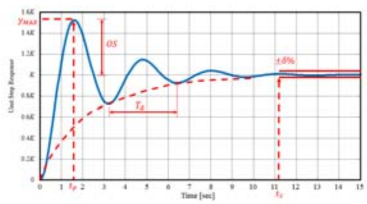
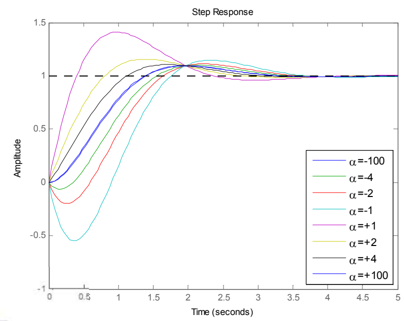
FROM LAST TIME…

Dynamic Response

- 1st and 2nd order step response
- System identification (time response)
- Effects of system zeros
- Transient and steady-state





$$y(t) = \boxed{(A_1 + A_4)e^{-3t} + (A_2 + A_5)e^{-t}} + \boxed{A_3}$$

Transient Response   Steady-State Response

FREQUENCY RESPONSE

Topics

- Frequency response of LTI systems
- Bode plots
- System identification (frequency response)
- Modeling errors in Bode plots

At the end of this section, students should be able to:

- Differentiate between first and second order systems using Bode plot.
- Determine steady-state sinusoidal response from Bode plot.
- Plot modeling errors in frequency domain using MATLAB.

VIDEO

Car Shaker

http://www.youtube.com/watch?v=LmoEcDFSiZY

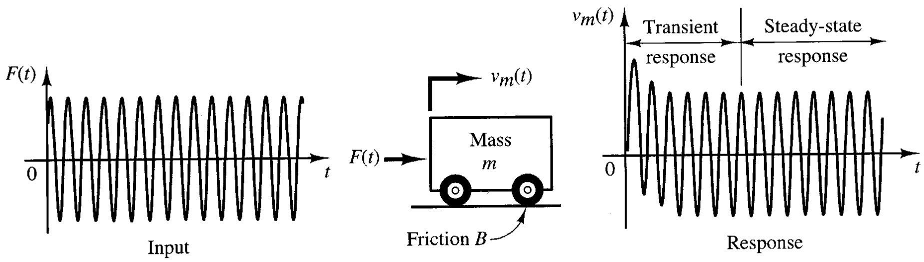
Questions to consider

- Why are they shaking the car?
- What type of input is applied to the car?
- How do we quantify the response of the suspension?
- Why does this matter?

FREQUENCY RESPONSE

FREQUENCY RESPONSE

APPLYING A SINUSOIDAL INPUT WILL RESULT IN A SINUSOIDAL OUTPUT



- During transient response, output will vary
- During steady-state response, output is sinusoidal

EX: FIND THE FORCED RESPONSE OF A STABLE FIRST ORDER SYSTEM TO A SINUSOIDAL INPUT

$$\dot{y} + 5y = 10u \qquad u(t) = \sin(2t)$$

$$Y(s) = G(s) \cdot U(s)$$

where $G(s) = \_\_\_\_\_\_\_\_$ and $U(s) = L[\sin(2t)]$

$$Y(s) = \_\_\_\_\_\_\_\_\_\_\_\_$$

- This is a frequency response (in s-domain) at $2\ \mathrm{rad/sec}$
- What is the solution in t-domain?

APPLY PARTIAL FRACTION EXPANSION AND SOLVE FOR COEFFICIENTS

$$Y(s) = \frac{}{}$$

$$Y(s) = \frac{A_1}{} + \frac{A_2 \cdot}{} + \frac{A_3 \cdot}{}$$

- How do we convert this back to t-domain?
- What are the forced and free responses?
- What are the transient and steady-state responses?

APPLY INVERSE LAPLACE TRANSFORM TO FIND $y(t)$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\cdot \frac{}{} + \cdot \frac{}{} + \cdot \frac{}{} \right]$$

$$\boxed{\begin{aligned} A\sin(\omega \cdot t) + B\cos(\omega \cdot t) &= M\sin(\omega \cdot t + \phi) \\ M &= \sqrt{A^2 + B^2} \\ \phi &= \operatorname{atan2}(B, A) = \angle(A \end{aligned}}$$

$$y(t) = \cdot e^{-5t} + \cdot \sin(2t + \phi)$$

- This was tedious, how can we easily repeat for all frequencies?

LET'S GENERALIZE THE SAME EXAMPLE WITH AN ARBITRARY FREQUENCY ω RAD/S

$$\dot{y} + 5y = 10u \qquad u(t) = A\sin(\omega t)$$

$$Y(s) = G(s) \cdot U(s) \quad = \frac{10}{s+5} \cdot \frac{A\omega}{s^2 + \omega^2} \quad = \frac{10}{s+5} \cdot \frac{A\omega}{(s - j\omega)(s + j\omega)}$$

$$= \frac{B_1}{s+5} + \frac{B_2}{s - j\omega} + \frac{B_3}{s + j\omega}$$

- What are the forced and free responses?
- What are the transient and steady-state responses?

USE THE RESIDUE FORMULA TO FIND COEFFICIENTS

$$B_1 = (s+5)Y(s)\Big|_{s=-5} = (s+5)\frac{10}{(s+5)}\frac{A\omega}{s^2+\omega^2}\Big|_{s=-5} =$$

$$B_2 = (s-j\omega)Y(s)\Big|_{s=j\omega} = (s-j\omega)\frac{10}{(s+5)}\frac{A\omega}{s^2+\omega^2}\Big|_{s=j\omega} =$$

$$B_3 = (s+j\omega)Y(s)\Big|_{s=-j\omega} = (s+j\omega)\frac{10}{(s+5)}\frac{A\omega}{s^2+\omega^2}\Big|_{s=-j\omega} =$$

$$B_1 = \frac{10A\omega}{5^2 + \omega^2}$$

$$B_2 = \frac{A}{2j} \cdot \frac{10}{j\omega + 5} = \frac{A}{2j} \cdot G(j\omega) = \frac{A}{2j} \cdot |G(j\omega)|e^{j\phi}$$

$$B_3 = \frac{-A}{2j} \cdot \frac{10}{-j\omega + 5} = \frac{-A}{2j} \cdot G(-j\omega) = \frac{-A}{2j} \cdot |G(j\omega)|e^{-j\phi}$$

APPLY SOME MATHEMAGIC TO DETERMINE THE STEADY-STATE RESPONSE

$$Y(s) = \frac{B_1}{s+5} + \frac{B_2}{s-j\omega} + \frac{B_3}{s+j\omega}$$

$$Y_{SS}(s) = \frac{B_2}{s-j\omega} + \frac{B_3}{s+j\omega}$$

$$y_{ss}(t) = L^{-1}[Y_{SS}(s)] = B_2 \cdot e^{j\omega t} + B_3 \cdot e^{-j\omega t}$$

$$y_{ss}(t) = |G(j\omega)| \cdot \sin\big(\omega t + \angle G(j\omega)\big)$$

THE FREQUENCY RESPONSE FUNCTION $G(j\omega)$ DESCRIBES THE FREQUENCY RESPONSE

$$G(j\omega) = G(s)\Big|_{s=j\omega} = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}$$

- Describes steady−state response due to sinusoidal input
- $G(j\omega)$ is the complex number obtained by substituting $j\omega$ for s in the transfer function $G(s)$

STEADY-STATE OUTPUT CAN BE EASILY GENERATED FROM THE FRF

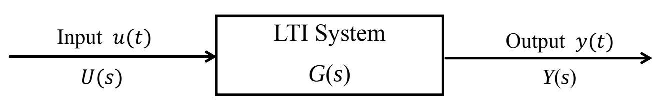
$$u(t) = A\sin(\omega t)$$

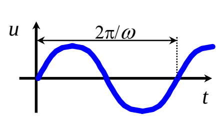
$$G(j\omega)$$

$$y_{SS}(t) = |G(j\omega)| \cdot A\sin\big(\omega t + \angle G(j\omega)\big)$$

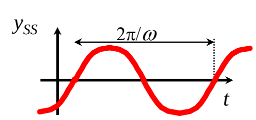
- Steady-state response is scaled in magnitude and shifted in phase by the frequency response function $G(j\omega)$
- Transient response is not considered

FREQUENCY RESPONSE





$\Rightarrow$



$u(t) = A_u\sin(\omega t)$ $\qquad$ $y_{SS}(t) = |G(j\omega)| \cdot A_u\sin\big(\omega t + \angle G(j\omega)\big)$

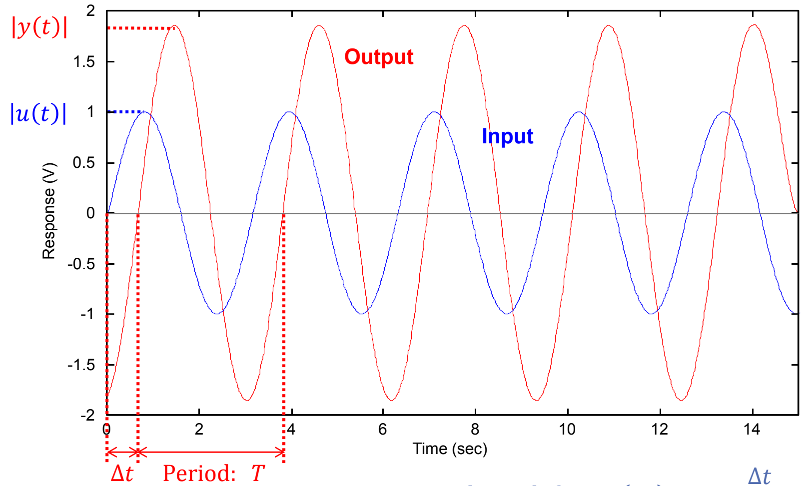
- A different perspective of the role of the transfer function:

$$|G(j\omega)| = \frac{\text{Amplitude of the steady state sinusoidal output}}{\text{Amplitude of the sinusoidal input}}$$

$$\angle G(j\omega) = \text{Phase difference (shift) between } y_{SS}(t) \text{ and the sinusoidal output}$$

FREQUENCY RESPONSE

Amplitdue Ratio: $|G(j\omega)| = \dfrac{|y(t)|}{|u(t)|}$

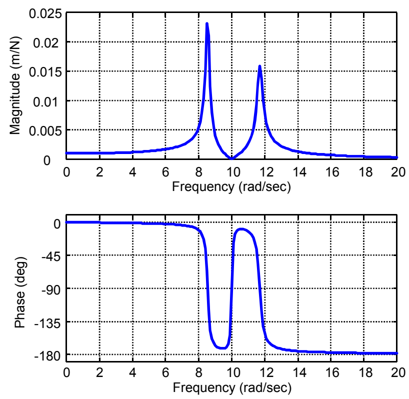


Phase Shift: $\angle G(j\omega) = -2\pi \dfrac{\Delta t}{T}$

FREQUENCY RESPONSE

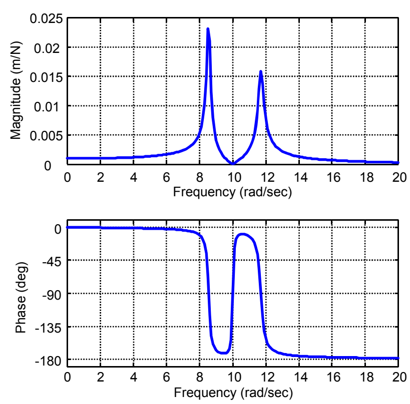
BODE PLOTS

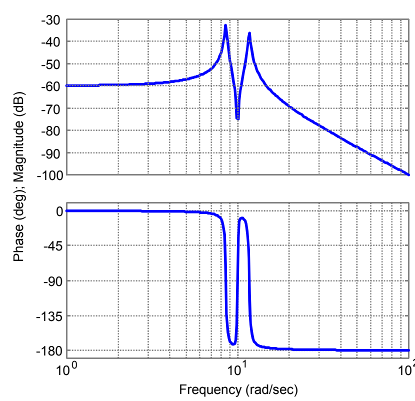
HOW DO WE REPRESENT A FREQUENCY RESPONSE VISUALLY?



- Recall: $G(j\omega)$ is a complex number which can be represented in Cartesian coordinates or magnitude-phase notation
- Plot magnitude and phase as functions of frequency
- What's happening at low frequencies?
- What's happening at high frequencies?
- Are these the same?

SAME EXAMPLE WITH LOGARITHMIC FREQUENCY AND MAGNITUDE SCALES





- High and low frequencies are now clearly different
- Characteristic shapes of system components are now clear

BODE PLOTS REPRESENT FREQUENCY RESPONSE VISUALLY WITH SOME STRETCHING FOR CLARITY

- Magnitude Plot: plots the magnitude of $G(j\omega)$ in decibels w.r.t. logarithmic frequency, i.e.

$$|G(j\omega)|_{dB} = 20\log_{10}|G(j\omega)| \quad vs. \quad \log_{10}\omega$$

- Phase Plot: plots the linear phase angle of $G(j\omega)$ w.r.t. logarithmic frequency, i.e.

$$\angle G(j\omega) \quad vs. \quad \log_{10}\omega$$

BODE DIAGRAMS

Transfer Function:

$$G(s) = \frac{b_m s^{(m)} + b_{m-1}s^{(m-1)} + \cdots + b_1 s + b_0}{s^{(n)} + a_{n-1}s^{(n-1)} + \cdots + a_1 s + a_0} = \frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Frequency Response

$$G(j\omega) = \frac{b_m(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_m)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_n)}$$

Bode Magnitude

$$\begin{aligned} 20\log_{10}(|G(j\omega)|) = \quad & 20\log_{10}(|b_m|) + \sum_{i=1}^{n} 20\log_{10}\left(\left|\frac{1}{(j\omega-p_i)}\right|\right) \\ & + \sum_{i=1}^{m} 20\log_{10}(|(j\omega-z_i)|) \end{aligned}$$

Bode Phase

$$\begin{aligned} \angle G(j\omega) = \; & = \angle b_m + \angle(j\omega-z_1) + \angle(j\omega-z_2) + \cdots + \angle(j\omega-z_m) \\ & -\angle(j\omega-p_1) - \angle(j\omega-p_2) - \cdots - \angle(j\omega-p_n) \end{aligned}$$

BODE DIAGRAM BUILDING BLOCKS

1st Order Real Poles

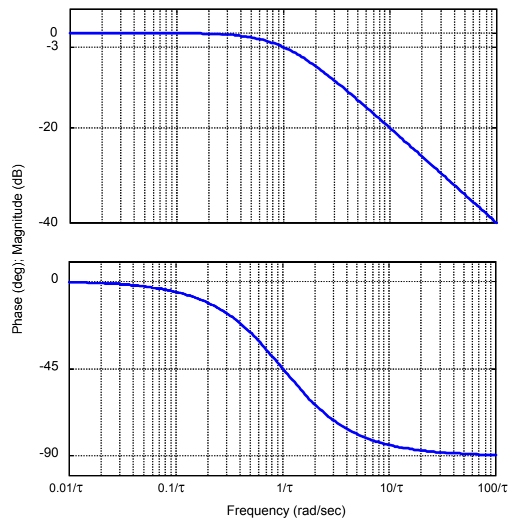
- Transfer Function:

$$G_{p1}(s) = \frac{1}{\tau s + 1} \qquad \tau > 0$$

- Frequency Response:

$$G_{p1}(j\omega) = \frac{1}{\tau j\omega + 1} \qquad \tau > 0$$

$$\left|G_{p1}(j\omega)\right| = \frac{1}{\sqrt{\tau^2\omega^2 + 1}}$$
$$\angle G_{p1}(j\omega) = -atan2(\tau\omega, 1)$$
$$= -\tan^{-1}\tau\omega$$



BODE DIAGRAM BUILDING BLOCKS

2nd Order Complex Poles

- Transfer Function:

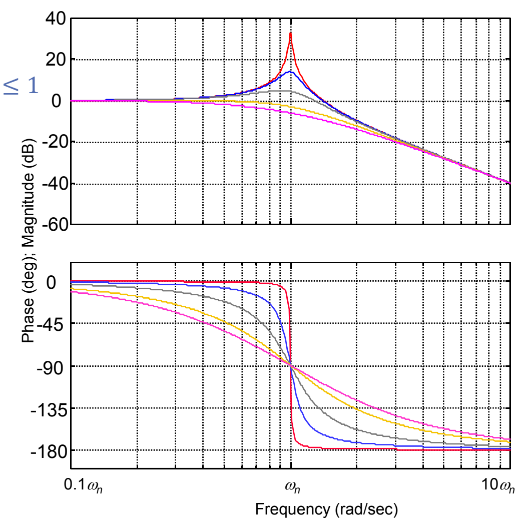
$$G_{p2}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 0 \le \zeta \le 1$$

- Frequency Response:

$$G_{p2}(j\omega) = \frac{1}{j\frac{2\zeta\omega}{\omega_n} + \left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

$$\left|G_{p2}(j\omega)\right| = \frac{1}{\sqrt{\frac{4\zeta^2\omega^2}{\omega_n^2} + \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}}$$

$$\angle G_{p2}(j\omega) = -\tan^{-1}\left[\frac{\frac{2\zeta\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}\right]$$



A FEW OBSERVATIONS

- Three different characteristic frequencies:
  - Natural Frequency (ω_n)
  - Damped Natural Frequency (ω_d): $\omega_d = \omega_n\sqrt{1-\zeta^2}$
  - Resonant (Peak) Frequency (ω_r): $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

$$\omega_r \leq \omega_d \leq \omega_n$$

- When the damping ratio $\zeta > 0.707$, there is no peak in the Bode magnitude plot. DO NOT confuse this with the condition for over-damped and under-damped systems: when $\zeta < 1$ the system is under-damped (has overshoot) and when $\zeta > 1$ the system is over-damped (no overshoot).

- As $\zeta \rightarrow 0$, $\omega_r \rightarrow \omega_n$ and $|G(j\omega)|_{MAX}$ increases; also the phase transition from 0° to -180° becomes sharper.

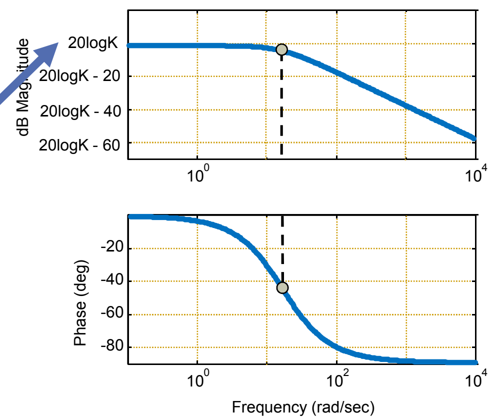
FREQUENCY RESPONSE

SYSTEM IDENTIFICATION (FREQUENCY RESPONSE)

STATIC GAIN, K, MAY BE READ DIRECTLY FROM THE BODE PLOT

$$G(j\omega) = \frac{K}{\tau j\omega + 1}$$

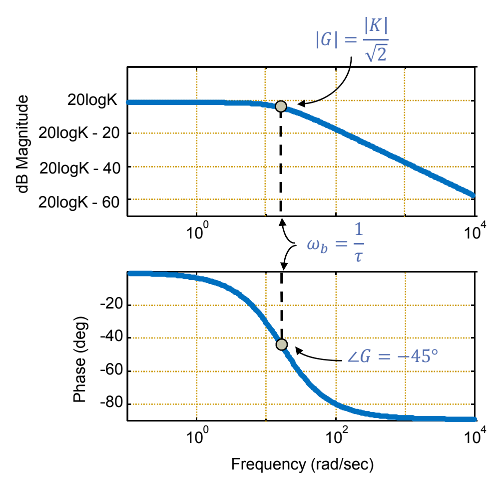
$$|G(j0)| = |K|$$
$$|G(j0)| = 20\log_{10}|K| \;\; dB$$



TIME CONSTANT, τ, IS FOUND FROM THE BREAK FREQUENCY

$$G(j\omega) = \frac{K}{\tau j\omega + 1}$$

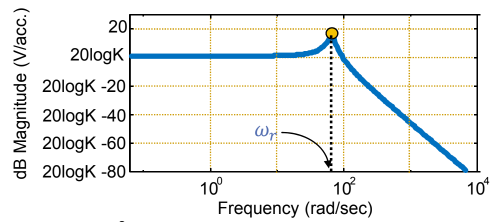
$$\angle G\left(j\frac{1}{\tau}\right) = -\frac{\pi}{4} = -45°$$

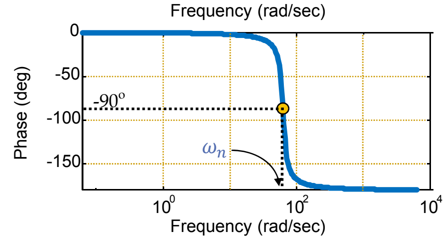


NATURAL FREQUENCY CAN BE FOUND FROM THE PHASE PLOT

$$G(j\omega) = \frac{K}{j\frac{2\zeta\omega}{\omega_n} + \left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

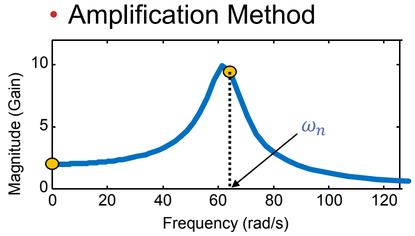
$$\angle G(j\omega_n) = -\frac{\pi}{2} = -90°$$





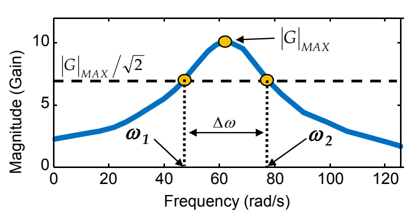
THERE ARE THREE METHODS FOR IDENTIFYING THE DAMPING RATIO

- Amplification Method



$$\frac{|y(t)|_{\omega=\omega_n}}{|y(t)|_{\omega=0}} = \frac{|G(j\omega_n)|}{|G(j0)|} = \frac{1}{2\zeta}$$

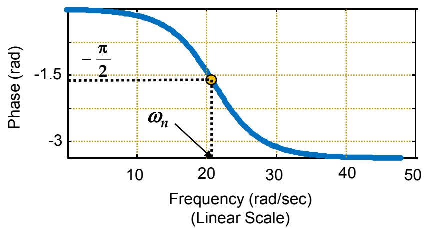
- Half Power Method



$$\zeta = \frac{\Delta\omega}{2\omega_n} = \frac{\omega_2 - \omega_1}{2\omega_n}$$

THERE ARE THREE METHODS FOR IDENTIFYING THE DAMPING RATIO

- Slope of Phase Angle Method



$$\zeta = -\frac{1}{\omega_n}\left[\frac{d\phi}{d\omega}\bigg|_{\omega=\omega_n}\right]^{-1}$$

FREQUENCY RESPONSE

MODEL ERRORS IN BODE PLOTS

RECALL THAT WE CAN DEFINE BOTH NOMINAL AND ACTUAL TRANSFER FUNCTION MODELS

Nominal System Model
$$Y(s) = G_o(s)U(s)$$

Actual (Calibration) System Model
$$Y(s) = G(s)U(s)$$

- Nominal model is reduced for simpler analysis or controller design
- How much do they deviate?
- How does the deviation appear in Bode plots?

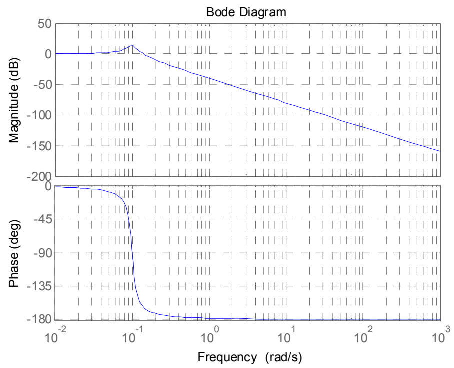
WE CAN QUANTIFY TRANSFER FUNCTION MODELING ERRORS

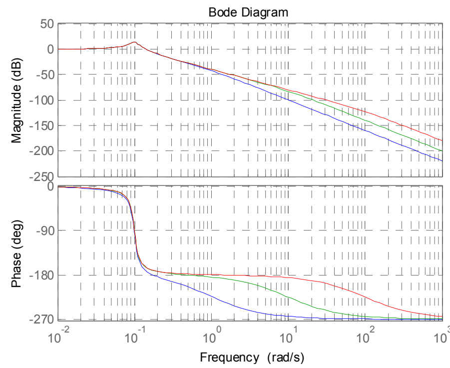
$$Y(s) = G(s)U(s) \quad = \big(G_o(s) + G_\varepsilon(s)\big)U(s)$$
$$= G_o(s)\big(1 + G_\Delta(s)\big)U(s)$$

$$|G_\Delta(j\omega)| = \varepsilon(\omega)$$

- $\left|G_{\Delta(j\omega)}\right|$ puts a frequency-dependent bounds on the model error
- This will be very useful for quantifying robustness

WHAT IF THE MODEL IS MISSING A HIGH-FREQUENCY POLE?

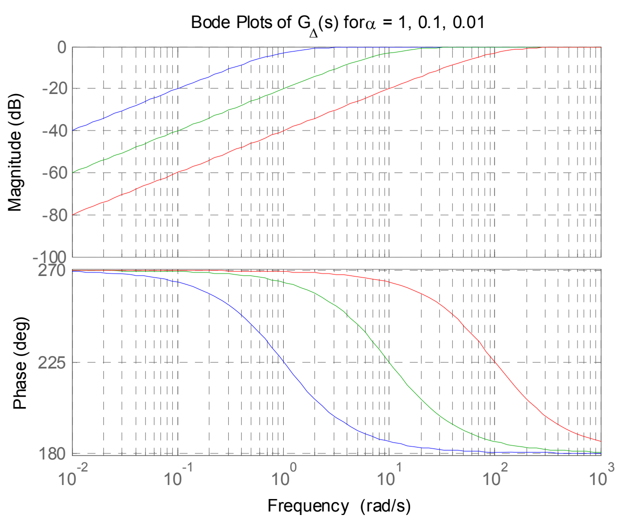




$$G_o(s) = F(s)$$

$$G(s) = \frac{1}{\alpha s + 1} F(s)$$

WHAT IF THE MODEL IS MISSING A HIGH-FREQUENCY POLE?

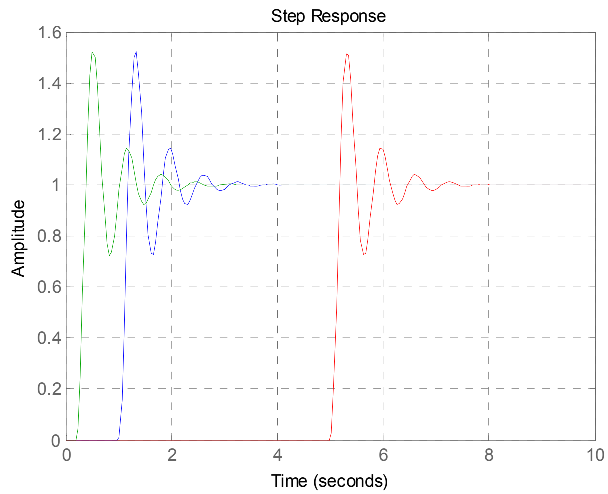


$$G_o(s) = F(s)$$

$$G(s) = \frac{1}{\alpha s + 1} F(s)$$

$$G_\Delta(s) = \frac{-\alpha s}{\alpha s + 1}$$

WHAT IF THE MODEL HAS THE WRONG TIME DELAY?



$$G_o(s) = e^{-\tau_o s}F(s)$$

$$G(s) = e^{-\tau s}F(s)$$

WHAT IF THE MODEL HAS THE WRONG TIME DELAY?

$$G_o(s) = e^{-\tau_o s}F(s)$$

$$G(s) = e^{-\tau s}F(s)$$

$$G_\Delta(s) = e^{-(\tau-\tau_o)s} - 1$$
$$= e^{\left(1-\frac{\tau}{\tau_o}\right)\tau_o s} - 1$$

$$e^{\left(1-\frac{\tau}{\tau_o}\right)\tau_o j\omega} = \cos\left[\left(1-\frac{\tau}{\tau_o}\right)\tau\omega\right] + j\sin\left[\left(1-\frac{\tau}{\tau_o}\right)\tau\omega\right]$$

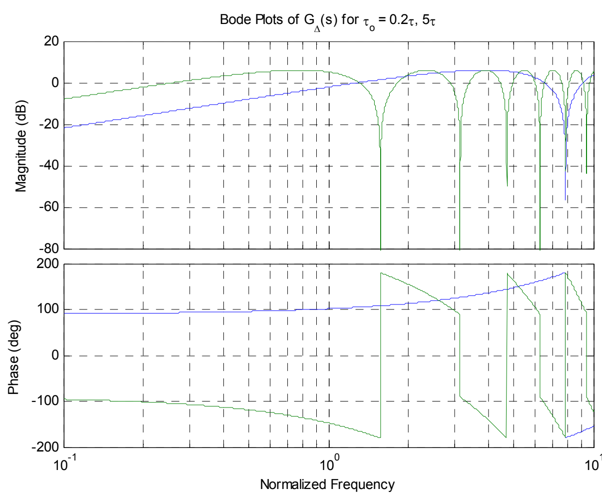
$$G_\Delta(j\omega) = e^{\left(1-\frac{\tau}{\tau_o}\right)\tau_o j\omega} - 1 = \left(\cos\left[\left(1-\frac{\tau}{\tau_o}\right)\tau\omega\right] - 1\right) + j\sin\left[\left(1-\frac{\tau}{\tau_o}\right)\tau\omega\right]$$

WHAT IF THE MODEL HAS THE WRONG TIME DELAY?

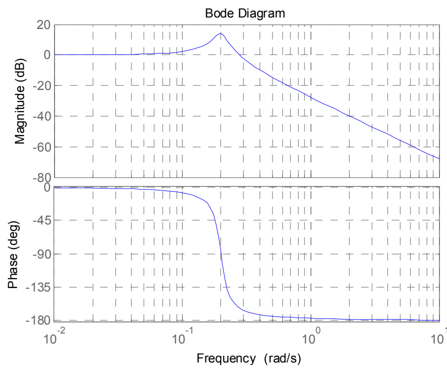
$$|G_\Delta(j\omega)| = 2\left|\sin\left(\frac{\omega\tau_o}{2}\left(1-\frac{\tau}{\tau_o}\right)\right)\right|$$

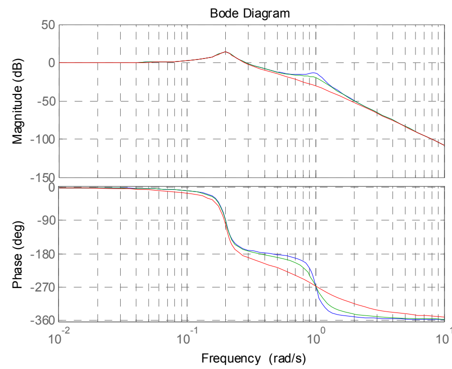
$$\angle G_\Delta(j\omega) = atan2\begin{bmatrix} \sin\left(\frac{\omega\tau_o}{2}\left(1-\frac{\tau}{\tau_o}\right)\right), \\ -1+\cos\left(\frac{\omega\tau_o}{2}\left(1-\frac{\tau}{\tau_o}\right)\right) \end{bmatrix}$$

WHAT IF THE MODEL HAS THE WRONG TIME DELAY?



WHAT IF THE MODEL IS MISSING A RESONANCE?

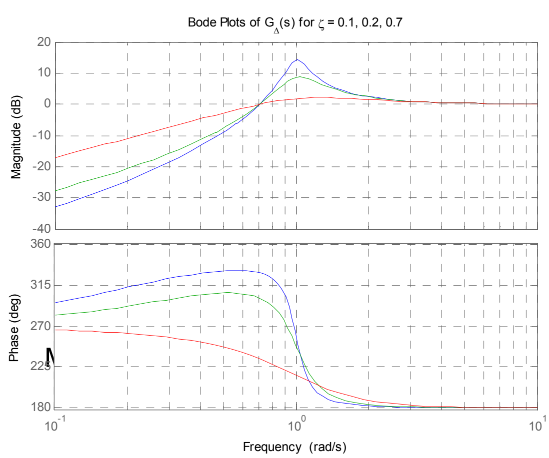




$$G_o(s) = F(s)$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s)$$

WHAT IF THE MODEL IS MISSING A RESONANCE?



$$G_o(s) = F(s)$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s)$$

$$G_\Delta(s) = \frac{-s(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

COMING UP…

Analysis of Feedback Systems

- Typical classical feedback controller structure
- Nominal sensitivity functions
- Stability of nominal feedback system
- Robust stability

Pole Placement Controller Design

- Pole placement design
- Controller with integration
- PID via pole placement