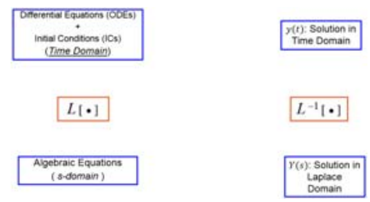
FROM LAST TIME…

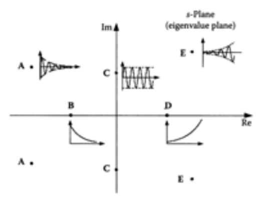
Transfer Functions

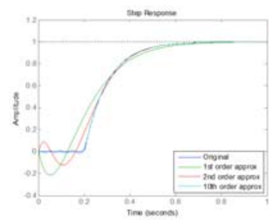
- Laplace transform and inverse Laplace transform
- Free vs. forced response
- Transfer function
- System stability
- Time delays



$$Y(s) = \boxed{[C(sI - A)^{-1}]x(0)} + \boxed{[C(sI - A)^{-1}B + D]U(s)}$$

Free Response   Forced Response





DYNAMIC RESPONSE OF LTI SYSTEMS

Topics

- What is a system response?
- Types of inputs
- Free vs. Forced Response
- Transient vs. Steady State Response

At the end of this section, students should be able to:

- Identify common input types.
- Distinguish between free and forced response.
- Distinguish between transient and steady-state response.

VIDEO

Robot Step Response

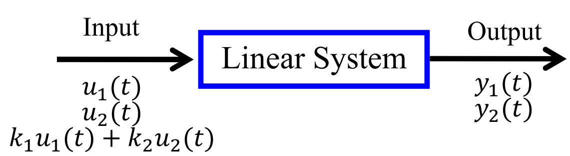
http://www.youtube.com/watch?v=DrTpdQl_Y9M

Questions to consider

- What is the purpose of the device?
- What type of input is being applied?
- How does the device move in response to the input?

RECALL: LINEAR SYSTEMS OBEY LINEAR SUPERPOSITION OF INPUTS

$$y^n + a_{n-1}y^{n-1} + \cdots + a_1\dot{y} + a_0 y = b_m u^m + b_{m-1}\, u^{m-1} + \cdots + b_1\dot{u} + b_0 u$$



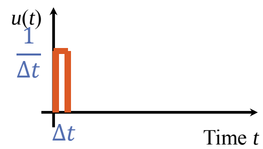
- The response of a linear system to a complicated input can be obtained by studying how the system responds to simple inputs, such as zero input, unit impulse, unit step, and sinusoidal inputs.

RECALL: INPUT TYPES

Initial Conditions

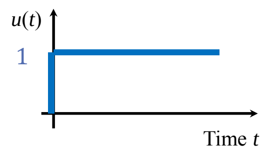
- Zero Input
- Some non-zero starting position/speed/velocity/etc.

Impulse



Step

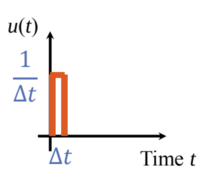
- Input is a constant that "turns on"



Sinusoidal

- Sine or Cosine

A TRANSFER FUNCTION IS THE LAPLACE TRANSFORM OF THE IMPULSE RESPONSE WITH ZERO ICS



Unit impulse is a Dirac delta:

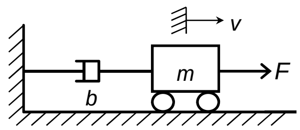
$$u(t) = \delta(t - c)$$

Let the signal begin at time 0:

$$U(s) = \int_0^{\infty} \delta(t - c) e^{-st} dt = e^{-cs} = 1$$

Giving the zero IC impulse response:

ACCELERATION OF A VEHICLE CAN BE MODELED AS THE STEP RESPONSE OF A FIRST ORDER SYSTEM



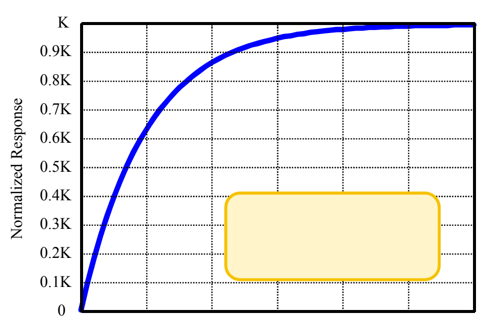
$$F(t) - bv = m\dot{v}$$

$$\frac{m}{b}\dot{v} + v = \frac{1}{b}F(t)$$

- Model the drivetrain and traction as a lumped force $F(t)$
- Model all sources of friction as a lumped damper b
- If $F(t)$ is an on-off condition, this is a *step response*

REWRITE OUR VEHICLE MODEL AS A "STANDARD" FIRST ORDER SYSTEM

$$\frac{m}{b}\dot{v} + v = \frac{1}{b}F(t) \quad \Longrightarrow \quad \tau\dot{v} + v = K \cdot f(t)$$



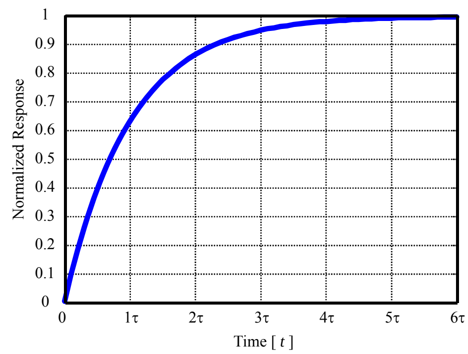
- τ : time constant
- K : static (steady-state, DC) gain

AFTER NORMALIZING, UNIT STEP RESPONSE CAN BE DISTINGUISHED BY SETTLING TIME

Normalize such that

$$t \to \infty, \qquad y_n \to 1$$

$$\Rightarrow y_n(t) = \frac{y(t)}{K} = \left(1 - e^{-\frac{t}{\tau}}\right)$$



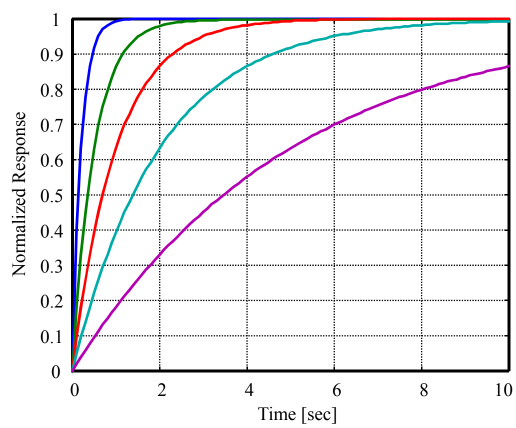
Time	τ	2τ	3τ	4τ	5τ
$(1 - e^{-t/\tau})$					

TIME CONSTANT AFFECTS BOTH SETTLING TIME AND INITIAL SLOPE

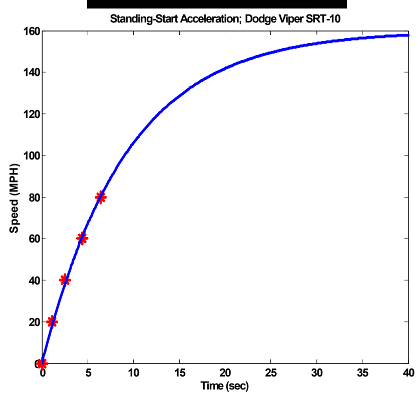
Slope at $t = 0$:

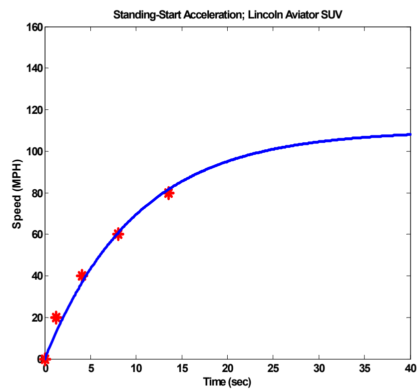
$$\frac{d}{dt} y_n(t) =$$

$$\frac{d}{dt} y_n(0) =$$

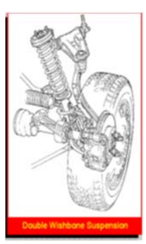


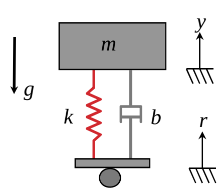
HOW DO THESE VEHICLES DIFFER BASED ON ACCELERATION DATA?



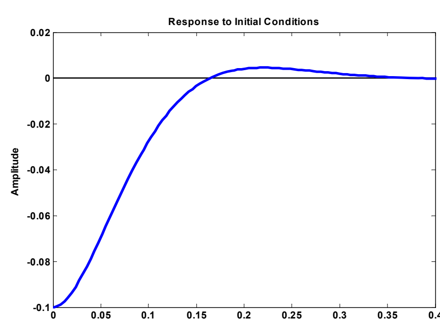


EXAMPLE: AUTOMOTIVE SUSPENSION





$$m\ddot{y} + b\dot{y} + ky = b\dot{r} + kr$$

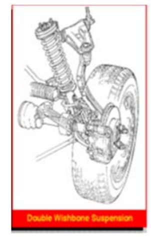


For free response:

$$m\ddot{y} + b\dot{y} + ky = 0$$
$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = 0$$
$$\ddot{y} + 28\dot{y} + 400y = 0$$

DYNAMIC RESPONSE OF 2ND ORDER SYSTEMS

Characteristic Equation:



$$\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{u} + b_0 u$$

$$s^2 + a_1 s + a_0 = 0$$

$$\Rightarrow \ s =$$

Free Response [$y_H(t)$]: ($u = 0$)

Determined by the roots of the characteristic equation:

- Real and Distinct [s_1 & s_2]: $y_H(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
- Real and Identical [$s_1 = s_2$]: $y_H(t) = A_1 e^{s_1 t} + A_2 t e^{s_1 t}$
- Complex [$s_{1,2} = \alpha \pm j\beta$]:

$$y_H(t) = e^{\alpha t}[A_1 \cos(\beta t) + A_2 \sin(\beta t)] = A e^{\alpha t}\cos(\beta t + \phi)$$

RESPONSE OF STABLE 2ND ORDER SYSTEM

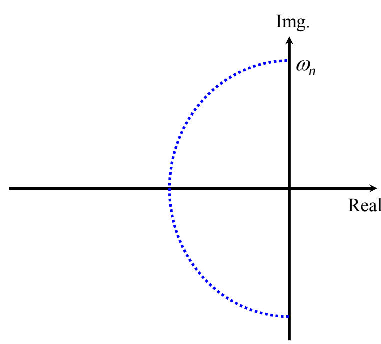
Stable 2nd Order System

$$\ddot{y} + a_1\dot{y} + a_0 y = bu \qquad \Rightarrow \qquad \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 u$$

- where $\omega_n > 0$: natural frequency [rad/s]
  $\zeta > 0$: damping ratio
  K : static (steady-state, DC) gain
- Pole locations

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\begin{cases} \zeta > 1: \\ \zeta = 1: \\ \zeta < 1: \end{cases}$$



RESPONSE OF STABLE 2ND ORDER SYSTEM

Unit Step response of under-damped 2nd order systems

(u = 1 and zero ICs)

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 u$$

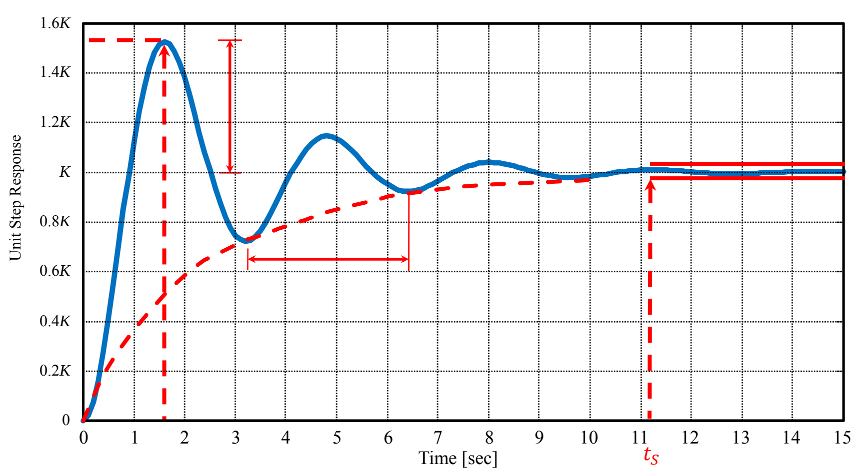
$$Y(s) = G(s)U(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$Y(s) = \frac{K}{s} + A \cdot \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} + B \cdot \frac{\omega_d^2}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$y(t) = K + Ae^{-\zeta\omega_n t}\cos(\omega_d t) + Be^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$\boxed{y(t) = K - \frac{K}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)}$$

UNIT STEP RESPONSE OF STABLE 2ND ORDER SYSTEM



STEP RESPONSE OF 2ND ORDER SYSTEM

Peak Time (t_P)

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

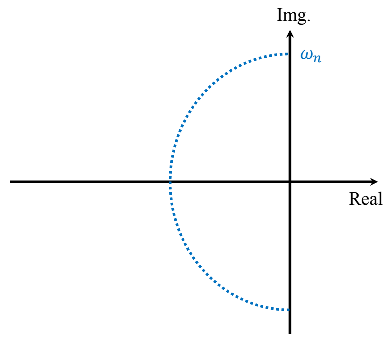
Percent Overshoot (%OS)

$$\%\mathrm{OS} = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Settling Time (t_S)

- Time required for the response to be within δ% of the final (steady-state) value:

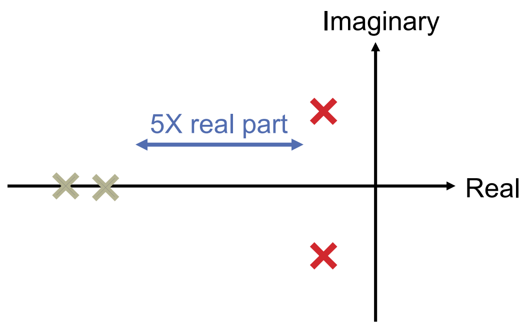
$$t_s = -\frac{1}{\zeta\omega_n}\ln\left(\frac{\delta}{100}\right)$$



DYNAMIC RESPONSE

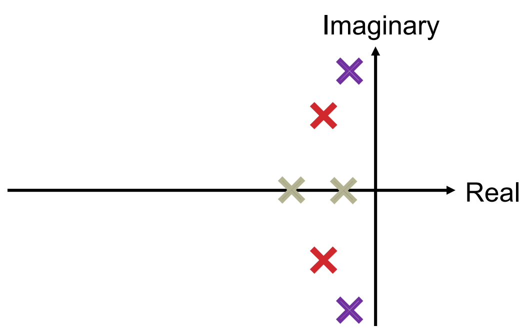
SYSTEM IDENTIFICATION (TIME RESPONSE)

IF A SYSTEM IS CLEARLY FIRST OR SECOND ORDER DOMINANT, SYSTEM ID IS EASY



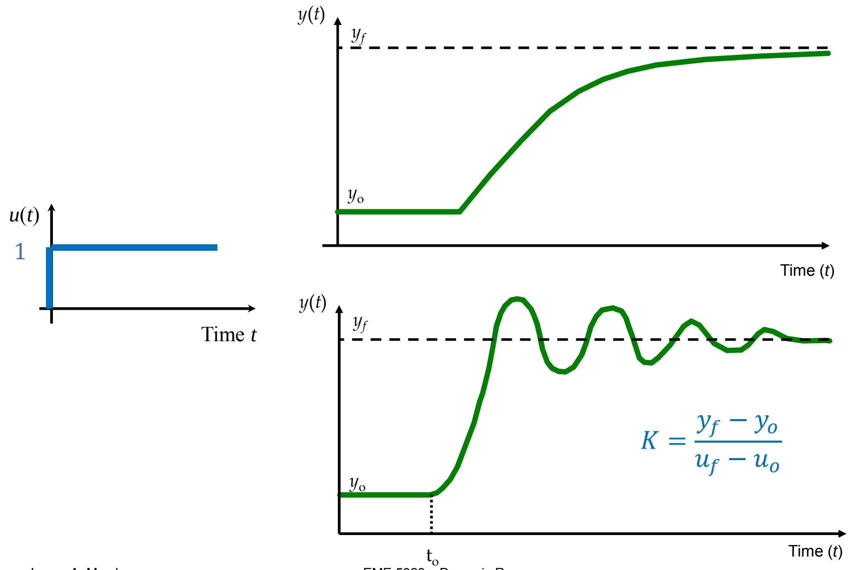
- Recall: for dominance, real part of poles must be separated by 5X
- If poles are dominant, remaining poles may be ignored (lumped into model error)

IF A SYSTEM IS NOT CLEARLY FIRST OR SECOND ORDER DOMINANT, SYSTEM ID IS NOT EASY



- Now what? More advanced techniques are needed

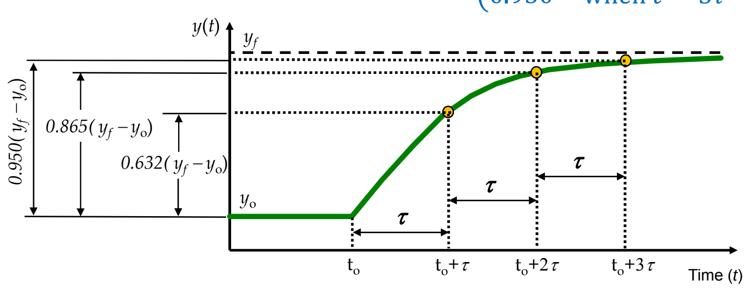
STATIC GAIN: RATIO OF STEADY-STATE CHANGE IN OUTPUT TO INPUT



$$K = \frac{y_f - y_o}{u_f - u_o}$$

FIRST ORDER STEP RESPONSE: 63.2% METHOD

$$\frac{\text{change in } y(t)}{\text{total change}} = \frac{y(t) - y_0}{y_f - y_0} = 1 - e^{-\left(\frac{t}{\tau}\right)} = \begin{cases} 0.632 & \text{when } t = \tau \\ 0.865 & \text{when } t = 2\tau \\ 0.950 & \text{when } t = 3\tau \end{cases}$$



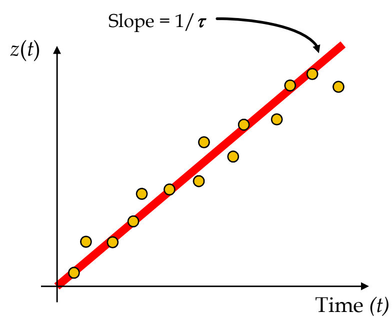
- Measure step response
- Determine when output moves 63.2% from initial to final
- Time constant is time difference from initial time

FIRST ORDER STEP RESPONSE: LOG-LIN METHOD

$$y(t) = y_0 + (y_f - y_0)\left[1 - e^{-\left(\frac{t}{\tau}\right)}\right]$$

$$e^{-\left(\frac{t}{\tau}\right)} = \frac{y_f - y(t)}{y_f - y_0}$$

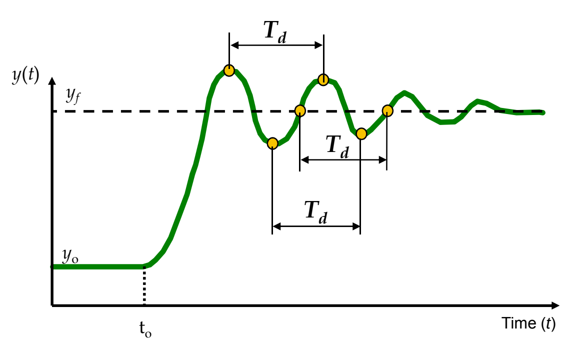
$$\left(-\frac{1}{\tau}\right)t = \ln\left(\frac{y_f - y(t)}{y_f - y_0}\right)$$



- Measure step response
- Plot $z(t)$ versus time and fit the line
- Slope is $1/\tau$

SECOND ORDER STEP RESPONSE: DAMPED NATURAL FREQUENCY

$$T_d \Rightarrow \qquad \omega_d = \frac{2\pi}{T_d} = \omega_n\sqrt{1-\zeta^2}$$

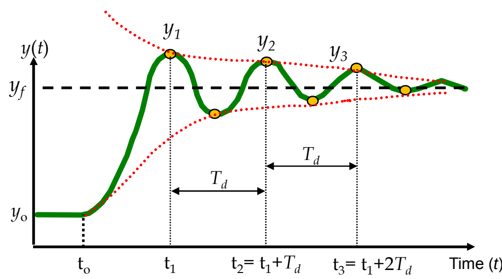


SECOND ORDER STEP RESPONSE: LOG DECREMENT METHOD

$$\begin{cases} y_i - y_f = \dfrac{(y_f - y_o)}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_i} \\ y_{i+n} - y_f = \dfrac{(y_f - y_o)}{\sqrt{1-\zeta^2}} e_n^{-\zeta\omega_n(t_i+nT_d)} \end{cases}$$

$$\delta = \frac{1}{n}\ln\left(\frac{y_i - y_f}{y_{i+n} - y_f}\right) = \zeta\omega_n T_d$$

$$= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$



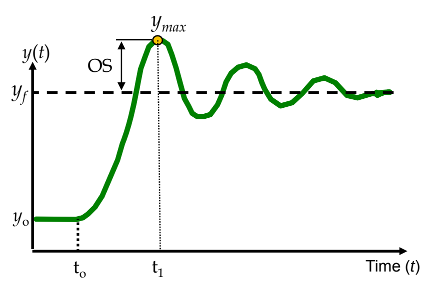
- Measure step response
- Determine deviation from steady-state at two peaks
- Solve for ζ

SECOND ORDER STEP RESPONSE: PERCENT OVERSHOOT METHOD

$$\mathrm{OS} = y_{max} - y_f = (y_f - y_o)\frac{e^{-\zeta\omega_n\frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin(\phi)$$

$$\mathrm{OS} = (y_f - y_o)e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{1}{\pi}\ln\left(\frac{y_f - y_o}{\mathrm{OS}}\right)$$

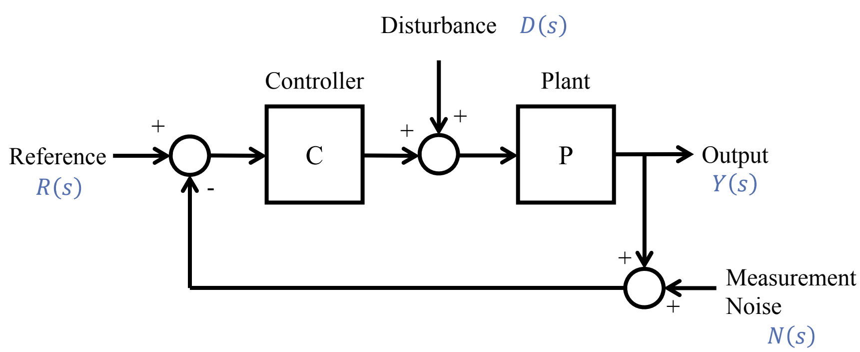


- Measure step response
- Measure overshoot
- Solve for ζ

DYNAMIC RESPONSE

EFFECTS OF SYSTEM ZEROS

RECALL OUR 1-DOF UNITY-FEEDBACK CONTROL LOOP



$$Y(s) = \frac{CP}{1+CP}R(s) + \frac{P}{1+CP}D(s) - \frac{CP}{1+CP}N(s)$$

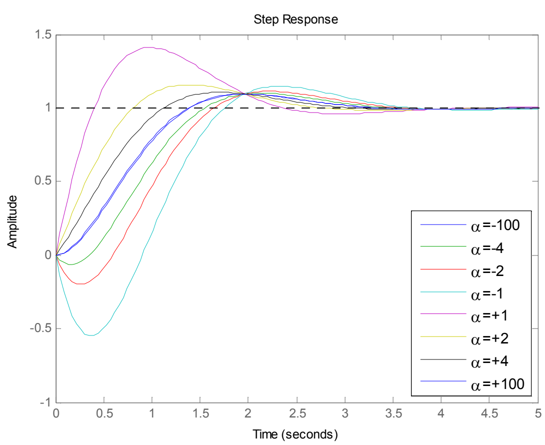
SPLIT CONTROLLER AND PLANT INTO NUMERATOR AND DENOMINATOR

$$Y(s) = \frac{CP}{1+CP}R(s) + \frac{P}{1+CP}D(s) - \frac{CP}{1+CP}N(s)$$

$$Y(s) = \frac{N_C N_P}{D_C D_P + N_C N_P}R(s) + \frac{N_P}{D_C D_P + N_C N_P}D(s) - \frac{N_C N_P}{D_C D_P + N_C N_P}N(s)$$

- Poles of closed-loop system are a combination of plant and controller poles (as desired)
- Zeros of closed-loop system are original plant and controller zeros

HOW DO ZEROS AFFECT UNIT STEP RESPONSE?



$$G(s) = \frac{\frac{1}{1.2\alpha}s + 1}{\frac{1}{4}s^2 + 0.6s + 1}$$

$$\Rightarrow \omega_n = 2, \quad \zeta = 0.6$$

poles:
$p_{1,2} = -1.2 \pm j1.6$

zeros:
$z_1 = -1.2\alpha$

Fast zero:

Slow zero:

RHP zero:

LHP zero:

USING MATLAB, TEST IT FOR YOURSELF

```
% clear the old variables and figures
close all
clear all
% setup the plant model
wn = 2;
ze = 0.6;
a = -1;
SYS = tf([1/(1.2*a) 1],[0.25 0.6 1]);
% draw the plant step response
figure;
step(SYS,5);
% add a controller (not a good design, just a
demonstration)
Kp = 10;
CL = feedback(Kp*SYS,1,-1);
% compare the step responses
figure;
step(SYS,CL);
% compare the poles and zeros
figure;
pzmap(SYS,CL);
```


RHP ZEROS AND UNDERSHOOT

Lemma 4.2

Assume a stable LTI system has TF with unity static gain and a RHP zero at $s = c > 0$. Let t_s be the $\delta\%$ band settling time of the system. Then, the unit step response exhibits an undershoot M_u satisfying

$$M_u = \frac{1-\delta}{e^{ct_s} - 1}$$

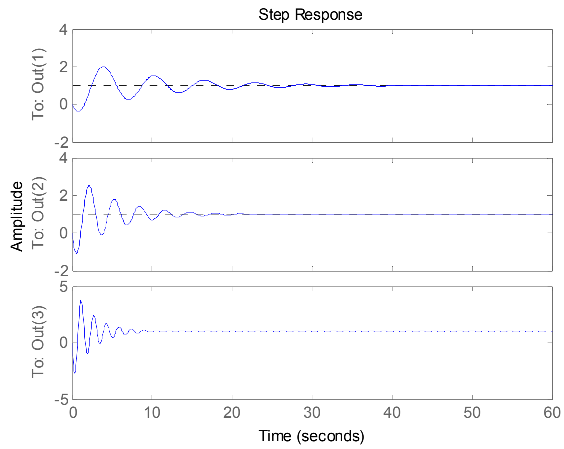
Proof: (See textbook)

Lemma 4.2 shows that, when a system has nonminimum-phase (RHP) zeros, there is a trade-off between having a fast step response and having small undershoot!

EXAMPLE OF LEMMA 4.2

$$G(s) = \frac{(-s+1)\omega_n^2}{s^2 + 0.2\omega_n s + \omega_n^2}$$

ω_n [rad/s]	t_s [sec]	M_u
1	39	1%
2	19	7%
4	10	26%



SLOW LHP ZEROS AND OVERSHOOT

Lemma 4.3

Assume a stable LTI system has TF with unity static gain and a LHP zero at $s = c < 0$. Further, assume that

- A1: The system has a dominant pole(s) with real part of $-p,\ p > 0$
- A2: The zero is much slower than the dominant pole(s), i.e., $\eta = \left|\frac{c}{p}\right| \ll 1$
- A3: Let K be a positive scalar satisfying $|v(t)| = |1 - y(t)| < Ke^{-pt}$, for all $t \geq t_s$ where y(t) is the unit step response

Then, the unit step response has an overshoot bounded below by

$$M_p \geq \frac{1}{e^{-ct_s} - 1}\left(1 - \frac{K\eta}{1-\eta}\right)$$

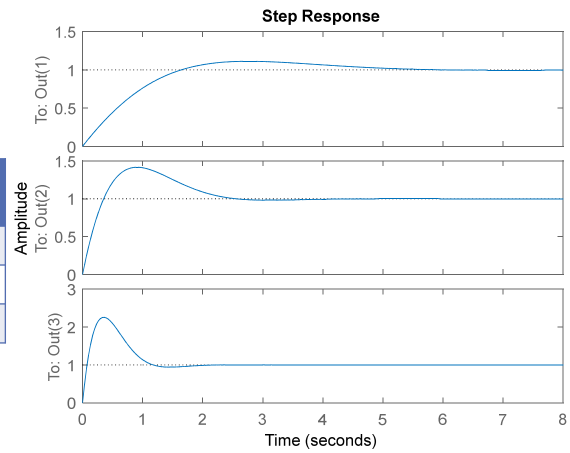
Proof: (see textbook)

Lemma 4.3 shows that, when a system has LHP slow zeros, there is a trade-off between having a fast step response and having small overshoot!

EXAMPLE OF LEMMA 4.3

$$G(s) = \frac{(s+1)\omega_n^2}{s^2 + 0.2\omega_n s + \omega_n^2}$$

ω_n [rad/s]	η	t_s [sec]	M_u
1	1.43	5.1	11%
2	0.71	2.4	42%
4	0.36	1.8	126%



DYNAMIC RESPONSE

TRANSIENT AND STEADY-STATE

TOTAL RESPONSE OF A STABLE LTI SYSTEM CAN BE DECOMPOSED INTO TWO PARTS

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 \dot{y} + a_0 y = b_m u^m + b_{m-1}\, u^{m-1} + \cdots + b_1 \dot{u} + b_0 u$$

$$\Downarrow$$

$$y(t) = \underbrace{y_T(t)}_{\substack{\text{Transient}\\ \text{Response}}} + \underbrace{y_{SS}(t)}_{\substack{\text{Steady}-\text{State}\\ \text{Response}}}$$

TRANSIENT AND STEADY STATE RESPONSES

Transient Response ($y_T(t)$)

- Contains the free response $y_{FREE}(t)$ of the system plus a portion of the forced response.
- Will decay to zero at a rate that is determined by the characteristic roots (poles) of the system.

Steady State Response ($y_{SS}(t)$)

- Will take the same form as the forcing input.
- Specifically, for a sinusoidal input, the steady state response will be a sinusoidal signal with the same frequency as the input but with different magnitude and phase.

EXAMPLE: TOTAL RESPONSE OF A STABLE SECOND ORDER SYSTEM TO STEP INPUT WITH ICS

$$\ddot{y} + 4\dot{y} + 3y = 6u \qquad \begin{array}{l} u(t) = 5 \\ \dot{y}(0) = 0 \\ y(0) = 2 \end{array}$$

$$[s^2Y(s) - sy(0) - \dot{y}(0)] + 4[sY(s) - y(0)] + 3Y(s) = 4U(s)$$

$$Y(s) = \frac{6}{s^2 + 4s + 3} \cdot U(s) + \frac{s + 4}{s^2 + 4s + 3} \cdot y(0)$$

$$Y(s) = \boxed{\frac{6}{(s+3)(s+1)} \cdot \frac{5}{s}} + \boxed{\frac{(s+4)}{(s+3)(s+1)} \cdot 2}$$

CONVERT THE TOTAL RESPONSE BACK TO TIME DOMAIN

$$Y(s) = \frac{6}{(s+3)(s+1)} \cdot \frac{5}{s} + \frac{(s+4)}{(s+3)(s+1)} \cdot 2$$

$$Y(s) = \frac{A_1}{s+3} + \frac{A_2}{s+1} + \frac{A_3}{s} + \frac{A_4}{s+3} + \frac{A_5}{s+1}$$

$$y(t) = \underbrace{\boxed{(A_1 + A_5)e^{-3t} + (A_2 + A_5)e^{-t}}}_{\text{Transient Response}} + \underbrace{\boxed{A_3}}_{\text{Steady-State Response}}$$

- Transient response combines free response and part of forced response
- Steady state contains only part of forced response

STEADY STATE RESPONSE

Final Value Theorem (FVT)

- Given a signal's LT $F(s)$, if **the poles of $sF(s)$ all lie in the LHP (stable region)**, then $f(t)$ converges to a constant value $f(\infty)$. $f(\infty)$ can be obtained without knowing $f(t)$ by using the FVT:

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Ex: A model of a linear system is determined to be:

$$\ddot{y} + 4\dot{y} + 12y = 4\dot{u} + 3u$$

- If a constant input $u = 5$ is applied at $t = 0$, will the output $y(t)$ converge to a constant value?
- If the output converges, what will be its steady state value?

COMING UP…

Frequency Response

- Forced response to sinusoidal inputs
- Frequency response of LTI systems
- Bode plots
- Modeling errors in frequency domain

Analysis of Feedback Systems

- Classical feedback controller structure
- Nominal sensitivity functions
- Stability of nominal feedback system