FROM LAST TIME...

Intro to State-Space Models

- Review of State Space
- Transfer Functions and State Space
- Canonical Forms

State Matrix

Input Matrix

$$\dot{x} = Ax(t) + Bu(t)$$

State Vector

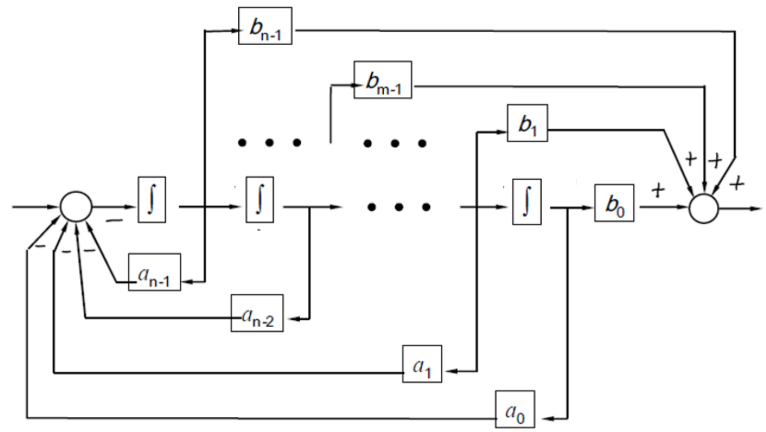
Input Vector

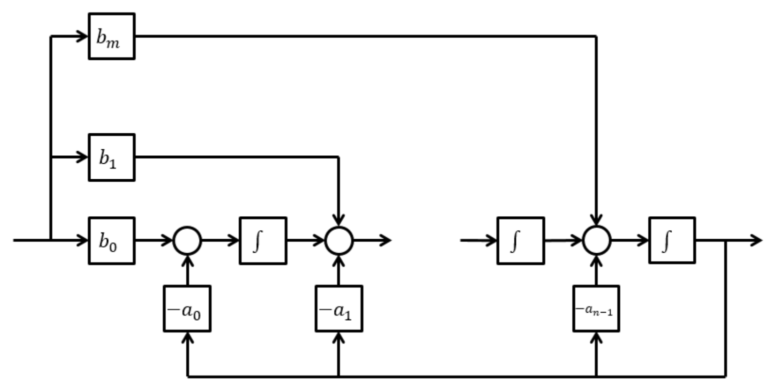
Output Vector

$$y = Cx(t) + Du(t)$$

Output Matrix

Feedthrough Matrix





LINEAR ALGEBRA PRIMER

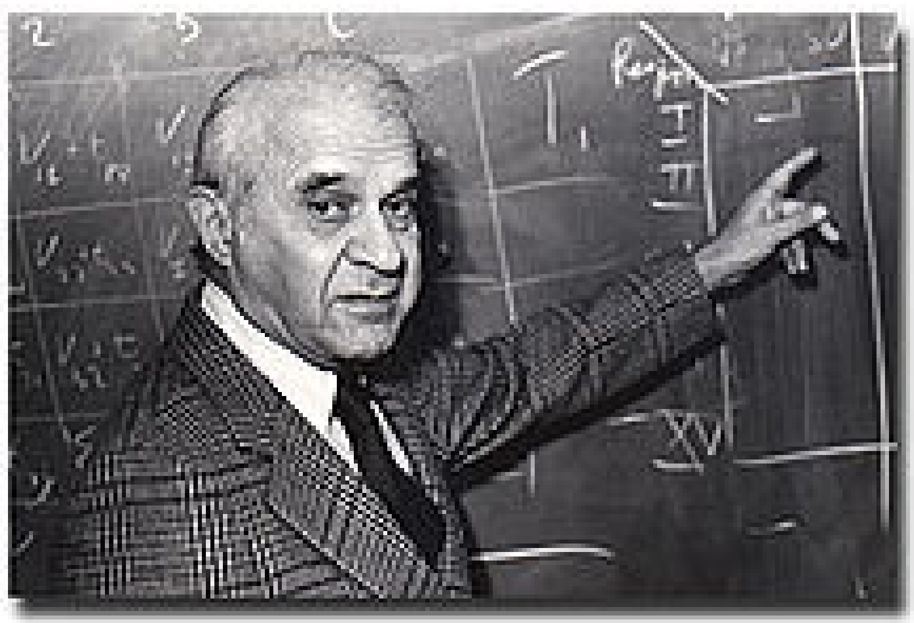
Topics

- Matrix Inverses
- Eigenvalues and Eigenvectors
- Jordan Canonical Form

At the end of this sections, students should be able to:

- Compute matrix inverses (up to 3x3).
- Describe eigenvalues and eigenvectors.
- Transform matrices to Jordan Canonical Form.

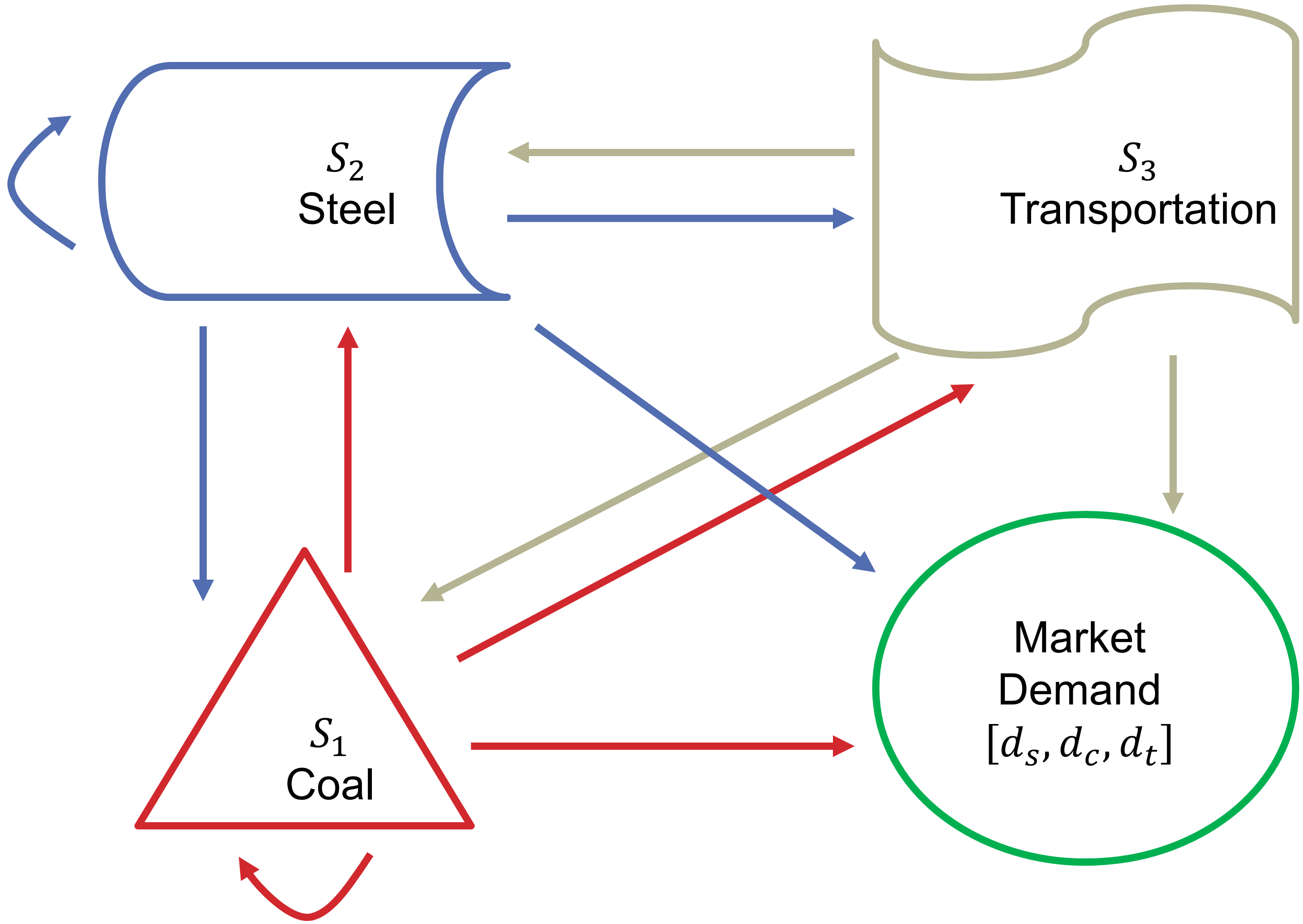
WASSILY LEONTIEF WON THE NOBEL MEMORIAL PRIZE IN ECONOMIC SCIENCES IN 1973



- Input-output tables analyze the process by which inputs from one industry produce outputs for consumption or for inputs for another industry

- With the input-output table, one can estimate the change in demand for inputs resulting from a change in production of the final good

CONSIDER A SIMPLIFIED ECONOMY WITH FOUR SECTORS

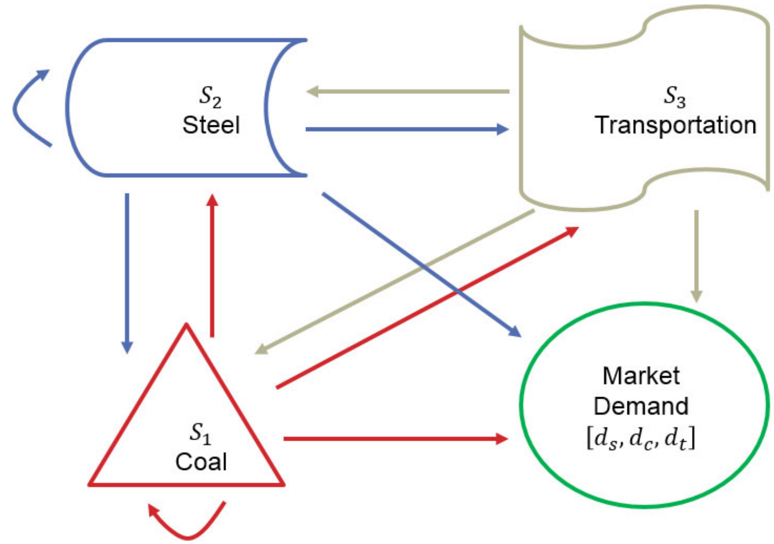


WE CAN WRITE THESE INPUT-OUTPUT RELATIONSHIPS AS LINEAR EQUATIONS…

$$c = 0.02c + 2s + 0.01t + D_c$$

$$s = 0.1c + 0.01s + 0.1t + D_s$$

$$t = 0.3c + 0.5s + 0t + D_t$$



- c is the amount of coal (in tons)
- s is the amount of steel (in tons)
- t is the amount of transportation

...WHICH CAN BE WRITTEN AS A MATRIX EQUATION!

$$\begin{bmatrix} c \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0.02 & 2 & 0.01 \\ 0.1 & 0.01 & 0.1 \\ 0.3 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} c \\ s \\ t \end{bmatrix} + \begin{bmatrix} D_c \\ D_s \\ D_t \end{bmatrix}$$

- We can solve for the required amount of each good using the matrix inverse

$$\begin{bmatrix} c \\ s \\ t \end{bmatrix} = \left(I_3 - \begin{bmatrix} 0.02 & 2 & 0.01 \\ 0.1 & 0.01 & 0.1 \\ 0.3 & 0.5 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} D_c \\ D_s \\ D_t \end{bmatrix}$$

- ∴ Matrix inverses are worth at least US$1.2 million (current Nobel Prize monetary value)

WHY DO WE CARE ABOUT LINEAR ALGEBRA FOR CONTROLLER DESIGN?

System properties

- Poles (eigenvalues)
- Controllability/Observability (rank)

System response

- SS to TF (matrix inverse)
- Free response (matrix exponential)

Controller design

- Observer/Controller design (eigenvalues, inverse, matrix multiplication)
- Optimal control (calculus of variations, transpose, inverse)

GENERAL MATRIX STUFF

THE DIMENSIONS OF A MATRIX ARE GIVEN IN ROWS X COLUMNS

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -3 & 9 & 11 \end{bmatrix}$$

Dimension of A is 2x3

- MATLAB: [r,c] = size(A)

THE RANK OF A MATRIX IS THE NUMBER OF LINEARLY INDEPENDENT ROWS OR COLUMNS

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix} \quad \Rightarrow A_{ref} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rank(A) = 2$$

- MATLAB: [k] = rank(A)
- If all rows/columns are linearly independent, the matrix is full rank

↳ CAN BE SOLVED

MATRIX MULTIPLICATION IS ASSOCIATIVE AND DISTRIBUTIVE BUT NOT COMMUTATIVE

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

$$A(B + C) = AB + AC$$

$$A \cdot B \neq B \cdot A$$

- MATLAB: D = A*(B*C)

THE MATRIX TRANSPOSE IS FLIPPED OVER THE DIAGONAL

Column → Row

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

- Symmetric: $A = A^T$
- Skew-symmetric: $A = -A^T$
- Orthogonal: $A^{-1} = A^T$

MATLAB A'

MATRIX INVERSE

RECALL THE CALCULATION OF A MATRIX INVERSE

ADJOINT

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{[c_{ij}]^T}{|A|}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

Minor: Determinant of submatrix that is formed by deleting the i^{th} row and j^{th} column

- Matrix is not invertible (singular) if the determinant is zero

LET'S TRY A SIMPLE EXAMPLE

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

1. For the each term, find the minor

$$M_{11} = \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix} = -6 - (-4) = -2$$

2. Now complete the c matrix

$i \rightarrow$ column
$j \rightarrow$ row

$$c_{11} = \overbrace{(-1)^{1+1}}^{1} M_{11} = -2$$

$$c = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

$(-1)^{1+2} M_{12} = -1(-3) = 3$

$\begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix} = 0 + 3 = 3$

CONTINUING THE EXAMPLE

3. Completed c matrix

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \qquad c = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

4. Find determinant of A

$$|A| = \begin{vmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{vmatrix} = 1$$

5. Apply formula to determine inverse of A

$$A^{-1} = \frac{[c_{ij}]^T}{|A|} = \frac{1}{1}\begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}^T = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

DENOTES DETERMINANT

OUR INTEREST IS WITH A VERY SPECIAL MATRIX INVERSE

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$\frac{Y(s)}{U(s)} = \frac{C \operatorname{adj}(sI - A)\, B}{|sI - A|} + D$$

A - λI

CHARECTERISTIC

Poles are determined by $|sI - A| = 0$ which is an eigenvalue problem!

WE CAN APPLY MATRIX INVERSES TO CONVERTING STATE-SPACE MODELS INTO TRANSFER FUNCTIONS

$\dot{x} = Ax + BU \longrightarrow$ INPUT

$y = Cx + DU \longrightarrow$ OUTPUT

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0$$

IF D IS NOT A ZERO, MULTIPLY IT BY A LOW PASS FILTER, OF HIGH VALUE

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$\lambda I - A = \begin{bmatrix} \lambda I - 0 & 1 \\ -k/m & \lambda I + \frac{b}{m} \end{bmatrix} \rightarrow \begin{bmatrix} \lambda & -1 \\ k/m & \lambda + b/m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ k/m & s + B/m \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$\begin{bmatrix} s & -1 \\ k/m & s+b/m \end{bmatrix} \quad |A| = s(s+b/m) - (-1)(k/m)$$

$$C_{ij} = (-1)^{i+j} M_{ij} \rightarrow C_{11} = (-1)^{1+1}(s+b/m)$$

$$= \begin{bmatrix} s+b/m & -k/m \\ 1 & s \end{bmatrix}^T$$

$$C_{12} = -(1)^{1+2}(k/m)$$

$$C_{21} = -(1)^{2+1}(-1)$$

$$\begin{vmatrix} s & -1 \\ k/m & s+b/m \end{vmatrix} = s(s+b/m) - (-1)(k/m)$$

$$C_{22} = -(1)^{2+2}(s)$$

$$s^2 + sb/m + k/m$$

$$\frac{Y(s)}{U(s)} = \frac{[1 \;\; 0]\begin{bmatrix} s+b/m & 1 \\ -k/m & s \end{bmatrix}\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}{s^2 + sb/m + k/m}$$

THE MATRIX INVERSION LEMMA IS OFTEN USEFUL WHEN PROVING STATE-SPACE FORMULATIONS

$$= \frac{1/m}{s^2 + b/m\, s + k/m} \cdot \frac{1}{ms^2 + bs + k}$$

$$(A + BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}$$

NOT GOING TO USE THIS

- Also called the Sherman–Morrison–Woodbury formula or Woodbury formula
- Used in Schur complement and Kalman filter

↳ WON'T DO THIS ↳ WILL TALK ABOUT THIS

LINEAR ALGEBRA PRIMER

EIGENVALUES AND EIGENVECTORS

RECALL THE SOLUTION OF A FIRST ORDER ODE

$$\dot{x} = ax \quad \Longrightarrow \quad x = ce^{at}$$

Now, for a system of first order differential equations,

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1 &= v_1 e^{\lambda t} \\ x_2 &= v_2 e^{\lambda t} \end{aligned}$$

$\dot{x}$

$$\begin{aligned} \lambda v_1 e^{\lambda t} &= e^{\lambda t}(a_{11}v_1 + a_{12}v_2) \\ \lambda v_2 e^{\lambda t} &= e^{\lambda t}(a_{21}v_1 + a_{22}v_2) \end{aligned}$$

$$\begin{aligned} \lambda v_1 &= a_{11}v_1 + a_{12}v_2 \\ \lambda v_2 &= a_{21}v_1 + a_{22}v_2 \end{aligned}$$

$$\lambda \boldsymbol{v} = A\boldsymbol{v}$$

Eigen vectors & Eigenvalues

THE EIGENVALUES OF STATE MATRIX A ARE THE ROOTS OF THE CHARACTERISTIC EQUATION

$$Ax = \lambda x$$

$$\lambda I x - Ax = 0$$

$$(\lambda I - A)x = 0$$

$$x = (\lambda I - A)^{-1} 0$$

$$|\lambda I - A| = 0$$

DETERMINENT

FREE RESPONSE $B = C = D = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda I - 0 & 0 - 1 \\ 0 - -2 & \lambda I - -3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix} X = 0$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0 \qquad \lambda(\lambda+3) + 2 = (\lambda+1)(\lambda+2) = 0$$

2 poles @ -1 & -2, THE SYSTEM IS STABLE

↳ 1ST ORDER NO IMAG. POLES

$$\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$

$\lambda_1 = -1$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_1 = -2$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

EIGENVECTORS

$AX' = \lambda X'$

PUT SYS. IN NATURAL MODE $\rightarrow \omega_n$

When initial conditions align with an eigenvector, the solution stays on that eigenvector, i.e., if the system starts in a given mode, it follows that mode to equilibrium.

$\hookrightarrow$ G. TOWARD ZERO FOR FREE RESPONSE

$\hookrightarrow$ EIGEN VECTOR REP. THE MODE

$\hookrightarrow$ FREQ

Eigenvectors diagonalize the A matrix.

$A = T \Lambda T^{-1}$

$$\hookrightarrow \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \end{bmatrix}$$

EIGENVECTORS

State vector written in terms of state variables:

$$x = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Represent state vector in terms of eigenvectors:

$$x = \alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_n x^n$$

Assume: $\|x^i\| = 1$: unit length

WEIGTED SUM OF THE VECTORS

EIGENVECTORS

$$\dot{x} = Ax$$

$$\frac{d}{dt}x = Ax \quad = \alpha_1 A x^1 + \alpha_2 A x^2 + \cdots + \alpha_n A x^n$$

$$Ax^i = \lambda_i x^i$$

$$\dot{x} = \lambda_1 \alpha_1 x^1 + \lambda_2 \alpha_2 x^2 + \cdots + \lambda_n \alpha_n x^n$$

If $x = \alpha_i x^i$ STATE IS ALLIGNED W/ THE EIGENVECTOR

$\dot{x} = \alpha_i \lambda_i x^i$ STATE VELOCITY IS ALLIGNED W/ THE EIGENVECTOR

LINEAR ALGEBRA PRIMER

MATRIX EXPONENTIAL

THE MATRIX EXPONENTIAL WILL BE USED TO CALCULATE FORCED AND FREE RESPONSE

$$\underbrace{\dot{x} = Ax \qquad x(t) = e^{At}x(0)}_{\text{FREE RESPONSE}}$$

- More detail coming next section

TO CALCULATE THE MATRIX EXPONENTIAL, ONE TRICK IS TO DIAGONALIZE THE MATRIX

← EIGENVALUE DIAG. MATRIX

$$A = T\Lambda T^{-1} = T\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} T^{-1}$$

$$\Downarrow$$

$$e^{At} = e^{T\Lambda T^{-1}t} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

- So how do we diagonalize a matrix?

WE NEED A STATE TRANSFORMATION TO DIAGONALIZE THE STATE MATRIX…

$$\dot{x} = Ax \qquad\qquad x = Tz$$

$$T\dot{z} = ATz$$
$$\dot{z} = T^{-1}ATz$$

- Choose T such that

$$T^{-1}AT = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- This is called a modal representation, poles appear on the diagonal and modes are decoupled

BUT HOW DO WE CHOOSE THE TRANSFORMATION MATRIX TO MAKE THIS WORK?

$$T^{-1}AT = \Lambda$$

$$\underbrace{AT = T\Lambda}$$

$$A\begin{bmatrix} v^1 & v^2 & \cdots & v^n \end{bmatrix} = \begin{bmatrix} v^1 & v^2 & \cdots & v^n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$T = \begin{bmatrix} v^1 & v^2 & \cdots & v^n \end{bmatrix}$$

- Assuming that the eigenvalues are real and distinct, choose T from the eigenvectors!

RECALL THE PREVIOUS EXAMPLE

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\lambda_1 = -1, \qquad x^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = -2, \qquad x^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$T = \begin{array}{c} \begin{array}{cc} X^1 & X^2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \end{array} \qquad T^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

flip & change sign

1 2

$$T^{-1}AT = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \left(\begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} ? \right)$$

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

?

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \Lambda$$

EIGENVALUES

DECAY @ e^{-1t} e^{-2t}

$$\dot{z} = \Lambda z$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$z_1(t) = z_{10} e^{-t}$$

$$z_2(t) = z_{20} e^{-2t}$$

$$x = Tz = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} =$$

$$x_1(t) = z_{10} e^{-t} + z_{20} e^{-2t}$$

$$z = T^{-1}x = \begin{bmatrix} 2x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix}$$

IN TERMS OF ICs

$$x_1(t) = (2x_{10} + x_{20})e^{-t} + (-x_{10} - x_{20})e^{-2t}$$

DOES THIS ALWAYS WORK?

Definitions

- The multiplicity of an eigenvalue is called the algebraic multiplicity
- The number of linearly independent eigenvectors corresponding to a single eigenvalue is called the geometric multiplicity

A matrix that has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity is called defective

- A defective matrix doesn't have enough eigenvectors
- A defective matrix is not completely diagonalizable
- But… a defective matrix can be almost diagonalized

CONSIDER A GENERALIZED EIGENVECTOR

$$AT = T\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$T = [v^1 \quad v^2]$$

$$A[v^1 \quad v^2] = [v^1 \quad v^2]\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = [\lambda v^1 \quad v^1 + \lambda v^2]$$

$$Av^1 = \lambda v^1$$

$$Av^2 = v^1 + \lambda v^2 \qquad \Rightarrow \quad (A - \lambda I)v^2 = v^1$$

THEN THE JORDAN CANONICAL FORM ALLOWS ALMOST DIAGONALIZATION

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

$$J = \begin{bmatrix} J_1(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_p(\lambda_p) \end{bmatrix}$$

$$l_i = m_i - (n - r_i) + 1$$

$$\boxed{A = T^{-1} J T}$$

EXAMPLE

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\lambda = -2, -2, -2$

X^1

X^2

X^3

THIS RESULTS IN THE FOLLOWING TRANSFORMATION MATRIX

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2X2 JORDAN BLOCK

$$J = T^{-1}AT = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$

1X1 JORDAN BLOCK

A MATRIX IS ALMOST DIAGONALIZED, DOES THIS WORK FOR THE MATRIX EXPONENTIAL?

$$e^{At} = e^{T\Lambda T^{-1}t} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

$$e^{At} = e^{TJT^{-1}t} = Te^{Jt}T^{-1} \quad = T\begin{bmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_n t} \end{bmatrix} T^{-1}$$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \cdots & \frac{t^{k-1}}{(k-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \cdots & \vdots \\ 0 & 0 & e^{\lambda t} & \cdots & \frac{t^2}{2}e^{\lambda t} \\ \vdots & \vdots & \vdots & \ddots & te^{\lambda t} \\ 0 & 0 & 0 & \cdots & e^{\lambda t} \end{bmatrix}$$

RETURN TO THE PREVIOUS EXAMPLE

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad J = T^{-1}AT = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$e^{At} = Te^{Jt}T^{-1}$$

$$= T\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}T^{-1}$$

COMING UP…

Solution of LTI State Equations

- State Transition Matrix
- Free Response
- Forced Response

Controllability

- Definition of Controllability
- Controllable Canonical Form
- Controllable Decomposition
- Stabilizability