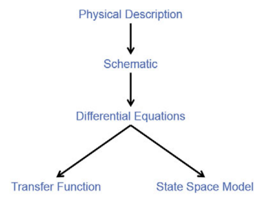
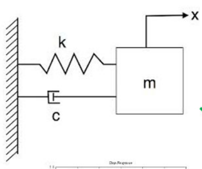
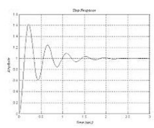
FROM LAST TIME…

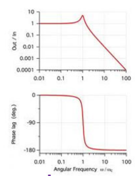


Modeling Physical Systems

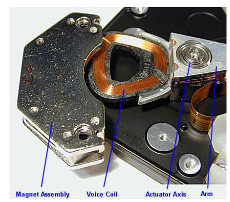
- Why we model dynamic systems
- How we model dynamic systems

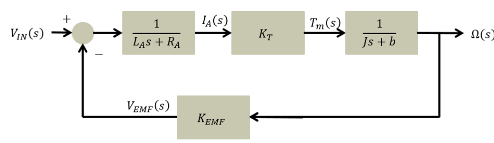






$$G(s) = \frac{K_s\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$





TRANSFER FUNCTIONS

Topics

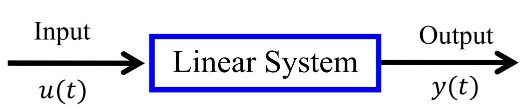
- Free vs. Forced Response
- Transfer Function
- System Stability

At the end of this section, students should be able to:

- Distinguish between free and forced response.
- Identify poles and zeros of transfer functions.
- Determine stability of transfer functions.
- Model systems with time delays using transfer functions.

SOMETHING FUN

IN CONTROL THEORY, OUR PRIMARY CONCERN IS HOW A SYSTEM RESPONDS TO SELECTED INPUTS



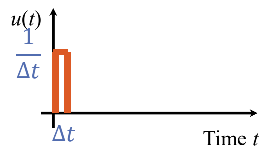
- Input is selected for application
  (system identification, reference tracking, etc.)
- Output is the solution of a set of differential equations
  (our system model)

INPUT / RESPONSE TYPES

Initial Conditions

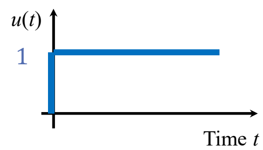
- Zero Input
- Some non-zero starting position/speed/velocity/etc.

Impulse



Step

- Input is a constant that "turns on"



Sinusoidal

- Sine or Cosine

FREE & FORCED RESPONSES

Free Response ($u(t) = 0$ & nonzero ICs)

- The response of a system to *zero input* and *nonzero initial conditions*
- To solve: let $u(t) = 0$ and use LT and ILT to solve for the free response

Forced Response (zero ICs & nonzero $u(t)$)

- The response of a system to *nonzero input* and *zero initial conditions*
- To solve: assume zero ICs and use LT and ILT to solve for the forced response (replace differentiation with s in the I/O ODE)

TRANSFER FUNCTIONS RELATE INPUT TO FORCED RESPONSE

Given a general nth order system model:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}\, u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

The forced response (zero ICs) of the system due to input $u(t)$ is:

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s)$$
$$= b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \cdots + b_1 s U(s) + b_0 U(s)$$

$$[a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0] \cdot Y(s) = [b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0] \cdot U(s)$$

$$Y_{FORCED}(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \cdot U(s)$$

WE CAN FACTOR THE TRANSFER FUNCTION NUMERATOR AND DENOMINATOR INTO ROOTS AND GAIN

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

$$G(s) = \frac{N(s)}{D(s)} = \rule{6cm}{0.4pt}$$

We call these roots the poles and zeros of the system

We call the gain ratio the static gain of the system

POLES ARE THE ROOTS OF THE DENOMINATOR POLYNOMIAL

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

$$\begin{aligned} D(s) &= a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ &= a_n(s - p_1)(s - p_2)\cdots(s - p_n) = 0 \end{aligned}$$

$$\Rightarrow p_1, p_2, \cdots, p_n\text{:} \;\; n \text{ poles of the TF}$$

Poles effect the time response, frequency response, and stability of the system

ZEROS ARE THE ROOTS OF THE NUMERATOR POLYNOMIAL

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

$$\begin{aligned} N(s) &= b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 \\ &= b_m(s - z_1)(s - z_2)\cdots(s - z_m) = 0 \end{aligned}$$

$$\Rightarrow z_1, z_2, \cdots, z_m\text{:} \;\; m \text{ zeros of the TF}$$

Zeros affect the time response and frequency response of the system

STATIC GAIN, $G(0)$, IS THE VALUE OF THE TRANSFER FUNCTION WHEN $s = 0$

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

$$K_S =$$

The static gain K_S can be interpreted as the steady state value of the unit step response.

FREE RESPONSE IS ONLY DUE TO INITIAL CONDITIONS

Given a general nth order system model:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}\, u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

The free response (zero input) of the system due to ICs is:

$$a_n[s^n Y(s) - s^{n-1} y(0) - \cdots - y^{n-1}(0)] + a_{n-1}[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{n-2}(0)]$$
$$+ \cdots + a_1[sY(s) - y(0)] + a_0 Y(s) = 0$$

$$[a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0] Y_{FREE}(s)$$
$$= [a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_1] y(0) + \cdots + y^{n-1}(0)$$

$$Y_{FREE}(s) = \frac{F(s)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

STATE SPACE MODELS ALSO HAVE FREE AND FORCED RESPONSES

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\Big\downarrow \mathcal{L}$$

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

TRANSFER FUNCTIONS

SYSTEM STABILITY

COMPARE THE FREE AND FORCED RESPONSES

$$Y_{FORCED}(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \cdot U(s)$$

$$Y_{FREE}(s) = \frac{F(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

- Same poles
- Different zeros

FREE RESPONSE CAN BE REPRESENTED IN TIME DOMAIN BY PARTIAL FRACTION EXPANSION

$$Y_{FREE}(s) = \frac{F(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
$$= \frac{F(s)}{a_n (s - p_1)(s - p_2)\cdots(s - p_n)}$$
$$= \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \cdots + \frac{A_n}{s - p_n}$$

Assume $p_1 \neq p_2 \neq \cdots \neq p_n$ i.e. n distinct poles

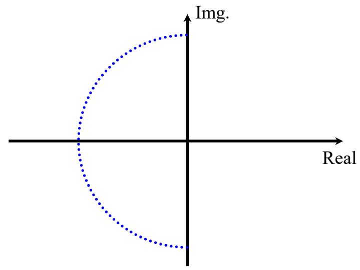
$$y_{FREE}(t) = L^{-1}[Y_{FREE}(s)] \quad = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}$$

HOW DO POLE LOCATIONS AFFECT THE TIME-DOMAIN FREE RESPONSE?

$$y_{FREE}(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}$$

p_i is real
$p_i < 0 \Rightarrow$
$p_i = 0 \Rightarrow$
$p_i > 0 \Rightarrow$

$p_i = \sigma + j\beta$
$\sigma < 0 \Rightarrow$
$\sigma = 0 \Rightarrow$
$\sigma > 0 \Rightarrow$

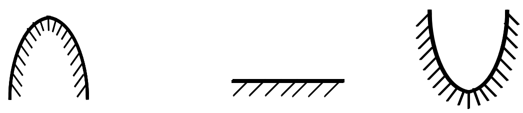


- Response due to poles in open left-half plane (LHP) decays to zero
- Response due to poles in open right-half plane (RHP) explodes

A SYSTEM IS *STABLE* IF ITS FREE RESPONSE CONVERGES TO ZERO

Hanging pendulum returns to equilibrium (stable)

Inverted pendulum does not return to equilibrium (unstable)



Stable System

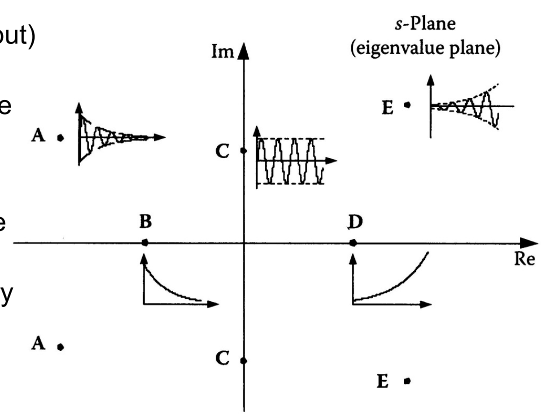
- Free response (zero input) decays to zero
- Real portion of poles are negative (LHP)

Marginally Stable System

- Will continue to oscillate within bounds
- Zero or strictly imaginary pair of poles

Unstable System

- Will grow unbounded
- Real portion of poles are positive (RHP)



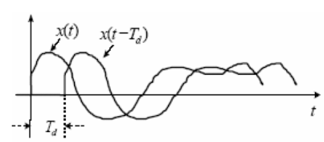
TRANSFER FUNCTIONS

TIME DELAYS

WHAT IF OUR SYSTEM HAS A TIME DELAY?

Mathematical Model in Time-Domain

$$z(t) = \begin{cases} 0 & t < T_d \\ x(t-T_d) & t > T_d \end{cases}$$



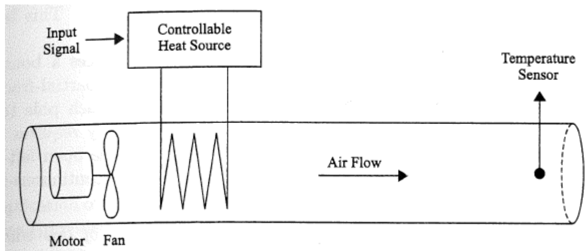
Mathematical Model in s-Domain

$$Z(s) = \int_0^\infty z(t)e^{-st}dt = \int_0^{T_d} 0 \cdot e^{-st}dt + \int_{T_d}^\infty x(t-T_d)e^{-st}dt$$

$$= \int_{T_d}^\infty x(t-T_d)e^{-s(t-T_d)}e^{-sT_d}d(t-T_d)$$

Transfer Function for Pure Delay of T_d sec

EXAMPLE: FORCED AIR HEATING SYSTEM



Temperature sensor is a thermocouple with first-order response

Temperature measurement is also effected by time required for air to move from heat source to sensor

$$H(s) = e^{-sT_d} \frac{K_s}{\tau s + 1}$$

TIME DELAYS CAN BE APPROXIMATED AS AN LTI SYSTEM (PADE APPROX.)

First order approximation:

$$e^{-sT_d} \approx \frac{2 - sT_d}{2 + sT_d}$$

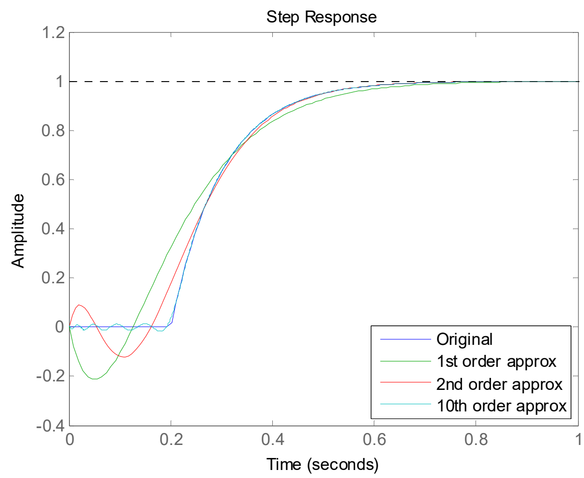
Second order approximation:

$$e^{-sT_d} \approx \frac{1 - \frac{T_d}{2} s + \frac{T_d^2}{12} s^2}{1 + \frac{T_d}{2} s + \frac{T_d^2}{12} s^2}$$

Higher order approximations are available!

USE MATLAB TO COMPARE APPROXIMATION ACCURACY

$$H(s) = e^{-s0.2}\frac{1}{0.1s+1}$$



COMING UP…

Dynamic Response

- 1^{st} and 2^{nd} order step response
- System identification (time response)
- Effects of system zeros
- Transient and steady state

Frequency Response

- Frequency response of LTI systems
- Bode plots
- System identification (frequency response)
- Modeling errors in Bode plots