observer performance independent of r
2. Observer acts on tracking error $(r - y)$

CASE 1: OBSERVER PERFORMANCE INDEPENDENT OF r

STATE ESTIMATION ERROR $e = x - \hat{x}$

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$
$$= [Ax + \cancel{B(k_r r} - \cancel{K\hat{x}})] - [(A - LC - \cancel{BK)\hat{x}} + \cancel{Bk_r r} - Ly - Mr]$$

$$\dot{e} = (A - LC)e - Mr$$

- Choose $M = 0$ IF $M=0$ IT IS SIMILAR TO A PRE-FILTER DESIGN
- Set k_r for steady-state error spec

CASE 1: DETERMINE THE CONTROL ACTION

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$
$$u = k_r r - K\hat{x}$$

$$\dot{\hat{x}} = (A - LC - BK)\hat{x} + Bk_r r + Ly$$

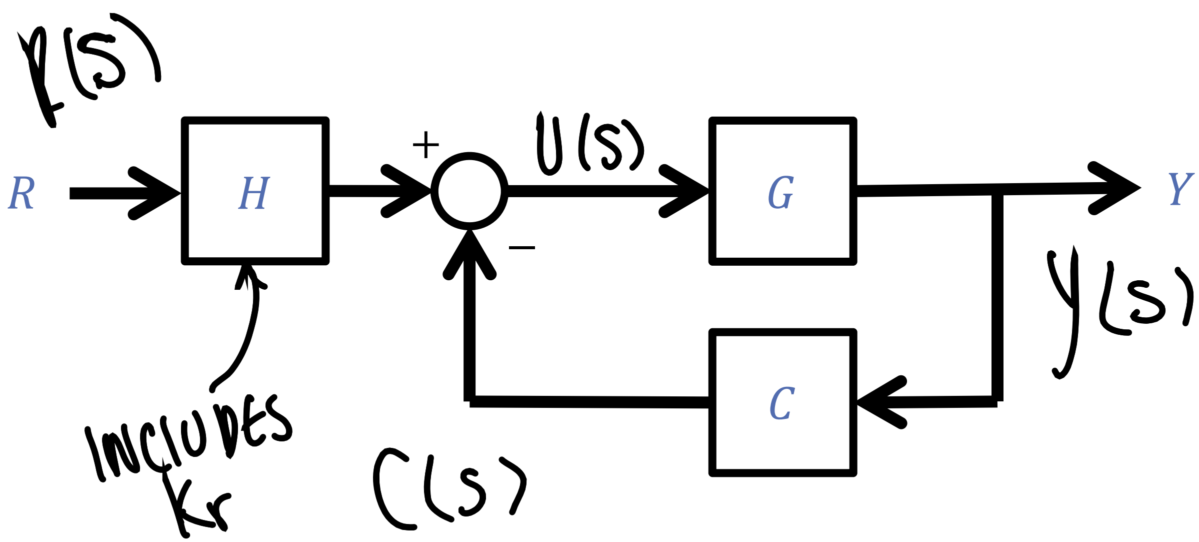
$sIX(s) = \ldots\ldots X(s) + \ldots\ldots$

$$\hat{X}(s) = (sI - A + LC + BK)^{-1}[Bk_r R(s) + LY(s)]$$
$$U(s) = k_r R(s) - K\hat{X}(s)$$

$$U(s) = \underbrace{k_r[1 - K(sI - A + LC + BK)^{-1}B]}_{H(s)}R(s) - \underbrace{K(sI - A + LC + BK)^{-1}L}_{C(s)}Y(s)$$

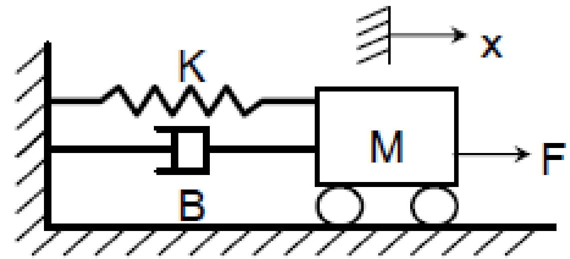
$$U(s) = H(s)R(s) - C(s)Y(s)$$

THIS IS SIMILAR TO OUR PREVIOUS WORK WITH THE 2-DOF CONTROLLER

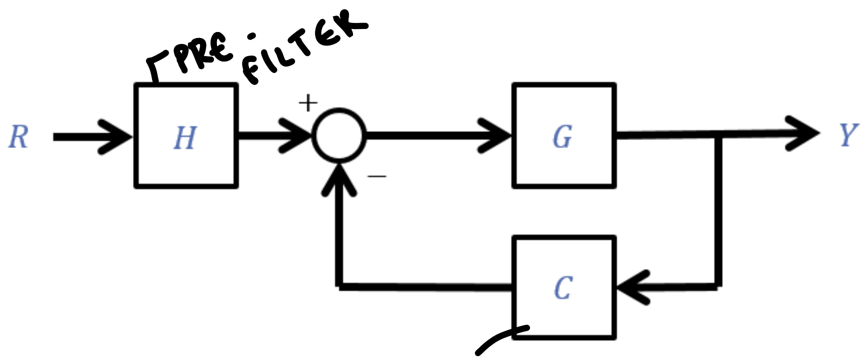


APPLY TO THE MASS-SPRING-DAMPER EXAMPLE FROM BEFORE

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



$$K = \begin{bmatrix} 24 & 6 \end{bmatrix} \qquad L = \begin{bmatrix} 35 \\ 288 \end{bmatrix}$$



$$U(s) = H(s)R(s) - C(s)Y(s)$$

OBSERVER + STATE FEEDBACK CONTROLLER

- Choose observer independent of r
- Design $H(s)$ and $C(s)$ as transfer functions

SOLVE FOR $H(s)$ AND $C(s)$, THEN $T(s)$

$$H(s) = k_r[1 - K(sI - A + LC + BK)^{-1}B]$$

$$= k_r \frac{s^2 + 36s + 324}{s^2 + 42s + 558}$$

$$G(s) = \frac{1}{s^2 + s + 1}$$

$$C(s) = K(sI - A + LC + BK)^{-1}L = \frac{2568s + 7542}{s^2 + 42s + 558}$$

$$T(s) = \frac{Y(s)}{R(s)} = H(s)\frac{G(s)}{1 + C(s)G(s)} = \frac{k_r\,(s^2 + 36s + 324)}{(s^2 + 42s + 558)\underbrace{(s + 18)^2}_{\text{OBSERVER poles}}}$$

DC GAIN $T(0) = \dfrac{k_r}{25}$

CHOOSE $k_r = 25$

CASE 2: OBSERVER ACTS ON TRACKING ERROR $(r-y)$

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly + Mr$$
$$u = k_r r - K\hat{x}$$

MATRIX

SCALAR #

Set $M = -L$, $k_r = 0$

$$\dot{\hat{x}} = (A - LC - BK)\hat{x} - L(r - y)$$
$$u = -K\hat{x}$$

$$U(s) = [K(sI - A + LC + BK)^{-1}L][R(s) - Y(s)]$$

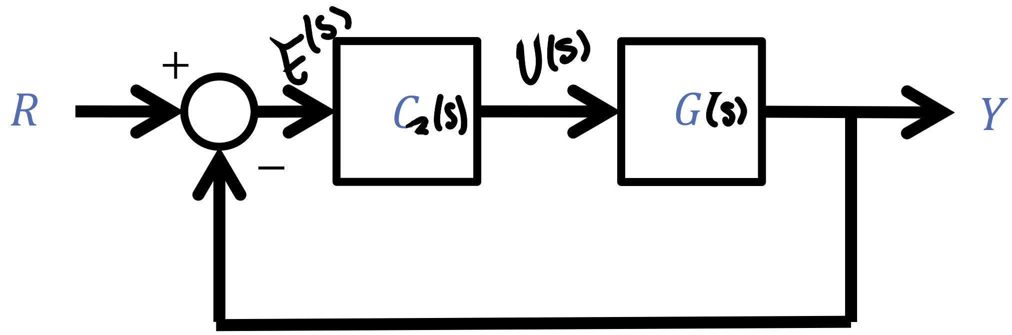
$C_2(s)$

SECOND CASE

TRACKING ERROR

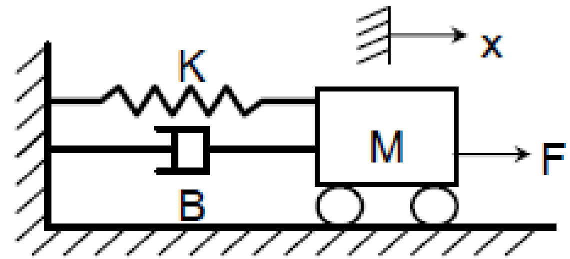
DESIRED - WHERE WE ARE

THIS IS SIMILAR TO OUR PREVIOUS WORK 1-DOF FORWARD PATH CONTROLLER

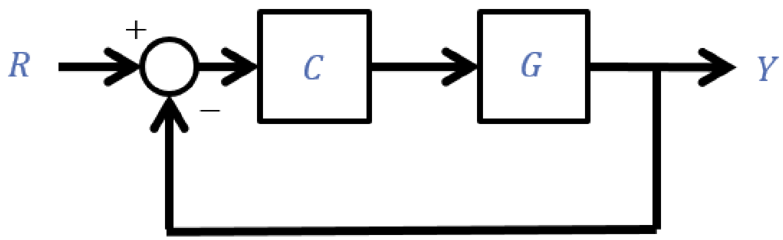


APPLY TO THE MASS-SPRING-DAMPER EXAMPLE FROM BEFORE

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



$$K = \begin{bmatrix} 24 & 6 \end{bmatrix} \qquad L = \begin{bmatrix} 35 \\ 288 \end{bmatrix}$$



$$U(s) = C(s)\big(R(s) - Y(s)\big)$$

- Choose observer acting on $(r - y)$ ~ ERROR B/T INPUT & OUTPUT
- Design $C(s)$ as a transfer functions

SOLVE FOR $C(s)$, THEN $T(s)$

$$C_2(s) = K(sI - A + LC + BK)^{-1}L$$

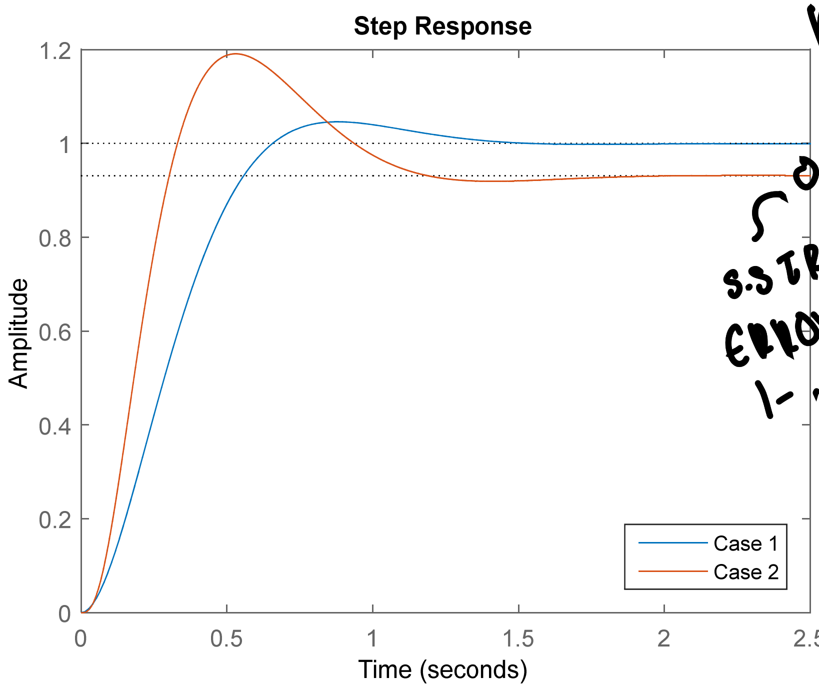
$$= \frac{2568s + 7542}{s^2 + 42s + 538}$$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C_2(s)G(s)}{1 + C_2(s)G(s)} = \frac{(2568s + 7542)}{(s^2 + 42s + 558)(s + 18)^2}$$

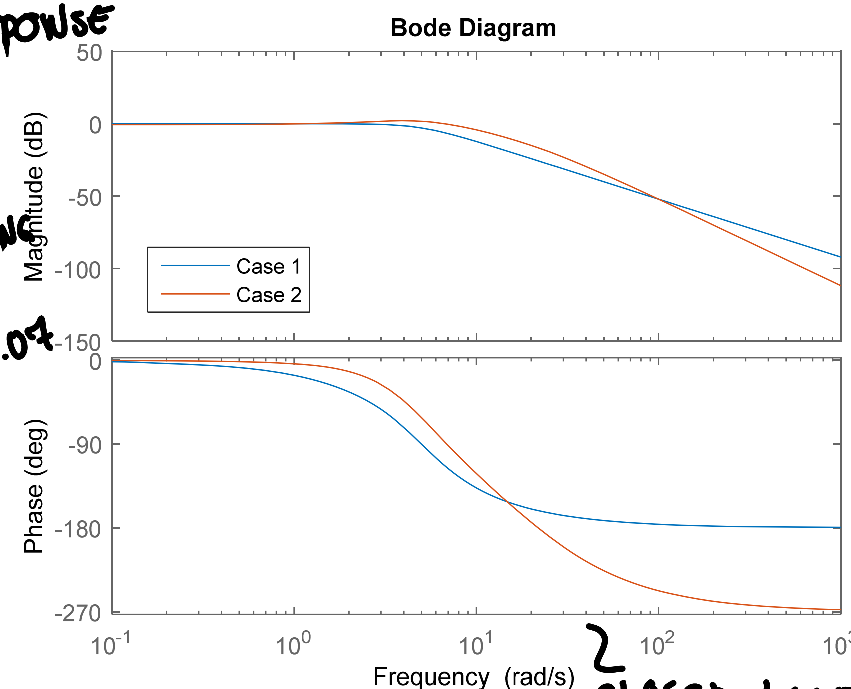
$$T(0) = .93$$

COMPARE THE RESULTS OF THE TWO METHODS

ZEROS CONTROL TRANSIENT RESPONSE



S.S TRACKING ERROR
1 - .93 = .07



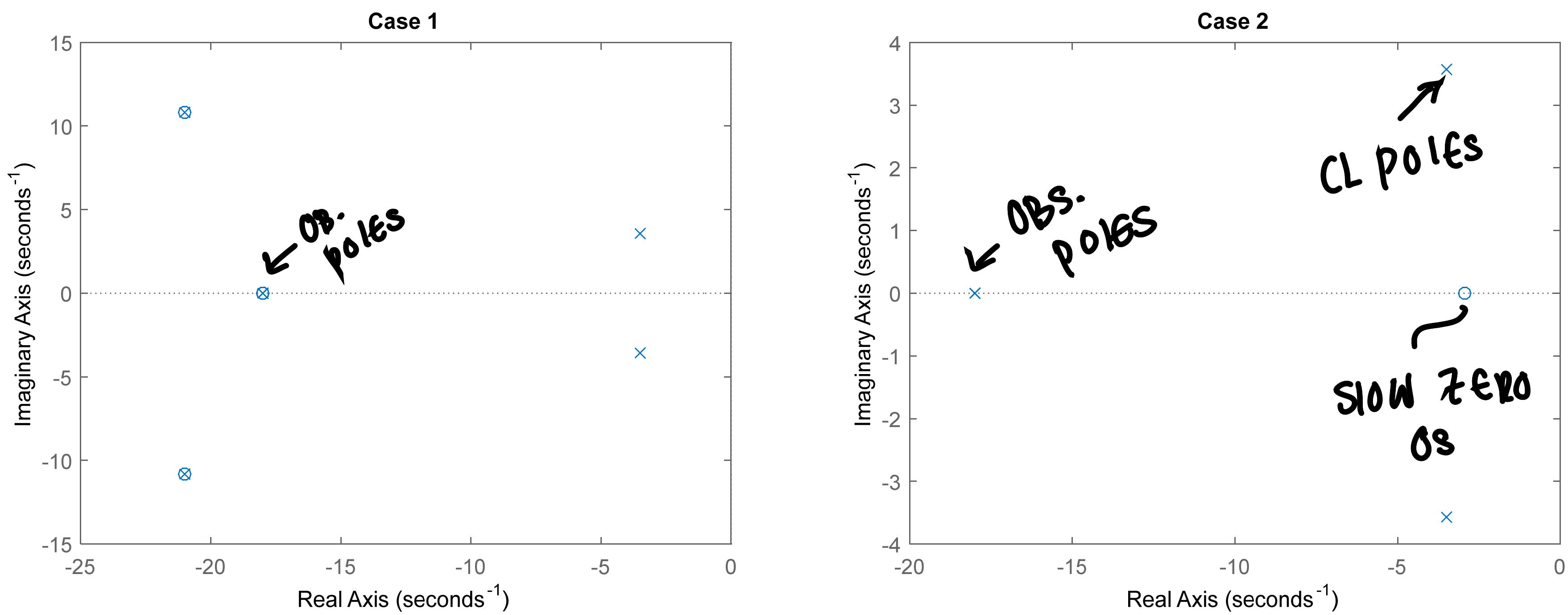
CLOSED LOOP

PHASE MARGINS READ FROM OPEN LOOP!

- As expected, Case 2 results in steady-state tracking error
- What causes the differences in transient behavior?

CL ZEROS

COMPARE THE POLES AND ZEROS OF THE TWO CASES



- In Case 1, the prefilter is cancelling the closed-loop poles due to the state observer
- In Case 2, an extra zero is introduced by the location of the controller in the feedforward path

INTEGRAL CONTROL

WHAT IF WE NEED INTEGRATION IN OUR CONTROLLER DESIGN?

PID

$$u = -Kx = K_I \int (r - y)dt \quad K_I \cdot \frac{1}{s} e$$

$$-\dot{x} = r - y$$

ENSURES ZERO S.S ERROR (FREE INTEGRATOR)

~ADDING STATES

- Augment state space by a new integral state
- Design state feedback as usual for augmented state vector

AUGMENT THE STATE VECTOR WITH THE NEW INTEGRATION STATE

ORIGINAL　　　　ORIGINAL

$$x_a = \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x \\ -\int r - y \end{bmatrix}$$

NEW　　　　INTEGRAL OF TRACKING ERROR

$$u = -Kx - K_I x_{n+1}$$

$$\dot{x} = Ax + Bu = (A - BK)x - BK_I x_{n+1}$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} - \begin{bmatrix} BK & BK_I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

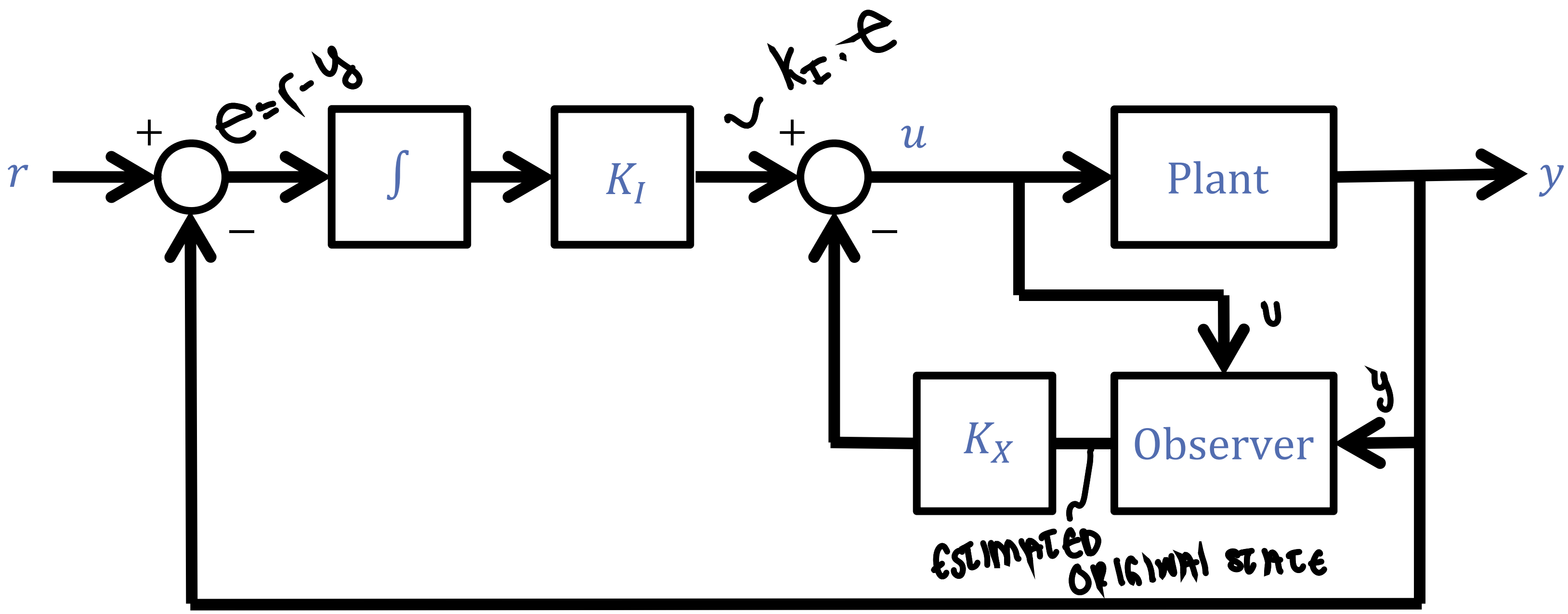
$\dot{x}_{n+1} = -r + y$

$\dot{x} = Ax + Bu$
$y = Cx + \cancel{D}$

- Design the state feedback gains $\mathrm{K} = [\underbrace{K_X}_{\text{ORIGINAL STATES}} \quad \underbrace{K_I}_{\text{INTEGRAL STATES}}]$ as usual for the desired closed-loop poles

TO IMPLEMENT, APPLY THE INTEGRATOR IN THE LOOP

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{n+1} \end{bmatrix} = A_a \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} - BK_a \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$



RECALL THAT ADDING AN INTEGRAL STATE IS A SPECIAL CASE OF THE INTERNAL MODEL PRINCIPLE

STEADY-STATE DISTURBANCE COMPENSATION
REQUIRES THAT GENERATING POLYNOMIAL OF DISTURBANCES BE INCLUDED AS PART OF THE CONTROLLER DENOMINATOR.

- Can we do something similar in state-space? Of course!

SUPPOSE YOU WANT TO CANCEL AN INPUT DISTURBANCE

$$\dot{x} = Ax + B(u + d)$$
$$y = Cx$$

- Let the input disturbance $d(t)$ be the free response of the disturbance generating system

$$\dot{x}_d = A_d x_d$$
$$d = C_d x_d$$

$$\begin{bmatrix} \dot{X} \\ \dot{X}_d \end{bmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} X \\ X_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} X \\ X_d \end{bmatrix} + \ldots$$

- Augmented the plant with the disturbance generating system

PUT THE DISTURBANCE INTO THE PLANT

CHECK CONTROLLABILITY AND OBSERVABILITY OF THE AUGMENTED SYSTEM

$u = -K\hat{x} - C_d\hat{x}_d$

$\dot{x} = Ax + B(u+d)$

$= Ax + B(-K\hat{x} - C_d\hat{x}_d + d)$

$= Ax - BK\hat{x} + BC_d x_d + BC_d\hat{x}_d$

$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}}_d \end{bmatrix} = \begin{bmatrix} A-BK-LC & 0 \\ -L_2C & A_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_d \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y$

$u = [K \quad C_d] \begin{bmatrix} \hat{x} \\ \hat{x}_d \end{bmatrix}$

$\frac{U(s)}{Y(s)} = (K \quad C_d) \begin{bmatrix} sI - A + LC + BK & 0 \\ L_2C & sI - A_d \end{bmatrix}^{-1} [\;]$

$$W_C = \begin{bmatrix} B & AB \\ 0 & 0 \end{bmatrix}$$

$B \quad AB$

$W_c = [B \;\; AB \;\; A^2B, \;\; ..]$

NON-FULL RANK

$$W_O = \begin{bmatrix} C & 0 \\ CA & CBC_d \end{bmatrix}$$

$W_o = [C \;\; AC \;\; A^2C]^T$

C

AC

- The disturbance state is uncontrollable (as expected)
- All states are observable (assuming original system observable)

DESIGN A STATE OBSERVER AS USUAL, THEN STATE FEEDBACK

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}}_d \end{bmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (y - C\hat{x})$$

$$|SI - A| = \begin{bmatrix} \lambda - A & -BC_d \\ 0 & \lambda - A_d \end{bmatrix}$$

$$\det(SI - A) = (\lambda - A)(\lambda - A_d) = \lambda^2 - \lambda(A + A_d) + AA_d$$

desired $P_{1,2} = -6$ $\quad (s+6)^2 = s^2 + 12s + 36$

$$K = [(12 + (A + A_d)), 36 - AA_d]T^{-1}$$

OBSERVE THE CONTROLLER WHEN CONVERTED TO A TRANSFER FUNCTION

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}}_d \end{bmatrix} = \begin{bmatrix} A - BK - L_1 C & 0 \\ -L_2 C & A_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_d \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y$$

$$u = -\begin{bmatrix} K & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_d \end{bmatrix}$$

$$\frac{U(s)}{Y(s)} = \begin{bmatrix} K & C_d \end{bmatrix} \begin{bmatrix} A - BK - L_1 C & 0 \\ -L_2 C & A_d \end{bmatrix}^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

- The denominator of the controller is given by

- The denominator of the controller includes the disturbance generating polynomial!

COMING UP…

Optimal Control

- Calculus of Variations
- Optimal Control
- Linear Quadratic Regulator